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ORDER STRUCTURES AND FIXED POINTS

by

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Contents

Introduction

It is the purpose of these lectures to exhibit the importance of order structures in nonlinear functional analysis and some of its applications. To do this, we prove in the first section an almost trivial fixed point theorem for increasing maps in rather general ordered sets. In Section 2 we show how this general result can be used to simplify and unify the proofs of a variety of well known and seemingly unrelated fixed point theorems.

In Section 3 we mix the order with topological structures and prove some new fixed point theorems for not necessarily continuous maps which have relatively compact image or are condensing. In Section 4 we demonstrate the usefulness of these results for applications. As an example we prove a general existence theorem for quasilinear two-point boundary value problems with discontinuous nonlinearities.

In Section 5, finally, we give some bibliographical references and add some remarks concerning extensions and related results.

It is not the purpose of these lectures to give the most general results or the best possible applications. In particular, in Section 2 we have usually restricted ourselves to simple situations although more general results are known and could be obtained along the given lines. Moreover, it is certainly possible to use the basic fixed point theorems in ordered sets, given in Section 1 and 3, in a variety of other situations, both to simplify and unify known proofs and to obtain new results. We leave this to the interested reader (whose attention we direct to the remarks and hints in Section 5).

1. Fixed Point Theorems in Ordered Sets

Let X be a nonempty set. A binary relation $\leq$ in X is called an order on X provided it is reflexive, transitive, and antisymmetric, that is,

(i) $x \leq x$ for all $x \in X$,

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$,

(iii) $x \leq y$ and $y \leq x$ implies $x = y$.

A nonempty set X together with an order $\leq$ on X is called an ordered set and denoted by $(X, \leq)$, or simply by X.

We often write $x \geq y$ instead of $y \leq x$, and $x < y$ or $y > x$ means that $x \leq y$ and $x \neq y$.

Let X be an ordered set. An element $m \in X$ is called a maximal element of X if $x \geq m$ implies $x = m$, and $g \in X$ is a greatest element of X if $x \leq g$ for all $x \in X$. It is clear that there exists at most one greatest element of X, and that every greatest element is a maximal element (but not conversely). Minimal elements and the least element of X are defined by reversing the above inequalities.

Let A be a nonempty subset of some ordered set X. An element $x \in X$ is called an upper bound for A if $a \leq x$ for all $a \in A$, and x is called a lower bound for A if $x \leq a$ for all $a \in A$. If x is an upper [resp., lower] bound for A, then we often write $A \leq x$ [resp., $x \leq A$]. The set A

is said to be bounded above [resp., bounded below] if there exists an upper bound [resp., lower bound] for A. It is called order bounded if it is bounded above and below.

If A is bounded above [resp., below] and if the set of all upper [resp., lower] bounds for A possesses a least [resp., greatest] element, then this (necessarily unique) element is called the supremum of A [resp., infimum of A] and denoted by $\sup(A)$ [resp., $\inf(A)$].

Example 1: For every set X, we denote by $\mathcal{P}(X)$ the power set of X, that is, the set of all subsets of X. For every nonempty subset $\mathcal{A}$ of $\mathcal{P}(X)$, we define the natural order on $\mathcal{A}$ by " $A \leq B$ iff $A \subset B$ ". It is easily verified that this is in fact an order on $\mathcal{A}$, and $\mathcal{A}$ is often said to be ordered by inclusion.

It is clear that $\emptyset$ is the least and X is the greatest element of $\mathcal{P}(X)$. Moreover, every nonempty subset $\mathcal{B}$ of $\mathcal{P}(X)$ has an infimum and a supremum in $\mathcal{P}(X)$, namely $\inf(\mathcal{B}) = \cap\mathcal{B} := \cap\{B \mid B \in \mathcal{B}\}$ and $\sup(\mathcal{B}) = \cup\mathcal{B}$ $:= \cup\{B \mid B \in \mathcal{B}\}$. □

Example 2: Let X be a set containing at least two elements, and let $\mathcal{P}^*(X)$ be the family of all nonempty subsets of X, ordered by inclusion. Then X is the greatest element of $\mathcal{P}^*(X)$, but $\mathcal{P}^*(X)$ does not possess a least element. On the other hand, for every $x \in X$, the singleton $\{x\}$ is a minimal element of $\mathcal{P}^*(X)$. If $A,B \subset X$ are nonempty and disjoint, then the subset $\mathcal{A} := \{A,B\}$ of $\mathcal{P}^*(X)$ has a supremum, namely $\sup(\mathcal{A}) = A \cup B$, but no infimum, in fact, not even a lower bound in $\mathcal{P}^*(X)$. □

A nonempty subset C of an ordered set X is called a <u>chain</u> if it is <u>totally ordered</u>, that is, for every pair $x,y \in C$, either $x \le y$ or $y \le x$.

The basic existence theorem for maximal elements is

<u>Zorn's Lemma:</u> *If every chain in an ordered set X has an upper bound, then X has at least one maximal element.*

Clearly, by defining a new order "$\precsim$" on X by letting $x \precsim y$ iff $y \le x$, it follows that an ordered set X has at least one minimal element provided each chain in X has an infimum. Finally, we recall that Zorn's lemma is equivalent to the axiom of choice and to Zermelo's well-ordering theorem (e.g. Dugundji [1]).

Let X be an ordered set. For every $a \in X$, we let

$$S_+(a) := \{x \in X \mid a \le x\}$$

and

$$S_-(a) := \{x \in X \mid x \le a\} .$$

The sets $S_+(a)$ and $S_-(a)$ are called the <u>right</u> and the <u>left</u> <u>section at a</u>, respectively. Moreover, for every pair $a,b \in X$, the order interval $[a,b]$ (between a and b) is defined by

$$[a,b] := S_+(a) \cap S_-(b) = \{x \in X \mid a \le x \le b\} \quad .$$

Thus $[a,b] \neq \emptyset$ iff $a \le b$.

Let X and Y be ordered sets and denote the order in either set by the same symbol $\le$. A function $f : X \to Y$ is called increasing if $x \le y$ implies $f(x) \le f(y)$, and decreasing if $x \le y$ implies $f(x) \ge f(y)$. Moreover, f is called strictly increasing [resp., strictly decreasing] if $x < y$ implies $f(x) < f(y)$ [resp., $f(x) > f(y)$].

The following trivial observation will frequently be used in the remainder of these notes.

(1.1) *Let* $f : X \to X$ *be increasing. Then*

$$x_o \le f(x_o) \quad \textit{implies} \quad f(S_+(x_o)) \subset S_+(x_o)$$

and

$$f(x_o) \le x_o \quad \textit{implies} \quad f(S_-(x_o)) \subset S_-(x_o) \quad .$$

Thus, $a \le f(a)$ *and* $f(b) \le b$ *imply*

$$f([a,b]) \subset [a,b] \quad .$$

After these preparations we turn to the proofs of the basic fixed point theorems for ordered sets.

(1.2) *Proposition: Let* X *be an ordered set such that every chain has an upper bound [resp., lower bound], and let* $f : X \to X$ *be a map such that* $x \le f(x)$ [*resp.,* $f(x) \le x$] *for every* $x \in X$. *Then* f *has at least one fixed point, that* *is, there exists an element* $y \in X$ *such that* $f(y) = y$.

Proof: Zorn's lemma implies the existence of a maximal [resp., minimal] element m of X . Then $m \le f(m)$ [resp., $f(m) \le m$] and, consequently, $m = f(m)$. □

(1.3) *Remark:* Suppose that X has the stronger property, that every chain has a supremum [resp., infimum] . Then the existence of a fixed point of f (under the assumption that $x \le f(x)$ [resp., $f(x) \le x$] for all $x \in X$, of course) can be proved without using Zorn's lemma or, equivalently, the axiom of choice. This is Zermelo's fixed point theorem on which he based his second proof of the well-ordering theorem (cf. Zermelo [1] or also Theorem I.2.5 in Dunford-Schwartz [1]).It should be observed that for the proof of our basic result, namely the following Theorem (1.4), Zermelo's fixed point theorem is strong enough. Hence, by using in the following proof Zermelo's fixed point theorem instead of Proposition (1.2), one obtains "constructive" proofs for all of the results given in these notes. Here "constructive" means: without using the axiom of choice. Since we do not try to avoid the axiom of choice, we have based our proofs on Proposition (1.2) which is a trivial consequence of Zorn's lemma. □

The following theorem is the fundamental result of this section. In fact, all fixed point theorems given in these notes will be based on it.

(1.4) *Theorem: Let X be an ordered set such that every chain has an infimum [resp., supremum], and let $f : X \to X$ be increasing. Suppose that there exists an element $x_o \in X$ such that $f(x_o) \le x_o$ [resp., $x_o \le f(x_o)$]. Then f has a greatest [resp., least] fixed point in $S_-(x_o)$ [resp., $S_+(x_o)$].*

Proof: We consider only the first case. The second case is treated analogously.

(a) Existence of a fixed point in $S_-(x_o)$: Let

$$X_- := \{x \in X \mid f(x) \le x\} \cap S_-(x_o) \quad ,$$

and observe that X_- is not empty ($x_o \in X$) and $f(X_-) \subset X_-$. Moreover, if C is a chain in X_-, it has an infimum a in X. Then $a \le C$ implies $f(a) \le f(c) \le c$ for all $c \in C$, that is, $f(a) \le C$. Thus $f(a) \le a$, since a is the greatest lower bound of C in X. This shows that every chain in X_- has an infimum in X_-, and Proposition (1.2), applied to X_-, gives

$$F_- := \{x \in X \mid f(x) = x\} \cap S_-(x_o) \neq \emptyset \quad .$$

(b) Existence of a greatest fixed point in $S_-(x_o)$: Let

$$Y_- := \{y \in X_- \mid F_- \le y\} \quad ,$$

and observe that Y_- is well-defined (by (a)), nonempty ($x_o \in Y_-$), and $f(Y_-) \subset Y_-$. Moreover, it is easily verified that every chain in Y_- has an infimum in Y_-. Hence Proposition (1.2) implies the existence of a fixed point of f in Y_- which, by the very definition of Y_-, is the greatest fixed point of f in $S_-(x_o)$. □

In the following, an ordered set is called chain complete if every chain has an infimum and a supremum.

(1.5) *Corollary: Let* X *be a chain complete ordered set and let* $f : X \to X$ *be increasing. Suppose that there exist points* $\check{y}, \hat{y} \in X$ *such that* $\check{y} \leq \hat{y}$, $\check{y} \leq f(\check{y})$ *, and* $f(\hat{y}) \leq \hat{y}$ *. Then* f *has a least and a greatest fixed point in* $[\check{y}, \hat{y}]$.

Proof: Apply Theorem (1.4) to the chain complete ordered set $[\check{y}, \hat{y}]$. □

2. Some Applications

A. Tarski's fixed point theorem

An ordered set X is called a lattice if, for every pair $x,y \in X$, there exist $\sup\{x,y\}$ and $\inf\{x,y\}$. A lattice is called complete if every nonempty subset possesses a supremum and an infimum. It should be observed that every complete lattice possesses a least and a greatest element.

Example 1: The power set $\mathcal{P}(X)$ of an arbitrary set X is a complete lattice with respect to the natural order. □

Example 2: The natural numbers $\mathbb{R}$ with their natural order form a lattice such that every order bounded subset has an infimum and a supremum, that is, $\mathbb{R}$ is a Dedekind complete lattice. Hence every order interval (that is, every closed bounded interval) in $\mathbb{R}$ is a complete lattice. □

Example 3: Let $(X,\mathcal{A},\mu)$ be a σ-finite measure space and denote by $L_p(X,\mathcal{A},\mu)$, $1 \le p \le \infty$, the standard Lebesgue spaces of (equivalence classes of) measurable real-valued functions on X. Define the natural order in $L_p(X,\mathcal{A},\mu)$ by: "$f \le g$ iff $f(x) \le g(x)$ for μ-almost all $x \in X$". Then $L_p(X,\mathcal{A},\mu)$ is a Dedekind complete lattice (e.g. Dunford-Schwartz [1]). In particular, every order interval in $L_p(X,\mathcal{A},\mu)$ is a complete lattice. □

As an immediate consequence of Corollary (1.5), we obtain the following fixed point theorem, due to Tarski, and sometimes called the Birkhoff-Tarski theorem.

(2.1) *Theorem (Tarski): Let* X *be a complete lattice and let* $f : X \to X$ *be increasing. Then* f *possesses a least and a greatest fixed point.*

B. Generalized Contractions

Let $X = (X,d)$ be a metric space. A map $f : X \to X$ is called a generalized contraction if there exists a constant $\gamma < 1$ such that $\operatorname{diam}(f(M)) \leq \gamma \operatorname{diam}(M)$ for every bounded subset M of X with $f(M) \subset M$. Here $\operatorname{diam}(M)$ denotes the diameter of M, that is, $\operatorname{diam}(M) = \sup\{d(x,y) \mid x,y \in M\}$.

Example 1: Every Banach contraction, that is, every function $f : X \to X$ satisfying $d(f(x),f(y)) \leq \gamma\, d(x,y)$ for all $x,y \in X$ and some $\gamma < 1$, is a generalized contraction. □

Example 2: Every quasi-contraction (as studied for example by Lj. B. Ćirić [1]) is a generalized contraction, where $f : X \to X$ is called a quasi-contraction if there exists a constant $\gamma < 1$ such that

$$d(f(x),f(y)) \leq \gamma \max\{d(x,y),d(x,f(x)),d(y,f(y)),d(x,f(y)),d(y,f(x))\}$$

for all $x,y \in X$.

More generally, if there exists a constant $\gamma < 1$ and a natural number n such that

$$d(f(x),f(y)) \leq \gamma \max_{i,j,k,l \leq n} \{d(f^i(x),x), d(f^j(y),y), d(f^k(x),y), d(f^l(y),x)\}$$

for all $x,y \in X$, then f is a generalized contraction. □

(2.2) Theorem: *Every generalized contraction on a nonempty, bounded, complete metric space has a unique fixed point.*

Proof: Since $Fix(f)$, the set of all fixed points of f , is invariant under f , it follows that $diam(Fix(f)) = 0$. Hence f has at most one fixed point.

Let

$$\mathcal{X} := \{A \subset X \mid \emptyset \neq A = \bar{A} \text{ and } f(A) \subset A\} ,$$

and define a binary relation $\leq$ on $\mathcal{X}$ by

$$A \leq B \text{ if either } A = B \text{ or } A \subset \overline{f(B)} .$$

Then it is easily verified that $\mathcal{X} = (\mathcal{X},\leq)$ is an ordered set $(X \in \mathcal{X})$.

We claim now that every chain $\mathcal{C}$ in $\mathcal{X}$ has an infimum. Since this is obvious if $\mathcal{C}$ has a least element, we can assume that $\mathcal{C}$ does not possess a least element. Then, for every $C \in \mathcal{C}$, there exists a $D \in \mathcal{C}$ such that $D < C$, hence $D \subset \overline{f(C)}$. Consequently,

$$\begin{aligned} diam(D) &\leq diam\ \overline{f(C)} = diam(f(C)) \\ &\leq \gamma\ diam(C) \leq \gamma\ diam(X) , \end{aligned}$$

and, by induction, for each $n \in \mathbb{N}$, we can find a set $C_n \in \mathscr{C}$ such that

$$\operatorname{diam}(C_n) \leq \gamma^n \operatorname{diam}(X) .$$

This implies that $\mathscr{C}$ is the base of a Cauchy filter. Thus, by completeness, there exists an element $x \in X$ such that $\mathscr{C} \to x$, which implies, in particular, that $x \in \cap \mathscr{C}$ (since each $C \in \mathscr{C}$ is closed). Moreover, $\cap \mathscr{C} = \{x\}$ since $\operatorname{diam}(C_n) \to 0$ as $n \to \infty$. Hence

$$\{f(x)\} = f(\cap \mathscr{C}) \subset \cap\{f(C) \mid C \in \mathscr{C}\} \subset \cap \mathscr{C} = \{x\} ,$$

which shows that

$$f(\cap \mathscr{C}) = \cap \mathscr{C} \in \mathscr{K} .$$

Since, for every $C \in \mathscr{C}$, there exists a $D \in \mathscr{C}$ with $D \subset \overline{f(C)}$, it follows that

$$\cap \mathscr{C} \subset D \subset \overline{f(C)}$$

for all $C \in \mathscr{C}$, that is, $\cap \mathscr{C} \leq \mathscr{C}$. On the other hand, if $B \in \mathscr{K}$ is a lower bound for $\mathscr{C}$, it follows that $B \subset \overline{f(C)}$ for all $C \in \mathscr{C}$, and consequently,

$$B \subset \cap \{\overline{f(C)} \mid C \in \mathscr{C}\} \subset \cap \mathscr{C} = f(\cap \mathscr{C}) ,$$

that is, $B \leq \cap \mathscr{C}$. Thus $\cap \mathscr{C} = \inf(\mathscr{C})$ in $\mathscr{K}$, and every chain in $\mathscr{K}$ has an infimum.

For each $A \in \mathscr{K}$, let $\phi(A) := \overline{f(A)}$. Since $\overline{f(A)} \subset A$ (by the closedness of A), it follows that

$$f(\phi(A)) = f(\overline{f(A)}) \subset f(A) \subset \overline{f(A)} = \phi(A) .$$

Thus ϕ maps $\mathscr{K}$ into itself. Moreover, if $A < B$, then $A \subset \overline{f(B)}$ implies

$$\phi(A) = \overline{f(A)} \subset \overline{f(\overline{f(B)})} = \overline{f(\phi(B))} ,$$

that is, $\phi(A) \leq \phi(B)$. Thus ϕ is increasing.

Consequently, since obviously $\phi(X) \leq X$, we can apply Theorem (1.4) to deduce the existence of a nonempty subset A of X such that $A = \overline{f(A)}$. Hence

$$\mathrm{diam}(A) = \mathrm{diam}(\overline{f(A)}) = \mathrm{diam}(f(A)) \leq \gamma\ \mathrm{diam}(A) ,$$

and, consequently, $\mathrm{diam}(A) = 0$. Thus $A = \{a\} = \overline{f(\{a\})} = \{f(a)\}$, which shows that a is a fixed point of f. □

C. Condensing Maps

Let $X = (X,d)$ be a metric space. The (Kuratowski) measure of noncompactness α in X is defined, for every nonempty bounded subset B of X, by

$$\alpha(B) := \inf\{\varepsilon > o \mid B \text{ can be covered by finitely many subsets of } X \text{ of diameter} \leq \varepsilon\} .$$

It is not too difficult to show that α has the following properties (where A and B denote nonempty bounded subsets of X):

(i) $\alpha(A) = o$ iff A is totally bounded.

(ii) $\alpha(A) = \alpha(\overline{A})$.

(iii) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$.

(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(v) If X is a normed vector space, then $\alpha(\mathrm{co}(A)) = \alpha(A)$, where $\mathrm{co}(A)$ denotes the convex hull of A.

A map $f : X \to X$ is called condensing if, for every bounded subset B of X with $\alpha(B) > 0$,

$$\alpha(f(B)) < \alpha(B) \quad .$$

(It should be observed that f is not supposed to be continuous!)

Example 1: Every Banach contraction is condensing.

Example 2: Let X be a Banach space and suppose that $f : X \to X$ satisfies $f = g + h$, where g is a Banach contraction and h is relatively compact on bounded sets (that is, for every bounded set $B \subset X$, the set $\overline{h(B)}$ is compact). Then f is condensing. (For a proof of these facts and for many related results cf. the papers by Sadovskii [1] and Browder [1].)

The following theorem is the basic fixed point theorem in the theory of condensing maps. It generalizes, in some sense, Banach's contraction mapping principle as well as Schauder's fixed point theorem.

(2.3) Theorem (Sadovskii): *Let X be a nonempty, closed, bounded, and convex subset of some Banach space, and suppose that $f : X \to X$ is condensing and continuous. Then f has a fixed point.*

Proof: (a) Let $x_o \in M$ be arbitrary and let $M := \{f^k(x_o) \mid k \in \mathbb{N}\}$. Then $f(M) \cup \{x_o\} = M$ and, consequently,

$\alpha(M) = \alpha(f(M))$, which shows that $\alpha(M) = 0$, that is, M is relatively compact. Since $f(M) \subset M$, the continuity of f implies $f(\bar{M}) \subset \bar{M}$.

Let

$$\mathcal{K}_0 := \{A \subset X \mid \emptyset \neq A = \bar{A} \subset \bar{M}\}$$

and order $\mathcal{K}_0$ by inclusion. Then, due to the compactness of $\bar{M}$, every chain $\mathcal{C}$ in $\mathcal{K}_0$ has an infimum, namely $\cap \mathcal{C}$ (by the finite intersection property).

For every $A \in \mathcal{K}_0$, let $\phi(A) := \overline{f(\bar{A})}$. Then ϕ maps $\mathcal{K}_0$ into itself, is increasing, and satisfies $\phi(\bar{M}) \leq \bar{M}$. Hence Theorem (1.4) implies the existence of a nonempty subset B_0 of X such that $B_0 = \overline{f(B_0)}$.

(b) Let now

$$\mathcal{K} := \{A \subset X \mid B_0 \subset A = \bar{A}\} \quad ,$$

ordered by inclusion. Then $\mathcal{K} \neq \emptyset$, and every chain $\mathcal{C}$ in $\mathcal{K}$ has an infimum (namely $\cap \mathcal{C}$) . For every $A \in \mathcal{K}$, let $\psi(A) := \overline{co}\, f(A)$, where $\overline{co}(\ldots)$ denotes the closed convex hull. Since

$$\psi(B_0) = \overline{co\, f(B_0)} \geq \overline{f(B_0)} = B_0 \quad .$$

and $\psi(X) \leq X$, it is easily seen that ψ is an increasing self-map of $\mathcal{K}$. Hence, by Theorem (1.4), there exists a nonempty set $A \subset X$ such that $A = \overline{co}\, f(A)$. Consequently,

$$\alpha(A) = \alpha(\overline{co}\, f(A)) = \alpha(f(A)) \quad ,$$

which shows that A is compact and convex. Since, moreover, $f(A) \subset A$, we can apply Schauder's fixed point theorem to $f|A$. This proves the existence of a fixed point of f . □

D. Nonexpansive Maps

Let X be a nonempty subset of some Banach space E. A map $f : X \to E$ is called nonexpansive if

$$\|f(x)-f(y)\| \leq \|x-y\|$$

for all $x,y \in X$.

The following example shows that, in general, a nonexpansive selfmap of the closed unit ball of a Banach space does not have a fixed point.

Example 1: Let $E = c_o$, the Banach space of all o-sequences, endowed with the usual maximum norm, that is, $\|x\| = \max_{k \in \mathbb{N}} |x_k|$, where $x = (x_k)$ and $x_k \to o$ as $k \to \infty$. For every $x \in E$ with $\|x\| \leq 1$, let

$$f(x) := (1-\|x\|, x_o, x_1, x_2, \ldots) \quad ,$$

and observe that $\|f(x)\| \leq 1$ and that f is nonexpansive. Suppose that $x = f(x)$. Then it follows that $x_k = 1 - \|x\|$ for all $k \in \mathbb{N}$, which is impossible. □

Consequently, in order to guarantee the existence of a fixed point of a nonexpansive map f, one has to require some additional structur for the domain of f. For this purpose we introduce the following definition: Let X be a nonempty, closed convex subset of some Banach space. Then X is said to have normal structure if, for every closed convex subset M of X containing at least two points, there exists a point $m_o \in M$ such that

$$\sup_{m \in M} \|m-m_o\| < \operatorname{diam}(M) \quad .$$

Example 2 (Browder, Edelstein): Every nonempty, closed, bounded, convex subset of a uniformly convex Banach space has normal structure.

Proof: Recall that a Banach space E is uniformly convex if, for every $\varepsilon > o$, there exists a positive number $\delta(\varepsilon)$ such that $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$ imply $\|x+y\| \leq 2(1-\delta(\varepsilon))$.

Suppose now that M is closed and convex such that $d := \mathrm{diam}(M) > o$. Let $x,y \in M$ with $x \neq y$ be fixed, and let $\varepsilon := \|x-y\|/d$. Then $m_o := (x+y)/2 \in M$ and

$$\frac{m-m_o}{d} = \frac{1}{2}\left\{\frac{m-x}{d} + \frac{m-y}{d}\right\}$$

for every $m \in M$. Since $\|(m-x)/d\| \leq 1$, $\|(m-y)/d\| \leq 1$, and $\|\frac{m-x}{d} - \frac{m-y}{d}\| = \|\frac{x-y}{d}\| = \varepsilon$, the uniform convexity implies the existence of a positive number $\delta(\varepsilon)$ such that

$$\|m-m_o\| \leq d(1-\delta(\varepsilon)) < \mathrm{diam}(M)$$

for all $m \in M$. □

We use now Theorem (1.4) to give a simple proof of the basic fixed point theorem for condensing maps.

(2.4) Theorem (Kirk): *Every nonexpansive selfmap of a nonempty, weakly compact, convex subset X of a Banach space has a fixed point, provided X has normal structure.*

Proof: (a) Let K be a nonempty, closed, convex subset of X. For every $x \in K$, let

$$r_x(K) := \sup_{y \in K} \|x-y\| ,$$

and let

$$r(K) := \inf_{x \in K} r_x(K) .$$

Then $r_x(K)$ is the radius of the smallest closed ball with center at x, containing K, and $r(K)$ is the radius of the smallest closed ball with center in K, containing K. Finally, let

$$K_c := \{x \in K \mid r_x(K) = r(K)\} .$$

It is clear that, for every $m \in \mathbb{N}$, there exists a point $x_m \in K$ such that

$$\|x-x_m\| \leq r(K) + 1/m$$

for all $x \in K$. Since K is closed and convex, it is weakly closed, hence weakly compact, since $K \subset X$ and X is weakly compact. Therefore, by the Eberlein-Smulian theorem (e.g. Dunford-Schwartz [1]), K is weakly sequentially compact. Hence there exists a subsequence (x_{m_k}) of (x_m) converging weakly to some $x^* \in K$ as $k \to \infty$. Since for every $x \in K$, the function $y \to \|x-y\|$ is weakly lower sequentially continuous, it follows that

$$\|x^*-x\| \leq r(K) \quad \text{for all } x \in K .$$

Hence $x^* \in K_c$ and $K_c \neq \emptyset$.

Observe that, for every $y \in K$, the set

$$K(y) := \{x \in K \mid \|x-y\| \leq r(K)\}$$

is closed and convex. Hence K_c is also closed and convex since, obviously,

$$K_c = \bigcap_{y \in K} K(y) .$$

Finally we observe that $\operatorname{diam}(K) > 0$ implies $\operatorname{diam}(K_c) < \operatorname{diam}(K)$. Indeed,

$$\operatorname{diam}(K_c) = \sup_{x,y \in K_c} \|x-y\| \le r(K) < \operatorname{diam}(K) ,$$

by normal structure.

(b) Let

$$\mathcal{K} := \{A \subset X \mid \emptyset \neq A = \overline{co}(A)\} ,$$

ordered by inclusion. Then $\mathcal{K}$ is an ordered set such that every chain $\mathcal{C}$ has an infimum, namely $\cap \mathcal{C}$ (by the weak compactness of X and the finite intersection property of $\mathcal{C}$).

For every $A \in \mathcal{K}$, let

$$\phi(A) := [\overline{co}(f(A))]_c .$$

Then, by part (a), ϕ is an increasing selfmap of $\mathcal{K}$, and $\phi(X) \le X$. Hence Theorem (1.4) implies the existence of a nonempty set $K \subset X$ such that

$$K = [\overline{co}(f(K))]_c .$$

Suppose now that $\operatorname{diam}(f(K)) > 0$. Then, again by part (a),

$$\operatorname{diam}(K) = \operatorname{diam}[\overline{co}(f(K))]_c < \operatorname{diam}(\overline{co}\, f(K))$$

$$= \operatorname{diam} f(K) \le \operatorname{diam}(K) ,$$

where the last inequality is a consequence of the nonexpansiveness of f . This contradiction implies $f(K) = \{a\}$. Hence $[\overline{co}\, f(K)]_c = \{a\}$, and, therefore, $K = \{a\}$. This shows that $a = f(a)$. □

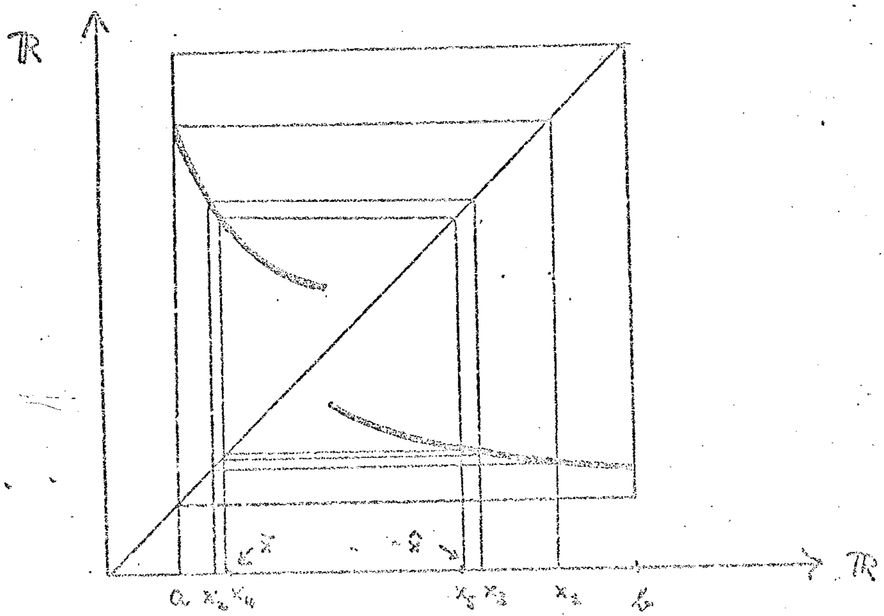
(2.5) <u>*Corollary (Browder, Göhde)*</u>: *Let* X *be a nonempty, closed, convex, bounded subset of a uniformly convex Banach space. Then every nonexpansive selfmap of* X *has a fixed point.*

Proof: Since every uniformly convex space is reflexive, it follows that X is weakly compact (since a closed and convex set is weakly closed). By Example 2, X has also normal structure. Hence the assertion follows from Theorem (2.4). □

E. Decreasing Maps

Let X be an ordered set and suppose that $f : X \to X$ is decreasing. Then simple one-dimensional examples show that, in general, f does not have a fixed point.

Example 1: Let $X = [a,b] \subset \mathbb{R}$ and let $f : [a,b] \to [a,b]$ be decreasing and discontinuous, such that its graph is as in the following picture.



Consider now the iteration scheme

$$x_0 = a$$
$$x_{k+1} = f(x_k) \quad , \qquad k \in \mathbb{N} .$$

Then it is easily verified that the sequence (x_k) has the property that

$$x_o \le x_2 \le x_4 \le \dots \quad \dots \le x_5 \le x_3 \le x_1 \ .$$

Hence $x_{2k} = f(x_{2k-1}) \uparrow \bar{x}$ and $x_{2k+1} = f(x_{2k}) \downarrow \hat{x}$, where $\bar{x} \le \hat{x}$. Consequently, if f is continuous at $\bar{x}$ and $\hat{x}$, it follows that $\bar{x} = f(\hat{x}) \le f(\bar{x}) = \hat{x}$. Moreover, if f has a fixed point, then it is contained in the interval $[\bar{x},\hat{x}]$. □

In the following we shall show that the above example is, in some sense, typical. In fact, we shall show that there exist two points $\bar{x},\hat{x} \in X$ such that $\bar{x} = f(\hat{x}) \le f(\bar{x}) = \hat{x}$, and that all possible fixed points of f are contained in $[\bar{x},\hat{x}]$. Consequently, if $x < y$ implies either $f(x) \neq y$ or $f(y) \neq x$, it follows that f has precisely one fixed point.

We begin with a much more general result concerning so-called intertwined.

(2.6) <u>*Lemma:*</u> *Let* X *be a chain complete ordered set possessi a least and a greatest element. Let* $g : X \times X \to X$ *be a map su that*

(i) $g(.,y) : X \to X$ *is increasing for every* $y \in X$;

(ii) $g(x,.) : X \to X$ *is decreasing for every* $x \in X$.

Then there exist two points $\bar{x},\hat{x} \in X$ *such that* $\bar{x} \le \hat{x}$, $g(\bar{x},\hat{x}) = \bar{x}$, *and* $g(\hat{x},\bar{x}) = \hat{x}$. *Moreover, if* $f(x) := g(x,x)$ *fo all* $x \in X$, *then*

$$\mathrm{Fix}(f) \subset [\check{x}, \hat{x}] \;,$$

where $\mathrm{Fix}(f)$ *denotes the set of all fixed points of* f *.*

Proof: Let $Y := X \times X$, and denote the points of Y by $u = (u^1, u^2)$, $v = (v^1, v^2)$, ... Define an order $\leq$ in Y by

$$u \leq v \quad \text{iff} \quad u^1 \leq v^1 \quad \text{and} \quad u^2 \geq v^2 \;.$$

Moreover, let $U := \{u \in Y \mid u^1 \leq u^2\}$, and observe that U is an ordered set (with the induced order) possessing a least element $u_o = (\check{m}, \hat{m})$, where $\check{m}, \hat{m} \in X$ satisfy $\check{m} \leq X \leq \hat{m}$.

Finally, let

$$G(u) := (g(u^1, u^2), g(u^2, u^1))$$

and observe that G is an increasing selfmap of U . Since $u_o \leq G(u_o)$ and every chain in U has a supremum, Theorem (1.4) implies the existence of a least fixed point $u^* = (\check{x}, \hat{x})$ of G . Thus $\check{x} \leq \hat{x}$, $g(\check{x}, \hat{x}) = \check{x}$, and $g(\hat{x}, \check{x}) = \hat{x}$.

Suppose now that $x \in \mathrm{Fix}(f)$. Then $(x,x) \in U$ is a fixed point of G and, consequently, $u^* \leq (x,x)$, that is, $\check{x} \leq x$ and $\hat{x} \geq x$. □

(2.7) <u>*Corollary:*</u> *Let the hypotheses of Lemma (2.6) be satisfied. If* $x < y$ *implies either* $g(x,y) \neq x$ *or* $g(y,x) \neq y$ *, then* f *has a unique fixed point.*

Proof: Lemma (2.6) and the hypotheses imply $\check{x} = \hat{x}$. □

(2.8) <u>*Theorem:*</u> *Let* X *be a chain complete ordered set possessing a least or a greatest element, and let* $f : X \to X$

be decreasing. Then there exist points $\bar{x}, \hat{x} \in X$ *with* $\bar{x} = f(\hat{x}) \le f(\bar{x}) = \hat{x}$ *such that* $\mathrm{Fix}(f) \subset [\bar{x}, \hat{x}]$.

Moreover, if

(i) *either* $x < y$ *implies* $f(x) \neq y$ *or* $f(y) \neq x$,

(ii) *or* f^2 *has at most one fixed point,*

then f *has a unique fixed point.*

Proof: Let $\bar{m}$ be the least element of X [resp., let $\hat{m}$ be the greatest element of X] and let $\hat{m} := f(\bar{m})$ [resp., $\bar{m} := f(\hat{m})$] . Then $X_o := [\bar{m}, \hat{m}]$ is a chain complete ordered set possessing a least and a greatest element. Since $f(X) \subset X_o$, it follows that $\mathrm{Fix}(f) \subset X_o$. Hence, by letting $g(x,y) := f(y)$ for all $x, y \in X_o$ and applying Lemma (2.6) to X_o , we obtain the first part of the assertion.

If condition (i) is satisfied, then the remaining part of the assertion is trivial. Hence suppose that f^2 has at most one fixed point. Since f^2 is increasing, Theorem (1.4) implies that $\mathrm{Fix}(f^2) = \{x_o\}$. By applying f to the equation $f^2(x_o) = x_o$, we find that $f(f^2(x_o)) = f^2(f(x_o)) = f(x_o)$, that is, $f(x_o) \in \mathrm{Fix}(f^2) = \{x_o\}$. Hence $f(x_o) = x_o$, and the assertion follows from the fact that $\mathrm{Fix}(f) \subset \mathrm{Fix}(f^2)$. □

3. Fixed Points in Ordered Topological Spaces

Let X be an ordered set and a topological space. Then X is called an ordered topological space (OTS) provided each of the sections $S_+(x)$ and $S_-(x)$, $x \in X$, is closed. Consequently, every order interval in an OTS is closed.

Example 1: Let E be a real topological vector space. A subset P of E is called a cone if $P + P \subset P$, $\mathbb{R}_+ P \subset P$, $P \cap (-P) = \{o\}$, and $P = \overline{P}$. It is easily verified that every cone is nonempty $(o \in P)$ and convex. Given a cone P in E, we let $x \leq y$ iff $y - x \in P$. Then it is easily verified that $\leq$ is an order on E having the additional properties that $x \leq y$ implies $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for all $z \in E$ and $\lambda \in \mathbb{R}_+$ (that is, $\leq$ is a vector order on E). This order is said to be induced by P and $(E,\leq)$ is called an ordered topological vector space (OTVS). We write often E or (E,P) instead of $(E,\leq)$, and P is called the positive cone of the OTVS. Since, for every $x \in E$, we have obviously $S_\pm(x) = x \pm P$, it is clear that $S_\pm(x)$ is closed. *Hence every OTVS is an OTS.* □

Example 2: Let X be a locally compact Hausdorff space and denote by $C(X)$ the vector space of all real-valued continuous functions on X, endowed with the (locally convex) topology of compact convergence. (Recall that a defining family of seminorms is given by

$p_K(f) := \max_{x \in K} |f(x)|$, where K runs through all compact subsets of X .) The positive cone inducing the natural order on $C(X)$ is denoted by $C_+(X)$ and defined by

$$C_+(X) := \{f \in C(X) \mid f(x) \geq o \text{ for all } x \in X\} .$$

It is easily verified that $C(X)$ is an OTVS, in fact, an ordered locally convex topological vector space with the natural order. Moreover, if X is compact, then $C(X)$ is an ordered Banach space (OBS). □

Example 3: Let $(X,\mathcal{A},\mu)$ be a σ-finite measure space and let $E = L_p(X,\mathcal{A},\mu)$, $1 \leq p \leq \infty$ (cf. Example 3 of Section 2.A). Then the cone

$L_p^+(X,\mathcal{A},\mu) := \{f \in E \mid f(x) \geq o \text{ for } \mu\text{-a.a. } x \in X\}$ defines an OBS. □

Example 4: Let Ω be an open subset of $\mathbb{R}^n$ and denote by $W_p^k(\Omega)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$, the standard Sobolev spaces of real-valued distributions whose derivatives up to the order k belong to $L_p(\Omega)$. Then the cone $W_p^k(\Omega) \cap L_p^+(\Omega)$ induces the natural order in $W_p^k(\Omega)$, and $W_p^k(\Omega)$ is an OBS. □

The following elementary lemma implies the fundamental fact that every compact OTS is chain complete.

(3.1) Lemma: *Every relatively compact chain in an OTS has an infimum and a supremum.*

Proof: Let C be a relatively compact chain in some OTS. For every $c \in C$, let

(1) $$A_{\pm}(c) := \overline{S_{\pm}(c) \cap C} \subset S_{\pm}(c) \cap \bar{C} \quad .$$

Hence $A_{\pm}(c)$ is compact, and it is easily verified that the family $\{A_{\pm}(c) \mid c \in C\}$ has the finite intersection property. Hence there exist elements m_{+} and m_{-} such that

$$m_{\pm} \in \cap \{A_{\pm}(c) \mid c \in C\} \subset \bar{C} \quad .$$

Thus (1) implies $m_{\pm} \in A_{\pm}(c) \subset S_{\pm}(c)$ for every $c \in C$, that is, $m_{-} \leq c \leq m_{+}$ for every $c \in C$. Suppose that $x \geq C$, that is, $C \subset S_{-}(x)$. Then $\bar{C} \subset S_{-}(x)$ and, since $m_{+} \in \bar{C}$, it follows that $m_{+} \leq x$. This shows that $m_{+} = \sup(C)$. Similarly it follows that $m_{-} = \inf(C)$. □

(3.2) <u>Corollary:</u> *Every compact OTS is chain complete.*

Let S be a nonempty set and X a topological space. Then a map $f : S \to X$ is called <u>relatively compact</u> if $\overline{f(S)}$ is compact in X.

After these preparations it is now easy to prove the following fixed point theorem for (not necessarily continuous) relatively compact increasing maps.

(3.3) <u>Theorem:</u> *Let X be an OTS and let $f : X \to X$ be relatively compact and increasing. Suppose that there exists an element $x_o \in X$ such that $x_o \leq f(x_o)$ [resp., $f(x_o) \leq x_o$]. Then f has a least [resp., greatest] fixed point in $S_{+}(x_o)$ [resp., $S_{-}(x_o)$].*

Proof: Observe that $Y := \overline{f(X)}$ is a compact OTS (with the induced order, of course), and that $f_Y := f|Y$ maps Y into itself, is increasing, and has the property that $\mathrm{Fix}(f_Y) = \mathrm{Fix}(f)$.

Let $y_o := f(x_o) \in Y$ and observe that $y_o \le f_Y(y_o)$ [resp., $f_Y(y_o) \le y_o$] . Hence the assertion follows by applying Theorem (1.4) and Corollary (3.2) to the map f_Y on Y . □

(3.4) Corollary: *Let* X *be an OTS and let* $f : X \to X$ *be increasing and relatively compact on order intervals. Suppose that there exist points* $\bar{y}, \hat{y} \in X$ *such that* $\bar{y} \le \hat{y}$, $\bar{y} \le f(\bar{y})$, *and* $f(\hat{y}) \le \hat{y}$. *Then* f *has a least and a greatest fixed point in* $[\bar{y}, \hat{y}]$.

Proof: Apply Theorem (3.3) to the OTS $[\bar{y}, \hat{y}]$. □

Of course, it is easy to prove a fixed point theorem for relatively compact intertwined maps which corresponds to Corollary (2.7). We leave this to the reader, but we include a result concerning decreasing maps.

(3.5) Theorem: *Let* X *be an OTS having a least or a greatest element, and let* $f : X \to X$ *be decreasing and relatively compact. If either*

(i) $x < y$ *implies* $f(x) \nleq y$ *or* $f(y) \nleq x$,

or

(ii) f^2 *has at most one fixed point,*

then f *has a unique fixed point.*

Proof: Let $\bar{m} \in X$ satisfy $\bar{m} \leq X$ [resp., let $\hat{m} \in X$ satisfy $X \leq \hat{m}$], and let $Y := \overline{f(X)}$. Then Y is a compact OTS and $f_Y := f|Y$ is a decreasing selfmap of Y. Moreover, Y has a greatest [resp., least] element, namely $f(\bar{m})$ [resp., $f(\hat{m})$]. Hence the assertion follows from Theorem (2.8) and Corollary (3.2). □

We extend now the basic fixed point theorem (Theorem (3.3)) to the case of (not necessarily continuous) condensing maps.

(3.6) Theorem: *Let X be a bounded, complete, ordered metric space, and let $f : X \to X$ be increasing and condensing. If there exists a point $x_o \in X$ such that $x_o \leq f(x_o)$ [resp., $f(x_o) \leq x_o$] then f has a least [resp., greatest] fixed point in $S_+(x_o)$ [resp., $S_-(x_o)$].*

Proof: We consider the case that $x_o \leq f(x_o)$. The other case is treated analogously.

Let

$$\mathcal{X} := \{A \subset X \mid x_o \in A = \bar{A} \text{ and } \mathrm{Fix}(f) \cap S_+(x_o) \subset A\} ,$$

ordered by inclusion, and observe that every chain $\mathcal{C}$ in $\mathcal{X}$ has an infimum (namely $\cap \mathcal{C}$). Let $\phi(A) := \overline{f(A)} \cup \{x_o\}$. Then ϕ is an increasing selfmap of $\mathcal{X}$ which satisfies $\phi(X) \leq X$. Hence, by Theorem (1.4), there exists a subset B of X such that

$$B = \overline{f(B)} \cup \{x_o\} .$$

This implies that B has the following properties:

(i) $f(B) \subset B$;

(ii) $B \supset \mathrm{Fix}(f) \cap S_+(x_o)$;

(iii) $x_o \in B$ and $x_o \leq f(x_o)$;

(iv) B is compact, since

$$\alpha(B) = \alpha(\overline{f(B)} \cup \{x_o\}) = \alpha(f(B))$$

(where α denotes the measure of noncompactness) implies $\alpha(B) = o$ (f being condensing). The assertion follows now by applying Theorem (3.3) to $f|B$. □

We leave it to the reader to formulate and prove the extensions of Corollary (3.4) and Theorem (3.5), as well as the corresponding result for intertwined maps, to the case of condensing functions.

4. Applications

A. An Abstract Fixed Point Theorem

As an easy consequence of Corollary (3.4) we prove the following general fixed point theorem for increasing maps in ordered locally convex topological vector spaces.

(4.1) *Theorem: Let* E *be a reflexive, locally convex, ordered, topological vector space, and let* $\bar{y}, \hat{y} \in E$ *satisfy* $\bar{y} \leq \hat{y}$. *Suppose that* $f : [\bar{y}, \hat{y}] \to E$ *is increasing such that*

(i) $\bar{y} \leq f(\bar{y})$ *and* $f(\hat{y}) \leq \hat{y}$;

(ii) $f([\bar{y}, \hat{y}])$ *is bounded.*

Then f *has a least and a greatest fixed point.*

Proof: Since E is reflexive, every bounded set is relatively weakly compact (that is, relatively compact with respect to the topology $\sigma(E,E')$). Since the positive cone of E is closed and convex, it is weakly closed. This implies that $X := [\bar{y}, \hat{y}]$, endowed with the induced weak topology, is an OTS. Moreover, $f : X \to X$ (by (i)) and f is relatively compact (by (ii)). Hence the assertion follows from Corollary (3.4). □

(4.2) Remark: Since (i) and the fact that f is increasing imply that $f([\bar{y}, \hat{y}]) \subset [\bar{y}, \hat{y}]$, condition (ii) is automatically satisfied if the order intervals in E are bounded. This is easily seen to be the case if the positive cone of E is normal (which is the case iff there exists a generating family of seminorms Q on E such that $o \leq x \leq y$ implies $q(x) \leq q(y)$ for all $q \in Q$, that is, iff each seminorm in Q is monotone). In particular, if E is an OBS, then its positive cone is normal iff there exists a positive constant γ such that $o \leq x \leq y$ implies $\|x\| \leq \gamma \|y\|$.

B. Nonlinear Two-Point Boundary Value Problems

In this subsection we consider boundary value problems (BVPs) with possibly discontinuous nonlinearities. For this purpose, let Ω be a measurable subset of $\mathbb{R}^n$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$. Then, for every function $u : \Omega \to \mathbb{R}$, let

$$F(u)(x) := f(x,u(x))$$

for all $x \in \Omega$. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be admissible if, for every $u \in C(\Omega)$, the function $F(u)$ is measurable.

Example 1: Suppose that, for every $\xi \in \mathbb{R}$, the function $f(.,\xi) : \Omega \to \mathbb{R}$ is measurable. Moreover, suppose that there exist points $-\infty < \xi_o < \xi_1 < \ldots < \xi_m < \infty$ such that, for a.a. $x \in \Omega$, the function $f(x,.)$ is continuous in $\mathbb{R} \setminus \bigcup_{j=o}^{m} \{\xi_j\}$ and left continuous at the points ξ_j, $j = o,\ldots,m$. Then f is admissible.

Indeed, by replacing f by $f - f(.,o)$, we can assume without loss of generality that $f(.,o) = o$. Suppose that $u \in C(\Omega)$ and let ψ_k be the characteristic function of the measurable set $u^{-1}((\xi_{k-1},\xi_k])$, $k = o,\ldots,m+1$, where $\xi_{-1} := -\infty$ and $\xi_{m+1} := \infty$. Then

$$F(u) = F(\sum_k u\psi_k) = \sum_k F(u)\psi_k .$$

Consequently, it suffices to show that $F(u)$ is measurable on $u^{-1}((\xi_{k-1},\xi_k])$ for $k = o,\ldots,m+1$.

By standard arguments one deduces the existence of a sequence $(s_{k,j})_{j \in \mathbb{N}}$ of simple functions such that $s_{k,j} \to u|(\xi_{k-1},\xi_k]$ pointwise as $j \to \infty$, and such that $s_{k,j}(x) \in (\xi_{k-1},\xi_k)$ for every $x \in u^{-1}((\xi_{k-1},\xi_k])$, $k = o,\ldots,m$, and $j \in \mathbb{N}$. Hence, by the continuity of $f(x,.)$ on $(\xi_{k-1},\xi_k]$, it follows that

$$F(s_{k,j}) \to F(u) \quad \text{on} \quad u^{-1}((\xi_{k-1},\xi_k])$$

as $j \to \infty$. Therefore it remains to show that $F(s)$ is measurable, where $s = \sum \alpha_j \chi_{A_j}$ is a simple function. But this follows from

$$F(s) = \sum F(\alpha_j)\chi_{A_j}$$

and the presupposed measurability of $f(.,\xi)$. □

Suppose now that $p \in C^1(\mathbb{R},(o,\infty))$ and that $f:[o,1] \times \mathbb{R} \to \mathbb{R}$ is admissible. Then we consider the BVP

$$(1) \qquad \begin{cases} -[p(u)u']' = f(x,u) & \text{in } (o,1), \\ u(o) = u(1) = o . \end{cases}$$

By a <u>solution</u> of (1) we mean a function u belonging to

$$AC^1[o,1] := \{u \in C^1[o,1] \mid u' \text{ absolutely continuous}\}$$

such that $u(o) = u(1) = o$ and

$$-[p(u(x))u'(x)]' = f(x,u(x))$$

for a.a. $x \in (o,1)$. A function v is called a <u>subsolution of (1)</u> if $v \in AC^1[o,1]$, $v(o) \le o$, $v(1) \le o$, and

$$-[p(v(x))v'(x)]' \le f(x,v(x))$$

for a.a. $x \in (o,1)$. <u>Supersolutions</u> are defined by reversing the above inequalities.

Example 2: Suppose that $f(x,\xi) \ge o$ for all $(x,\xi) \in [o,1] \times \mathbb{R}$. Then $v = o$ is a subsolution for (1). If, for some $\xi_o \in \mathbb{R}_+$, $f(x,\xi_o) \le o$ for a.a. $x \in (o,1)$, then the constant function $v = \xi_o$ is a supersolution for (1). □

After these preparations we prove now the following existence theorem for the BVP (1).

(4.2) *<u>Theorem</u>: Suppose that* $\bar{v}$ *is a subsolution and* $\hat{v}$ *is a supersolution for (1) such that* $\bar{v} \le \hat{v}$ *(pointwise). Moreover, suppose that there exists a constant* $\alpha \ge o$ *such that*

$$f(x,\xi) - f(x,\eta) \geq -\alpha(\xi-\eta)$$

for all $x \in [o,1]$ *and* $\bar{\xi} := \min \bar{v} \le \eta \le \xi \le \max \hat{v} =: \hat{\xi}$. *Then the BVP (1) has a least and a greatest solution in the order interval* $[\bar{v},\hat{v}]$ *(with respect to the pointwise order).*

Proof: For every $\xi \in \mathbb{R}$, let $P(\xi) := \int_o^\xi p(\eta)d\eta$ and observe that $P(o) = o$, $P \in C^2(\mathbb{R})$, and P is strictly increasing.

Hence P possesses a strictly increasing inverse $Q := P^{-1} \in C^2(I)$, where $I := P(\mathbb{R}) \subset \mathbb{R}$, such that $Q(o) = o$.

Let $g(x,\xi) := f(x,Q(\xi))$ for $(x,\xi) \in [o,1] \times I$. Then (1) is equivalent to the BVP

$$(2) \qquad \begin{cases} -v'' = g(x,v) \quad \text{in} \quad (o,1) , \\ v(o) = v(1) = o , \end{cases}$$

which is easily verified by means of the transformation $v := P(u)$. Moreover, $\bar{w} := P(\bar{v})$ is a subsolution for (2), $\hat{w} := P(\hat{v})$ is a supersolution for (2), and $\bar{w} \le \hat{w}$.

Let $\bar{\eta} := P(\bar{\xi})$, $\hat{\eta} := P(\hat{\xi})$, and

$$\beta := \max_{\bar{\eta} \le \eta \le \hat{\eta}} Q'(\eta) = \max_{\bar{\xi} \le \xi \le \hat{\xi}} \frac{1}{p(\xi)} > o .$$

Then it follows that, for all $x \in [o,1]$ and $\bar{\eta} \le \eta < \xi \le \hat{\eta}$,

$$\begin{aligned} g(x,\xi) - g(x,\eta) &= f(x,Q(\xi)) - f(x,Q(\eta)) \\ &\ge -\alpha(Q(\xi)-Q(\eta)) \ge -\alpha\beta(\xi-\eta) . \end{aligned}$$

Thus, for every $x \in [o,1]$, the function

$$\xi \to h(x,\xi) := g(x,\xi) + \alpha\beta\xi$$

is increasing on $[\bar{\eta},\hat{\eta}]$ and the BVP (2) is equivalent to

$$(3) \qquad \begin{cases} -v'' + \alpha\beta v = h(x,v) \quad \text{in} \quad (o,1) , \\ v(o) = v(1) = o . \end{cases}$$

Clearly, $\bar{w}$ is a subsolution for (3) and $\hat{w}$ is a supersolution for (3) such that $\bar{w} \le \hat{w}$. Moreover, if $H(v)(x) := h(x,v(x))$ for all $x \in [o,1]$, then $H : C[o,1] \to L_\infty(o,1)$ and H is increasing on $[\bar{w},\hat{w}]$ (with respect to the natural orders in $C[o,1]$ and $L_\infty(o,1)$).

It is well known that the linear BVP

$$(4)\qquad \begin{aligned} -v'' + \alpha\beta v &= a \quad \text{in} \quad (o,1) \ , \\ v(o) = v(1) &= o \end{aligned}$$

possesses a unique Green's function $G \in C([o,1] \times [o,1])$ with $G \geq o$ such that, for every $a \in L_\infty(o,1)$, the unique solution $v \in AC^1[o,1]$ of (4) is given by

$$v(x) = \int_o^1 G(x,y)a(y)dy \ , \quad x \in [o,1] \ .$$

Using these facts, it is easily verified that the BVP (3) [and hence (1)] is equivalent to the Hammerstein integral equation

$$v(x) = \int_o^1 G(x,y)h(y,v(y))dy \ , \quad o \leq x \leq 1 \ ,$$

in $C[o,1]$.

Let

$$T(v) := \int_o^1 G(.,y)h(y,v(y))dy$$

for all $v \in [\bar{w},\hat{w}] \subset C[o,1]$. Then

$$T : [\bar{w},\hat{w}] \to C[o,1] \ ,$$

and T is increasing and relatively compact (due to the Arzela-Ascoli theorem) but, of course, in general not continuous. Hence the assertion follows immediately from Corollary (3.4), provided we show that

$$\bar{w} \leq T(\bar{w}) \quad \text{and} \quad T(\hat{w}) \leq \hat{w} \ .$$

To verify the first inequality let $\bar{u} := \bar{w} - T(\bar{w})$, and suppose that $\gamma := \max_{o \leq x \leq 1} \bar{u}(x) > o$. Observe that $\bar{u}(o) \leq o$, $\bar{u}(1) \leq o$, and

$$-\bar{u}'' + \alpha\beta\bar{u} = -\bar{w}'' + \alpha\beta\bar{w} - h(.,\bar{w}(.)) \leq o$$

for a.a. $x \in (o,1)$, since $\bar{w}$ is a subsolution for (3). Hence there exist a point $x_o \in (o,1)$ such that $\bar{u}(x_o) = \gamma$ and a neighborhood $I_o \subset (o,1)$ of x_o such that $\bar{u}(x) \geq \gamma/2$ for all

$x \in I_o$. Let $x,y \in I$ satisfy $x < y$ and integrate the above inequality between x and y . Then

$$o \geq \int_x^y (-\bar{\bar{u}}''+\alpha\beta\bar{\bar{u}}) \geq -[\bar{\bar{u}}'(y)-\bar{\bar{u}}'(x)] + \frac{\alpha\beta\gamma}{2}(y-x) ,$$

which shows that $\bar{\bar{u}}'$ is strictly increasing on I_o . Hence $\bar{\bar{u}}$ is strictly convex on I_o , which contradicts the fact that $\bar{\bar{u}}$ attains its global maximum at $x_o \in I$. Hence $\bar{\bar{u}} = \bar{\bar{w}} - T(\bar{\bar{w}}) \leq o$. A similar argument shows that $T(\hat{w}) \leq \hat{w}$. □

5. Notes and Remarks

Section 1:

Our basic result, namely Theorem (1.4), does not seem to be in the literature in its given form. However it is closely related to some earlier results by Kolodner [1], Tartar [1,2], and Bakhtin [1]. In fact, Kolodner [1] has observed that an increasing function mapping an order interval into itself has at least one fixed point, provided each chain has a supremum (or each chain has an infimum).

Tartar [1,2] has shown that an increasing selfmap of an order interval has a least fixed point if each chain has a supremum. If, in addition, for every pair of points a,b, there exists $\sup\{a,b\}$, then there exists also a greatest fixed point.

Bakhtin [1] considers a commuting family F of increasing maps such that $x \leq f(x)$ for all $x \in X$ and all $f \in F$. By using transfinite induction and a hypothesis which is close to the assumption that every chain has a supremum, he proves the existence of a least common fixed point for the family F.

We refer to the papers by Tartar [1,2] for some applications of his results to problems in nonlinear differential equations.

Section 2:

The applications of Theorem (1.4) in Subsections A–D have been motivated by a recent paper of Fuchssteiner [1] . This author deduces these fixed point theorems (as well as some others) also from a general result for a selfmap f of an ordered set X such that $f(x) \leq x$ for all $x \in X$. However the two approaches are different.

It should be remarked that recently Brézis and Browder [1] have given a general abstract principle involving order structures, which has many applications to problems in nonlinear functional analysis. We do not discuss the relation of their principle with the results of the present paper. However it is clear that order structures play a rather important rôle in nonlinear functional analysis and further studies of these aspects may lead to important insights and new results.

Subsection A

Theorem (2.1) has been proved by Tarski in 1939 but not published by him until 1955 (Tarski [1]). This theorem had been reproduced in the 1948 edition of G. Birkhoff's book on lattice theory, where "a proper historical reference had been omitted by mistake". For this reason, Theorem (2.1) is often called the "Birkhoff-Tarski theorem".

For applications of Tarski's theorem to quasi-variational inequalities we refer to the survey article by Mosco [1] , where further references can be found.

Tarski showed also that Fix(f) is a complete lattice, though not necessarily a sublattice of X . Davis [1] proved the following converse to Tarski's theorem: "If every increasing map in a lattice X has a fixed point, then X is a complete lattice." Hence Theorem (1.4) implies the known fact that a *lattice is complete iff every chain possesses a supremum (resp., infimum).*

Subsection B

We do not discuss the numerous generalizations of Banach's contraction mapping principle. But we should like to mention that Theorem (2.2) is contained in Fuchssteiner [1].

Subsection C

For the basic facts about measures of noncompactness, set contractions, and condensing maps we refer to Sadovskii [1], Darbo [1] , and also Browder [1] , where further examples of condensing maps can be found (in particular, so-called intertwined representations). Condensing maps occuring in the theory of differential equations have been studied by Stuart [3] , for example.

Subsection D

Example 1 is well-known (and probably due to R. Beals). For Example 2 see Browder [2] and Edelstein [1] . The original proof of Kirk's theorem is given in Kirk [1] . Corollary (2.5) has been proved earlier and independently by Browder [2] and Göhde [1] . For many more results on nonexpansive maps we refer to Browder [1] .

Subsection E

The rather trivial results of this subsection are new. It should be pointed out that Condition (i) in Theorem (2.8) is, of course, only a sufficient condition for the existence of a fixed point.

In the special case that X is an order interval in some OBS and $f : X \to X$ is continuous and has relatively compact image, the existence of a fixed point of f follows, of course, from Schauder's fixed point theorem without any further restriction on f. If, however, f is only continuous (and not compact) then the existence of a fixed point cannot be guaranteed without further restrictions [like conditions (i) and (ii) of Theorem (2.8)], even if f has the rather strong property of being decreasing. It would be of interest to obtain better results for decreasing maps which could bridge the apparent gap between Theorem (2.8) and the situation where Schauder's fixed point theorem applies.

The results concerning "intertwined" maps (e.g. Corollary (2.7)) and, in particular, decreasing maps, should be applicable to nonlinear integral and differential equations, namely to situations where Schauder's fixed point theorem does not apply. (In this connection cf. also Theorem (3.5) and the papers of Stuart [1] and Kuiper [2]).

Section 3:

The fixed point theorems of this section are new. Some results which are closely related to Theorem (3.3) in the context of OBSs are contained in Bakhtin [1]. The existence of at least one fixed point under the hypotheses of Theorem (3.6), but with the stronger assumption that f is a strict set contraction (that is, there exists a constant $\gamma < 1$ such that $\alpha(f(B)) \leq \gamma\alpha(B)$ for all bounded subsets B of X) has been proved earlier and by a different method by Leggett [1]. However Leggett did not obtain the existence of a least [resp., greatest] fixed point.

The fact that there exists a least and/or a greatest fixed point can be very helpful for deducing further information, e.g. uniqueness results. We do not go into details but refer the reader to the survey article Amann [1] where many results of this type for *continuous* maps are given. By using the above results, it is easy to extend a considerable amount of the theorems given there to discontinuous maps (but, of course, not the topological results which have been proved by fixed point index arguments). In particular, by using minorants and majorants, or by imposing conditions concerning the asymptotic behaviour of the maps (e.g., asymptotic linearity) it is also possible to deduce the existence of a second fixed point in cases where already one fixed point is explicitly known.

Section 4

As already mentioned in the introduction, it is the purpose of this section to demonstrate the applicability of the results of Section 3. For this purpose we have restricted ourselves to relatively simple settings. In particular, we consider only relatively compact maps. But of course, there are also applications to problems involving condensing maps (e.g. in the case of differential equations in unbounded domains, cf. Stuart [3]).

Subsection A

Of course, the existence of at least one fixed point under the hypotheses of Theorem (4.1) follows from the Schauder-Tychonoff theorem, provided f is weakly continuous. But it is the advantage of Theorem (4.1) that no continuity assumption has been made. This fact facilitates its application to concrete situations considerably (e.g. to the study of weak solutions of nonlinear elliptic differential equations in unbounded domains).

Subsection B

For simplicity we have restricted our considerations to Dirichlet boundary conditions, but it is obvious that the same method applies to any kind of Sturm-Liouville boundary conditions.

The given example has been motivated by some papers of Chandra and Fleishman [1] and Fleishman and Mahar [1] . In these papers the authors study two-point boundary value problems with discontinuous nonlinearities and they give also some physical motivations for their research. By using an iteration scheme they establish the existence of a least and a greatest solution between a sub- and a supersolution. However, in order to prove the convergence of the iteration scheme to a solution, they have to single out a class of functions on which the nonlinear operator (corresponding to T in our case) is actually continous. This is achieved by restricting the nonlinear operator to increasing functions, which implies essentially that Green's function has to be increasing in its first variable. It is obvious that this requirement is rather restrictive and that onl a particular type of boundary conditions can be handled by thi method. In particular, these authors are unable to treat the rather natural case of Dirichlet boundary conditions without imposing certain symmetry conditions.

Our approach which is based on the general fixed point theorem proved in Section 3 (Theorem (3.3)) does not need any hypotheses of this type and is much more flexible. In particular, it is also applicable to rather general nonlinear ellipt boundary value problems of the form

$$Au = f(x,u) \quad \text{in} \quad \Omega \subset \mathbb{R}^n ,$$

where A is a strongly uniformly elliptic differential opera of second order (e.g., $Au = -\Delta u$), together with boundary coditions of the form

$$u = o \quad \text{on} \quad \partial\Omega$$

or

$$\frac{\partial u}{\partial \beta} = g(x,u) \quad \text{on} \quad \partial\Omega ,$$

where β is an outward pointing, nowhere tangent, smooth ve field on $\partial\Omega$. In the case of the Dirichlet boundary conditi or if g is linear in u, a result corresponding to Theorem (4.2) can be obtained by means of Tarski's fixed point theorem. Indeed, in this case the above BVP can be reduced to an equivalent fixed point equation of the form

$$u = KF(u)$$

in $L_p(\Omega)$, $1 < p < \infty$. Here K is a positive compact linear operator on $L_p(\Omega)$, and F is an increasing (possibly discontinuous) map on $L_p(\Omega)$. Moreover, if $\bar{v}$ is a subsolution and $\hat{v}$ is a supersolution for the corresponding BVP, it follows that $\bar{v} \leq KF(\bar{v})$ and $KF(\hat{v}) \leq \hat{v}$. (For details in the case of continuous maps cf. Amann [1] and the bibliography therein.)

In the case of nonlinear boundary conditions, the problem of finding (appropriate weak) solutions to the elliptic BVP can be reduced to an equivalent fixed point equation $u = T(u)$ in $C(\bar{\Omega})$. Here T is an increasing, relatively compact map of the form

$$T(u) = S(F(u), G(\tau(u))),$$

where $S : C(\bar{\Omega}) \times C(\partial\Omega) \to C(\bar{\Omega})$ is a compact positive linear operator, namely essentially the solution operator for the linear BVP

$$Au = f \qquad \text{in} \quad \Omega ,$$

$$\frac{\partial u}{\partial \beta} = g \qquad \text{on} \quad \partial\Omega .$$

Moreover, F corresponds to the nonlinearity $f(x,u)$, G to the nonlinearity $g(x,u)$, and $\tau : C(\bar{\Omega}) \to C(\partial\Omega)$ is the trace operator. In this case (that is, in the case of nonlinear boundary conditions), the full strength of Theorem (3.3) (or Corollary

(3.4)) is needed, since the underlying ordered set is not a complete lattice. (We refer to Amann [2] for the precise definition of S, F, G , and τ and their properties in the continuous case. It is not too difficult to handle the discontinuous case by making use of the results given in that paper.)

It should be remarked that general elliptic BVPs of the above type can not be handled by the method used by Fleishman and coworkers, since there is no natural definition of an increasing map in several variables which can be used to prove that the iteration scheme is convergent. (For a very special case (which is, in fact, reduced to an ordinary differential equation) and which is rather technical, we refer to Fleishman [1] .)

Finally we should like to mention that the fixed point theorems of this paper are also applicable to elliptic boundary value problems in unbounded domains, as well as to parabolic initial boundary value problems. For further studies of BVPs with discontinuities we refer to Kuiper [1] and Stuart [2] .

General Remark

Almost all the results of Sections 3 and 4 are nearly trivial if the maps are continuous. In this case the least [resp., greatest] fixed point of f can be obtained by the monotone iteration scheme $x_{k+1} = f(x_k)$, $k \in \mathbb{N}$. For this fact and many more results we refer to Amann [1] . -

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