Anisotropic Function Spaces on Singular Manifolds

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A rather complete investigation of anisotropic Bessel potential, Besov, and Hölder spaces on cylinders over (possibly) noncompact Riemannian manifolds with boundary is carried out. The geometry of the underlying manifold near its 'ends' is determined by a singularity function which leads naturally to the study of weighted function spaces. Besides of the derivation of Sobolev-type embedding results, sharp trace theorems, point-wise multiplier properties, and interpolation characterizations particular emphasize is put on spaces distinguished by boundary conditions. This work is the fundament for the analysis of time-dependent partial differential equations on singular manifolds.

1 Introduction

In [5] we have performed an in-depth study of Sobolev, Bessel potential, and Besov spaces of functions and tensor fields on Riemannian manifolds which may have a boundary and may be noncompact and noncomplete. That as well as the present research is motivated by — and provides the basis for — the study of elliptic and parabolic boundary value problems on piece-wise smooth manifolds, on domains in $\mathbb{R}^m$ with a piece-wise smooth boundary in particular.

A singular manifold M is to a large extent determined by a 'singularity function' $\rho \in C^\infty\big(M,(0,\infty)\big)$. The behavior of ρ at the 'singular ends' of M, that is, near that parts of M at which ρ gets either arbitrarily small or arbitrarily large, reflects the singular structure of M.

The basic building blocks for a useful theory of function spaces on singular manifolds are weighted Sobolev spaces based on the singularity function ρ. More precisely, we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. Then, given $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and $p \in (1,\infty)$, the weighted Sobolev space $W_p^{k,\lambda}(M) = W_p^{k,\lambda}(M,\mathbb{K})$ is the completion of $\mathcal{D}(M)$, the space of smooth functions with compact support in M, in $L_{1,\mathrm{loc}}(M)$ with respect to the norm

$$u \mapsto \Big(\sum_{i=0}^{k} \big\|\rho^{\lambda+i}\,|\nabla^i u|_g\big\|_p^p\Big)^{1/p}. \tag{1.1}$$

Here ∇ denotes the Levi-Civita covariant derivative and $|\nabla^i u|_g$ is the 'length' of the covariant tensor field $\nabla^i u$ naturally derived from the Riemannian metric g of M. Of course, integration is carried out with respect to the volume measure of M. It turns out that $W_p^{k,\lambda}(M)$ is well-defined, independently — in the sense of equivalent norms — of the representation of the singularity structure of M by means of the specific singularity function.

A very special and simple example of a singular manifold is provided by a bounded smooth domain whose boundary possesses a conical point. More precisely, suppose Ω is a bounded domain in $\mathbb{R}^m$ whose topological boundary, $\mathrm{bdry}(\Omega)$, contains the origin, and $\Gamma := \mathrm{bdry}(\Omega)\backslash\{0\}$ is a smooth $(m-1)$-dimensional submanifold of $\mathbb{R}^m$ lying locally on one side of Ω. Also suppose that $\Omega \cup \Gamma$ is near 0 diffeomorphic to a cone

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$\{\, ry\,;\, 0<r<1,\ y\in B\,\}$, where B is a smooth compact submanifold of the unit sphere in $\mathbb{R}^m$. Then, endowing $M:=\Omega\cup\Gamma$ with the Euclidean metric, we get a singular manifold with a single conical singularity, as considered in [35] and [27], for example. In this case the weighted norm (1.1) is equivalent to

$$u\mapsto\Big(\sum_{|\alpha|\le k}\|r^{\lambda+|\alpha|}\partial^\alpha u\|_{L_p(\Omega)}^p\Big)^{1/p},$$

where $r(x)$ is the Euclidean distance from $x\in M$ to the origin. Moreover, $W_p^{k,\lambda}(M)$ coincides with the space $V_{p,\lambda+k}^k(\Omega)$ employed by S.A. Nazarov and B.A. Plamenevsky [35, p. 319] and, in the case $p=2$, by V.A. Kozlov, V.G. Maz′ya, and J. Rossmann (see Section 6.2 of [27], for instance).

In [5] we have exhibited a number of examples of singular manifolds. For more general classes, comprising notably manifolds with corners and non-smooth cusps, we refer to H. Amann [6]. It is worthwhile to point out that our concept of singular manifolds encompasses, as a very particular case, manifolds with bounded geometry (that is, Riemannian manifolds without boundary possessing a positive injectivity radius and having all covariant derivatives of the curvature tensor bounded). In this case we can set $\rho=\mathbf{1}$, the function constantly equal to 1, so that $W_p^{k,\lambda}(M)$ is independent of λ and equal to the standard Sobolev space $W_p^k(M)$.

The weighted Sobolev spaces $W_p^{k,\lambda}(M)$ and their fractional order relatives, that is, Bessel potential and Besov spaces, come up naturally in, and are especially useful for, the study of elliptic boundary value problems for differential and pseudodifferential operators in non-smooth settings. This is known since the seminal work of V.A. Kondrat′ev [22] and has since been exploited and amplified by numerous authors in various levels of generality, predominantly however in the Hilbertian case $p=2$ (see [5] for further bibliographical remarks).

For an efficient study of evolution equations on singular manifolds we have to have a good understanding of function spaces on space-time cylinders $M\times J$ with $J\in\{\mathbb{R},\mathbb{R}^+\}$, where $\mathbb{R}^+=[0,\infty)$. Then, in general, the functions (or distributions) under consideration posses different regularity properties with respect to the space and time variables. Thus we are led to study anisotropic Sobolev spaces and their fractional order relatives.

Anisotropic weighted Sobolev spaces depend on two additional parameters, namely $r\in\mathbb{N}^\times:=\mathbb{N}\backslash\{0\}$ and $\mu\in\mathbb{R}$. More precisely, we denote throughout by $\partial=\partial_t$ the vector-valued distributional 'time' derivative. Then, given $k\in\mathbb{N}^\times$,

$W_p^{(kr,k),(\lambda,\mu)}(M\times J)$ is the linear subspace of $L_{1,\mathrm{loc}}(M\times J)$ consisting of all u satisfying

$$\rho^{\lambda+i+j\mu}\,|\nabla^i\partial^j u|_g\in L_p(M\times J)\quad\text{for}\quad i+jr\le kr, \tag{1.2}$$

endowed with its natural norm.

It is a Banach space, a Hilbert space if $p=2$.

Spaces of this type, as well as fractional order versions thereof, provide the natural domain for an L_p-theory of linear differential operators of the form

$$\sum_{i+jr\le kr}a_{ij}\cdot\nabla^i\partial^j,$$

where a_{ij} is a time-dependent contravariant tensor field of order i and $\cdot$ indicates complete contraction. In this connection the values $\mu=0$, $\mu=1$, and $\mu=r$ are of particular importance. If $\mu=1$, then space and time derivatives carry the same weight. If also $r=1$, then we get isotropic weighted Sobolev spaces on $M\times J$.

If $\mu=0$, then the intersection space characterization

$$W_p^{(kr,k),(\lambda,0)}(M\times J)\doteq L_p\big(J,W_p^{kr,\lambda}(M)\big)\cap W_p^k\big(J,L_p^\lambda(M)\big)$$

is valid, where $\doteq$ means: equal except for equivalent norms. Spaces of this type (with $k=1$) have been used by S. Coriasco, E. Schrohe, and J. Seiler [9], [10] for studying parabolic equations on manifolds with conical points. In this case ρ is (equivalent to) the distance from the singular points. Anisotropic spaces with $\mu=0$ are also important for certain classes of degenerate parabolic boundary value problems (see [6]).

The spaces $W_p^{(kr,k),(\lambda,r)}(M\times J)$ constitute, perhaps, the most natural extension of the 'stationary' spaces $W_p^{k,\lambda}(M)$ to the space-time cylinder $M\times J$. They have been employed by V.A. Kozlov [23]–[26] — in the Hilbertian setting $p=2$ — for the study of general parabolic boundary value problems on a cone M. (Kozlov, as well as the authors mentioned below, write $W_{\lambda+kr}^{(kr,k)}$ for $W_2^{(kr,k),(\lambda,r)}$.) The space $W_p^{(2,1),(\lambda,2)}(M\times J)$ occurs in the works on second order parabolic equations on smooth infinite wedges by V.A. Solonnikov [45] and A.I. Nazarov [34] (also see V.A. Solonnikov and E.V. Frolova [46], [47]), as well as in the studies of W.M. Zajączkowski [52]–[55], A. Kubica and W.M. Zajączkowski [28], [29], and K. Pileckas [36]–[38] (see the references in these papers for earlier work) on Stokes and Navier-Stokes equations. In all these papers, except the ones of Pileckas, ρ is the distance to the singularity set, where in Zajączkowski's publications M is obtained from a smooth subdomain of $\mathbb{R}^m$ by eliminating a line segment. Pileckas considers subdomains of $\mathbb{R}^m$ with outlets to infinity and ρ having possibly polynomial or exponential growth.

In this work we carry out a detailed study of anisotropic Sobolev, Bessel potential, Besov, and Hölder spaces on singular manifolds and their interrelations. Besides of this introduction, the paper is structured by the following sections on whose principal content we comment below.

We have already pointed out in [5] that it is not sufficient to study function spaces on singular manifolds since spaces of tensor fields occur naturally in applications. In order to pave the way for a study of *systems* of differential and pseudodifferential operators it is even necessary to deal with tensor fields taking their values in general vector bundles. This framework is adopted here.

Sections 2 and 3 are of preparatory character. In the former, besides of fixing notation and introducing conventions used throughout, we present the background material on vector bundles on which this paper is based. We emphasize, in particular, duality properties and local representations which are fundamental for our approach.

Since we are primarily interested in noncompact manifolds we have to impose suitable regularity conditions 'at infinity'. This is done in Section 3 where we introduce the class of 'fully uniformly regular' vector bundles. They constitute the 'image bundles' for the tensor fields on the singular manifolds which we consider here.

After these preparations, singular manifolds are introduced in Section 4. There we also install the geometrical frame which we use from thereon without further mention.

Although we study spaces of tensor fields taking their values in uniformly regular vector bundles, the vector bundles generated by these tensor fields are *not* uniformly regular themselves, in general. In fact, their metric and their covariant derivative depend on the metric g of the underlying singular Riemannian manifold. Since the singularity behavior of g is controlled by the singularity function ρ, due to our very definition of a singular manifold, we have to study carefully the dependence of all relevant parameters on ρ as well. This is done in Section 5. On its basis we can show in later sections that the various function spaces are independent of particular representations; they depend on the underlying geometric structure only.

Having settled these preparatory problems we can then turn to the main subject of this paper, the study of function spaces (more precisely, spaces of vector-bundle-valued tensor fields) on singular manifolds. We begin in Sections 6 and 7 by recalling and amplifying some results from our previous paper [5] on isotropic spaces. On the one hand this allows us to introduce some basic concepts and on the other hand we can point out the changes which have to be made to cover the more general setting of of vector-bundle-valued tensor fields.

The actual study of anisotropic weighted function spaces begins in Section 8. First we introduce Sobolev spaces which can be easily described invariantly. They form the building blocks for the theory of anisotropic weighted Bessel potential and Besov spaces. The latter are invariantly defined by interpolation between Sobolev spaces and by duality.

This being done, it has to be shown that these spaces coincide in the most simple situation in which M is either the Euclidean space $\mathbb{R}^m$ or a closed half-space $\mathbb{H}^m$ thereof with the ‘usual’ anisotropic Bessel potential and Besov spaces, respectively. In the Euclidean model setting a thorough investigation has been carried out in H. Amann [4] by means of Fourier analytic techniques. That work is the fundament upon which the present research is built. The basic result which settles this identification and is fundamental for the whole theory as well as for the study of evolution equations is Theorem 9.3. In particular, it establishes isomorphisms between the function spaces on $M \times J$ and certain countable products of corresponding spaces on model manifolds. By these isomorphisms we can transfer the known properties of the ‘elementary’ spaces on $\mathbb{R}^m \times J$ and $\mathbb{H}^m \times J$ to $M \times J$. With this method we establish the most fundamental properties of anisotropic Bessel potential and Besov spaces which are already stated in Section 8.

In Section 10 we take advantage of the fact that the anisotropic spaces we consider live in cylinders over M so that the ‘time variable’ plays a distinguished role. This allows us to introduce some useful semi-explicit equivalent norms for Besov spaces.

It is well-known that spaces of Hölder continuous functions are intimately related to the theory of partial differential equations on Euclidean spaces. They occur naturally, even in the L_p-theory, as point-wise multiplier spaces, in particular as coefficient spaces for differential operators. Although it is fairly easy to study Hölder continuous functions on subsets of $\mathbb{R}^m$, it is surprisingly difficult to do this on manifolds. Our approach to this problem is similar to the way in which we defined Bessel potential and (L_p-based) Besov spaces on manifolds. Namely, first we introduce spaces of bounded and continuously differentiable functions. Then we define Hölder spaces, more generally Besov-Hölder spaces, by interpolation. This is not straightforward since we can only interpolate between spaces of bounded C^k-functions whose derivatives are uniformly continuous. Due to the fact that we are mainly interested in noncompact manifolds, the concept of uniform continuity is not a priori clear and has to be clarified first. Then the next problem is to show that Hölder spaces introduced in this invariant way can be described locally by their standard anisotropic counterparts on $\mathbb{R}^m \times J$ and $\mathbb{H}^m \times J$. Such representations in local coordinates are, of course, fundamental for the study of concrete equations, for example.

In order to achieve these goals we set up the preliminary Section 11 in which we establish the needed properties of (vector-valued) Hölder and Bessel-Hölder spaces in Euclidean settings. In Section 12 we can then settle the problems alluded to above. It should be mentioned that in these two sections we consider time-independent isotropic as well as time-dependent anisotropic spaces, thus complementing the somewhat ad hoc results on Hölder spaces in [5].

Having introduced all these spaces and established their basic properties we proceed now to more refined features. In Section 13 we show that, similarly as in the Euclidean setting, Hölder spaces are universal point-wise multiplier spaces for Bessel potential and Besov spaces modeled on L_p. For this we establish the rather general (almost) optimal Theorem 13.5.

In practice point-wise multiplications occur, as a rule, through contractions of tensor fields. For this reason we carry out in Section 14 a detailed study of mapping properties of contractions of tensor fields, one factor belonging to a Hölder space and the other one to a Bessel potential or a Besov space, in particular. It should be noted that we impose minimal regularity assumptions for the multiplier space. The larger part of Section 14 is, however, devoted to the problem of the existence of a continuous right inverse for a multiplier operator induced by a complete contraction. The main result of this section thus is Theorem 14.9. It is basic for the theory of boundary value problems.

Section 15 contains general Sobolev-type embedding theorems for parameter-dependent weighted Bessel potential and Besov spaces. They are natural extensions of the corresponding classical results in the Euclidean setting.

Making use of our point-wise multiplier and Sobolev-type embedding theorems we study in Section 16 mapping properties of differential operators in anisotropic spaces. In view of applications to quasilinear equations we strive for minimal regularity requirements for the coefficient tensors.

All results established up to this point hold both for $J = \mathbb{R}$ and $J = \mathbb{R}^+$. In contrast, Section 17 is specifically concerned with anisotropic spaces on the half-line $\mathbb{R}^+$. It is shown that in many cases properties of function spaces on $\mathbb{R}^+$ can be derived from the corresponding results on the whole line $\mathbb{R}$. This can simplify the situation since $M \times \mathbb{R}$ is a usual manifold (with boundary), whereas $M \times \mathbb{R}^+$ has corners if $\partial M \neq \emptyset$.

In Section 18 we consider the important case where M has a nonempty boundary and establish the fundamental trace theorem for anisotropic weighted Bessel potential and Besov spaces, both on the 'lateral boundary' $\partial M \times J$ and on the 'initial boundary' $M \times \{0\}$ if $J = \mathbb{R}^+$.

In the next section we characterize spaces of functions having vanishing initial traces. Section 20 is devoted to extending the boundary values. Here we rely, besides the trace theorem, in particular on the 'right inverse theorem' established in Section 14. The results of this section are of great importance in the theory of boundary value problems.

Section 21 describes the behavior of anisotropic weighted Bessel potential, Besov, and Hölder spaces under interpolation. In addition to this, we also derive interpolation theorems for 'spaces with vanishing boundary conditions'. These results are needed for a 'weak L_p-theory' of parabolic evolution equations.

Our investigation of weighted anisotropic function spaces is greatly simplified by the fact that we consider full and half-cylinders over M. In this case we can take advantage of the dilation invariance of J. In practice, cylinders of finite height come up naturally and are of considerable importance. For this reason it is shown, in the last section, that all embedding, interpolation, trace theorems, etc. are equally valid if J is replaced by $[0,T]$ for some $T \in (0,\infty)$.

In order to cover the many possibilities due to the (unavoidably) large set of parameters our spaces depend upon, and to eliminate repetitive arguments, we use rather condensed notation in which we exhibit the locally relevant information only. This requires a great deal of concentration on the part of the reader. However, everything simplifies drastically in the important special case of Riemannian manifolds with bounded geometry. In that case there are no singularities and all spaces are parameter-independent. Readers interested in this situation only can simply ignore all mention of the parameters λ, μ, and $\vec{\omega}$ and set $\rho = \mathbf{1}$. Needless to say that even in this 'simple' situation the results of this paper are new.

2 Vector Bundles

First we introduce some notation and conventions from functional analysis. Then we recall some relevant facts from the theory of vector bundles. It is the main purpose of this preparatory section to create a firm basis for the following. We emphasize in particular duality properties and local representations, for which we cannot refer to the literature. Background material on manifolds and vector bundles is found in J. Dieudonné [12] or J. Jost [21], for example.

Given locally convex (Hausdorff topological vector) spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of continuous linear maps from $\mathcal{X}$ into $\mathcal{Y}$, and $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X},\mathcal{X})$. By $\mathcal{L}\mathrm{is}(\mathcal{X},\mathcal{Y})$ we mean the set of all isomorphisms in $\mathcal{L}(\mathcal{X},\mathcal{Y})$, and $\mathcal{L}\mathrm{aut}(\mathcal{X}) := \mathcal{L}\mathrm{is}(\mathcal{X},\mathcal{X})$ is the automorphism group in $\mathcal{L}(X)$. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is endowed with the uniform operator norm. In this situation $\mathcal{L}\mathrm{is}(\mathcal{X},\mathcal{Y})$ is open in $\mathcal{L}(\mathcal{X},\mathcal{Y})$. We write $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ for the duality pairing between $\mathcal{X}' := \mathcal{L}(\mathcal{X},\mathbb{K})$ and $\mathcal{X}$, that is, $\langle x',x\rangle_{\mathcal{X}}$ is the value of $x' \in \mathcal{X}'$ at $x \in \mathcal{X}$.

Let $H = \big(H,(\cdot\,|\,\cdot)\big)$ be a Hilbert space. Then the *Riesz isomorphism* is the conjugate linear isometric isomorphism $\vartheta = \vartheta_H : H \to H'$ defined by

$$\langle \vartheta x, y\rangle = (y\,|\,x), \qquad x,y \in H, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_H$. Then

$$(x'\,|\,y')^* := (\vartheta^{-1}y'\,|\,\vartheta^{-1}x'), \qquad x',y' \in H', \tag{2.2}$$

defines the *adjoint* inner product on H', and $H^* := \big(H',(\cdot\,|\,\cdot)^*\big)$ is the adjoint Hilbert space. Denoting by $\|\cdot\|$ and $\|\cdot\|^*$ the inner product norms associated with $(\cdot\,|\,\cdot)$ and $(\cdot\,|\,\cdot)^*$, respectively, we obtain from (2.1) and (2.2)

$$|\langle x',x\rangle| \le \|x'\|^*\,\|x\|, \qquad x' \in H', \quad x \in H. \tag{2.3}$$

It follows from (2.1)–(2.3) and the fact that ϑ is an isometry that $\|x'\|^* = \sup\{\, |\langle x', x\rangle|,\ \|x\| \le 1 \,\}$ for $x' \in H'$. Thus $\|\cdot\|^*$ is the norm in $H' = \mathcal{L}(H, \mathbb{K})$, the dual norm. In other words, $H' = H^*$ as Banach spaces. For this and historical reasons we use the 'star notation' for the dual space in the finite-dimensional setting and in connection with vector bundles, whereas the 'prime notation' is more appropriate in functional analytical considerations.

If H_1 and H_2 are Hilbert spaces and $A \in \mathcal{L}(H_1, H_2)$, then it has to be carefully distinguished between the dual $A' \in \mathcal{L}(H_2', H_1')$, defined by $\langle A'x_2', x_1\rangle_{H_1} = \langle x_2', Ax_1\rangle_{H_2}$, and the adjoint $A^* \in \mathcal{L}(H_2, H_1)$, given by $(A^*x_2 \,|\, x_1)_{H_1} = (x_2 \,|\, Ax_1)_{H_2}$ for $x_i \in H_i$ and $x_2' \in H_2'$.

Suppose H_1 and H_2 are finite-dimensional. Then $\mathcal{L}(H_1, H_2)$ is a Hilbert space with the Hilbert-Schmidt inner product $(\cdot|\cdot)_{HS}$ defined by $(A|B)_{HS} := \mathfrak{tr}(B^*A)$ for $A, B \in \mathcal{L}(H_1, H_2)$, where $\mathfrak{tr}$ denotes the trace. The corresponding norm $|\cdot|_{HS}$ is the Hilbert-Schmidt norm $A \mapsto \sqrt{\mathfrak{tr}(A^*A)}$.

Throughout this paper, we use the summation convention for indices labeling coordinates or bases. This means that such a repeated index, which appears once as a superscript and once as a subscript, implies summation over its whole range.

By a *manifold* we always mean a smooth, that is, C^∞ manifold with (possibly empty) boundary such that its underlying topological space is separable and metrizable. Thus, in the context of manifolds, we work in the smooth category. A manifold need not be connected, but all connected components are of the same dimension.

Let M be an m-dimensional manifold and $V = (V, \pi, M)$ a $\mathbb{K}$ vector bundle of rank n over M. For a nonempty subset S of M we denote by V_S, or $V_{|S}$, the restriction $\pi^{-1}(S)$ of V to S. If S is a submanifold, or $S = \partial M$, then V_S is a vector bundle of rank n over S. As usual, $V_p := V_{\{p\}}$ is the fibre $\pi^{-1}(p)$ of V over p. Occasionally, we use the symbolic notation $V = \bigcup_{p \in M} V_p$.

By $\Gamma(S, V)$ we mean the $\mathbb{K}^S$ module of all sections of V over S (no smoothness). If S is a submanifold, or $S = \partial M$, then $C^k(S, V)$ is for $k \in \mathbb{N} \cup \{\infty\}$ the Fréchet space of C^k sections over S. It is a $C^k(S) := C^k(S, \mathbb{K})$ module. In the case of a trivial bundle $M \times E = (M \times E, \mathrm{pr}_1, M)$ for some n-dimensional Banach space E, a section over S is a map from S into E, that is, $\Gamma(S, M \times E) = E^S$. Accordingly $C^k(S, M \times E) = C^k(S, E)$ is the Fréchet space of all C^k maps from S into E. As usual, pr_i denotes the natural projection onto the i-th factor of a Cartesian product (of sets).

Let $\widetilde{V} = (\widetilde{V}, \widetilde{\pi}, \widetilde{M})$ be a vector bundle over a manifold $\widetilde{M}$. A C^k map $(f_0, f) : (M, V) \to (\widetilde{M}, \widetilde{V})$, that is, $f_0 \in C^k(M, \widetilde{M})$ and $f \in C^k(V, \widetilde{V})$, is a C^k *bundle morphism* if the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ \ f\ \ } & \widetilde{V} \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\pi}} \\
M & \xrightarrow{\ \ f_0\ \ } & \widetilde{M}
\end{array}
$$

is commuting, and $f \,|\, V_p \in \mathcal{L}(V_p, V_{f_0(p)})$ for $p \in M$. It is a *conjugate linear* bundle morphism if $f \,|\, V_p$ is a conjugate linear map. By defining compositions of bundle morphisms in the obvious way one gets, in particular, the category of smooth, that is C^∞, bundles in which we work. Thus a *bundle isomorphism* is an isomorphism in the category of smooth vector bundles. If $M = \widetilde{M}$, then f is called *bundle morphism* if (id_M, f) is one.

A *bundle metric* on V is a smooth section h of the tensor product $V^* \otimes V^*$ such that $h(p)$ is an inner product on V_p for $p \in M$. Then the continuous map

$$|\cdot|_h : V \to C(M), \quad v \mapsto \sqrt{h(v, v)}$$

is the *bundle norm* derived from h.

Suppose $V = (V, h)$ is a *metric vector bundle*, that is, V is endowed with a bundle metric h. Then V_p is an n-dimensional Hilbert space with inner product $h(p)$. Hence $V_p^* = (V_p', h^*(p))$, where $h^*(p)$ is the adjoint inner product on V_p', equals V_p' as a Banach space. The dual bundle $V^* = \bigcup_{p \in M} V_p^*$ is endowed with the adjoint bundle metric h^* satisfying $h^* \,|\, (V^* \oplus V^*)_p = h^*(p)$ for $p \in M$, where $\oplus$ is the Whitney sum.

The (*bundle*) *duality pairing* $\langle\cdot,\cdot\rangle_V$ is the smooth section of $V \otimes V^*$ defined by $\langle\cdot,\cdot\rangle_V(p) = \langle\cdot,\cdot\rangle_{V_p}$ for $p \in M$. It follows

$$|\langle v^*, v\rangle_V| \le |v^*|_{h^*}\,|v|_h, \quad (v^*, v) \in \Gamma(M, V^* \oplus V).$$

We denote by $h_\flat(p): V_p \to V_p^*$ the Riesz isomorphism for $\big(V_p, h(p)\big)$ and by $h^\sharp(p)$ its inverse. This defines the $C^\infty(M)$-conjugate linear (*bundle*) *Riesz isomorphism* $h_\flat : V \to V^*$ and its inverse $h^\sharp : V^* \to V$, given by $h_\flat\,|\,V_p = h_\flat(p)$ and $h^\sharp\,|\,V_p^* = h^\sharp(p)$, respectively, for $p \in M$. Thus

$$\langle h_\flat v, w\rangle_V = h(w, v), \quad (v, w) \in \Gamma(M, V \oplus V).$$

The canonical identification of V_p^{**} with V_p implies

$$V^{**} = V, \quad \langle v, v^*\rangle_{V^*} = \langle v^*, v\rangle_V, \qquad (v, v^*) \in \Gamma(M, V \oplus V^*).$$

We fix an n-dimensional Hilbert space $E = \big(E, (\cdot|\cdot)_E\big)$, a *model fiber for* V. We also fix a basis $(e_1, \dots, e_n)$ of E and denote by $(\varepsilon^1, \dots, \varepsilon^n)$ the dual basis. Of course, without loss of generality we could set $E = \mathbb{K}^n$. However, for notational simplicity it is more convenient to use coordinate-free settings.

Let U be open in M. A *local chart for* V *over* U is a map

$$\kappa\ltimes\varphi : V_U \to \kappa(U) \times E, \quad v_p \mapsto \big(\kappa(p), \varphi(p)v_p\big), \qquad v_p \in V_p, \quad p \in U,$$

such that $(\kappa, \kappa\ltimes\varphi) : (U, V_U) \to \big(\kappa(U), \kappa(U) \times E\big)$ is a bundle isomorphism, where $\kappa(U)$ is open in the closed half-space $\mathbb{H}^m := \mathbb{R}^+ \times \mathbb{R}^{m-1}$ of $\mathbb{R}^m$ (and $\mathbb{R}^0 := \{0\}$). In particular, κ is a local chart for M.

Suppose $\kappa\ltimes\varphi$ and $\widetilde{\kappa}\ltimes\widetilde{\varphi}$ are local charts of V over U and $\widetilde{U}$, respectively. Then the *coordinate change*

$$(\widetilde{\kappa}\ltimes\widetilde{\varphi}) \circ (\kappa \circ \varphi)^{-1} : \kappa(U \cap \widetilde{U}) \times E \to \widetilde{\kappa}(U \cap \widetilde{U}) \times E$$

is given by $(x, \xi) \mapsto \big(\widetilde{\kappa} \circ \kappa^{-1}(x), \varphi_{\kappa\widetilde{\kappa}}(x)\xi\big)$, where

$$\varphi_{\kappa\widetilde{\kappa}} \in C^\infty\big(\kappa(U \cap \widetilde{U}), \mathcal{L}\mathrm{aut}(E)\big)$$

is the corresponding *bundle transition map*. It follows

$$\varphi_{\widetilde{\kappa}\widehat{\kappa}}\varphi_{\kappa\widetilde{\kappa}} = \varphi_{\kappa\widehat{\kappa}}, \quad \varphi_{\kappa\kappa} = 1_E, \tag{2.4}$$

1_E being the identity in $\mathcal{L}(E)$. We set

$$\varphi^{-\top}(p) := \big(\varphi^{-1}(p)\big)' \in \mathcal{L}\mathrm{is}(V_p^*, E^*), \qquad p \in U.$$

Then $\kappa\ltimes\varphi^{-\top} : V_U^* \to \kappa(U) \times E^*$ is the local chart for V^* over U *dual* to $\kappa\ltimes\varphi$.

In the following, we use standard notation for the pull-back and push-forward of functions, that is, $\kappa^* f = f \circ \kappa$ and $\kappa_* f = f \circ \kappa^{-1}$. The *push-forward by* $\kappa\ltimes\varphi$ is the vector space isomorphism

$$(\kappa\ltimes\varphi)_* : \Gamma(U, V) \to E^{\kappa(U)}, \quad v \mapsto \Big(x \mapsto \varphi\big(\kappa^{-1}(x)\big)v\big(\kappa^{-1}(x)\big)\Big).$$

Its inverse is the *pull-back*, defined by

$$(\kappa\ltimes\varphi)^* : E^{\kappa(U)} \to \Gamma(U, V), \quad \xi \mapsto \Big(p \mapsto \big(\varphi(p)\big)^{-1}\xi\big(\kappa(p)\big)\Big).$$

It follows that $(\kappa\ltimes\varphi)_*$ is a vector space isomorphism from $C^\infty(U, V)$ onto $C^\infty\big(\kappa(U), E\big)$, and

$$(\widetilde{\kappa}\ltimes\widetilde{\varphi})_*(\kappa\ltimes\varphi)^*\xi = \varphi_{\kappa\widetilde{\kappa}}\big(\xi \circ (\widetilde{\kappa} \circ \kappa^{-1})\big), \qquad \xi \in E^{\widetilde{\kappa}(U_\kappa \cap U_{\widetilde{\kappa}})}. \tag{2.5}$$

Furthermore,

$$\kappa_*\big(\langle v^*, v\rangle_V\big) = \big\langle (\kappa\ltimes\varphi^{-\top})_* v^*, (\kappa\ltimes\varphi)_* v\big\rangle_E, \qquad (v^*, v) \in \Gamma(U, V^* \oplus V). \tag{2.6}$$

In addition,

$$(\kappa\ltimes\varphi)_*(fv)=(\kappa_* f)(\kappa\ltimes\varphi)_* v,\qquad f\in\mathbb{K}^U,\quad v\in\Gamma(U,V). \tag{2.7}$$

We define the *coordinate frame* $(b_1,\dots,b_n)$ for V over U *associated with* $\kappa\ltimes\varphi$ by

$$b_\nu:=(\kappa\ltimes\varphi)^* e_\nu,\qquad 1\le\nu\le n.$$

Then

$$\beta^\nu:=(\kappa\ltimes\varphi^{-\top})^*\varepsilon^\nu,\qquad 1\le\nu\le n,$$

defines the *dual coordinate frame* for V^* over U. In fact, it follows from (2.6) that

$$\langle\beta^\mu,b_\nu\rangle_V=\kappa^*\bigl(\langle\varepsilon^\mu,e_\nu\rangle_E\bigr)=\delta^\mu_\nu,\qquad 1\le\mu,\nu\le n.$$

Let $(\widetilde{b}_1,\dots,\widetilde{b}_n)$ be the coordinate frame for V over $\widetilde{U}$ associated with $\widetilde{\kappa}\ltimes\widetilde{\varphi}$. Then (2.5) and (2.6) imply

$$\kappa_*\langle\beta^\mu,\widetilde{b}_\nu\rangle_V=\bigl\langle\varepsilon^\mu,(\kappa\ltimes\varphi)_*(\widetilde{\kappa}\ltimes\widetilde{\varphi})^* e_\nu\bigr\rangle_E=\langle\varepsilon^\mu,\varphi_{\widetilde{\kappa}\kappa}e_\nu\rangle_E=:(\varphi_{\widetilde{\kappa}\kappa})^\mu_\nu\in C^\infty\bigl(\kappa(U\cap\widetilde{U})\bigr).$$

Hence we infer from $\widetilde{b}_\nu=\langle\beta^\mu,\widetilde{b}_\nu\rangle_V b_\mu$ on $U\cap\widetilde{U}$ and (2.7) that

$$(\kappa\ltimes\varphi)_*\widetilde{b}_\nu=(\varphi_{\widetilde{\kappa}\kappa})^\mu_\nu e_\mu,\qquad 1\le\nu\le n. \tag{2.8}$$

The push-forward of the bundle metric h is the bundle metric $(\kappa\ltimes\varphi)_*h$ on $\kappa(U)\times E$ defined by

$$(\kappa\ltimes\varphi)_*h(\xi,\eta):=\kappa_*\Bigl(h\bigl((\kappa\ltimes\varphi)^*\xi,(\kappa\ltimes\varphi)^*\eta\bigr)\Bigr),\qquad \xi,\eta\in E^{\kappa(U)}. \tag{2.9}$$

Since h is a smooth section of $V^*\otimes V^*$ it has a local representation with respect to the dual coordinate frame:

$$h=h_{\mu\nu}\beta^\mu\otimes\beta^\nu,\qquad h_{\mu\nu}=h(b_\mu,b_\nu)\in C^\infty(U). \tag{2.10}$$

In the following, we endow $\mathbb{K}^{r\times s}$ with the Hilbert-Schmidt norm by identifying it with $\mathcal{L}(\mathbb{K}^s,\mathbb{K}^r)$ by means of the standard bases. Then we call $[h]:=[h_{\mu\nu}]\in C^\infty(U,\mathbb{K}^{n\times n})$ *representation matrix* of h with respect to the local coordinate frame $(b_1,\dots,b_n)$. Let $\widetilde{[h]}$ be the representation matrix of h with respect to the local coordinate frame associated with $\widetilde{\kappa}\ltimes\widetilde{\varphi}$. It follows from (2.8) that

$$\kappa_*\widetilde{[h]}=[\varphi_{\widetilde{\kappa}\kappa}]^\top\kappa_*[h]\overline{[\varphi_{\widetilde{\kappa}\kappa}]}\ \text{on}\ \kappa(U\cap\widetilde{U}), \tag{2.11}$$

where $[\varphi_{\widetilde{\kappa}\kappa}]$ is the representation matrix of $\varphi_{\widetilde{\kappa}\kappa}\in C^\infty\bigl(U,\mathcal{L}(E)\bigr)$ with respect to $(e_1,\dots,e_n)$ and $a^\top$ is the transposed of the matrix a.

It should also be noted that (2.9) implies

$$\kappa_*(|v|_h)=|(\kappa\ltimes\varphi)_*v|_{(\kappa\ltimes\varphi)_*h},\qquad v\in\Gamma(U,V). \tag{2.12}$$

Let $[h^*]$ be the representation matrix of h^* with respect to the dual coordinate frame on U. Denote by $[h^{\mu\nu}]$ the inverse of $[h]$. It is a consequence of $\langle b_\nu,h_\flat b_\mu\rangle_{V^*}=\langle h_\flat b_\mu,b_\nu\rangle_V=h(b_\nu,b_\mu)=h_{\nu\mu}$ that

$$h_\flat b_\mu=\langle b_\nu,h_\flat b_\mu\rangle_{V^*}\beta^\nu=h_{\nu\mu}\beta^\nu=\overline{h_{\mu\nu}}\beta^\nu.$$

Hence $h^\sharp\beta^\nu=h^{\nu\rho}b_\rho$. This implies $h^{*\mu\nu}=h^*(\beta^\mu,\beta^\nu)=h(h^\sharp\beta^\nu,h^\sharp\beta^\mu)=h^{\nu\rho}\overline{h^{\mu\sigma}}h_{\rho\sigma}=\overline{h^{\mu\nu}}$, that is,

$$[h]^{-1}=\overline{[h^*]}. \tag{2.13}$$

Let $V_i=(V_i,h_i)$ be a metric vector bundle of rank n_i over M, where $i=1,2$. Assume U is open in M and $\kappa\ltimes\varphi_i$ is a local chart for V_i over U. Denote by $(b^i_1,\dots,b^i_{n_i})$ the coordinate frame for V_i over U associated with $\kappa\ltimes\varphi_i$ and by $(\beta^1_i,\dots,\beta^{n_i}_i)$ its dual frame. Suppose $a\in\Gamma\bigl(U,\mathrm{Hom}(V_1,V_2)\bigr)$. Then

$$a=a^{\nu_2}_{\nu_1}b^2_{\nu_2}\otimes\beta^{\nu_1}_1,\qquad a^{\nu_2}_{\nu_1}=\langle\beta^{\nu_2}_2,ab^1_{\nu_1}\rangle_{V_2}\in\mathbb{K}^U. \tag{2.14}$$

Hence, given $u_i = u_i^{\nu_i} b_{\nu_i}^i \in \Gamma(U, V_i)$, it follows from (2.10) that

$$h_2(au_1, u_2) = a_{\nu_1}^{\nu_2} h_{2,\nu_2\widetilde{\nu}_2} u_1^{\nu_1} \overline{u_2^{\widetilde{\nu}_2}}.$$

For the adjoint section $a^* = a_{\nu_2}^{*\nu_1} b_{\nu_1}^1 \otimes \beta_2^{\nu_2} \in \Gamma\big(U, \mathrm{Hom}(V_2, V_1)\big)$ we find analogously

$$h_1(u_1, a^* u_2) = \overline{a_{\widetilde{\nu}_2}^{*\widetilde{\nu}_1}} h_{1,\nu_1\widetilde{\nu}_1} u_1^{\nu_1} \overline{u_2^{\widetilde{\nu}_2}}.$$

From $h_2(au_1, u_2) = h_1(u_1, a^*u_2)$ for all u_i in $\Gamma(U, V_i)$ we thus get $a_{\nu_1}^{\nu_2} h_{2,\nu_2\widetilde{\nu}_2} = \overline{a_{\widetilde{\nu}_2}^{*\widetilde{\nu}_1}} h_{1,\nu_1\widetilde{\nu}_1}$. Hence it follows from (2.13)

$$a_{\nu_2}^{*\nu_1} = h_1^{*\nu_1\widetilde{\nu}_1} \overline{a_{\widetilde{\nu}_1}^{\widetilde{\nu}_2}}\, \overline{h_{2,\widetilde{\nu}_2\nu_2}}, \qquad 1 \le \nu_i \le n_i. \tag{2.15}$$

The following well-known basic examples of vector bundles are included for later reference and to fix notation.

Examples 2.1 **(a)** (Trivial bundles) Consider the trivial vector bundle $V = (V, h) := \big(M \times E, (\cdot|\cdot)_E\big)$ with the usual identification of the inner product of E with the bundle metric $M \times E$. For any local chart κ of M, the *trivial bundle chart over* κ is given by $\kappa \ltimes 1_E$. Thus $(\kappa \ltimes 1_E)_* v = \kappa_* v$ for $v \in \Gamma\big(\mathrm{dom}(\kappa), M \times E\big) = E^{\mathrm{dom}(\kappa)}$.

(b) (Tangent bundles) Let $M = (M, g)$ be an m-dimensional Riemannian manifold. Throughout this paper we denote by TM the tangent bundle if $\mathbb{K} = \mathbb{R}$ and the complexified tangent bundle if $\mathbb{K} = \mathbb{C}$. Then g, respectively its complexification, is a bundle metric on TM (also denoted by g if $\mathbb{K} = \mathbb{C}$). Thus

$$T^*M := (TM)^* = (T^*M, g^*)$$

is the (complexified, if $\mathbb{K} = \mathbb{C}$) cotangent bundle of M.

We use $\mathbb{K}^m$ as the model fiber for TM and choose for $(e_1, \ldots, e_m)$ the standard basis $e_j^i = \delta_j^i$, $1 \le i, j \le m$. Furthermore, $(\cdot|\cdot) = (\cdot|\cdot)_{\mathbb{K}^m}$ is the Euclidean (Hermitean) inner product on $\mathbb{K}^m$ and $|\cdot| = |\cdot|_{\mathbb{K}^m}$ the corresponding norm. We identify $(\mathbb{K}^m)^*$ with $\mathbb{K}^m$ by means of the duality pairing

$$\langle \eta, \xi \rangle = \langle \eta, \xi \rangle_{\mathbb{K}^m} := \eta_i \xi^j, \qquad \eta = \eta_i \varepsilon^i, \quad \xi = \xi^j e_j, \tag{2.16}$$

so that $\varepsilon^i = e_i$ for $1 \le i \le m$.

Suppose κ is a local chart for M and set $U := \mathrm{dom}(\kappa)$. Denote by $T\kappa : T_U M = (TM)_U \to \kappa(U) \times \mathbb{K}^m$ the (complexified, if $\mathbb{K} = \mathbb{C}$) tangent map of κ. Then $\kappa \ltimes T\kappa$ is a local chart for TM over U, the *canonical chart* for TM over κ. It is completely determined by κ. For this reason $(\kappa \ltimes T\kappa)_* v$ is denoted, as usual, by $\kappa_* v$ for $v \in \Gamma(U, TM)$. Then the push-forward $\big(\kappa \ltimes (T\kappa)^{-\top}\big)_* w$ of a covector field $w \in \Gamma(U, T^*M)$ is the usual push-forward of w, denoted by $\kappa_* w$ also.

Note that the bundle transition map for the coordinate change $(\widetilde{\kappa} \ltimes T\widetilde{\kappa}) \circ (\kappa \ltimes T\kappa)^{-1}$ equals $\partial_x(\widetilde{\kappa} \circ \kappa^{-1})$, where ∂_x denotes the (Fréchet) derivative (on $\mathbb{R}^m$).

The coordinate frame for TM on U associated with κ, that is, with $\kappa \ltimes T\kappa$, equals $(\partial/\partial x^1, \ldots, \partial/\partial x^m)$. Its dual frame is $(dx^1, \ldots, dx^m)$. The representation matrix of g with respect to this frame is the *fundamental matrix* $[g_{ij}] \in C^\infty(U, \mathbb{K}^{m \times m})$ of M on U.

For abbreviation, we set $\mathcal{T}M := C^\infty(M, TM)$ and $\mathcal{T}^*M := C^\infty(M, T^*M)$. Then $\mathcal{T}M$, respectively $\mathcal{T}^*M$, is the $C^\infty(M)$ module of all (complexified, if $\mathbb{K} = \mathbb{C}$) smooth vector, respectively covector, fields on M. □

Let $V_i = (V_i, h_i)$ be a metric vector bundle over M for $i = 1, 2$. Then the dual $(V_1 \otimes V_2)^*$ of the tensor product $V_1 \otimes V_2$ is identified with $V_1^* \otimes V_2^*$ by means of the duality pairing $\langle \cdot, \cdot \rangle_{V_1 \otimes V_2}$ defined by

$$\langle v_1^* \otimes v_2^*, v_1 \otimes v_2 \rangle_{V_1 \otimes V_2} := \langle v_1^*, v_1 \rangle_{V_1} \langle v_2^*, v_2 \rangle_{V_2}, \qquad (v_i^*, v_i) \in \Gamma(M, V_i^* \oplus V_i). \tag{2.17}$$

By $h_1 \otimes h_2$ we denote the bundle metric for $V_1 \otimes V_2$, given by

$$h_1 \otimes h_2(v_1 \otimes v_2, w_1 \otimes w_2) := h_1(v_1, w_1) h_2(v_2, w_2), \qquad (v_i, w_i) \in \Gamma(M, V_i \oplus V_i). \tag{2.18}$$

We always equip $V_1 \otimes V_2$ with this metric.

Suppose that κ is a local chart for M and $\kappa\ltimes\varphi_i$ is a local chart for V_i over $\mathrm{dom}(\kappa)$. Then $\kappa\ltimes(\varphi_1\otimes\varphi_2)$ denotes the local chart for $V_1\otimes V_2$ over $\mathrm{dom}(\kappa)$ induced by $\kappa\ltimes\varphi_i$, $i=1,2$, that is,

$$\big(\kappa\ltimes(\varphi_1\otimes\varphi_2)\big)_*(v_1\otimes v_2)=(\kappa\ltimes\varphi_1)_*v_1\otimes(\kappa\ltimes\varphi_2)_*v_2,\qquad (v_1,v_2)\in\Gamma(M,V_1\oplus V_2). \tag{2.19}$$

It is obvious how these concepts generalize to tensor products of more than two vector bundles over M.

A *connection* on V is a map

$$\nabla:\ \mathcal{T}M\times C^\infty(M,V)\to C^\infty(M,V),\qquad (X,v)\mapsto\nabla_X v$$

which is $C^\infty(M)$ linear in the first argument, additive in its second, and satisfies the ‘product rule’

$$\nabla_X(fv)=(Xf)v+f\nabla_X v,\qquad X\in\mathcal{T}M,\quad v\in C^\infty(M,V),\quad f\in C^\infty(M), \tag{2.20}$$

where $Xf:=df(X)=\langle df,X\rangle:=\langle df,X\rangle_{TM}$. Equivalently, ∇ is considered as a $\mathbb{K}$ linear map,

$$\nabla:\ C^\infty(M,V)\to\mathcal{T}^*M\otimes C^\infty(M,V),$$

called *covariant derivative*, defined by

$$\langle\nabla v,X\otimes v^*\rangle_{TM\otimes V^*}=\langle v^*,\nabla_X v\rangle_V,\qquad v^*\in C^\infty(M,V^*),\quad v\in C^\infty(M,V),\quad X\in\mathcal{T}M, \tag{2.21}$$

and satisfying the product rule. Here and in similar situations, TM is identified with the ‘real’ subbundle of the complexification $TM+iTM$ if $\mathbb{K}=\mathbb{C}$. (In other words: We consider ‘real derivatives’ of complex-valued sections.)

A connection is *metric* if it satisfies

$$Xh(v,w)=h(\nabla_X v,w)+h(v,\nabla_X w),\qquad X\in\mathcal{T}M,\quad v,w\in C^\infty(M,V). \tag{2.22}$$

Let ∇ be a metric connection on V. Then we define a connection on V^*, again denoted by ∇, by

$$\langle\nabla_X v^*,v\rangle_V:=X\langle v^*,v\rangle_V-\langle v^*,\nabla_X v\rangle_V \tag{2.23}$$

for $v^*\in C^\infty(M,V^*)$, $v\in C^\infty(M,V)$, and $X\in\mathcal{T}M$. It follows for $v,w\in C^\infty(M,V)$ and $X\in\mathcal{T}M$ that, due to (2.22),

$$\begin{aligned}Xh(v,w)&=X\langle h_\flat w,v\rangle_V=\big\langle\nabla_X(h_\flat w),v\big\rangle_V+\langle h_\flat w,\nabla_X v\rangle_V\\&=\big\langle\nabla_X(h_\flat w),v\big\rangle_V+h(\nabla_X v,w)=\big\langle\nabla_X(h_\flat w),v\big\rangle_V+Xh(v,w)-h(v,\nabla_X w)\\&=\big\langle\nabla_X(h_\flat w),v\big\rangle_V+Xh(v,w)-\big\langle h_\flat(\nabla_X w),v\big\rangle_V.\end{aligned}$$

This and $h^\sharp=(h_\flat)^{-1}$ imply

$$\nabla\circ h_\flat=h_\flat\circ\nabla,\quad h^\sharp\circ\nabla=\nabla\circ h^\sharp.$$

Consequently,

$$\begin{aligned}Xh^*(v^*,w^*)&=Xh(h^\sharp w^*,h^\sharp v^*)=h(h^\sharp\nabla_X w^*,h^\sharp v^*)+h(h^\sharp w^*,h^\sharp\nabla_X v^*)\\&=h^*(\nabla_X v^*,w^*)+h^*(v^*,\nabla_X w^*)\end{aligned}$$

for $v^*,w^*\in C^\infty(M,V^*)$. This shows that ∇ is a metric connection on (V^*,h^*).

Let (V_i,h_i) be a metric vector bundle over M for $i=1,2$. Suppose ∇_i is a metric connection on V_i. Then

$$\nabla_X(v_1\otimes v_2):=\nabla_{1X}v_1\otimes v_2+v_1\otimes\nabla_{2X}v_2,\qquad v_i\in C^\infty(M,V_i),\quad X\in\mathcal{T}M, \tag{2.24}$$

defines a metric connection $\nabla=\nabla(\nabla_1,\nabla_2)$ on $V_1\otimes V_2$, the connection *induced* by ∇_1 and ∇_2. In the particular case where either $V_2=V_1$ or $V_2=V_1^*$ and $\nabla_2=\nabla_1$, we write again ∇_1 for $\nabla(\nabla_1,\nabla_1)$.

Let ∇ be a connection on V. Suppose $\kappa\ltimes\varphi$ is a local chart for V over U. The *Christoffel symbols* $\Gamma_{i\mu}^{\nu}$, $1\le i\le m,\ 1\le \mu,\nu\le n$, of ∇ with respect to $\kappa\ltimes\varphi$ are defined by

$$\nabla_{\partial/\partial x^i} b_\mu = \Gamma_{i\mu}^{\nu} b_\nu. \tag{2.25}$$

Here and in similar situations, it is understood that Latin indices run from 1 to m and Greek ones from 1 to n. It follows

$$\nabla v = \Bigl(\frac{\partial v^\nu}{\partial x^i} + \Gamma_{i\mu}^{\nu} v^\mu\Bigr)\, dx^i \otimes b_\nu, \qquad v = v^\nu b_\nu \in C^\infty(U,V). \tag{2.26}$$

Let V_1 and V_2 be metric vector bundles over M with metric connections ∇_1 and ∇_2, respectively. For a smooth section a of $\mathrm{Hom}(V_1,V_2)$ we define

$$(\nabla_{12}a)u := \nabla_2(au) - a\nabla_1 u, \qquad u\in C^\infty(M,V_1). \tag{2.27}$$

Then ∇_{12} is a metric connection on $\mathrm{Hom}(V_1,V_2)$, the one *induced* by ∇_1 and ∇_2, where $\mathrm{Hom}(V_1,V_2)$ is endowed with the (fiber-wise defined) Hilbert-Schmidt inner product. It is verified that this definition is consistent with (2.14) and (2.24). Hence we also write $\nabla(\nabla_1,\nabla_2)$ for ∇_{12}.

3 Uniform Regularity

Let M be an m-dimensional manifold. We set $Q := (-1,1)\subset\mathbb{R}$. If κ is a local chart for M, then we write U_κ for the corresponding coordinate patch $\mathrm{dom}(\kappa)$. A local chart κ is *normalized* if $\kappa(U_\kappa) = Q^m$ whenever $U_\kappa\subset \mathring{M}$, the interior of M, whereas $\kappa(U_\kappa) = Q^m\cap\mathbb{H}^m$ if $U_\kappa\cap\partial M\ne\emptyset$. We put $Q_\kappa^m := \kappa(U_\kappa)$ if κ is normalized.

An atlas $\mathfrak{K}$ for M has *finite multiplicity* if there exists $k\in\mathbb{N}$ such that any intersection of more than k coordinate patches is empty. In this case

$$\mathfrak{N}(\kappa) := \{\,\widetilde{\kappa}\in\mathfrak{K}\,;\ U_{\widetilde{\kappa}}\cap U_\kappa\ne\emptyset\,\}$$

has cardinality $\le k$ for each $\kappa\in\mathfrak{K}$. An atlas is *uniformly shrinkable* if it consists of normalized charts and there exists $r\in(0,1)$ such that $\bigl\{\,\kappa^{-1}(rQ_\kappa^m)\ ;\ \kappa\in\mathfrak{K}\,\bigr\}$ is a cover of M.

Given an open subset X of $\mathbb{R}^m$ or $\mathbb{H}^m$ and a Banach space $\mathcal{X}$ over $\mathbb{K}$, we write $\|\cdot\|_{k,\infty}$ for the usual norm of $BC^k(X,\mathcal{X})$, the Banach space of all $u\in C^k(X,\mathcal{X})$ such that $|\partial^\alpha u|_{\mathcal{X}}$ is uniformly bounded for $\alpha\in\mathbb{N}^m$ with $|\alpha|\le k$ (see Section 11).

By c we denote constants ≥ 1 whose numerical value may vary from occurrence to occurrence; but c is always independent of the free variables in a given formula, unless an explicit dependence is indicated.

Let S be a nonempty set. On $\mathbb{R}^S$ we introduce an equivalence relation $\sim$ by setting $f\sim g$ iff there exists $c\ge 1$ such that $f/c\le g\le cf$. Inequalities between bundle metrics have to be understood in the sense of quadratic forms.

An atlas $\mathfrak{K}$ for M is *uniformly regular* if

$$\begin{aligned} &\text{(i)} && \mathfrak{K} \text{ is uniformly shrinkable and has finite multiplicity.}\\ &\text{(ii)} && \|\widetilde{\kappa}\circ\kappa^{-1}\|_{k,\infty}\le c(k),\ \ \kappa,\widetilde{\kappa}\in\mathfrak{K},\ \ k\in\mathbb{N}. \end{aligned} \tag{3.1}$$

In (ii) and in similar situations it is understood that only $\kappa,\widetilde{\kappa}\in\mathfrak{K}$ with $U_\kappa\cap U_{\widetilde{\kappa}}\ne\emptyset$ are being considered. Two uniformly regular atlases $\mathfrak{K}$ and $\widetilde{\mathfrak{K}}$ are *equivalent*, $\mathfrak{K}\approx\widetilde{\mathfrak{K}}$, if

$$\begin{aligned} &\text{(i)} && \mathrm{card}\{\,\widetilde{\kappa}\in\widetilde{\mathfrak{K}}\,;\ U_{\widetilde{\kappa}}\cap U_\kappa\ne\emptyset\,\}\le c,\ \ \kappa\in\mathfrak{K}.\\ &\text{(ii)} && \|\widetilde{\kappa}\circ\kappa^{-1}\|_{k,\infty}\le c(k),\ \ \kappa\in\mathfrak{K},\ \ \widetilde{\kappa}\in\widetilde{\mathfrak{K}},\ \ k\in\mathbb{N}. \end{aligned} \tag{3.2}$$

Let V be a vector bundle of rank n over M with model fiber E. Suppose $\mathfrak{K}$ is an atlas for M and $\kappa\ltimes\varphi$ is for each $\kappa\in\mathfrak{K}$ a local chart for V over U_κ. Then $\mathfrak{K}\ltimes\Phi := \{\,\kappa\ltimes\varphi\ ;\ \kappa\in\mathfrak{K}\,\}$ is an *atlas for* V *over* $\mathfrak{K}$. It is *uniformly regular* if

$$\begin{aligned} &\text{(i)} && \mathfrak{K} \text{ is uniformly regular;}\\ &\text{(ii)} && \|\varphi_{\kappa\widetilde{\kappa}}\|_{k,\infty}\le c(k),\quad \kappa\ltimes\varphi,\widetilde{\kappa}\ltimes\widetilde{\varphi}\in\mathfrak{K}\ltimes\Phi,\ \ k\in\mathbb{N}, \end{aligned} \tag{3.3}$$

where $\varphi_{\kappa\widetilde{\kappa}}$ is the bundle transition map corresponding to the coordinate change $(\widetilde{\kappa}\ltimes\widetilde{\varphi})\circ(\kappa\ltimes\varphi)^{-1}$. Two atlases $\mathfrak{K}\ltimes\Phi$ and $\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}$ for V over $\mathfrak{K}$ and $\widetilde{\mathfrak{K}}$, respectively, are equivalent, $\mathfrak{K}\ltimes\Phi\approx\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}$, if

$$\begin{aligned} &\text{(i)} && \mathfrak{K}\approx\widetilde{\mathfrak{K}};\\ &\text{(ii)} && \|\varphi_{\kappa\widetilde{\kappa}}\|_{k,\infty}\le c(k),\ \ \kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi,\ \ \widetilde{\kappa}\ltimes\widetilde{\varphi}\in\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi},\ \ k\in\mathbb{N}. \end{aligned} \tag{3.4}$$

Suppose h is a bundle metric for V. Let $\mathfrak{K}\ltimes\Phi$ be a uniformly regular atlas for V over $\mathfrak{K}$. Then h is *uniformly regular over* $\mathfrak{K}\ltimes\Phi$ if

$$\begin{aligned} &\text{(i)} && (\kappa\ltimes\varphi)_*h\sim(\cdot|\cdot)_E,\ \ \kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi;\\ &\text{(ii)} && \|(\kappa\ltimes\varphi)_*h\|_{k,\infty}\le c(k),\ \ \kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi,\ \ k\in\mathbb{N}. \end{aligned} \tag{3.5}$$

Let $[h]_{\kappa\ltimes\varphi}=[h_{\mu\nu}]_{\kappa\ltimes\varphi}$ be the representation matrix of h with respect to the local coordinate frame associated with $\kappa\ltimes\varphi$. Then it follows from (2.10) that

$$\kappa_*\big([h]_{\kappa\ltimes\varphi}\big)=[\kappa_*h_{\mu\nu}]=\big[(\kappa\ltimes\varphi)_*h\big]. \tag{3.6}$$

Hence (3.5)(i) is equivalent to

$$|\zeta|^2/c\le\kappa_*h_{\mu\nu}(x)\zeta^\mu\overline{\zeta^\nu}\le c\,|\zeta|^2,,\qquad x\in Q_\kappa^m,\quad \zeta\in\mathbb{K}^n,\quad \kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi.$$

If $\mathfrak{K}\ltimes\Phi\approx\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}$ and h is uniformly regular over $\mathfrak{K}$, then we see from (2.4) and (2.11) that h is uniformly regular over $\widetilde{\mathfrak{K}}$.

Assume ∇ is a connection on V. Let $\mathfrak{K}\ltimes\Phi$ be an atlas for V over $\mathfrak{K}$. For $\kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi$ we denote by $\Gamma_{i\mu}^\nu[\kappa\ltimes\varphi]$ the Christoffel symbols of ∇ with respect to the coordinate frame for V over U_κ induced by $\kappa\ltimes\varphi$. Then ∇ is *uniformly regular over* $\mathfrak{K}\ltimes\Phi$ if

$$\begin{aligned} &\text{(i)} && \mathfrak{K}\ltimes\Phi\text{ is uniformly regular;}\\ &\text{(ii)} && \big\|\kappa_*\big(\Gamma_{i\mu}^\nu[\kappa\ltimes\varphi]\big)\big\|_{k,\infty}\le c(k),\ \ 1\le i\le m,\ \ 1\le\mu,\nu\le n,\ \ \kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi,\ \ k\in\mathbb{N}. \end{aligned}$$

Suppose ∇ is uniformly regular over $\mathfrak{K}\ltimes\Phi$ and $\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}\approx\mathfrak{K}\ltimes\Phi$. Then it follows from (2.8), (2.26), (3.2), and (3.4) that ∇ is uniformly regular over $\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}$.

A *uniformly regular structure* for M is a maximal family of equivalent uniformly regular atlases for it. We say M is a **uniformly regular manifold** if it is endowed with a uniformly regular structure. In this case it is understood that each uniformly regular atlas under consideration belongs to this uniformly regular structure.

Let M be uniformly regular and V a vector bundle over M. A *uniformly regular bundle structure* for V is a maximal family of equivalent uniformly regular atlases for V. Then V is a *uniformly regular vector bundle over* M, if it is equipped with a uniformly regular bundle structure. Again it is understood that in this case each atlas for V belongs to the given uniformly regular bundle structure. A *uniformly regular metric vector bundle* is a uniformly regular vector bundle endowed with a uniformly regular bundle metric. By a **fully uniformly regular vector bundle** $V=(V,h_V,\nabla_V)$ over M we mean a uniformly regular vector bundle V over M equipped with a uniformly regular bundle metric h_V and a uniformly regular metric connection ∇_V.

As earlier, it is the main purpose of the following examples to fix notation and to prepare the setting for further investigations.

Examples 3.1 **(a)** (Trivial bundles) Let $E=\big(E,(\cdot|\cdot)_E\big)$ be an n–dimensional Hilbert space. Suppose M is a uniformly regular manifold. It is obvious from Example 2.1(a) that the trivial bundle $M\times E$ is uniformly regular over M and $(\cdot|\cdot)_E$ is a uniformly regular bundle metric.

We consider E as a manifold of dimension n if $\mathbb{K}=\mathbb{R}$, and of dimension $2n$ if $\mathbb{K}=\mathbb{C}$ (using the standard identification of $\mathbb{C}=\mathbb{R}+i\mathbb{R}$ with $\mathbb{R}^2$) whose smooth structure is induced by the trivial chart 1_E. We identify TE canonically with $E\times E$. Then $Tv:\,TM\to TE=E\times E$, the tangential of $v\in C^\infty(M,E)$, is well-defined. We set

$$d_X\,v:=\mathrm{pr}_2\circ Tv(X),\qquad X\in\mathcal{T}M,\quad v\in C^\infty(M,E).$$

Then

$$d: \mathcal{T}M \times C^\infty(M,E) \to C^\infty(M,E), \quad (X,v) \mapsto d_X v$$

is a connection on $M \times E$, the E*-valued differential* on M.

Let $(e_1, \dots, e_n)$ be a basis for E and use the same symbol for the constant frame $p \mapsto (e_1, \dots, e_n)$ of $M \times E$. Then it follows that

$$df = df^\nu e_\nu, \qquad f = f^\nu e_\nu \in C^\infty(M,E).$$

Thus, since all Christoffel symbols are identically zero, d is trivially uniformly regular.

(b) (Subbundles) Let V be a vector bundle of rank n over a manifold M, endowed with a bundle metric h and a metric connection ∇. Suppose W is a subbundle of rank ℓ. Denote by $\iota: W \hookrightarrow V$ the canonical injection. Let $h_W := \iota^* h$ be the pull-back metric on W. We write P for the the orthogonal projection onto W in V. Then $P \in C^\infty\bigl(M, \mathrm{Hom}(V,V)\bigr)$ and it is verified that

$$\nabla_W: \mathcal{T}M \times C^\infty(M,W) \to C^\infty(M,W), \quad (X,w) \mapsto P\nabla_X\bigl(\iota(w)\bigr)$$

is a metric connection on (W, h_W), the one induced by ∇.

Let E be a model fiber of V and $(e_1, \dots, e_n)$ a basis for it. Suppose V is uniformly regular and there exists an atlas $\mathfrak{K} \ltimes \Phi$ for V such that $(\kappa \ltimes \varphi)^*(e_1, \dots, e_\ell)$ is for each $\kappa \in \mathfrak{K}$ a frame for W over U_κ. Then it is checked that $W = (W, h_W, \nabla_W)$ is a fully uniformly regular vector bundle over M.

Suppose $V_i = (V_i, h_i, \nabla_i)$, $i = 1,2$, are fully uniformly regular vector bundles over M. Set

$$(h_1 \oplus h_2)(v_1 \oplus v_2,\ \widetilde{v}_1 \oplus \widetilde{v}_2) := h_1(v_1, \widetilde{v}_1) + h_2(v_2, \widetilde{v}_2), \qquad (v_i, \widetilde{v}_i) \in \Gamma(M, V_i \oplus V_i),$$

and

$$(\nabla_1 \oplus \nabla_2)(v_1 \oplus v_2) := \nabla_1 v_1 \oplus \nabla_2 v_2, \qquad (v_1, v_2) \in C^\infty(M, V_1 \oplus V_2).$$

Then $(V_1 \oplus V_2,\ h_1 \oplus h_2,\ \nabla_1 \oplus \nabla_2)$ is a fully uniformly regular vector bundle over M. Furthermore, V_i is for $i = 1,2$ a fully uniformly regular subbundle of V.

(c) (Riemannian manifolds) Let $M = (M,g)$ be an m-dimensional Riemannian manifold. We denote by $g_m = (dx^1)^2 + \cdots + (dx^m)^2$ the Euclidean metric on $\mathbb{R}^m$ and use the same symbol for its complexification as well as for the restriction thereof to open subsets of $\mathbb{R}^m$ and $\mathbb{H}^m$. Then M is a **uniformly regular Riemannian manifold**, if TM is uniformly regular and g is a uniformly regular bundle metric on TM. It follows from Example 2.1(b) that M is a uniformly regular Riemannian manifold iff

$$\begin{aligned} &\text{(i)} && M \text{ is uniformly regular;} \\ &\text{(ii)} && \kappa_* g \sim g_m, \quad \kappa \in \mathfrak{K}; \\ &\text{(iii)} && \|\kappa_* g\|_{k,\infty} \le c(k), \quad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}, \end{aligned} \tag{3.7}$$

for some uniformly regular atlas $\mathfrak{K}$ for M. Of course, $\kappa_* g := (\kappa \ltimes T\kappa)_* g$ in conformity with standard usage.

We denote by ∇_g the (complexified, if $\mathbb{K} = \mathbb{C}$) Levi-Civita connection for M, that is, for TM. Its Christoffel symbols with respect to the coordinate frame $(\partial/\partial x^1, \dots, \partial/\partial x^m)$ over U_κ admit the representation

$$2\Gamma_{ij}^k = g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{\ell i} - 2\partial_\ell g_{ij}), \tag{3.8}$$

where $\partial_i := \partial/\partial x^i$. From this and (3.7)(ii) and (iii) it follows that ∇_g is uniformly regular if (M,g) is a uniformly regular Riemannian manifold. In addition, ∇_g is metric and $\Gamma_{ij}^k = \Gamma_{ji}^k$.

(d) Every compact Riemannian manifold is a uniformly regular Riemannian manifold.

(e) It has been shown in Example 2.1(c) of [5] that $\mathbb{R}^m = (\mathbb{R}^m, g_m)$ and $\mathbb{H}^m = (\mathbb{H}^m, g_m)$ are uniformly regular Riemannian manifolds.

(f) (Homomorphism bundles) For $i = 1,2$ let (V_i, h_i) be a uniformly regular metric vector bundle of rank n_i over M. We denote by (V_{12}, h_{12}) the homomorphism bundle $V_{12} := \mathrm{Hom}(V_1, V_2)$ endowed with the Hilbert-Schmidt bundle metric $h_{12} = (\cdot|\cdot)_{HS}$.

Assume $\mathfrak{K}\ltimes\Phi_i$ is a uniformly regular atlas for V_i, and E_i is a model fiber for V_i with basis $(e_1^i,\dots,e_{n_i}^i)$ and dual basis $(\varepsilon_i^1,\dots,\varepsilon_i^{n_i})$. For $\kappa\ltimes\varphi_i\in\mathfrak{K}\ltimes\Phi_i$ we define a bundle isomorphism

$$(\kappa,\kappa\ltimes\varphi_i):\ \bigl(U_\kappa,(V_{12})_{U_\kappa}\bigr)\to\bigl(\kappa(U_\kappa),\kappa(U_\kappa)\times\mathcal{L}(E_1,E_2)\bigr)$$

by setting $(\kappa\ltimes\varphi_{12})a_p:=\bigl(\kappa(p),\varphi_{12}(p)a_p\bigr)$ for $p\in U_\kappa$ and $a_p\in(V_{12})_p$, where

$$\varphi_{12}(p)a_p(x):=\varphi_2(p)a_p\varphi_1^{-1}(x),\qquad x=\kappa(p).$$

It follows

$$(\widetilde{\kappa}\ltimes\widetilde{\varphi}_{12})_*(\kappa\ltimes\varphi_{12})^*b=(\widetilde{\kappa}\ltimes\widetilde{\varphi}_2)_*(\kappa\ltimes\varphi_2)^*b(\kappa\ltimes\varphi_1)_*(\widetilde{\kappa}\ltimes\widetilde{\varphi}_1)^*,\qquad b\in\mathcal{L}(E_1,E_2),$$

if $\widetilde{\kappa}\ltimes\widetilde{\varphi}_i$ belongs to a uniformly regular atlas for V_i. From this we deduce that

$$\mathfrak{K}_{12}:=\{\,\kappa\ltimes\varphi_{12}\ ;\ \kappa\ltimes\varphi_i\in\mathfrak{K}\ltimes\Phi_i,\ i=1,2\,\}$$

is a uniformly regular atlas for V_{12} and that any two such atlases are equivalent. Hence V_{12} is a uniformly regular vector bundle over M.

The coordinate frame of V_{12} over U_κ associated with $\kappa\ltimes\varphi_{12}$ is given by

$$\{\,b_{\nu_2}^2\otimes\beta_1^{\nu_1}\ ;\ 1\le\nu_i\le n_i,\ i=1,2\,\},\tag{3.9}$$

where $(b_1^i,\dots,b_{n_i}^i)$ is the coordinate frame of V_i over U_κ associated with $\kappa\ltimes\varphi_i$ and $(\beta_i^1,\dots,\beta_i^{n_i})$ is its dual frame. By (2.15) and (3.9) we find

$$[h_{12}]=\bigl[h_1^{*\nu_1\widetilde{\nu}_1}\,\overline{h_{2,\widetilde{\nu}_2\nu_2}}\,\bigr].\tag{3.10}$$

From this, (2.10), (3.5), and (3.6) we deduce

$$(\kappa\ltimes\varphi_{12})_*h_{12}(a,a)=\kappa_*h_1^{*\nu_1\widetilde{\nu}_1}\kappa_*h_{2,\nu_2\widetilde{\nu}_2}a_{\nu_1}^{\nu_2}\overline{a_{\widetilde{\nu}_1}^{\widetilde{\nu}_2}}\sim\sum_{\nu_2}\kappa_*h_1^{*\nu_1\widetilde{\nu}_1}a_{\nu_1}^{\nu_2}\overline{a_{\widetilde{\nu}_1}^{\nu_2}}\sim\sum_{\nu_1,\nu_2}a_{\nu_1}^{\nu_2}\overline{a_{\nu_1}^{\nu_2}}=(a,a)_{HS}$$

for $a\in\mathcal{L}(E_1,E_2)$, as well as $\|(\kappa\ltimes\varphi_{12})_*h_{12}\|_{k,\infty}\le c(k)$ for $\kappa\ltimes\varphi_{12}\in\mathfrak{K}\ltimes\Phi_{12}$ and $k\in\mathbb{N}$. Hence (V_{12},h_{12}) is a uniformly regular metric vector bundle over M.

Suppose ∇_i is a uniformly regular metric connection on V_i. Then it is a consequence of the consistency of (2.27) with (2.24) that ∇_{12} is a uniformly regular metric connection on V_{12}.

(g) (Tensor products) Let (V_i,h_i), $i=1,2$, be uniformly regular metric vector bundles over M. Then it follows from (2.17)–(2.19) that $(V_1\otimes V_2,\ h_1\otimes h_2)$ is a uniformly regular metric vector bundle over M. If ∇_i is a uniformly regular metric connection on V_i, then we see from (2.24) that $\nabla(\nabla_1,\nabla_2)$ is a uniformly regular metric connection on $V_1\otimes V_2$. □

4 Singular Manifolds

Let $M=(M,g)$ be an m-dimensional Riemannian manifold. Suppose $\rho\in C^\infty\bigl(M,(0,\infty)\bigr)$. Then $(\rho,\mathfrak{K})$ is a *singularity datum* for M if

$$\begin{aligned}
&\text{(i)}\quad (M,g/\rho^2)\text{ is a uniformly regular Riemannian manifold.}\\
&\text{(ii)}\quad \mathfrak{K}\text{ is a uniformly regular atlas for } M \text{ which is orientation preserving if } M \text{ is oriented.}\\
&\text{(iii)}\quad \|\kappa_*\rho\|_{k,\infty}\le c(k)\rho_\kappa,\ \ \kappa\in\mathfrak{K},\ \ k\in\mathbb{N},\ \text{where } \rho_\kappa:=\kappa_*\rho(0)=\rho\bigl(\kappa^{-1}(0)\bigr).\\
&\text{(iv)}\quad \rho_\kappa/c\le\rho(p)\le c\rho_\kappa,\ \ p\in U_\kappa,\ \ \kappa\in\mathfrak{K}.
\end{aligned}\tag{4.1}$$

Two singularity data $(\rho,\mathfrak{K})$ and $(\widetilde{\rho},\widetilde{\mathfrak{K}})$ are *equivalent*, $(\rho,\mathfrak{K})\approx(\widetilde{\rho},\widetilde{\mathfrak{K}})$, if

$$\rho\sim\widetilde{\rho}\quad\text{and}\quad\mathfrak{K}\approx\widetilde{\mathfrak{K}}.\tag{4.2}$$

Note that (4.1)(iv) and (4.2) imply

$$1/c \le \rho_\kappa/\rho_{\widetilde{\kappa}} \le c, \qquad \kappa \in \mathfrak{K}, \quad \widetilde{\kappa} \in \widetilde{\mathfrak{K}}, \quad U_\kappa \cap U_{\widetilde{\kappa}} \ne \emptyset. \tag{4.3}$$

A *singularity structure*, $\mathfrak{S}(M)$, for M is a maximal family of equivalent singularity data. A *singularity function* for M is a function $\rho \in C^\infty\big(M,(0,\infty)\big)$ such that there exists an atlas $\mathfrak{K}$ with $(\rho,\mathfrak{K}) \in \mathfrak{S}(M)$. The set of all singularity functions is the *singularity type*, $\mathfrak{T}(M)$, of M. By a **singular manifold** we mean a Riemannian manifold M endowed with a singularity structure $\mathfrak{S}(M)$. Then M is said to be *singular of type* $\mathfrak{T}(M)$. If $\rho \in \mathfrak{T}(M)$, then it is convenient to set $[\![\rho]\!] := \mathfrak{T}(M)$.

Let M be singular of type $[\![\rho]\!]$. Then M is a uniformly regular Riemannian manifold iff $\rho \sim \mathbf{1}$. If $\rho \not\sim \mathbf{1}$, then either $\inf \rho = 0$ or $\sup \rho = \infty$, or both. Hence M is not compact but has singular ends. It follows from (4.1) that the diameter of the coordinate patches converges either to zero or to infinity near the singular ends in a manner controlled by the singularity type $\mathfrak{T}(M)$.

We refer to [5] and [6] for examples of singular manifolds which are not uniformly regular Riemannian manifolds.

Throughout the rest of this paper we assume

$$\boxed{\begin{array}{l} M = (M,g) \text{ is an } m\text{-dimensional singular manifold.} \\ W = (W, h_W, D) \text{ is a fully uniformly regular vector bundle of rank } n \text{ over } M. \\ \sigma,\tau \in \mathbb{N}. \end{array}} \tag{4.4}$$

It follows from the preceding section that the uniform regularity of W, h_W, and D is independent of the particular choice of the singularity datum $(\rho,\mathfrak{K})$.

Henceforth, TM and T^*M have to be interpreted as the complexified tangent and cotangent bundles, respectively, if $\mathbb{K} = \mathbb{C}$. Accordingly, $\langle\cdot,\cdot\rangle_{TM}$, g, and ∇_g are then the complexified duality pairing, Riemannian metric, and Levi-Civita connection, respectively.

As usual, $T^\sigma_\tau M = TM^{\otimes\sigma} \otimes T^*M^{\otimes\tau}$ is the (σ,τ)-tensor bundle, that is, the vector bundle of all $\mathbb{K}$-valued tensors on M being contravariant of order σ and covariant of order τ. In particular, $T^1_0 M = TM$, $T^0_1 M = T^*M$, and $T^0_0 M = M \times \mathbb{K}$. Then

$$V = V^\sigma_\tau(W) = T^\sigma_\tau(M,W) := T^\sigma_\tau M \otimes W$$

is the vector bundle of *W-valued (σ,τ)-tensors on M*.

If $W = M \times E$ with an n-dimensional Hilbert space E, then we write $T^\sigma_\tau(M,E)$ for $T^\sigma_\tau(M, M\times E)$ and call its elements E-valued (σ,τ)-tensors. Furthermore, $T^\sigma_\tau(M,\mathbb{K})$ is naturally identified with $T^\sigma_\tau M$. For abbreviation, we set

$$\mathcal{T}^\sigma_\tau(M,W) := C^\infty\big(M, T^\sigma_\tau(M,W)\big).$$

It is the $C^\infty(M)$ module of smooth *W-valued (σ,τ)-tensor fields on M*.

The canonical identification of $(T^\sigma_\tau M)^*$ with $T^\tau_\sigma M$ leads to $T^\sigma_\tau(M,W)^* = T^\tau_\sigma(M,W^*)$ with respect to the (bundle) duality pairing

$$\langle\cdot,\cdot\rangle_V := \langle\cdot,\cdot\rangle_{T^\sigma_\tau M} \otimes \langle\cdot,\cdot\rangle_W.$$

We endow V with the bundle metric

$$h := (\cdot|\cdot)^\tau_\sigma \otimes h_W, \tag{4.5}$$

where $(\cdot|\cdot)^\tau_\sigma := g^{\otimes\sigma} \otimes g^{*\otimes\tau}$ is the bundle metric on $T^\sigma_\tau M$ induced by g (denoted by $(\cdot|\cdot)_g$ in Section 3 of [5]).

Finally, we equip V with the metric connection

$$\nabla := \nabla(\nabla_g, D)$$

induced by the Levi-Civita connection of M and connection D of W. In summary, in addition to (4.4),

$$\boxed{V = (V, h, \nabla) := \big(T^\sigma_\tau(M,W),\ (\cdot|\cdot)^\tau_\sigma \otimes h_W,\ \nabla(\nabla_g, D)\big)}$$

is a standing assumption. In particular, ∇ is a $\mathbb{K}$-linear map from $\mathcal{T}^\sigma_\tau(M,W)$ into $\mathcal{T}^\sigma_{\tau+1}(M,W)$. We set $\nabla^0 := \mathrm{id}$ and $\nabla^{k+1} := \nabla \circ \nabla^k$ for $k \in \mathbb{N}$. Note $\nabla u = Du$ for $u \in \mathcal{T}^0_0(M,W) = C^\infty(M,W)$.

5 Local Representations

Although W is a fully uniformly regular vector bundle over M this is not true for V, due to the fact that h involves the singular Riemannian metric g. For this reason we have to study carefully the dependence of various local representations on the singularity datum. This is done in the present section.

For a subset S of M and a normalized atlas $\mathfrak{K}$ we let $\mathfrak{K}_S := \{\, \kappa \in \mathfrak{K}\ ;\ U_\kappa \cap S \neq \emptyset \,\}$; hence $\mathfrak{K}_\emptyset = \emptyset$. Then, given $\kappa \in \mathfrak{K}$,

$$\mathbb{X}_\kappa := \begin{cases} \mathbb{R}^m & \text{if } \kappa \in \mathfrak{K}\backslash\mathfrak{K}_{\partial M}, \\ \mathbb{H}^m & \text{otherwise}, \end{cases} \tag{5.1}$$

considered as an m-dimensional uniformly regular Riemannian manifold with the Euclidean metric. Furthermore, Q^m_κ is an open Riemannian submanifold of $\mathbb{X}_\kappa$.

Let F be a finite-dimensional Hilbert space. Then, using standard identifications,

$$T^\sigma_\tau(Q^m_\kappa, F) = (\mathbb{K}^m)^{\otimes\sigma} \otimes \big((\mathbb{K}^m)^*\big)^{\otimes\tau} \otimes F.$$

Of course, we identify $(\mathbb{K}^m)^*$ with $\mathbb{K}^m$ by means of (2.16), but continue to denote it by $(\mathbb{K}^m)^*$ for clarity. We endow $T^\sigma_\tau(Q^m_\kappa, F)$ with the inner product

$$(\cdot|\cdot)_{T^\sigma_\tau(Q^m_\kappa,F)} := (\cdot|\cdot)^{\otimes\sigma}_{\mathbb{K}^m} \otimes (\cdot|\cdot)^{\otimes\tau}_{(\mathbb{K}^m)^*} \otimes (\cdot|\cdot)_F. \tag{5.2}$$

For $\nu \in \mathbb{N}^\times$ we set $\mathbb{J}_\nu := \{1,\dots,m\}^\nu$ and denote its general point by $(i) = (i_1,\dots,i_\nu)$. The standard basis $(\breve{e}_1,\dots,\breve{e}_m)$ of $\mathbb{K}^m$, that is, $\breve{e}^i_j = \delta^i_j$, and its dual basis $(\breve{\varepsilon}^1,\dots,\breve{\varepsilon}^m)$ induce the *standard basis*

$$\big\{\, \breve{e}_{(i)} \otimes \breve{\varepsilon}^{(j)}\ ;\ (i) \in \mathbb{J}_\sigma,\ (j) \in \mathbb{J}_\tau \,\big\}$$

of $T^\sigma_\tau Q^m_\kappa$, where $\breve{e}_{(i)} = \breve{e}_{i_1} \otimes \cdots \otimes \breve{e}_{i_\sigma}$ and $\breve{\varepsilon}^{(j)} = \breve{\varepsilon}^{j_1} \otimes \cdots \otimes \breve{\varepsilon}^{j_\tau}$. Then

$$a \in T^\sigma_\tau(Q^m_\kappa, F) = \mathcal{L}\big(\big((\mathbb{K}^m)^*\big)^{\otimes\sigma} \otimes (\mathbb{K}^m)^{\otimes\tau}, F\big)$$

has the representation matrix $\big[a^{(i)}_{(j)}\big] \in F^{m^\sigma \times m^\tau}$. We endow $F^{m^\sigma \times m^\tau}$ with the inner product

$$\big(\big[a^{(i)}_{(j)}\big] \,\big|\, \big[b^{(\tilde{i})}_{(\tilde{j})}\big]\big)_{HS,F} := \sum_{(i)\in\mathbb{J}_\sigma,\ (j)\in\mathbb{J}_\tau} \big(a^{(i)}_{(j)} \,\big|\, b^{(i)}_{(j)}\big)_F$$

which coincides with the Hilbert-Schmidt inner product if $F = \mathbb{K}$. For abbreviation, we set

$$E = E^\sigma_\tau = E^\sigma_\tau(F) := F^{m^\sigma \times m^\tau}, \quad (\cdot|\cdot)_E := (\cdot|\cdot)_{HS,F}. \tag{5.3}$$

It follows from (5.2) that $a \mapsto \big[a^{(i)}_{(j)}\big]$ defines an isometric isomorphism by which

$$\text{we identify } \big(T^\sigma_\tau(Q^m_\kappa, F), (\cdot|\cdot)_{T^\sigma_\tau(Q^m_\kappa,F)}\big) \text{ with } \big(E, (\cdot|\cdot)_E\big).$$

We assume

- $(\rho, \mathfrak{K})$ is a singularity datum for M;
- $\mathfrak{K} \ltimes \Phi$ is a uniformly regular atlas for W over $\mathfrak{K}$; (5.4)
- $F = \big(F, (\cdot|\cdot)_F\big)$ is a model fiber for W with basis $(e_1,\dots,e_n)$.

Suppose $\kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi$ and $\kappa=(x^1,\dots,x^m)$. Then

$$\kappa\ltimes\varphi^\sigma_\tau:\ V_{U_\kappa}\to Q^m_\kappa\times E,\quad v_p\mapsto\big(\kappa(p),\varphi^\sigma_\tau(p)v_p\big),\quad v_p\in V_p,\quad p\in U_\kappa,$$

the *local chart for V over U_κ induced by $\kappa\ltimes\varphi$*, is defined by

$$\varphi^\sigma_\tau(p)v_p:=(T_p\kappa)X^1_p\otimes\cdots\otimes(T_p\kappa)X^\sigma_p\otimes(T_p\kappa)^{-\top}\alpha_{1,p}\otimes\cdots\otimes(T_p\kappa)^{-\top}\alpha_{\tau,p}\otimes\varphi(p)w_p$$

for $v_p=X^1_p\otimes\cdots\otimes X^\sigma_p\otimes\alpha_{1,p}\otimes\cdots\otimes\alpha_{\tau,p}\otimes w_p\in T^\sigma_\tau(M,W)_p$ with $X^i_p\in T_pM,\ \alpha_{j,p}\in T^*_pM$, and w_p belonging to W_p.

Set

$$\frac{\partial}{\partial x^{(i)}}:=\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_\sigma}},\quad dx^{(j)}:=dx^{j_1}\otimes\cdots\otimes dx^{j_\tau},\qquad (i)\in\mathbb{J}_\sigma,\quad (j)\in\mathbb{J}_\tau.$$

Furthermore, let $(b_1,\dots,b_n)$ be the coordinate frame for W over U_κ associated with $\kappa\ltimes\varphi$ and $(\beta^1,\dots,\beta^n)$ its dual frame. Then

$$\Big\{\frac{\partial}{\partial x^{(i)}}\otimes dx^{(j)}\otimes b_\nu\ ;\ (i)\in\mathbb{J}_\sigma,\ (j)\in\mathbb{J}_\tau,\ 1\le\nu\le n\Big\}\tag{5.5}$$

is the coordinate frame for V over U_κ associated with $\kappa\ltimes\varphi^\sigma_\tau$. Hence $v\in\Gamma(U_\kappa,V)$ has the local representation

$$v=v^{(i),\nu}_{(j)}\frac{\partial}{\partial x^{(i)}}\otimes dx^{(j)}\otimes b_\nu$$

and

$$\varphi^\sigma_\tau v(x)=\big[v^{(i),\nu}_{(j)}\big(\kappa^{-1}(x)\big)e_\nu\big]\in F^{m^\sigma\times m^\tau}=E,\qquad x\in Q^m_\kappa.$$

Assume $\widetilde{\kappa}\ltimes\widetilde{\varphi}\in\mathfrak{K}\ltimes\Phi$. Then $(\widetilde{\kappa}\ltimes\widetilde{\varphi}^\sigma_\tau)\circ(\kappa\ltimes\varphi^\sigma_\tau)^{-1}=\big(\widetilde{\kappa}\circ\kappa^{-1},\ (\varphi^\sigma_\tau)_{\kappa\widetilde{\kappa}}\big)$, where

$$\big((\varphi^\sigma_\tau)_{\kappa\widetilde{\kappa}}\xi\big)^{(i),\nu}_{(j)}=A^{(i)}_{(\widetilde{\imath})}B^{(\widetilde{\jmath})}_{(j)}(\varphi_{\kappa\widetilde{\kappa}})^\nu_{\widetilde{\nu}}\xi^{(\widetilde{\imath}),\widetilde{\nu}}_{(\widetilde{\jmath})},\qquad \xi\in E,\tag{5.6}$$

with $A^{(i)}_{(\widetilde{\imath})}=A^{i_1}_{\widetilde{\imath}_1}\cdots A^{i_\sigma}_{\widetilde{\imath}_\sigma}$ and $B^{(\widetilde{\jmath})}_{(j)}=B^{\widetilde{\jmath}_1}_{j_1}\cdots B^{\widetilde{\jmath}_\tau}_{j_\tau}$, and

$$A^i_{\widetilde{\imath}}=\frac{\partial(\widetilde{\kappa}\circ\kappa^{-1})^i}{\partial x^{\widetilde{\imath}}},\quad B^{\widetilde{\jmath}}_j=\frac{\partial(\kappa\circ\widetilde{\kappa}^{-1})^{\widetilde{\jmath}}}{\partial y^j}\circ(\widetilde{\kappa}\circ\kappa^{-1})\tag{5.7}$$

for $1\le i,\widetilde{\imath},j,\widetilde{\jmath}\le n$ and $y=\widetilde{\kappa}\circ\kappa^{-1}(x)$. Hence (3.1), (3.3), and assumption (4.4) imply that

$$\mathfrak{K}\ltimes\Phi^\sigma_\tau:=\{\,\kappa\ltimes\varphi^\sigma_\tau\ ;\ \kappa\ltimes\varphi\in\mathfrak{K}\ltimes\Phi\,\}$$

is a uniformly regular atlas for V over $\mathfrak{K}$. From (3.2) and (3.4) we also infer that

$$\mathfrak{K}\ltimes\Phi\approx\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}\quad\Rightarrow\quad\mathfrak{K}\ltimes\Phi^\sigma_\tau\approx\widetilde{\mathfrak{K}}\ltimes\widetilde{\Phi}^\sigma_\tau.$$

The local chart $\kappa\ltimes\varphi^\sigma_\tau$ is completely determined by $\kappa\ltimes\varphi$. For this reason, and to simplify notation, we denote the push-forward and pull-back by $\kappa\ltimes\varphi^\sigma_\tau$ simply by $(\kappa\ltimes\varphi)_*$ and $(\kappa\ltimes\varphi)^*$, respectively. This is consistent with the use of κ_* for the push-forward of vector fields by $\kappa\ltimes\varphi$ (see Example 2.1(b)).

We set

$$g^{(j)(\ell)}_{(i)(k)}:=g_{i_1k_1}\cdots g_{i_\sigma k_\sigma}g^{j_1\ell_1}\cdots g^{j_\tau\ell_\tau}$$

with (i), (k) running through $\mathbb{J}_\sigma$ and (j), (ℓ) through $\mathbb{J}_\tau$. Then (4.5) and (2.13) imply

$$h(u,v)=g^{(j)(\ell)}_{(i)(k)}u^{(i),\nu}_{(j)}\overline{v^{(k),\mu}_{(\ell)}}h_W(b_\nu,b_\mu),\qquad u,v\in\Gamma(U_\kappa,V).$$

Hence, setting $u_\kappa:=(\kappa\ltimes\varphi)_*u$ etc., we get from (2.9)

$$(\kappa\ltimes\varphi)_*h(u_\kappa,v_\kappa)=\kappa_*g^{(j)(\ell)}_{(i)(k)}\kappa_*u^{(i),\nu}_{(j)}\kappa_*\overline{v^{(k),\mu}_{(\ell)}}\kappa_*h_{W_{\nu\mu}}.\tag{5.8}$$

Lemma 3.1 of [5] guarantees

$$\kappa_* g \sim \rho_\kappa^2 g_m, \quad \kappa_* g^* \sim \rho_\kappa^{-2} g_m, \qquad \kappa \in \mathfrak{K}, \tag{5.9}$$

and

$$\rho_\kappa^{-2} \, \|\kappa_* g\|_{k,\infty} + \rho_\kappa^2 \, \|\kappa_* g^*\|_{k,\infty} \le c(k), \qquad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}. \tag{5.10}$$

From (2.12), the uniform regularity of h_W over $\mathfrak{K} \ltimes \Phi$, (5.8), and (5.9) we deduce

$$\kappa_*(|u|_h) = |(\kappa\ltimes\varphi)_* u|_{(\kappa\ltimes\varphi)_* h} \sim \rho_\kappa^{\sigma-\tau} \, |(\kappa\ltimes\varphi)_* u|_{E_\tau^\sigma}, \qquad \kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi, \quad u \in \Gamma(M,V). \tag{5.11}$$

Suppose $u \in \mathcal{T}_\tau^\sigma(M,V)$ has the local representation

$$u = u_{(j)}^{(i),\nu} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \otimes b_\nu.$$

Then it follows from (2.20), (2.21), (2.23), (2.24), and (2.25), denoting by $D_{k\mu}^\nu$ the Christoffel symbols of D, that

$$\begin{aligned} \nabla u = {} & \frac{\partial u_{(j)}^{(i),\nu}}{\partial x^k} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \otimes dx^k \otimes b_\nu \\ & + \sum_{s=1}^{\sigma} u_{(j)}^{(i_1,\ldots,i_s,\ldots,i_\sigma),\nu} \, \Gamma_{k i_s}^{\ell} \frac{\partial}{\partial x^{(i_1,\ldots,\ell,\ldots,i_\sigma)}} \otimes dx^{(j)} \otimes dx^k \otimes b_\nu \\ & - \sum_{t=1}^{\tau} u_{(j_1,\ldots,j_t,\ldots,j_\tau)}^{(i),\nu} \, \Gamma_{k\ell}^{j_t} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j_1,\ldots,\ell,\ldots,j_\tau)} \otimes dx^k \otimes b_\nu \\ & + u_{(j)}^{(i),\mu} D_{k\mu}^{\nu} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \otimes dx^k \otimes b_\nu, \end{aligned} \tag{5.12}$$

with ℓ being at position s in $(i_1,\ldots,\ell,\ldots,i_\sigma)$ and position t in $(j_1,\ldots,\ell,\ldots,j_\tau)$.

We endow the trivial bundle $Q_\kappa^m \times E_\tau^\sigma$ with the Euclidean connection, denoted by ∂_x and being naturally identified with the Fréchet derivative. Thus, given $v \in C^\infty(Q_\kappa^m, E_\tau^\sigma)$,

$$\partial_x^\ell v \in C^\infty\bigl(Q_\kappa^m, \mathcal{L}^\ell(\mathbb{R}^m; E_\tau^\sigma)\bigr), \qquad \ell \in \mathbb{N}^\times,$$

where $\mathcal{L}^\ell(\mathbb{R}^m; E_\tau^\sigma)$ is the space of ℓ-linear maps from $\mathbb{R}^m$ into E_τ^σ. If $v = \bigl[v_{(j)}^{(i)}\bigr] : Q_\kappa^m \to F^{m^\sigma \times m^\tau}$, then, setting $\partial_{(k)} := \partial_{k_\ell} \circ \cdots \circ \partial_{k_1}$ for $(k) \in \mathbb{J}_\ell$ with $\partial_i = \partial/\partial x^i$,

$$\partial_x^\ell v = \bigl[\partial_{(k)} v_{(j)}^{(i)}\bigr] : Q_\kappa^m \to F^{m^\sigma \times m^{\tau+\ell}}. \tag{5.13}$$

Hence, using the latter interpretation,

$$\partial_x^\ell \in \mathcal{L}^\ell\bigl(C^\infty(Q_\kappa^m, E_\tau^\sigma), C^\infty(Q_\kappa^m, E_{\tau+\ell}^\sigma)\bigr), \qquad \ell \in \mathbb{N}, \tag{5.14}$$

where $\partial_x^0 := \mathrm{id}$.

We define the push-forward

$$(\kappa\ltimes\varphi)_* \nabla^\ell : C^\infty(Q_\kappa^m, E_\tau^\sigma) \to C^\infty(Q_\kappa^m, E_{\tau+\ell}^\sigma)$$

of ∇^ℓ by $\kappa\ltimes\varphi$ by

$$(\kappa\ltimes\varphi)_* \nabla^\ell := (\kappa\ltimes\varphi)_* \circ \nabla^\ell \circ (\kappa\ltimes\varphi)^*$$

for $\ell \in \mathbb{N}$. Then $(\kappa\ltimes\varphi)_* \nabla$ is a metric connection on $\bigl(T_\tau^\sigma(Q_\kappa^m, F), (\kappa\ltimes\varphi)_* h\bigr)$ and

$$(\kappa\ltimes\varphi)_* \nabla^{\ell+1} = \bigl((\kappa\ltimes\varphi)_* \nabla\bigr) \circ (\kappa\ltimes\varphi)_* \nabla^\ell, \qquad \ell \in \mathbb{N}.$$

Suppose $r \in \mathbb{N}^\times$ and $u \in C^r(M,V)$. Set $v := (\kappa \ltimes \varphi)_* u \in C^r(Q_\kappa^m, E_\tau^\sigma)$. Then we infer from (5.12) by induction, and from (5.13) and (5.14) that there exist

$$a_\ell \in C^\infty\big(Q_\kappa^m, \mathcal{L}(E_{\tau+\ell}^\sigma, E_{\tau+r}^\sigma)\big), \qquad 0 \le \ell \le r-1,$$

such that

$$(\kappa \ltimes \varphi)_* \nabla^r v = \partial_x^r v + \sum_{\ell=0}^{r-1} a_\ell \partial_x^\ell v. \tag{5.15}$$

More precisely, the entries of the matrix representation of a_ℓ are polynomials in the derivatives of order at most $r-\ell-1$ of the Christoffel symbols of ∇_g and D. Hence assumption (4.4) implies

$$\|a_\ell\|_{k,\infty} \le c(k), \qquad 0 \le \ell \le r-1, \quad \kappa \ltimes \varphi \in \mathfrak{K} \ltimes \Phi, \tag{5.16}$$

due to (3.8), (5.9), and (5.10). By solving system (5.15) for $0 \le \ell \le r$ 'from the bottom' we find

$$\partial_x^r v = (\kappa \ltimes \varphi)_* \nabla^r v + \sum_{\ell=0}^{r-1} \widetilde{a}_\ell (\kappa \ltimes \varphi)_* \nabla^\ell v, \tag{5.17}$$

where $\widetilde{a}_\ell \in C^\infty\big(Q_\kappa^m, \mathcal{L}(E_{\tau+\ell}^\sigma, E_{\tau+r}^\sigma)\big)$ satisfy

$$\|\widetilde{a}_\ell\|_{k,\infty} \le c(k), \qquad 0 \le \ell \le r-1, \quad \kappa \ltimes \varphi \in \mathfrak{K} \ltimes \Phi. \tag{5.18}$$

From (5.15)–(5.18) we infer that, given $r \in \mathbb{N}^\times$,

$$\sum_{i=0}^{r} \big|(\kappa \ltimes \varphi)_* \nabla^i \big((\kappa \ltimes \varphi)_* u\big)\big|_{E_{\tau+i}^\sigma} \sim \sum_{|\alpha| \le r} \big|\partial_x^\alpha \big((\kappa \ltimes \varphi)_* u\big)\big|_{E_\tau^\sigma} \tag{5.19}$$

for $\kappa \ltimes \varphi \in \mathfrak{K} \ltimes \Phi$ and $u \in C^r(M,V)$.

6 Isotropic Bessel Potential and Besov Spaces

Weighted (isotropic) function spaces on singular manifolds have been studied in detail in [5], where, however, only scalar-valued tensor fields are considered. In this and the next section we recall the basic definitions and notation on which we shall build in the anisotropic case, and describe the needed extensions to the case of vector-bundle-valued tensor fields.

We denote by $\mathring{\mathcal{D}} := \mathring{\mathcal{D}}(V) := \mathcal{D}(\mathring{M}, V)$, respectively $\mathcal{D} := \mathcal{D}(V) := \mathcal{D}(M,V)$, the LF-space of smooth sections of V which are compactly supported in $\mathring{M}$, respectively M. Then $\mathring{\mathcal{D}}' = \mathring{\mathcal{D}}'(V) := \mathring{\mathcal{D}}(V')'_{w^*}$ is the dual of $\mathring{\mathcal{D}}(V')$ endowed with the w^*-topology, the space of *distribution sections on* $\mathring{M}$, whereby $V' = T_\sigma^\tau(M, W')$. As usual, we identify $v \in L_{1,\mathrm{loc}}(\mathring{M}, V)$ with the distribution section $\big(u \mapsto \langle u, v\rangle_M\big) \in \mathring{\mathcal{D}}'$, where

$$\langle u, v\rangle_M := \int_M \langle u, v\rangle_V \, dV_g, \qquad u \in \mathcal{D}(\mathring{M}, V'), \quad v \in L_{1,\mathrm{loc}}(\mathring{M}, V),$$

and dV_g is the volume measure of M. Hence

$$\mathring{\mathcal{D}} \hookrightarrow \mathcal{D} \overset{d}{\hookrightarrow} L_{1,\mathrm{loc}}(M,V) \xrightarrow{u \mapsto u \,|\, \mathring{M}} L_{1,\mathrm{loc}}(\mathring{M}, V) \hookrightarrow \mathring{\mathcal{D}}',$$

where $\hookrightarrow$ means 'continuous' and $\overset{d}{\hookrightarrow}$ 'continuous and dense' embedding.

In addition to (4.4) we suppose throughout

$$\boxed{\rho \in \mathfrak{T}(M), \quad 1 < p < \infty, \quad \lambda \in \mathbb{R}.} \tag{6.1}$$

Assume $k \in \mathbb{N}$. The **weighted Sobolev space**

$$W_p^{k,\lambda} = W_p^{k,\lambda}(V) = W_p^{k,\lambda}(V;\rho)$$

of W-valued (σ,τ)-tensor fields on M is the completion of $\mathcal{D}$ in $L_{1,\mathrm{loc}}(V)$ with respect to the norm

$$u \mapsto \|u\|_{k,p;\lambda} := \Bigl(\sum_{i=0}^{k} \bigl\|\rho^{\lambda+\tau-\sigma+i}\,|\nabla^i u|_h\,\bigr\|_p^p\Bigr)^{1/p}.$$

It is independent of the particular choice of ρ in the sense that $W_p^{k,\lambda}(V;\rho') \doteq W_p^{k,\lambda}(V;\rho)$ for $\rho' \in [\![\rho]\!]$, where $\doteq$ means 'equal except for equivalent norms'.

For simplicity, we do not indicate the dependence of these norms, and of related ones to be introduced below, on (σ,τ). This has to be kept in mind.

Note that

$$W_p^{0,\lambda} = L_p^\lambda = L_p^\lambda(V) := \bigl(\{\, u \in L_{p,\mathrm{loc}}\ ;\ \|u\|_{p;\lambda} < \infty \,\},\ \|\cdot\|_{p;\lambda}\bigr),$$

where $\|\cdot\|_{p;\lambda} := \|\cdot\|_{0,p;\lambda}$. Also observe $W_p^{k,\lambda} \overset{d}{\hookrightarrow} W_p^{\ell,\lambda}$ for $k > \ell$.

Given $0 < \theta < 1$, we write $[\cdot,\cdot]_\theta$ for the complex, and $(\cdot,\cdot)_{\theta,q}$, $1 \le q \le \infty$, for the real interpolation functor of exponent θ (see [2, Section I.2] for definitions and a summary of the basic facts of interpolation theory of which we make free use). Then, given $k \in \mathbb{N}$,

$$H_p^{s,\lambda} = H_p^{s,\lambda}(V) := \begin{cases} [W_p^{k,\lambda}, W_p^{k+1,\lambda}]_{s-k}, & k < s < k+1,\\ W_p^{k,\lambda}, & s = k,\end{cases}$$

and

$$B_p^{s,\lambda} = B_p^{s,\lambda}(V) := \begin{cases} (W_p^{k,\lambda}, W_p^{k+1,\lambda})_{s-k,p}, & k < s < k+1,\\ (W_p^{k,\lambda}, W_p^{k+2,\lambda})_{1/2,p}, & s = k.\end{cases}$$

In favor of a unified treatment, throughout the rest of this paper

$$\boxed{\mathfrak{F} \in \{H, B\}, \quad \mathfrak{F}_p^{s,\lambda} := \mathfrak{F}_p^{s,\lambda}(V).}$$

We denote by $\mathring{\mathfrak{F}}_p^{s,\lambda}$ the closure of $\mathring{\mathcal{D}}$ in $\mathfrak{F}_p^{s,\lambda}$ for $s > 0$ and set

$$\mathfrak{F}_p^{-s,\lambda}(V) := \bigl(\mathring{\mathfrak{F}}_{p'}^{s,-\lambda}(V')\bigr)', \qquad s > 0,$$

with respect to the duality pairing induced by $\langle\cdot,\cdot\rangle_M$. We also set

$$B_p^{0,\lambda} := (W_p^{-1,\lambda}, W_p^{1,\lambda})_{1/2,p}.$$

This defines the **weighted Bessel potential space scale** $[\, H_p^{s,\lambda}\ ;\ s \in \mathbb{R}\,]$ and the **weighted Besov space scale** $[\, B_p^{s,\lambda}\ ;\ s \in \mathbb{R}\,]$.

It follows (see the next section) that $\mathfrak{F}_p^{s,\lambda}$ is for $s \in \mathbb{R}$ a reflexive Banach space, and

$$\mathcal{D} \overset{d}{\hookrightarrow} \mathfrak{F}_p^{s,\lambda} \overset{d}{\hookrightarrow} \mathfrak{F}_p^{t,\lambda} \overset{d}{\hookrightarrow} \mathcal{D}', \qquad -\infty < t < s < \infty.$$

Denoting, for *any* $s \in \mathbb{R}$, by $\mathring{\mathfrak{F}}_p^{s,\lambda}$ the closure of $\mathcal{D}$ in $\mathfrak{F}_p^{s,\lambda}$,

$$\mathring{\mathfrak{F}}_p^{s,\lambda} = \mathfrak{F}_p^{s,\lambda}, \qquad s < 1/p.$$

Thus, by reflexivity,

$$\mathring{\mathfrak{F}}_p^{s,\lambda}(V) = \big(\mathring{\mathfrak{F}}_{p'}^{-s,-\lambda}(V')\big)', \qquad s \in \mathbb{R},$$

with respect to $\langle\cdot,\cdot\rangle_M$.

If $\rho \sim \mathbf{1}$, then all these spaces are independent of λ. Furthermore, $\mathfrak{F}_p^{s,\lambda}$ reduces to the non-weighted (standard) Bessel potential space $H_p^s(V)$ and Besov space $B_p^s(V)$, respectively. Assume, in addition, $M = \mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$ with $g = g_m$, $V = \mathbb{X} \times E$, and $D = d_F$. Then $H_p^s(\mathbb{X}, E)$ is the classical (E-valued) Bessel potential space and $B_p^s(\mathbb{X}, E)$ the standard (E-valued) Besov space $B_{p,p}^s(\mathbb{X}, E)$. In the scalar case these spaces are well investigated (cf. H. Triebel [50], for example). Thus noting $\mathfrak{F}_p^s(\mathbb{X}, E) \simeq (\mathfrak{F}_p^s)^d$ with $d = \dim(E)$, we can make free use of their properties which we shall do without further reference.

7 The Isotropic Retraction Theorem

Let E_α be a locally convex space for each α in a countable index set. Then $\boldsymbol{E} := \prod_\alpha E_\alpha$ is endowed with the product topology. Now suppose that each E_α is a Banach space. Then we denote for $1 \le q \le \infty$ by $\ell_q(\boldsymbol{E})$ the linear subspace of $\boldsymbol{E}$ consisting of all $\boldsymbol{x} = (x_\alpha)$ such that

$$\|\boldsymbol{x}\|_{\ell_q(\boldsymbol{E})} := \begin{cases} \big(\sum_\alpha \|x_\alpha\|_{E_\alpha}^q\big)^{1/q}, & 1 \le q < \infty, \\ \sup_\alpha \|x_\alpha\|_{E_\alpha}, & q = \infty, \end{cases}$$

is finite. Then $\ell_q(\boldsymbol{E})$ is a Banach space with norm $\|\cdot\|_{\ell_q(\boldsymbol{E})}$, and

$$\ell_p(\boldsymbol{E}) \hookrightarrow \ell_q(\boldsymbol{E}), \qquad 1 \le p < q \le \infty. \tag{7.1}$$

We also set $c_c(\boldsymbol{E}) := \bigoplus_\alpha E_\alpha$, where $\bigoplus$ denotes the locally convex direct sum. Thus $\bigoplus_\alpha E_\alpha$ consists of all finitely supported sequences in $\boldsymbol{E}$ equipped with the finest locally convex topology for which all injections $E_\beta \to \bigoplus_\alpha E_\alpha$ are continuous. It follows

$$c_c(\boldsymbol{E}) \hookrightarrow \ell_q(\boldsymbol{E}), \quad 1 \le q \le \infty, \qquad c_c(\boldsymbol{E}) \overset{d}{\hookrightarrow} \ell_q(\boldsymbol{E}), \quad q < \infty. \tag{7.2}$$

Furthermore, $c_0(\boldsymbol{E})$ is the closure of $c_c(\boldsymbol{E})$ in $\ell_\infty(\boldsymbol{E})$.

If each E_α is reflexive, then $\ell_p(\boldsymbol{E})$ is reflexive as well, and $\ell_p(\boldsymbol{E})' = \ell_{p'}(\boldsymbol{E}')$ with respect to the duality pairing $\boldsymbol{\langle}\cdot,\cdot\boldsymbol{\rangle} := \sum_\alpha \langle\cdot,\cdot\rangle_\alpha$. Of course, $\boldsymbol{E}' := \prod_\alpha E_\alpha'$, and $\langle\cdot,\cdot\rangle_\alpha$ is the E_α-duality pairing.

Let assumption (5.4) be satisfied. A *localization system subordinate to* $\mathfrak{K}$ is a family $\big\{(\pi_\kappa, \chi_\kappa)\ ;\ \kappa \in \mathfrak{K}\big\}$ such that

$$\begin{aligned} &\text{(i)} && \pi_\kappa \in \mathcal{D}\big(U_\kappa, [0,1]\big) \text{ and } \{\pi_\kappa^2\ ;\ \kappa \in \mathfrak{K}\} \text{ is a partition of unity on } M \\ &&& \text{subordinate to the covering } \{U_\kappa\ ;\ \kappa \in \mathfrak{K}\}; \\ &\text{(ii)} && \chi_\kappa = \kappa^*\chi \text{ with } \chi \in \mathcal{D}\big(Q^m, [0,1]\big) \text{ and } \chi \,|\, \mathrm{supp}(\kappa_*\pi_\kappa) = \mathbf{1} \text{ for } \kappa \in \mathfrak{K}; \\ &\text{(iii)} && \|\kappa_*\pi_\kappa\|_{k,\infty} + \|\kappa_*\chi_\kappa\|_{k,\infty} \le c(k),\ \ \kappa \in \mathfrak{K},\ \ k \in \mathbb{N}. \end{aligned} \tag{7.3}$$

Lemma 3.2 of [5] guarantees the existence of such a localization system.

In addition to (5.4) we assume

$$\big\{(\pi_\kappa, \chi_\kappa)\ ;\ \kappa \in \mathfrak{K}\big\} \text{ is a localization system subordinate to } \mathfrak{K}.$$

For abbreviation, we put for $s \in \mathbb{R}$

$$W_{p,\kappa}^s := W_p^s(\mathbb{X}_\kappa, E), \quad \mathfrak{F}_{p,\kappa}^s := \mathfrak{F}_p^s(\mathbb{X}_\kappa, E), \qquad \kappa \in \mathfrak{K},$$

where $E = E_\tau^\sigma(F)$. Hence $\boldsymbol{W}_p^s = \prod_\kappa W_{p,\kappa}^s$ is well-defined, as is $\boldsymbol{\mathfrak{F}}_p^s$. We set

$$\mathcal{D}_\kappa := \mathcal{D}(\mathbb{X}_\kappa, E), \quad \mathring{\mathcal{D}}_\kappa := \mathcal{D}(\mathring{\mathbb{X}}_\kappa, E),$$

as well as

$$\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}(\mathbb{X}, E) := \bigoplus_\kappa \mathcal{D}_\kappa, \quad \mathring{\boldsymbol{\mathcal{D}}} = \boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E) := \bigoplus_\kappa \mathring{\mathcal{D}}_\kappa.$$

It should be noted that, due to (5.1), in $\boldsymbol{W}_p^s$, $\boldsymbol{\mathfrak{F}}_p^s$, $\boldsymbol{\mathcal{D}}$, and $\mathring{\boldsymbol{\mathcal{D}}}$ there occur at most two distinct function spaces.

Given $\kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi$, we put for $1 \le q \le \infty$

$$\varphi_{q,\kappa}^\lambda u := \rho_\kappa^{\lambda+m/q}(\kappa\ltimes\varphi)_*(\pi_\kappa u), \qquad u \in C(V),$$

and

$$\psi_{q,\kappa}^\lambda v := \rho_\kappa^{-\lambda-m/q}\pi_\kappa(\kappa\ltimes\varphi)^* v, \qquad v \in C(\mathbb{X}_\kappa, E).$$

Here and in similar situations it is understood that a partially defined and compactly supported section of a vector bundle is extended over the whole base manifold by identifying it with the zero section outside its original domain. In addition,

$$\varphi_q^\lambda u := (\varphi_{q,\kappa}^\lambda u) \in \prod_\kappa C(\mathbb{X}_\kappa, E), \qquad u \in C(V),$$

and

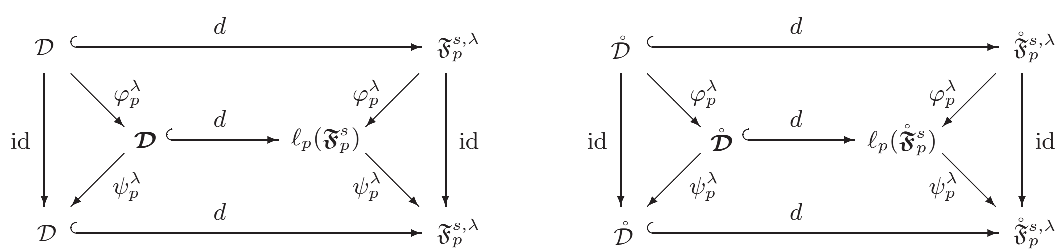
$$\psi_q^\lambda \boldsymbol{v} := \sum_\kappa \psi_{q,\kappa}^\lambda v_\kappa, \qquad \boldsymbol{v} = (v_\kappa) \in \prod_\kappa C(\mathbb{X}_\kappa, E).$$

A *retraction* from a locally convex space $\mathcal{X}$ onto a locally convex space $\mathcal{Y}$ is a map $R \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ possessing a right inverse $R^c \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, a coretraction.

If no confusion seems likely, we use the same symbol for a continuous linear map and its restriction to a linear subspace of its domain, respectively for a unique continuous linear extension of it. Furthermore, in a diagram arrows always represent continuous linear maps.

The following theorem shows that ψ_p^λ is a retraction from $\boldsymbol{\mathcal{D}}$ onto $\mathcal{D}$, and that φ_p^λ is a coretraction. Moreover, ψ_p^λ has a unique continuous linear extension to a retraction from $\ell_p(\boldsymbol{\mathfrak{F}}_p^s)$ onto $\mathfrak{F}_p^{s,\lambda}$, and φ_p^λ extends uniquely to a coretraction. This holds for any choice of $s \in \mathbb{R}$ and $p \in (1, \infty)$. Thus ψ_p^λ is a *universal* retraction from $\ell_p(\boldsymbol{\mathfrak{F}}_p^s)$ onto $\mathfrak{F}_p^{s,\lambda}$ in the sense that it is completely determined by its restriction to $\boldsymbol{\mathcal{D}}$. The same holds if $\mathcal{D}$ and $\mathfrak{F}_p^{s,\lambda}$ are replaced by $\mathring{\mathcal{D}}$ and $\mathring{\mathfrak{F}}_p^{s,\lambda}$, respectively.

Theorem 7.1 *Suppose $s \in \mathbb{R}$. Then the diagrams*



are commuting, where $s > 0$ in the second case.

P r o o f. (1) Suppose $W = M \times \mathbb{K}$ so that $V = T_\tau^\sigma(M, W) = T_\tau^\sigma M$. Also suppose $k \in \mathbb{N}$. Then Theorem 6.1 of [5] guarantees that

$$\psi_p^\lambda \text{ is a retraction from } \boldsymbol{\mathcal{D}} \text{ onto } \mathcal{D} \text{ and from } \ell_p(\boldsymbol{W}_p^k) \text{ onto } W_p^{k,\lambda}, \text{ and } \varphi_p^\lambda \text{ is a coretraction.} \tag{7.4}$$

Furthermore, set

$$\mathring{\varphi}_{p,\kappa}^\lambda := \rho_\kappa^{-m}\sqrt{\kappa_* g}\,\varphi_{p,\kappa}^\lambda, \quad \mathring{\psi}_{p,\kappa}^\lambda := \rho_\kappa^m(\sqrt{\kappa_* g})^{-1}\psi_{p,\kappa}^\lambda \tag{7.5}$$

and

$$\mathring{\varphi}_p^\lambda u := (\mathring{\varphi}_{p,\kappa}^\lambda u), \quad \mathring{\psi}_p^\lambda \boldsymbol{v} := \sum_\kappa \mathring{\psi}_{p,\kappa}^\lambda v_\kappa, \qquad u \in \mathring{\mathcal{D}}, \quad \boldsymbol{v} \in \mathring{\boldsymbol{\mathcal{D}}}.$$

Then it follows from Theorem 11.1 of [5] that $\mathring{\psi}_p^\lambda$ is a retraction from from $\mathring{\boldsymbol{\mathcal{D}}}$ onto $\mathring{\mathcal{D}}$ and from $\ell_p(\mathring{\boldsymbol{W}}_p^k)$ onto $\mathring{W}_p^{k,\lambda}$, and $\mathring{\varphi}_p^\lambda$ is a coretraction.

From step (2) of the proof of the latter theorem we know

$$\rho_\kappa^{-m}\sqrt{\kappa_* g} \sim \mathbf{1}, \quad \|\rho_\kappa^{-m}\sqrt{\kappa_* g}\|_{k,\infty} + \|\rho_\kappa^m(\sqrt{\kappa_* g})^{-1}\|_{k,\infty} \le c(k), \qquad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}. \tag{7.6}$$

This implies that we can replace $\mathring{\varphi}_{p,\kappa}$ and $\mathring{\psi}_{p,\kappa}$ in [5, Theorem 11.1] by $\varphi_{p,\kappa}$ and $\psi_{p,\kappa}$, respectively. Consequently,

$$\psi_p^\lambda \text{ is a retraction from } \mathring{\boldsymbol{\mathcal{D}}} \text{ onto } \mathring{\mathcal{D}} \text{ and from } \ell_p(\mathring{\boldsymbol{W}}_p^k) \text{ onto } \mathring{W}_p^{k,\lambda}, \text{ and } \varphi_p^\lambda \text{ is a coretraction.} \tag{7.7}$$

(2) Let now $W = (W, h_W, D)$ be an arbitrary fully uniformly regular vector bundle over M. Then (5.19) is the analogue of Lemma 3.1(iv) of [5]. Furthermore, (5.11) implies the analogue of [5, part (v) of Lemma 3.1]. If $W = M \times \mathbb{K}$, then the proofs of (7.4) and (7.7) are solely based on Lemma 3.1 of [5]. Hence, due to the preceding observations, they apply without change to the general case as well. Thus (7.4) and (7.7) hold if W is an arbitrary fully uniformly regular vector bundle over M.

(3) The assertions of the theorem are now deduced from (7.4) and (7.7) by interpolation and duality as in [5]. □

Let X and Y be Banach spaces, $R:\ X \to Y$ a retraction, and $R^c:\ Y \to X$ a coretraction. Then

$$\|y\|_Y = \|RR^c y\|_Y \le \|R\|\,\|R^c y\|_X \le \|R\|\,\|R^c\|\,\|y\|_Y, \qquad y \in Y.$$

Hence

$$\|\cdot\|_Y \sim \|R^c\cdot\|_X. \tag{7.8}$$

From this and Theorem 7.1 it follows that

$$u \mapsto \|\varphi_p^\lambda u\|_{\ell_p(\boldsymbol{\mathfrak{F}}_p^s)} \tag{7.9}$$

is a norm for $\mathfrak{F}_p^{s,\lambda}$. Furthermore, another choice of $\mathfrak{K}\ltimes\Phi$ and the localization system leads to an equivalent norm.

For $\kappa \in \mathfrak{K}$ and $\widetilde{\kappa} \in \mathfrak{N}(\kappa)$ we define a linear map

$$S_{\widetilde{\kappa}\kappa}:\ E^{\mathbb{X}_{\widetilde{\kappa}}} \to E^{\mathbb{X}_\kappa}, \quad v \mapsto (\kappa\ltimes\varphi)_*(\widetilde{\kappa}\ltimes\widetilde{\varphi})^*(\chi v). \tag{7.10}$$

The following lemma will be repeatedly useful.

Lemma 7.2 *Suppose $s \in \mathbb{R}^+$ with $s > 0$ if $\mathfrak{F} = B$. Then*

$$S_{\widetilde{\kappa}\kappa} \in \mathcal{L}(\mathfrak{F}_{p,\widetilde{\kappa}}^s, \mathfrak{F}_{p,\kappa}^s), \quad \|S_{\widetilde{\kappa}\kappa}\| \le c, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa \in \mathfrak{K}.$$

P r o o f. Note that, by (2.5) and our convention on $(\kappa\ltimes\varphi)_*$,

$$S_{\widetilde{\kappa}\kappa} v = (\varphi_\tau^\sigma)_{\widetilde{\kappa}\kappa}\big((\chi v) \circ (\kappa \circ \widetilde{\kappa}^{-1})\big).$$

Hence it follows from (3.1), (3.3), (5.6), (5.7), (7.3), and the product rule and Leibniz' formula that the assertion is true if $s \in \mathbb{N}$ and $\mathfrak{F} = H$, since $H_{p,\kappa}^s \doteq W_{p,\kappa}^s$ for $s \in \mathbb{N}$. Now we obtain the statement for general s by interpolation. □

It follows from Theorem 7.1 and the preceding consideration that

$$\begin{aligned}&\textit{all results proved in } [5] \textit{ for the Banach space scales } [\,\mathfrak{F}_p^{s,\lambda}\,;\ s \in \mathbb{R}\,]\\ &\textit{of scalar-valued } (\sigma,\tau)\textit{-tensor fields are likewise true for } W\textit{-valued } (\sigma,\tau)\textit{-tensor fields,}\end{aligned} \tag{7.11}$$

using obvious adaptions. Thus, in particular, the properties of $\mathfrak{F}_p^{s,\lambda}$ listed in Section 6 are valid. Henceforth, we use (7.11) without further ado and simply refer to [5].

8 Anisotropic Bessel Potential and Besov Spaces

Given subsets X and Y of a Hausdorff topological space, we write $X \Subset Y$ if $\overline{X}$ is compact and contained in the interior of Y.

Let I be an interval with nonempty interior and $\mathcal{X}$ a locally convex space. Suppose $\mathcal{Q}$ is a family of seminorms for $\mathcal{X}$ generating its topology. Then $C^\infty(I, \mathcal{X})$ is a locally convex space with respect to the topology induced by the family of seminorms

$$u \mapsto \sup_{t \in K} q\big(\partial^k u(t)\big), \qquad k \in \mathbb{N}, \quad K \Subset I, \quad q \in \mathcal{Q}.$$

This topology is independent of the particular choice of $\mathcal{Q}$.

For $K \Subset I$ we denote by $\mathcal{D}_K(I, \mathcal{X})$ the linear subspace of $C^\infty(I, \mathcal{X})$ consisting of those functions which are supported in K. We provide $\mathcal{D}_K(I, \mathcal{X})$ with the topology induced by $C^\infty(I, \mathcal{X})$. Then $\mathcal{D}(I, \mathcal{X})$, the vector space of smooth compactly supported $\mathcal{X}$-valued functions, is endowed with the inductive topology with respect to the spaces $\mathcal{D}_K(I, \mathcal{X})$ with $K \Subset I$. If $K \Subset K' \Subset I$, then $\mathcal{D}_{K'}(I, \mathcal{X})$ induces on $\mathcal{D}_K(I, \mathcal{X})$ its original topology. Note, however, that in general $\mathcal{D}(I, \mathcal{X})$ is not an LF-space since $\mathcal{D}_K(I, \mathcal{X})$ may not be a Fréchet space. Given a locally convex space $\mathcal{Y}$, a linear map $T : \mathcal{D}(I, \mathcal{X}) \to \mathcal{Y}$ is continuous iff its restriction to every subspace $\mathcal{D}_K(I, \mathcal{X})$ is continuous (e.g., Section 6 of H.H. Schaefer [40]).

From now on it is assumed, in addition to (4.4) and (6.1), that

$$\boxed{r \in \mathbb{N}^\times, \quad \mu \in \mathbb{R}, \quad J \in \{\mathbb{R}, \mathbb{R}^+\}.}$$

We set

- $1/\vec{r} := (1, 1/r) \in \mathbb{R}^2, \quad \vec{\omega} := (\lambda, \mu),$

so that $s/\vec{r} = (s, s/r)$ for $s \in \mathbb{R}$.

Suppose $k \in \mathbb{N}$. The **anisotropic weighted Sobolev space** of time-dependent W-valued (σ, τ)-tensor fields on M,

$W_p^{kr/\vec{r},\vec{\omega}} = W_p^{kr/\vec{r},\vec{\omega}}(J, V)$, is the linear subspace of $L_p(J, W_p^{kr,\lambda})$ consisting of all u satisfying $\partial^k u \in L_p(J, L_p^{\lambda+k\mu})$, endowed with the norm (8.1)

$$\|u\|_{kr/\vec{r},p;\vec{\omega}} := \big(\|u\|^p_{L_p(J, W_p^{kr,\lambda})} + \|\partial^k u\|_{L_p(J, L_p^{\lambda+k\mu})}\big)^{1/p}.$$

Thus $W_p^{0/\vec{r},\vec{\omega}} \doteq L_p(J, L_p^\lambda)$.

Theorem 8.1

(i) $W_p^{kr/\vec{r},\vec{\omega}}$ *is a reflexive Banach space.*

(ii) $\|u\|^\sim_{kr/\vec{r},p;\vec{\omega}} := \big(\|u\|^p_{L_p(J, W_p^{kr,\lambda})} + \sum_{j=0}^k \|\partial^j u\|^p_{L_p(J, W_p^{(k-j)r,\lambda+j\mu})}\big)^{1/p}$ *is an equivalent norm.*

(iii) $\mathcal{D}(J, \mathcal{D}) \overset{d}{\hookrightarrow} W_p^{kr/\vec{r},\vec{\omega}}$.

P r o o f. It follows from Theorem 9.3 below that $W_p^{kr/\vec{r},\vec{\omega}}$ is isomorphic to a closed linear subspace of a reflexive Banach space, hence it is complete and reflexive. Proofs for parts (ii) and (iii) are given in the next section. □

Observe

$$\|u\|^\sim_{kr/\vec{r},p;\vec{\omega}} = \Big(\int_J \sum_{i+jr \le kr} \big\|\rho^{\lambda+i+j\mu+\tau-\sigma}\, |\nabla^i \partial^j u|_h\, \big\|_p^p\, dt\Big)^{1/p} \tag{8.2}$$

and

$$W_p^{kr/\vec{r},(\lambda,0)} \doteq L_p(J, W_p^{kr,\lambda}) \cap W_p^k(J, L_p^\lambda).$$

Note that Theorem 8.1(ii) and (8.2) show that definition (8.1) coincides, except for equivalent norms, with (1.2). Also note that the reflexivity of L_p^λ implies

$$W_p^{0/\vec{r},\vec{\omega}} = L_p(J, L_p^\lambda) = \big(L_{p'}(J, L_{p'}^{-\lambda}(V'))\big)'$$

with respect to the duality pairing defined by

$$\langle u, v\rangle_{M\times J} := \int_J \big\langle u(t), v(t)\big\rangle_M \, dt.$$

Given $0 < \theta < 1$, we set

$$(\cdot,\cdot)_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if } \mathfrak{F} = H, \\ (\cdot,\cdot)_{\theta,p} & \text{if } \mathfrak{F} = B. \end{cases}$$

For $s > 0$ we define 'fractional order' spaces by

$$\mathfrak{F}_p^{s/\vec{r},\vec{\omega}} = \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J, V) := \begin{cases} (W_p^{kr/\vec{r},\vec{\omega}}, W_p^{(k+1)r/\vec{r},\vec{\omega}})_{(s-kr)/r}, & kr < s < (k+1)r, \\ (W_p^{kr/\vec{r},\vec{\omega}}, W_p^{(k+2)r/\vec{r},\vec{\omega}})_{1/2}, & s = (k+1)r. \end{cases} \tag{8.3}$$

We denote by

$$\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}} = \mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(J, V) \text{ the closure of } \mathcal{D}(\mathring{J}, \mathring{\mathcal{D}}) \text{ in } \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}. \tag{8.4}$$

Then negative order spaces are introduced by duality, that is,

$$\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}} = \mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J, V) := \big(\mathring{\mathfrak{F}}_{p'}^{s/\vec{r},-\vec{\omega}}(J, V')\big)', \qquad s > 0, \tag{8.5}$$

with respect to the duality pairing induced by $\langle\cdot,\cdot\rangle_{M\times J}$. We also set $s(p) := 1/2p$ and

$$H_p^{0/\vec{r},\vec{\omega}} := L_p(J, L_p^\lambda), \quad B_p^{0/\vec{r},\vec{\omega}} := (H_p^{-s(p)/\vec{r},\vec{\omega}}, H_p^{s(p)/\vec{r},\vec{\omega}})_{1/2,p}. \tag{8.6}$$

This defines the **weighted anisotropic Bessel potential space scale** $[\, H_p^{s/\vec{r},\vec{\omega}} \,;\, s \in \mathbb{R}\,]$ and the **weighted anisotropic Besov space scale** $[\, B_p^{s/\vec{r},\vec{\omega}} \,;\, s \in \mathbb{R}\,]$.

The proof of the following theorem, which describes the interrelations between these two scales and gives first interpolation results, is given in the next section. Henceforth, $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$ for $\xi_0, \xi_1 \in \mathbb{R}$ and $0 \le \theta \le 1$.

Theorem 8.2

(i) $H_p^{kr/\vec{r},\vec{\omega}} \doteq W_p^{kr/\vec{r},\vec{\omega}},\ k \in \mathbb{N}$.

(ii) $B_2^{s/\vec{r},\vec{\omega}} \doteq H_2^{s/\vec{r},\vec{\omega}},\ s \in \mathbb{R}$.

(iii) $(B_p^{s_0/\vec{r},\vec{\omega}}, B_p^{s_1/\vec{r},\vec{\omega}})_{\theta,p} \doteq B_p^{s_\theta/\vec{r},\vec{\omega}},\ 0 \le s_0 < s_1,\ 0 < \theta < 1$.

(iv) $[\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}}, \mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}}]_\theta \doteq \mathfrak{F}_p^{s_\theta/\vec{r},\vec{\omega}},\ 0 \le s_0 < s_1,\ 0 < \theta < 1$.

Next we prove, among other things, an elementary embedding theorem for anisotropic weighted Bessel potential and Besov spaces.

Theorem 8.3

(i) *Suppose* $-\infty < s_0 < s < s_1 < \infty$. *Then*

$$\mathcal{D}(J, \mathcal{D}) \overset{d}{\hookrightarrow} H_p^{s_1/\vec{r},\vec{\omega}} \overset{d}{\hookrightarrow} B_p^{s/\vec{r},\vec{\omega}} \overset{d}{\hookrightarrow} H_p^{s_0/\vec{r},\vec{\omega}}. \tag{8.7}$$

(ii) *Assume* $s < 1/p$ *if* $\partial M \ne \emptyset$, *and* $s < r(1+1/p)$ *if* $\partial M = \emptyset$ *and* $J = \mathbb{R}^+$. *Then* $\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}} = \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$.

P r o o f of (i) for $s\neq 0$. Using reiteration theorems, well-known density properties, and relations between the real and complex interpolation functor (e.g., [2, formula (I.2.5.2)] and Theorem 8.1(iii)), we see that (8.7) is true if $s_0\geq 0$.

Since $\mathcal{D}(\mathring{J},\mathring{\mathcal{D}})$ is dense in $H_p^{0/\vec r,\vec\omega}=L_p(J,L_p^\lambda)$ it follows

$$\mathring{H}_p^{s_1/\vec r,\vec\omega}\overset{d}{\hookrightarrow}\mathring{B}_p^{s/\vec r,\vec\omega}\overset{d}{\hookrightarrow}\mathring{H}_p^{s_0/\vec r,\vec\omega}\overset{d}{\hookrightarrow}L_p(J,L_p^\lambda),\qquad s_0\geq 0.$$

Hence the definition of the negative order spaces implies that (8.7) holds if $s_1\leq 0$, where the density of these embeddings follows by reflexivity. This implies assertion (i) if $s\neq 0$. The proofs for the case $s=0$ and for assertion (ii) are given in the next section. □

Corollary 8.4 *Suppose* $s\in\mathbb{R}$.

(i) $\mathfrak{F}_p^{s/\vec r,\vec\omega}$ *is a reflexive Banach space.*

(ii) *If* $s>0$, *then* $\mathring{\mathfrak{F}}_p^{s/\vec r,\vec\omega}=\big(\mathfrak{F}_{p'}^{-s/\vec r,-\vec\omega}(J,V')\big)'$ *with respect to* $\langle\cdot,\cdot\rangle_{M\times J}$.

(iii) $\mathfrak{F}_p^{s_1/\vec r,\vec\omega}\overset{d}{\hookrightarrow}\mathfrak{F}_p^{s_0/\vec r,\vec\omega}$ *if* $s_1>s_0$.

P r o o f. Assume $s>0$. Then assertion (i) follows from the reflexivity of $W_p^{kr/\vec r,\vec\omega}$ for $k\in\mathbb{N}$ and the duality properties of the real and complex interpolation functors. Hence $\mathring{\mathfrak{F}}_p^{s/\vec r,\vec\omega}(J,V')$, being a closed linear subspace of a reflexive Banach space, is reflexive. Thus $\mathfrak{F}_p^{-s/\vec r,\vec\omega}$ is reflexive since it is the dual of a reflexive Banach space. We have already seen that $H_p^{0/\vec r,\vec\omega}$ is reflexive. The reflexivity of $B_p^{0/\vec r,\vec\omega}$ follows by interpolation as well. This proves (i) for every $s\in\mathbb{R}$.

Assertion (ii) is a consequence of (i) and (8.5). Claim (iii) is immediate by (8.7). □

If M is uniformly regular, that is, $\mathfrak{T}(M)=[\![1]\!]$, then $\mathfrak{F}_p^{s/\vec r,\vec\omega}$ is independent of $\vec\omega$. These non-weighted spaces are denoted by $\mathfrak{F}_p^{s/\vec r}$, of course. If $W=M\times\mathbb{K}$, then we write $\mathfrak{F}_p^{s/\vec r,\vec\omega}(M\times J)$ for $\mathfrak{F}_p^{s/\vec r,\vec\omega}(J,V_0^0)$. Since $V_0^0=T_0^0M$ is in this case the trivial vector bundle $M\times\mathbb{K}$, whose sections are the $\mathbb{K}$-valued functions on M, this notation is consistent with usual identification of $L_p\big(J,L_p^\lambda(M)\big)$ with $L_p^\lambda(M\times J)$ via the identification of $u(t)$ with $u(\cdot,t)$.

9 The Anisotropic Retraction Theorem

Let $\{\,E_\alpha\ ;\ \alpha\in\mathsf{A}\,\}$ be a countable family of Banach spaces. We set $L_p(J,\boldsymbol{E}):=\prod_\alpha L_p(J,E_\alpha)$. Fubini's theorem implies

$$\ell_p\big(L_p(J,\boldsymbol{E})\big)=L_p\big(J,\ell_p(\boldsymbol{E})\big),\tag{9.1}$$

using obvious identifications. We also set $(\boldsymbol{E},\boldsymbol{F})_\theta:=\prod_\alpha(E_\alpha,F_\alpha)_\theta$ for $0<\theta<1$ if each (E_α,F_α) is an interpolation couple.

We presuppose as standing hypothesis

> $(\rho,\mathfrak{K})$ is a singularity datum for M.
> $\mathfrak{K}\ltimes\Phi$ is a uniformly regular atlas for W over $\mathfrak{K}$.
> $F=\big(F,(\cdot|\cdot)_F\big)$ is a model fiber for W with basis $(e_1,\dots,e_n)$.
> $\big\{\,(\pi_\kappa,\chi_\kappa)\ ;\ \kappa\in\mathfrak{K}\,\big\}$ is a localization system subordinate to $\mathfrak{K}$.

On the basis of (7.9) we can provide localized versions of the norms $\|\cdot\|_{kr/\vec r,p;\vec\omega}$ and $\|\cdot\|_{kr/\vec r,p;\vec\omega}^\sim$.

Theorem 9.1 *Suppose $k\in\mathbb{N}$. Set*

$$|||\cdot|||_{kr/\vec r,p;\vec\omega}:=\big(\|\varphi_p^\lambda u\|^p_{\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))}+\|\varphi_p^{\lambda+k\mu}(\partial^k u)\|^p_{\ell_p(L_p(J,\boldsymbol{L}_p))}\big)^{1/p}$$

and

$$|||\cdot|||^{\sim}_{kr/\vec r,p;\vec\omega}:=\Big(\|\varphi_p^\lambda u\|^p_{\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))}+\sum_{j=0}^{k}\|\varphi_p^{\lambda+j\mu}(\partial^j u)\|^p_{\ell_p(L_p(J,\boldsymbol{W}_p^{(k-j)r}))}\Big)^{1/p}.$$

Then $|||\cdot|||_{kr/\vec r,p;\vec\omega}\sim\|\cdot\|_{kr/\vec r,p;\vec\omega}$ and $|||\cdot|||^{\sim}_{kr/\vec r,p;\vec\omega}\sim\|\cdot\|^{\sim}_{kr/\vec r,p;\vec\omega}$.

Proof. This follows from (7.9) and (9.1). □

It is worthwhile to note

$$|||u|||^{\sim}_{kr/\vec r,p;\vec\omega}=\Big(\sum_\kappa\int_J\sum_{|\alpha|+jr\le kr}\big(\rho_\kappa^{\lambda+|\alpha|+j\mu+m/q}\,\|\partial_x^\alpha\partial^j(\kappa{\ltimes}\varphi)_*(\pi_\kappa u)\|_{p;E}\big)^p\,dt\Big)^{1/p}.$$

Together with Theorems 8.1(ii) and 9.1 this gives a rather explicit and practically useful local characterization of anisotropic Sobolev spaces.

For abbreviation, we set

- $\mathbb{Y}_\kappa:=\mathbb{X}_\kappa\times J,\quad \kappa\in\mathfrak{K}.$

Hence $\mathring{\mathbb{Y}}_\kappa=\mathring{\mathbb{X}}_\kappa\times\mathring{J}$ is the interior of $\mathbb{Y}_\kappa$ in $\mathbb{R}^{m+1}=\mathbb{R}^m\times\mathbb{R}$. We also put

$$\boldsymbol{\mathcal{D}}(\mathbb{Y},E):=\bigoplus_\kappa\mathcal{D}(\mathbb{Y}_\kappa,E),\quad \boldsymbol{\mathcal{D}}(\mathring{\mathbb{Y}},E):=\bigoplus_\kappa\mathcal{D}(\mathring{\mathbb{Y}},E)$$

and

$$W^{kr/\vec r}_{p,\kappa}:=W^{kr/\vec r}_p(\mathbb{Y}_\kappa,E),\quad \mathfrak{F}^{s/\vec r}_{p,\kappa}:=\mathfrak{F}^{s/\vec r}_p(\mathbb{Y}_\kappa,E),\qquad s\in\mathbb{R},\quad k\in\mathbb{N}.$$

More precisely, the 'local' spaces $W^{kr/\vec r}_{p,\kappa}$ and $\mathfrak{F}^{s/\vec r}_{p,\kappa}$ are special instances of $W^{kr/\vec r,\vec\omega}_p$ and $\mathfrak{F}^{s/\vec r,\vec\omega}_p$, respectively, namely with $M=(\mathbb{X}_\kappa,g_m)$, $\rho=\mathbf{1}$, $W=\mathbb{X}_\kappa\times F$, and $D=d_F$.

It is of fundamental importance that these spaces coincide with the anisotropic Sobolev, Bessel potential, and Besov spaces studied by means of Fourier analytical techniques in detail in H. Amann [4], therein denoted by $W^{k\nu/\boldsymbol{\nu}}_p(\mathbb{Y}_\kappa,E)$, $H^{s/\boldsymbol{\nu}}_p(\mathbb{Y}_\kappa,E)$, and $B^{s/\boldsymbol{\nu}}_p(\mathbb{Y}_\kappa,E)$, respectively, where $\nu:=r$ and $\boldsymbol{\nu}:=(1,r)$. For abbreviation, we set $W^{k\nu/\boldsymbol{\nu}}_{p,\kappa}:=W^{k\nu/\boldsymbol{\nu}}_p(\mathbb{Y}_\kappa,E)$ and $\mathfrak{F}^{s/\boldsymbol{\nu}}_{p,\kappa}:=\mathfrak{F}^{s/\boldsymbol{\nu}}_p(\mathbb{Y}_\kappa,E)$. Furthermore, we write $\widetilde{W}^{kr/\vec r}_{p,\kappa}$ for $L_p(J,W^{kr}_{p,\kappa})\cap W^k_p(J,L_{p,\kappa})$ endowed with the norm $\|\cdot\|^{\sim}_{kr/\vec r,p}$.

Lemma 9.2

(i) *If $k\in\mathbb{N}$, then $W^{kr/\vec r}_{p,\kappa}\doteq W^{k\nu/\boldsymbol{\nu}}_{p,\kappa}\doteq\widetilde{W}^{kr/\vec r}_{p,\kappa}$ for $\kappa\in\mathfrak{K}$.*

(ii) *If $s\in\mathbb{R}$, then $\mathfrak{F}^{s/\vec r}_{p,\kappa}\doteq\mathfrak{F}^{s/\boldsymbol{\nu}}_{p,\kappa}$ for $\kappa\in\mathfrak{K}$.*

Proof. (1) If $J=\mathbb{R}^+$ and $\kappa\in\mathfrak{K}_{\partial M}$, then $\mathbb{Y}_\kappa$ is isomorphic to the closed 2-corner $\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^{m-1}$ (in the sense of Section 4.3 of [4]) by a permutation isomorphism. Otherwise, $\mathbb{Y}_\kappa$ equals either the half-space $\mathbb{H}^{m+1}$ (except for a possible permutation) or $\mathbb{R}^{m+1}$.

(2) If $\mathbb{Y}_\kappa=\mathbb{R}^{m+1}$, then (i) follows from Theorem 2.3.8 of [4] and the definition of $W^{k\nu/\boldsymbol{\nu}}_{p,\kappa}$ in the first paragraph of [4, Section 3.5]. If $\mathbb{Y}_\kappa\ne\mathbb{R}^{m+1}$, then we obtain claim (i) by invoking [4, Theorem 4.4.3(i)].

(3) Suppose $\mathbb{Y}_\kappa=\mathbb{R}^{m+1}$. Then statement (ii) follows from [4, Theorem 3.7.1]. Let $\mathbb{Y}_\kappa\ne\mathbb{R}^{m+1}$ and $s\ne0$ if $\mathfrak{F}=B$. Then we get this claim by employing, in addition, [4, Theorems 4.4.1 and 4.4.4]. If $\mathbb{Y}_\kappa\ne\mathbb{R}^{m+1}$, $\mathfrak{F}=B$, and $s=0$, then we have to use [4, Theorem 4.7.1(ii) and Corollary 4.11.2] in addition. □

Due to this lemma we can apply the results of [4] to the local spaces $\mathfrak{F}_{p,\kappa}^{s/\vec{r}}$. This will be done in the following usually without referring to Lemma 9.2.

Let $\mathcal{X}$ be a locally convex space and $1 \le q \le \infty$. For $\kappa \in \mathfrak{K}$ we consider the linear map $\Theta_{q,\kappa}^{\mu} : \mathcal{X}^J \to \mathcal{X}^J$ defined by

$$\Theta_{q,\kappa}^{\mu} u(t) := \rho_\kappa^{\mu/q} u(\rho_\kappa^{\mu} t), \qquad u \in \mathcal{X}^J, \quad t \in J. \tag{9.2}$$

Note

$$\Theta_{q,\kappa}^{\mu} \circ \Theta_{q,\kappa}^{-\mu} = \Theta_{q,\kappa}^{0} = \mathrm{id} \tag{9.3}$$

and

$$\Theta_{q,\kappa}^{\mu}\big(C(J,\mathcal{X})\big) \subset C(J,\mathcal{X}). \tag{9.4}$$

Moreover,

$$\partial^k \circ \Theta_{q,\kappa}^{\mu} = \rho_\kappa^{k\mu} \Theta_{q,\kappa}^{\mu} \circ \partial, \qquad k \in \mathbb{N}, \tag{9.5}$$

and, if $\mathcal{X}$ is a Banach space,

$$\|\Theta_{q,\kappa}^{\mu} u\|_{L_q(J,\mathcal{X})} = \|u\|_{L_q(J,\mathcal{X})}. \tag{9.6}$$

We put

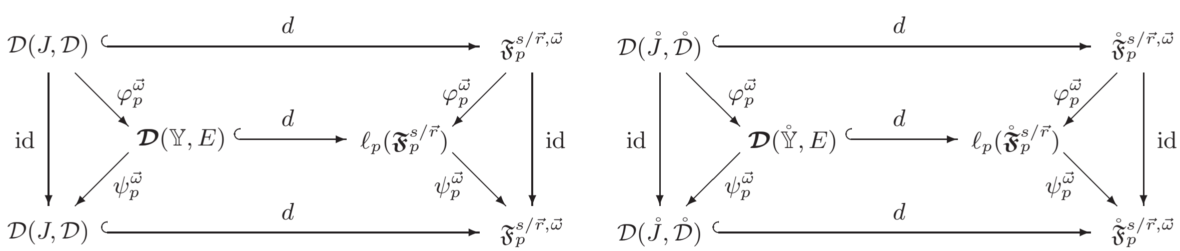
$$\varphi_{q,\kappa}^{\vec{\omega}} u := \Theta_{q,\kappa}^{\mu} \circ \varphi_{q,\kappa}^{\lambda} u, \quad \varphi_q^{\vec{\omega}} u := (\varphi_{q,\kappa}^{\vec{\omega}} u), \qquad u \in C\big(J, C(V)\big), \tag{9.7}$$

and

$$\psi_{q,\kappa}^{\vec{\omega}} v_\kappa := \Theta_{q,\kappa}^{-\mu} \circ \psi_{q,\kappa}^{\lambda} v_\kappa, \quad \psi_q^{\vec{\omega}} \boldsymbol{v} := \sum_\kappa \psi_{q,\kappa}^{\vec{\omega}} v_\kappa, \qquad \boldsymbol{v} = (v_\kappa) \in \bigoplus_\kappa C(\mathbb{Y}_\kappa, E). \tag{9.8}$$

After these preparations we can prove the following analogue to Theorem 7.1. Not only will it play a fundamental role in this paper but also be decisive for the study of parabolic equations on singular manifolds.

Theorem 9.3 *Suppose $s \in \mathbb{R}$. Then the diagrams*



are commuting, where $s > 0$ in the second case.

Proof. (1) It is not difficult to see that $\mathcal{D}(J, \mathcal{D}_\kappa) = \mathcal{D}(\mathbb{Y}_\kappa, E)$ by means of the identification $u(t) = u(\cdot, t)$ for $t \in J$ (see Corollary 1 in Section 40 of F. Treves [49], for example). Consequently,

$$\boldsymbol{\mathcal{D}}(\mathbb{Y}, E) = \bigoplus_\kappa \mathcal{D}(J, \mathcal{D}_\kappa).$$

Similarly, $\mathcal{D}(\mathring{J}, \mathring{\mathcal{D}}_\kappa) = \mathcal{D}(\mathring{\mathbb{Y}}_\kappa, E)$, and thus

$$\boldsymbol{\mathcal{D}}(\mathring{\mathbb{Y}}, E) = \bigoplus_\kappa \mathcal{D}(\mathring{J}, \mathring{\mathcal{D}}_\kappa).$$

Using this, (9.4), and (9.5), obvious modifications of the proof of Theorem 5.1 in [5] show that the assertions encoded in the respective left triangles of the diagrams are true.

(2) Suppose $k \in \mathbb{N}$. From (9.6) we get

$$\|\varphi_{p,\kappa}^{\vec{\omega}} u\|_{L_p(J, W_{p,\kappa}^{kr})} = \|\varphi_{p,\kappa}^{\lambda} u\|_{L_p(J, W_{p,\kappa}^{kr})}.$$

Hence, using (9.1)

$$\|\varphi_p^{\vec\omega}u\|_{\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))}=\|\varphi_p^{\lambda}u\|_{L_p(J,\ell_p(\boldsymbol{W}_p^{kr}))}.$$

From this and Theorem 7.1 we deduce

$$\|\varphi_p^{\vec\omega}u\|_{\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))}\le c\,\|u\|_{L_p(J,W_p^{kr,\lambda})},$$

that is,

$$\varphi_p^{\vec\omega}\in\mathcal{L}\big(L_p(J,W_p^{kr,\lambda}),\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))\big). \tag{9.9}$$

By means of (9.5) and (9.6) we obtain

$$\|\partial^j\varphi_{p,\kappa}^{\vec\omega}u\|_{L_p(J,W_{p,\kappa}^{k-j})}=\|\varphi_{p,\kappa}^{\lambda+j\mu}(\partial^j u)\|_{L_p},\qquad 0\le j\le k. \tag{9.10}$$

Consequently, invoking (9.1) and Theorem 7.1 once more,

$$\|\partial^k\varphi_p^{\vec\omega}u\|_{\ell_p(L_p(J,\boldsymbol{L}_p))}\le c\,\|\partial^k u\|_{L_p(J,L_p^{\lambda+k\mu})}.$$

This, together with (9.9), implies

$$\varphi_p^{\vec\omega}\in\mathcal{L}\big(W_p^{kr/\vec r,\vec\omega},\ell_p(\boldsymbol{W}_p^{kr/\vec r})\big). \tag{9.11}$$

(3) Note that

$$\varphi_{p,\kappa}^{\lambda}\psi_{p,\widetilde\kappa}^{\lambda}=a_{\widetilde\kappa\kappa}S_{\widetilde\kappa\kappa}, \tag{9.12}$$

where

$$a_{\widetilde\kappa\kappa}:=(\rho_\kappa/\rho_{\widetilde\kappa})^{\lambda+m/p}(\kappa_*\pi_\kappa)S_{\widetilde\kappa\kappa}(\widetilde\kappa_*\pi_{\widetilde\kappa}).$$

Lemma 7.2, estimate (4.3), and (7.3)(iii) imply

$$a_{\widetilde\kappa\kappa}\in BC^k(\mathbb{X}_\kappa),\quad \|a_{\widetilde\kappa\kappa}\|_{k,\infty}\le c(k),\qquad \widetilde\kappa\in\mathfrak{N}(\kappa),\quad \kappa\in\mathfrak{K},\quad k\in\mathbb{N}.$$

Hence we infer from (9.12) and Lemma 7.2

$$\varphi_{p,\kappa}^{\lambda}\psi_{p,\widetilde\kappa}^{\lambda}\in\mathcal{L}(W_{p,\widetilde\kappa}^k,W_{p,\kappa}^k),\quad \|\varphi_{p,\kappa}^{\lambda}\psi_{p,\widetilde\kappa}^{\lambda}\|\le c(k)$$

for $\widetilde\kappa\in\mathfrak{N}(\kappa)$, $\kappa\in\mathfrak{K}$, and $k\in\mathbb{N}$. By this and (9.6) we find

$$\|\varphi_{p,\kappa}^{\lambda}\psi_{p,\widetilde\kappa}^{\vec\omega}v\|_{L_p(J,W_{p,\kappa}^k)}=\|\varphi_{p,\kappa}^{\lambda}\psi_{p,\widetilde\kappa}^{\lambda}v\|_{L_p(J,W_{p,\kappa}^k)}\le c\,\|v\|_{L_p(J,W_{p,\widetilde\kappa}^k)} \tag{9.13}$$

for $\widetilde\kappa\in\mathfrak{N}(\kappa)$, $\kappa\in\mathfrak{K}$, and $k\in\mathbb{N}$. Similarly, using (9.5),

$$\big\|\varphi_{p,\kappa}^{\lambda+k\mu}\big(\partial^k(\psi_{p,\widetilde\kappa}^{\vec\omega}v)\big)\big\|_{L_p(J,L_{p,\kappa})}=\|\varphi_{p,\kappa}^{\lambda+k\mu}\psi_{p,\widetilde\kappa}^{(\lambda+k\mu,\mu)}(\partial^k v)\|_{L_p(J,L_{p,\kappa})}\le c\,\|\partial^k v\|_{L_p(J,L_{p,\widetilde\kappa})} \tag{9.14}$$

for $\widetilde\kappa\in\mathfrak{N}(\kappa)$, $\kappa\in\mathfrak{K}$, and $k\in\mathbb{N}$.

Observe

$$\varphi_{p,\kappa}^{\lambda}\psi_p^{\vec\omega}\boldsymbol{v}=\sum_{\widetilde\kappa\in\mathfrak{N}(\kappa)}\varphi_{p,\kappa}^{\lambda}\psi_{p,\widetilde\kappa}^{\vec\omega}v_{\widetilde\kappa}. \tag{9.15}$$

From (9.13)–(9.15) and the finite multiplicity of $\mathfrak{K}$ we infer

$$\|\varphi_p^{\lambda}(\psi_p^{\vec\omega}\boldsymbol{v})\|_{\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))}\le c\,\|\boldsymbol{v}\|_{\ell_p(L_p(J,\boldsymbol{W}_p^{kr}))} \tag{9.16}$$

and

$$\big\|\varphi_p^{\lambda+k\mu}\big(\partial^k(\psi_p^{\vec\omega}\boldsymbol{v})\big)\big\|_{\ell_p(L_p(J,\boldsymbol{L}_p))}\le c\,\|\partial^k\boldsymbol{v}\|_{\ell_p(L_p(J,\boldsymbol{L}_p))}.$$

Hence Theorem 9.1 implies

$$\|\psi_p^{\vec\omega}\boldsymbol{v}\|_{kr/\vec r,p;\vec\omega}\le c\,\|\boldsymbol{v}\|_{\ell_p(\boldsymbol{W}_p^{kr/\vec r})},$$

that is,

$$\psi_p^{\vec{\omega}} \in \mathcal{L}\bigl(\ell_p(\boldsymbol{W}_p^{kr/\vec{r}}), W_p^{kr/\vec{r},\vec{\omega}}\bigr). \tag{9.17}$$

It follows from $\psi_p^\lambda \varphi_p^\lambda = \mathrm{id}$ that $\psi_p^{\vec{\omega}} \varphi_p^{\vec{\omega}} = \mathrm{id}$. Thus we see from (9.11) and (9.17) that the diagram

$$\begin{array}{ccc} W_p^{kr/\vec{r},\vec{\omega}} & \xrightarrow{\quad \mathrm{id} \quad} & W_p^{kr/\vec{r},\vec{\omega}} \\ {\scriptstyle \varphi_p^{\vec{\omega}}} \searrow & & \nearrow {\scriptstyle \psi_p^{\vec{\omega}}} \\ & \ell_p(\boldsymbol{W}_p^{kr/\vec{r}}) & \end{array} \tag{9.18}$$

is commuting.

(4) It is a consequence of Lemma 9.2(i) and [4, Theorems 2.3.2(i) and 4.4.1] that $\mathcal{D}(J,\mathcal{D}_\kappa) = \mathcal{D}(\mathbb{Y}_\kappa, E)$ is dense in $W_{p,\kappa}^{kr/\vec{r}}$. This implies

$$\boldsymbol{\mathcal{D}}(\mathbb{Y},E) \overset{d}{\hookrightarrow} \bigoplus_\kappa W_{p,\kappa}^{kr/\vec{r}} = c_c(\boldsymbol{W}_p^{kr/\vec{r}}).$$

Hence, by (7.2),

$$\boldsymbol{\mathcal{D}}(\mathbb{Y},E) \overset{d}{\hookrightarrow} \ell_p(\boldsymbol{W}_p^{kr/\vec{r}}). \tag{9.19}$$

Thus we deduce from step (1) and (9.18) that

$$\begin{array}{ccccc} \mathcal{D}(J,\mathcal{D}) & & \hookrightarrow & & W_p^{kr/\vec{r},\vec{\omega}} \\ & {\scriptstyle \varphi_p^{\vec{\omega}}} \searrow & & \swarrow {\scriptstyle \varphi_p^{\vec{\omega}}} & \\ \mathrm{id} \downarrow & \boldsymbol{\mathcal{D}}(\mathbb{Y},E) & \overset{d}{\hookrightarrow} & \ell_p(\boldsymbol{W}_p^{kr/\vec{r}}) & \downarrow \mathrm{id} \\ & {\scriptstyle \psi_p^{\vec{\omega}}} \swarrow & & \searrow {\scriptstyle \psi_p^{\vec{\omega}}} & \\ \mathcal{D}(J,\mathcal{D}) & & \hookrightarrow & & W_p^{kr/\vec{r},\vec{\omega}} \end{array}$$

is a commuting diagram. From this and [4, Lemma 4.1.6] we obtain

$$\mathcal{D}(J,\mathcal{D}) \overset{d}{\hookrightarrow} W_p^{kr/\vec{r},\vec{\omega}}. \tag{9.20}$$

(5) Suppose $k \in \mathbb{N}$ and $kr < s \leq (k+1)r$. If $s < (k+1)r$, set $\theta := (s-kr)/r$ and $\ell := k+1$. Otherwise, $\theta := 1/2$ and $\ell := k+2$. Then we infer from (9.18) and (8.3) by interpolation that $\psi_p^{\vec{\omega}}$ is a retraction from

$$\bigl(\ell_p(\boldsymbol{W}_p^{kr/\vec{r}}), \ell_p(\boldsymbol{W}_p^{\ell r/\vec{r}})\bigr)_\theta \tag{9.21}$$

onto $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$. By Theorem 1.18.1 in H. Triebel [50], (9.21) equals $\ell_p\bigl((\boldsymbol{W}_p^{kr/\vec{r}}, \boldsymbol{W}_p^{\ell r/\vec{r}})_\theta\bigr)$, except for equivalent norms.

It follows from Lemma 9.2 and [4, Theorem 3.7.1(iv), formula (3.3.12), and Theorems 3.5.2 and 4.4.1] that $(W_{p,\kappa}^{kr/\vec{r}}, W_{p,\kappa}^{\ell r/\vec{r}})_\theta \doteq \mathfrak{F}_{p,\kappa}^{s/\vec{r}}$. This shows that the right triangle of the first diagram is commuting if $s > 0$. Furthermore, the density properties of the interpolation functor $(\cdot,\cdot)_\theta$, (9.19), and (9.20) imply that the 'horizontal embeddings' of the first diagram of the assertion are dense if $s > 0$. This proves the first assertion for $s > 0$.

(6) It is a consequence of what has just been shown and step (1) that the second part of the statement is true.

(7) Let X be a reflexive Banach space. Then

$$\langle v, \Theta_{p,\kappa}^\mu u\rangle_{L_p(J,X)} = \langle \Theta_{p',\kappa}^{-\mu} v, u\rangle_{L_p(J,X)}, \qquad u \in L_p(J,X), \quad v \in L_{p'}(J,X') = \bigl(L_p(J,X)\bigr)'.$$

We define $\mathring{\varphi}_p^{\vec{\omega}}$ and $\mathring{\psi}_p^{\vec{\omega}}$ by replacing $\varphi_{p,\kappa}^\lambda$ and $\psi_{p,\kappa}^\lambda$ in (9.7) and (9.8) by $\mathring{\varphi}_{p,\kappa}^\lambda$ and $\mathring{\psi}_{p,\kappa}^\lambda$, defined in (7.5), respectively. From this we infer (cf. the proof of Theorem 5.1 in [5])

$$\langle \mathring{\psi}_{p'}^{-\vec{\omega}} \boldsymbol{v}, u\rangle_{M\times J} = \boldsymbol{\langle} \boldsymbol{v}, \varphi_p^{\vec{\omega}} u \boldsymbol{\rangle}, \qquad \boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\mathring{\mathbb{Y}}, E), \quad u \in \mathcal{D}(J,\mathcal{D}), \tag{9.22}$$

and

$$\langle\!\langle \mathring{\varphi}_{p'}^{-\vec{\omega}} v, \boldsymbol{u} \rangle\!\rangle = \langle v, \psi_p^{\vec{\omega}} \boldsymbol{u} \rangle_{M\times J}, \qquad v \in \mathcal{D}(\mathring{J}, \mathring{\mathcal{D}}), \quad \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\mathbb{Y}, E). \tag{9.23}$$

Moreover, (8.5) implies for $s > 0$

$$\ell_p(\boldsymbol{\mathfrak{F}}_p^{-s/\vec{r}}) = \big(\ell_{p'}\big(\boldsymbol{\mathfrak{F}}_{p'}^{s/\vec{r}}(\mathbb{Y}, E')\big)\big)'.$$

It follows from (7.6) that $\mathring{\varphi}_p^{\vec{\omega}}$ and $\mathring{\psi}_p^{\vec{\omega}}$ possess the same mapping properties as $\varphi_p^{\vec{\omega}}$ and $\psi_p^{\vec{\omega}}$, respectively. Hence we deduce from (9.22) and (9.23) that, given $s > 0$,

$$\|\varphi_p^{\vec{\omega}} u\|_{\ell_p(\boldsymbol{\mathfrak{F}}_p^{-s/\vec{r}})} \le c\,\|u\|_{\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}}, \qquad u \in \mathcal{D}(J, \mathcal{D}),$$

and

$$\|\psi_p^{\vec{\omega}} \boldsymbol{u}\|_{\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}} \le c\,\|\boldsymbol{u}\|_{\ell_p(\boldsymbol{\mathfrak{F}}_p^{-s/\vec{r}})}, \qquad \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\mathbb{Y}, E). \tag{9.24}$$

We infer from (9.20), Theorem 8.3(i), and reflexivity that $\mathcal{D}(J, \mathcal{D})$ is dense in $\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}$. Hence

$$\varphi_p^{\vec{\omega}} \in \mathcal{L}\big(\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}, \ell_p(\boldsymbol{\mathfrak{F}}_p^{-s/\vec{r}})\big). \tag{9.25}$$

Since, as above, $\mathcal{D}(J, \mathcal{D}_\kappa) = \mathcal{D}(\mathbb{Y}_\kappa, E)$ is dense in $\mathfrak{F}_{p,\kappa}^{-s/\vec{r}}$ we see, by the arguments used to prove (9.19),

$$\boldsymbol{\mathcal{D}}(\mathbb{Y}, E) \stackrel{d}{\hookrightarrow} \ell_p(\boldsymbol{\mathfrak{F}}_p^{-s/\vec{r}}). \tag{9.26}$$

Thus (9.24) implies

$$\psi_p^{\vec{\omega}} \in \mathcal{L}\big(\ell_p(\boldsymbol{\mathfrak{F}}_p^{-s/\vec{r}}), \mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}\big). \tag{9.27}$$

From (9.25)–(9.27) and step (1) it now follows that the first statement is true if $s < 0$.

(8) Suppose $s = 0$. If $\mathfrak{F} = H$, then assertion (i) is contained in (9.18) (for $k = 0$). If $\mathfrak{F} = B$, then we deduce from Lemma 9.2(ii) and [4, Theorems 3.7.1, 4.4.1, 4.7.1(ii), and Corollary 4.11.2] that

$$(H_{p,\kappa}^{-s(p)/\vec{r}}, H_{p,\kappa}^{s(p)/\vec{r}})_{1/2,p} \doteq B_{p,\kappa}^{0/\vec{r}}, \qquad \kappa \in \mathfrak{K}.$$

Thus, as in step (5),

$$\big(\ell_p(\boldsymbol{H}_p^{-s(p)/\vec{r}}), \ell_p(\boldsymbol{H}_p^{s(p)/\vec{r}})\big)_{1/2,p} \doteq \ell_p(\boldsymbol{B}_p^{0/\vec{r}}).$$

Since we have already shown that $\psi_p^{\vec{\omega}}$ is a retraction from $\ell_p(\boldsymbol{H}_p^{\pm s(p)/\vec{r}})$ onto $H_p^{\pm s(p)/\vec{r}}$, it follows from definition (8.6) that it is a retraction from $\ell_p(\boldsymbol{B}_p^{0/\vec{r}})$ onto $B_p^{0/\vec{r},\vec{\omega}}$. This proves the theorem. □

Now we can supply the proofs left out in Section 8. First note that assertion (iii) of Theorem 8.1 has been shown in (9.20).

P r o o f of part (ii) of Theorem 8.1. It is a consequence of Lemma 9.2(i) that

$$\ell_p(\boldsymbol{W}_p^{kr/\vec{r}}) \doteq \ell_p(\widetilde{\boldsymbol{W}}_p^{kr/\vec{r}}).$$

Hence, due to (7.8) and (9.18),

$$\|\varphi_p^{\vec{\omega}}\cdot\|_{\ell_p(\widetilde{\boldsymbol{W}}_p^{kr/\vec{r}})} \sim \|\cdot\|_{kr/\vec{r},p;\vec{\omega}}.$$

Using (9.9) and (9.10) one verifies

$$\|\varphi_p^{\vec{\omega}}\cdot\|_{\ell_p(\widetilde{\boldsymbol{W}}_p^{kr/\vec{r}})} \sim |||\cdot|||_{kr/\vec{r},p;\vec{\omega}}^{\sim}.$$

Now the assertion follows from Theorem 9.1. □

P r o o f of Theorem 8.2. (1) Lemma 9.2 and [4, Theorems 3.7.1 and 4.4.3(i)] imply $H_{p,\kappa}^{kr/\vec{r}} \doteq W_{p,\kappa}^{kr/\vec{r}}$ for $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}$. Hence

$$\ell_p(\boldsymbol{H}_p^{kr/\vec{r}}) \doteq \ell_p(\boldsymbol{W}_p^{kr/\vec{r}}), \qquad k \in \mathbb{N},$$

and assertion (i) is a consequence of Theorem 9.3.

(2) In order to prove (ii) it suffices, due to Theorem 9.3 and Lemma 9.2, to show $H_2^{s/\boldsymbol{\nu}}(\mathbb{Y}_\kappa, E) \doteq B_2^{s/\vec{r}}(\mathbb{Y}_\kappa, E)$. By the results of Section 4.4 of [4] we can assume $\mathbb{Y}_\kappa = \mathbb{R}^{m+1}$.

Suppose $s > 0$ and write $H_2^s := H_2^s(\mathbb{R}^m, E)$, etc. Then [4, Theorem 3.7.2] asserts

$$H_2^{s/\boldsymbol{\nu}} = L_p(\mathbb{R}, H_2^s) \cap H_2^{s/\nu}(\mathbb{R}, L_2).$$

From Theorem 3.6.7 of [4] we get

$$B_2^{s/\boldsymbol{\nu}} = L_2(\mathbb{R}, B_2^s) \cap B_2^{s/\nu}(\mathbb{R}, L_2).$$

By Theorem 2.12 in [50] we know that $H_2^s \doteq B_2^s$. Remark 7 and Proposition 2(1) in H.-J. Schmeißer and W. Sickel [42] guarantee $H_2^{s/\nu}(\mathbb{R}, L_2) \doteq B_2^{s/\nu}(\mathbb{R}, L_2)$. This proves $H_2^{s/\boldsymbol{\nu}} \doteq B_2^{s/\boldsymbol{\nu}}$ for $s > 0$. The case $s < 0$ follows by duality.

From Lemma 9.2(ii) and [4, (3.4.1) and Theorem 3.7.1] we get $[\mathfrak{F}_2^{-s(p)/\vec{r}}, \mathfrak{F}_2^{s(p)/\vec{r}}]_{1/2} \doteq \mathfrak{F}_2^{0/\vec{r}}$. Thus, by what we already know,

$$B_2^{0/\vec{r}} \doteq [B_2^{-s(p)/\vec{r}}, B_2^{s(p)/\vec{r}}]_{1/2} \doteq [H_2^{-s(p)/\vec{r}}, H_2^{s(p)/\vec{r}}]_2 \doteq H_2^{0/\vec{r}}.$$

This settles the case $s = 0$ also.

(3) By [4, (3.3.12), (3.4.1), and Theorems 3.7.1(iv) and 4.4.1] we know that assertions (iii) and (iv) hold for the local spaces $\mathfrak{F}_{p,\kappa}^{s/\vec{r}}$. Thus we get (iii) and (iv) in the general case by the arguments of step (5) of the proof of Theorem 9.3. □

P r o o f of Theorem 8.3(i) for $s = 0$. Since (8.7) has already been established for $s \in \mathbb{R}\backslash\{0\}$ it remains to show that

$$H_p^{s_1/\vec{r},\vec{\omega}} \overset{d}{\hookrightarrow} B_p^{0/\vec{r},\vec{\omega}} \overset{d}{\hookrightarrow} H_p^{s_0/\vec{r},\vec{\omega}}$$

if $-1 + 1/p < s_0 < 0 < s_1 < 1/p$. By [4, Theorems 3.7.1(iii), 4.4.1, 4.7.1(ii), and Corollary 4.11.2]

$$H_{p,\kappa}^{s_1/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} B_{p,\kappa}^{0/\boldsymbol{\nu}} \overset{d}{\hookrightarrow} H_{p,\kappa}^{s_0/\boldsymbol{\nu}}.$$

From this and Lemma 9.2 we deduce

$$\ell_p(\boldsymbol{H}_p^{s_1/\vec{r}}) \overset{d}{\hookrightarrow} \ell_p(\boldsymbol{B}_p^{0/\vec{r}}) \overset{d}{\hookrightarrow} \ell_p(\boldsymbol{H}_p^{s_0/\vec{r}}).$$

Now the claim follows from Theorem 9.3. □

P r o o f of Theorem 8.3(ii). If $J = \mathbb{R}$ and $\partial M = \emptyset$, then the claim is obvious by (8.4), $\mathcal{D}(\mathring{J}, \mathring{\mathcal{D}}) = \mathcal{D}(J, \mathcal{D})$, and (i). Otherwise, we get from [4, Theorem 4.7.1 and Corollary 4.11.2], due to the stated restrictions for s, that $\mathring{\mathfrak{F}}_{p,\kappa}^{s/\vec{r}} = \mathfrak{F}_{p,\kappa}^{s/\vec{r}}$. Here we also used the fact that

$$\mathcal{D}(J, \mathcal{D}_\kappa) \overset{d}{\hookrightarrow} \mathfrak{F}_\kappa^{0/\vec{r}} \overset{d}{\hookrightarrow} \mathfrak{F}_\kappa^{-t/\vec{r}}, \qquad t > 0, \quad \kappa \in \mathfrak{K}.$$

Hence $\ell_p(\mathring{\boldsymbol{\mathfrak{F}}}_p^{s/\vec{r}}) = \ell_p(\boldsymbol{\mathfrak{F}}_p^{s/\vec{r}})$ and the claim follows from (the right triangles of the diagrams of) Theorem 9.3. □

10 Renorming of Besov Spaces

Let $\mathcal{X}$ be a Banach space and $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$. For $u : \mathbb{X} \to \mathcal{X}$ and $h \in \mathbb{H}^m \backslash \{0\}$ we put

$$\triangle_h u := u(\cdot + h) - u, \quad \triangle_h^{k+1} u := \triangle_h \triangle_h^k u, \qquad k \in \mathbb{N}, \quad \triangle_h^0 u := u.$$

Given $k \leq s < k+1$ with $s > 0$,

$$[u]_{s,p;\mathcal{X}} := \Big(\int_{\mathbb{X}} \Big(\frac{\|\triangle_h^{k+1} u\|_{p;\mathcal{X}}}{|h|^s} \Big)^p \frac{dh}{|h|^m} \Big)^{1/p},$$

where $\|\cdot\|_{p;\mathcal{X}} := \|\cdot\|_{L_p(\mathbb{X},\mathcal{X})}$. We set for $s > 0$

$$\|\cdot\|_{s,p;\mathcal{X}}^* := \big(\|\cdot\|_{p;\mathcal{X}}^p + [\cdot]_{s,p;\mathcal{X}}^p \big)^{1/p}.$$

Suppose $k \leq s < k+1$ with $k \in \mathbb{N}$ and $s > 0$. Then

$$\|u\|_{k,p;\mathcal{X}} := \Big(\sum_{|\alpha| \leq k} \|\partial_x^\alpha u\|_{p;\mathcal{X}}^p \Big)^{1/p}$$

is the norm of the *$\mathcal{X}$-valued Sobolev space* $W_p^k(\mathbb{X}, \mathcal{X})$ and

$$\|u\|_{s,p;\mathcal{X}}^{**} := \begin{cases} \Big(\|u\|_{k,p;\mathcal{X}}^p + \sum_{|\alpha|=k} [\partial_x^\alpha u]_{s-k,p;\mathcal{X}}^p \Big)^{1/p}, & k < s < k+1, \\ \Big(\|u\|_{k-1,p;\mathcal{X}}^p + \sum_{|\alpha|=k-1} [\partial_x^\alpha u]_{1,p;\mathcal{X}}^p \Big)^{1/p}, & s = k \in \mathbb{N}^\times. \end{cases}$$

Then, given $s > 0$,

$$B_p^s(\mathbb{X}, \mathcal{X}) := \big(\{ u \in L_p(\mathbb{X}, \mathcal{X}) \ ; \ [u]_{s,p;\mathcal{X}} < \infty \}, \ \|\cdot\|_{s,p;\mathcal{X}}^* \big)$$

is a Banach space, an *$\mathcal{X}$-valued Besov space*,

$$\|\cdot\|_{s,p;\mathcal{X}}^* \sim \|\cdot\|_{s,p;\mathcal{X}}^{**}, \tag{10.1}$$

and $\mathcal{D}(\mathbb{X}, \mathcal{X}) \overset{d}{\hookrightarrow} B_p^s(\mathbb{X}, \mathcal{X})$. These facts can be derived by modifying the corresponding well-known scalar-valued results (e.g., H.-J. Schmeißer [41] or H. Amann [3]).

Now we choose $\mathbb{X} = J$. Note that

$$\triangle_h^k \circ \Theta_{q,\kappa}^\mu = \Theta_{q,\kappa}^\mu \circ \triangle_{\rho_\kappa^\mu h}^k, \qquad 1 \leq q \leq \infty.$$

Hence (9.6) implies

$$[\Theta_{p,\kappa}^\mu u]_{s,p;\mathcal{X}} = \rho_\kappa^{\mu s} \, [u]_{s,p;\mathcal{X}}. \tag{10.2}$$

Suppose $s > 0$. Then

$$\|u\|_{s/\vec{r},p;\vec{\omega}}^* := \big(\|u\|_{p;B_p^{s,\lambda}}^p + [u]_{s/r,p;L_p^{\lambda+s\mu/r}}^p \big)^{1/p} \tag{10.3}$$

and, if $kr < s \leq (k+1)r$ with $k \in \mathbb{N}$,

$$\|u\|_{s/\vec{r},p;\vec{\omega}}^{**} := \Big(\|u\|_{p;B_p^{s,\lambda}}^p + \sum_{j \leq k} \|\partial^j u\|_{p;W_p^{(k-j)r,\lambda+\mu j}}^p + [\partial^k u]_{(s-kr)/r,p;L_p^{\lambda+s\mu/r}}^p \Big)^{1/p}. \tag{10.4}$$

Besides of these norms we introduce localized versions of them by

$$|||u|||_{s/\vec{r},p;\vec{\omega}}^* := \big(\|\varphi_p^\lambda u\|_{p;\ell_p(\boldsymbol{B}_p^s)}^p + [\varphi_p^{\lambda+s\mu/r} u]_{s/r,p;\ell_p(\boldsymbol{L}_p)}^p \big)^{1/p} \tag{10.5}$$

and, if $kr < s \le (k+1)r$,

$$|||u|||^{**}_{s/\vec r,p;\vec\omega} := \Bigl(\|\varphi_p^\lambda u\|^p_{p;\ell_p(\boldsymbol{B}_p^s)} + \sum_{j\le k} \|\partial^j \varphi_p^{\lambda+j\mu} u\|^p_{p;\ell_p(\boldsymbol{W}_p^{(k-j)r})} + [\partial^k \varphi_p^{\lambda+s\mu/r} u]^p_{(s-kr)/r,p;\ell_p(\boldsymbol{L}_p)}\Bigr)^{1/p}. \tag{10.6}$$

Theorem 10.1 *Suppose* $s > 0$. *Then* (10.3)–(10.6) *are equivalent norms for* $B_p^{s/\vec r,\vec\omega}$.

P r o o f. (1) It follows from (9.6) that

$$\|\varphi_p^\lambda u\|_{p;\ell_p(\boldsymbol{B}_p^s)} = \|\varphi_p^{\vec\omega} u\|_{p;\ell_p(\boldsymbol{B}_p^s)}. \tag{10.7}$$

Using (10.2) we get

$$[\varphi_{p,\kappa}^{\vec\omega} u]_{s/r,p;L_{p,\kappa}} = [\varphi_{p,\kappa}^{\lambda+s\mu/r} u]_{s/r,p;L_{p,\kappa}}. \tag{10.8}$$

Thus, by Fubini's theorem,

$$[\varphi_p^{\vec\omega} u]_{s/r,p;\ell_p(\boldsymbol{L}_p)} = [\varphi_p^{\lambda+s\mu/r} u]_{s/r,p;\ell_p(\boldsymbol{L}_p)}.$$

From this and (10.7) we obtain

$$|||u|||^{*}_{s/\vec r,p;\vec\omega} = \bigl(\|\varphi_p^{\vec\omega} u\|^p_{p;\ell_p(\boldsymbol{B}_p^s)} + [\varphi_p^{\vec\omega} u]^p_{s/r,p;\ell_p(\boldsymbol{L}_p)}\bigr)^{1/p}. \tag{10.9}$$

Similarly, invoking (9.5) as well,

$$|||u|||^{**}_{s/\vec r,p;\vec\omega} = \Bigl(\|\varphi_p^{\vec\omega} u\|^p_{p;\ell_p(\boldsymbol{B}_p^s)} + \sum_{j\le k} \|\partial^j \varphi_p^{\vec\omega} u\|^p_{p;\ell_p(\boldsymbol{W}_p^{(k-j)r})} + [\partial^k \varphi_p^{\vec\omega} u]^p_{(s-kr)/r,p;\ell_p(\boldsymbol{L}_p)}\Bigr)^{1/p}$$

if $kr < s \le (k+1)r$.

(2) Lemma 9.2 and [4, Theorems 3.6.3 and 4.4.3] imply

$$B_{p,\kappa}^{s/\vec r} \doteq L_p(J, B_{p,\kappa}^s) \cap B_p^{s/r}(J, L_{p,\kappa}), \qquad \kappa \in \mathfrak{K}.$$

Hence

$$\|\cdot\|_{B_{p,\kappa}^{s/\vec r}} \sim \|\cdot\|_{p;B_{p,\kappa}^s} + [\cdot]_{s/r,p;L_{p,\kappa}},$$

due to $B_{p,\kappa}^s \hookrightarrow L_{p,\kappa}$. From this, (10.9), and Fubini's theorem we deduce

$$|||\cdot|||^{*}_{s/\vec r,p;\vec\omega} \sim \|\varphi_p^{\vec\omega} \cdot\|_{\ell_p(\boldsymbol{B}_p^{s/\vec r})}.$$

Thus (7.8) and Theorem 9.3 guarantee that (10.5) is a norm for $B_p^{s/\vec r,\vec\omega}$. Similarly, using (10.1), we see that (10.6) is a norm for $B_p^{s/\vec r,\vec\omega}$.

(3) We set $\alpha := \lambda + s\mu/r$ and $\beta := \alpha + \tau - \sigma$. Then we deduce from (4.1)(iv), (5.11), (7.6), and [5, Lemma 3.1(iii)]

$$\begin{aligned}
[\varphi_{p,\kappa}^\alpha u]^p_{s/r,p;L_{p,\kappa}} &= \int_0^\infty \int_J \int_{\mathbb{X}_\kappa} \bigl(\rho_\kappa^{\alpha+m/p} \bigl|\triangle_\xi^{k+1}\bigl((\kappa\ltimes\varphi)_*(\pi_\kappa u)\bigr)\bigr|_E\bigr)^p \, dV_{g_m}\, dt\, \frac{d\xi}{\xi^{1+ps/r}} \\
&\sim \int_0^\infty \int_J \int_{\mathbb{X}_\kappa} \kappa_*\bigl((\rho^\beta \pi_\kappa \,|\triangle_\xi^{k+1} u|_h)^p \, dV_g\bigr)\, dt\, \frac{d\xi}{\xi^{1+ps/r}} \\
&= \int_0^\infty \int_J \int_{U_\kappa} (\rho^\beta \pi_\kappa \,|\triangle_\xi^{k+1} u|_h)^p \, dV_g\, dt\, \frac{d\xi}{\xi^{1+ps/r}}
\end{aligned}$$

for $u \in \mathcal{D}(J,\mathcal{D})$. We insert $\mathbf{1} = \sum_{\widetilde\kappa} \pi_{\widetilde\kappa}^2$ in the inner integral, sum over $\kappa \in \mathfrak{K}$, and interchange the order of summation. Then

$$[\varphi_p^\alpha u]^p_{s/r,p;\ell_p(\boldsymbol{L}_p)} \sim \sum_{\widetilde\kappa} \sum_{\kappa\in\mathfrak{N}(\widetilde\kappa)} \int_0^\infty \int_J \int_{U_{\widetilde\kappa}} \pi_{\widetilde\kappa}^2 (\rho^\beta \pi_\kappa \,|\triangle_\xi^{k+1} u|_h)^p \, dV_g\, dt\, \frac{d\xi}{\xi^{1+ps/r}}.$$

Using (7.3)(iii) and the finite multiplicity of $\mathfrak{K}$ we see that the last term can be bounded above by

$$c\sum_{\widetilde{\kappa}}\int_0^\infty\int_J\int_{U_{\widetilde{\kappa}}}\pi_{\widetilde{\kappa}}^2(\rho^\beta\,|\triangle_\xi^{k+1}u|_h)^p\,dV_g\,dt\,\frac{d\xi}{\xi^{1+ps/r}}$$
$$=c\int_0^\infty\int_J\int_M(\rho^\beta\,|\triangle_\xi^{k+1}u|_h)^p\,dV_g\,dt\,\frac{d\xi}{\xi^{1+ps/r}}=c\,[u]^p_{s/r,p;L_p^\alpha}.$$

Hence, recalling (10.8),

$$[\varphi_p^{\vec{\omega}}u]_{s/r,p;\ell_p(\boldsymbol{L}_p)}\le c\,[u]_{s/r,p;L_p^{\lambda+s\mu/r}},\qquad u\in\mathcal{D}(J,\mathcal{D}). \tag{10.10}$$

(4) It is a consequence of Theorem 7.1 that $\varphi_p^\lambda\in\mathcal{L}\big(B_p^{s,\lambda},\ell_p(\boldsymbol{B}_p^s)\big)$. This implies, due to (10.7),

$$\|\varphi_p^{\vec{\omega}}u\|_{p;\ell_p(\boldsymbol{B}_p^s)}=\|\varphi_p^\lambda u\|_{p;\ell_p(\boldsymbol{B}_p^s)}\le c\,\|u\|_{p;B_p^{s,\lambda}},\qquad u\in\mathcal{D}(J,\mathcal{D}). \tag{10.11}$$

Thus we obtain from (10.9), (10.10), and (10.11)

$$|||u|||^*_{s/\vec{r},p;\vec{\omega}}\le c\,\|u\|^*_{s/\vec{r},p;\vec{\omega}},\qquad u\in\mathcal{D}(J,\mathcal{D}). \tag{10.12}$$

We denote by $\overset{*}{B}{}_p^{s/\vec{r},\vec{\omega}}$ the completion of $\mathcal{D}(J,\mathcal{D})$ in $L_p(J,L_p^\lambda)$ with respect to the norm $\|\cdot\|^*_{s/\vec{r},p;\vec{\omega}}$. Then (10.12) and step (2) imply

$$\overset{*}{B}{}_p^{s/\vec{r},\vec{\omega}}\hookrightarrow B_p^{s/\vec{r},\vec{\omega}}.$$

(5) Observing $\psi_{p,\kappa}^{\vec{\omega}}=\chi_\kappa\psi_{p,\kappa}^{\vec{\omega}}$ and $0\le\chi_\kappa\le 1$, the finite multiplicity of $\mathfrak{K}$ implies

$$|\triangle_\xi^{k+1}\psi_p^{\vec{\omega}}\boldsymbol{v}|_h=\Big|\sum_\kappa\triangle_\xi^{k+1}\psi_{p,\kappa}^{\vec{\omega}}v_\kappa\Big|_h\le\Big(\sum_\kappa|\triangle_\xi^{k+1}\psi_{p,\kappa}^{\vec{\omega}}v_\kappa|_h^p\Big)^{1/p}\Big(\sum_\kappa\chi_\kappa\Big)^{1/p'}$$
$$\le c\Big(\sum_\kappa|\triangle_\xi^{k+1}\psi_{p,\kappa}^{\vec{\omega}}v_\kappa|_h^p\Big)^{1/p}$$

for $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathbb{Y},E)$. Hence, reasoning as in step (3),

$$[\psi_p^{\vec{\omega}}\boldsymbol{v}]^p_{s/r,p;L_p^{\lambda+s\mu/r}}\le c\int_0^\infty\int_J\int_M\rho^{\beta p}\sum_\kappa|\triangle_\xi^{k+1}\psi_{p,\kappa}^{\vec{\omega}}v_\kappa|_h^p\,dV_g\,dt\,\frac{d\xi}{\xi^{1+ps/r}}$$
$$\le c\int_0^\infty\int_J\int_{\mathbb{X}_\kappa}\sum_\kappa|\triangle_\xi^{k+1}(\pi_\kappa v_\kappa)|^p_{g_m}\,dV_{g_m}\,dt\,\frac{d\xi}{\xi^{1+ps/r}}$$
$$\le c\sum_\kappa[\pi_\kappa v_\kappa]^p_{s/r,p;L_{p,\kappa}}\le c\sum_\kappa[v_\kappa]^p_{s/r,p;L_{p,\kappa}}$$
$$\le c\sum_\kappa\|v_\kappa\|^p_{B_p^{s/r}(J,L_{p,\kappa})}$$

for $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathbb{Y},E)$.

(6) Theorem 7.1 and (7.9) guarantee that $\|\varphi_p^\lambda\cdot\|_{\ell_p(\boldsymbol{B}_p^s)}$ is an equivalent norm for $B_p^{s,\lambda}$. This implies

$$\|\psi_p^{\vec{\omega}}\boldsymbol{v}\|_{p;B_p^{s,\lambda}}\le c\,\|\varphi_p^\lambda\psi_p^{\vec{\omega}}\boldsymbol{v}\|_{p;\ell_p(\boldsymbol{B}_p^s)},\qquad\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathbb{Y},E). \tag{10.13}$$

From (9.16) we infer by interpolation, using the arguments of step (5) of the proof of Theorem 9.3, that

$$\|\varphi_p^\lambda\psi_p^{\vec{\omega}}\boldsymbol{v}\|_{\ell_p(L_p(J,\boldsymbol{B}_p^s))}\le c\,\|\boldsymbol{v}\|_{\ell_p(L_p(J,\boldsymbol{B}_p^s))},\qquad\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathbb{Y},E).$$

Hence (10.13) and (9.1) imply

$$\|\psi_p^{\vec{\omega}}\boldsymbol{v}\|_{p;B_p^{s,\lambda}}\le c\,\|\boldsymbol{v}\|_{p;\ell_p(\boldsymbol{B}_p^s)},\qquad\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathbb{Y},E).$$

By combining this with the result of step (5) we find, employing (9.1) once more,

$$\|\psi_p^{\vec{\omega}}\boldsymbol{v}\|_{s/\vec{r},p;\vec{\omega}}^{*} \le c\,\|\boldsymbol{v}\|_{\ell_p(\boldsymbol{B}_p^{s/\vec{r}})}, \qquad \boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathbb{Y},E).$$

Thus, by Theorem 9.3,

$$\|u\|_{s/\vec{r},p;\vec{\omega}}^{*} = \|\psi_p^{\vec{\omega}}(\varphi_p^{\vec{\omega}}u)\|_{s/\vec{r},p;\vec{\omega}}^{*} \le c\,\|\varphi_p^{\vec{\omega}}u\|_{\ell_p(\boldsymbol{B}_p^{s/\vec{r}})} = c\,|||u|||_{s/\vec{r},p;\vec{\omega}}^{*}, \qquad u\in\mathcal{D}(J,\mathcal{D}),$$

the last estimate being a consequence of (10.9). Since, by step (2), (10.5) is a norm for $B_p^{s/\vec{r},\vec{\omega}}$, we get

$$\|u\|_{s/\vec{r},p;\vec{\omega}}^{*} \le c\,\|u\|_{B_p^{s/\vec{r},\vec{\omega}}}, \qquad u\in\mathcal{D}(J,\mathcal{D}).$$

This implies $B_p^{s/\vec{r},\vec{\omega}} \hookrightarrow \overset{*}{B}{}_p^{s/\vec{r},\vec{\omega}}$. From this and step (4) it follows that (10.3) is a norm for $B_p^{s/\vec{r},\vec{\omega}}$.

(7) The proof of the fact that (10.4) is a norm for $B_p^{s/\vec{r},\vec{\omega}}$ is similar. □

Corollary 10.2 *If $s>0$, then $B_p^{s/\vec{r},(\lambda,0)} \doteq L_p(J,B_p^{s,\lambda})\cap B_p^{s/r}(J,L_p^{\lambda})$.*

11 Hölder Spaces in Euclidean Settings

In [5] it has been shown that isotropic weighted Hölder spaces are important point-wise multiplier spaces for weighted isotropic Bessel potential and Besov spaces. In Section 13 we shall show that similar results hold in the anisotropic case. For this reason we introduce and study anisotropic weighted Hölder spaces and establish the fundamental retraction theorem which allows for local characterizations. In order to achieve this we have to have a good understanding of Hölder spaces of Banach-space-valued functions on $\mathbb{R}^m$ and $\mathbb{H}^m$. In this section we derive those properties of such spaces which are needed to study weighted Hölder spaces on M.

Let $\mathcal{X}$ be a Banach space. Suppose $\mathbb{X}\in\{\mathbb{R}^m,\mathbb{H}^m\}$ and $X\in\{\mathbb{X},\ \mathbb{X}\times J\}$. Then $B=B(X,\mathcal{X})$ is the Banach space of all bounded $\mathcal{X}$-valued functions on X endowed with the supremum norm $\|\cdot\|_\infty=\|\cdot\|_{0,\infty}$.

Throughout this section, $k,k_0,k_1\in\mathbb{N}$. Then

$$BC^k=BC^k(X,\mathcal{X}) := \bigl(\{\,u\in C^k(X,\mathcal{X})\ ;\ \partial_x^\alpha u\in B(X,\mathcal{X}),\ |\alpha|\le k\,\},\ \|\cdot\|_{k,\infty}\bigr),$$

where

$$\|u\|_{k,\infty} := \max_{|\alpha|\le k}\|\partial_x^\alpha u\|_\infty,$$

is a Banach space. As usual, $BC=BC^0$. We write $\|\cdot\|_{k,\infty;\mathcal{X}}$ for $\|\cdot\|_{k,\infty}$ if it seems to be necessary to indicate the image space. Similar conventions apply to the other norms and seminorms introduced below.

Note that

$$BUC^k = \{\,u\in BC^k\ ;\ \partial_x^\alpha u \text{ is uniformly continuous for } |\alpha|\le k\,\}$$

is a closed linear subspace of BC^k. The mean value theorem implies the first embedding of

$$BC^{k+1}\hookrightarrow BUC^k\hookrightarrow BC^k. \tag{11.1}$$

Hence

$$BC^\infty := \textstyle\bigcap_k BC^k = \bigcap_k BUC^k. \tag{11.2}$$

It is a Fréchet space with the natural projective topology. Thus

$$BC^\infty\hookrightarrow BUC^k, \qquad k\in\mathbb{N}.$$

In fact, this embedding is dense. For this we recall that a *mollifier on* $\mathbb{R}^d$ is a family $\{\,w_\eta\ ;\ \eta>0\,\}$ of nonnegative compactly supported smooth functions on $\mathbb{R}^d$ such that $w_\eta(x)=\eta^{-d}w_1(x/\eta)$ for $x\in\mathbb{R}^d$ and $\int w_1\,dx=1$. Then, denoting by $w_\eta * u$ convolution,

$$w_\eta * u\in BC^\infty(\mathbb{R}^d,\mathcal{X}), \qquad u\in BC(\mathbb{R}^d,\mathcal{X}), \tag{11.3}$$

and

$$\lim_{\eta\to 0} w_\eta * u = u \text{ in } BUC^k(\mathbb{R}^d,\mathcal{X}), \qquad u\in BUC^k(\mathbb{R}^d,\mathcal{X}), \tag{11.4}$$

(cf. [7, Theorem X.7.11], for example, whose proof carries literally over to $\mathcal{X}$-valued spaces). From this we get

$$BC^\infty \overset{d}{\hookrightarrow} BUC^k \tag{11.5}$$

if $\mathbb{X}=\mathbb{R}^m$ and $J=\mathbb{R}$. In the other cases it follows by an additional extension and restriction argument based on the extension map (4.1.7) of [4] (also cf. Section 4.3 therein).

From now on $X=\mathbb{X}$. For $k\le s<k+1$, $0<\delta\le\infty$, and $u:\ \mathbb{X}\to\mathcal{X}$ we put

$$[u]^\delta_{s,\infty} := \sup_{h\in(0,\delta)^m} \frac{\|\triangle_h^{k+1}u\|_{\infty;\mathcal{X}}}{|h|^s}, \quad [\cdot]_{s,\infty}:=[\cdot]^\infty_{s,\infty}.$$

Furthermore,

$$\|\cdot\|^*_{s,\infty} := \|\cdot\|_\infty + [\cdot]_{s,\infty}, \qquad s>0.$$

Note that $h\in(0,\infty)^m\backslash(0,\delta)^m$ implies $\delta\le|h|_\infty\le|h|\le\sqrt{m}\,|h|_\infty$. Hence

$$[\cdot]_{\theta,\infty}\le[\cdot]^\delta_{\theta,\infty}+4\delta^{-\theta}\,\|\cdot\|_\infty, \qquad 0<\theta\le 1, \quad 0<\delta<\infty. \tag{11.6}$$

If $0<\theta_0<\theta\le 1$, then

$$[\cdot]^\delta_{\theta_0,\infty}\le\sqrt{m}\,\delta^{\theta-\theta_0}[\cdot]^\delta_{\theta,\infty}, \qquad 0<\delta<\infty. \tag{11.7}$$

Consequently,

$$[\cdot]_{\theta_0,\infty}\le\sqrt{m}\,[\cdot]^1_{\theta,\infty}+4\,\|\cdot\|_\infty\le\sqrt{m}\,[\cdot]_{\theta,\infty}+4\,\|\cdot\|_\infty.$$

This implies

$$\|\cdot\|^*_{\theta_0,\infty}\le c(m)\,\|\cdot\|^*_{\theta,\infty}, \qquad 0<\theta_0<\theta\le 1. \tag{11.8}$$

Suppose $u\in BC^k$ and denote by D the Fréchet derivative. Then, by the mean value theorem,

$$\triangle^k_h u(x)=\int_0^1\cdots\int_0^1 D^k u\bigl(x+(t_1+\cdots+t_k)h\bigr)[h]^k\,dt_1\cdots dt_k,$$

where $[h]^k:=(h,\ldots,h)\in\mathbb{X}^k$. From this we get

$$[u]^\delta_{\theta,\infty}\le m^{k/2}\delta^{k-\theta}\,\|u\|_{k,\infty}, \qquad 0<\theta\le 1, \quad \theta<k, \quad \delta>0, \quad u\in BC^k. \tag{11.9}$$

Thus, by (11.6),

$$\|\cdot\|^*_{\theta,\infty}\le c(m)\,\|\cdot\|_{1,\infty}, \qquad 0<\theta<1. \tag{11.10}$$

We also set for $k<s\le k+1$

$$\|u\|^{**}_{s,\infty}:=\|u\|_{k,\infty}+\max_{|\alpha|=k}[\partial^\alpha_x u]_{s-k,\infty}.$$

If $k<s<k+1$, then $\|\cdot\|_{s,\infty}:=\|\cdot\|^{**}_{s,\infty}$ and

$$BC^s=BC^s(\mathbb{X},\mathcal{X}):=\Bigl(\bigl\{\,u\in BC^k\ ;\ \max_{|\alpha|=k}[\partial^\alpha_x u]_{s-k,\infty}<\infty\,\bigr\},\ \|\cdot\|_{s,k}\Bigr), \qquad k<s<k+1,$$

is a *Hölder space* of order s.

Given $h=(h^1,\ldots,h^m)\in\mathbb{X}$, we set $h_j:=(0,\ldots,0,h^j,\ldots,h^m)$ for $1\le j\le m$, and $h_{m+1}:=0$. Then

$$\triangle_h u(x)=\sum_{j=1}^m\bigl(u(x+h_j)-u(x+h_{j+1})\bigr).$$

From this we infer for $0<\theta<1$ and $h^j\neq 0$ for $1\le j\le m$

$$\begin{aligned}\frac{\|\triangle_h u\|_\infty}{|h|^\theta} &\le \sum_{j=1}^m \frac{\|u(\cdot+h^je_j)-u\|_\infty}{|h^j|^\theta} \le \sum_{j=1}^m \sup_{h^j\neq 0}\frac{\|u(\cdot+h^je_j)-u\|_\infty}{|h^j|^\theta}\\ &= \sum_{j=1}^m \sup_{h^j>0}\frac{\|u(\cdot+h^je_j)-u\|_\infty}{(h^j)^\theta} \le m\,[u]_{\theta,\infty}.\end{aligned}$$

Consequently,

$$[u]_{\theta,\infty}\le \sup_{h\neq 0}\frac{\|\triangle_h u\|_\infty}{|h|^\theta}\le m\,[u]_{\theta,\infty},\qquad 0<\theta<1. \tag{11.11}$$

This shows that BC^s coincides, except for equivalent norms, with the usual Hölder space of order s if $s\in\mathbb{R}^+\setminus\mathbb{N}$. From (11.11) we read off that the last embedding of

$$BC^{k+1}\hookrightarrow BC^s\hookrightarrow BC^{s_0}\hookrightarrow BUC^k,\qquad k<s_0<s<k+1, \tag{11.12}$$

is valid. The other two follow from (11.8) and (11.10).

We introduce the **Besov-Hölder space scale** $[\,B_\infty^s\,;\ s>0\,]$ by

$$B_\infty^s := \begin{cases}(BUC^k,BUC^{k+1})_{s-k,\infty}, & k<s<k+1,\\ (BUC^k,BUC^{k+2})_{1/2,\infty}, & s=k+1.\end{cases}$$

Theorem 11.1

(i) $\|\cdot\|_{s,\infty}^*$ *and* $\|\cdot\|_{s,\infty}^{**}$ *are norms for* B_∞^s.

(ii) $B_\infty^s \doteq (BUC^{k_0},BUC^{k_1})_{(s-k_0)/(k_1-k_0),\infty}$ *for* $k_0<s<k_1$.

(iii) *If* $0<s_0<s_1$ *and* $0<\theta<1$, *then* $(B_\infty^{s_0},B_\infty^{s_1})_{\theta,\infty}\doteq B_\infty^{s_\theta}\doteq [B_\infty^{s_0},B_\infty^{s_1}]_\theta$.

P r o o f. (1) For $s>0$ we denote by $B_{\infty,\infty}^s=B_{\infty,\infty}^s(\mathbb{X},\mathcal{X})$ the 'standard' Besov space modeled on L_∞ for whose precise definition we refer to [4] (choosing the trivial weight vector therein).

It is a consequence of [4, (3.3.12), (3.5.2), and Theorem 4.4.1] that

$$B_{\infty,\infty}^s \doteq (BUC^{k_0},BUC^{k_1})_{(s-k_0)/(k_1-k_0),\infty},\qquad k_0<s<k_1.$$

This implies

$$B_\infty^s\doteq B_{\infty,\infty}^s \tag{11.13}$$

and, consequently, statement (ii).

(2) The first part of (iii) follows by reiteration from (ii).
For $\xi\in\mathbb{R}^m$ we set $\Lambda(\xi):=(1+|\xi|^2)^{1/2}$. Given $s\in\mathbb{R}$, we put $\boldsymbol{\Lambda}^s:=\mathcal{F}^{-1}\Lambda^s\mathcal{F}$, where $\mathcal{F}=\mathcal{F}_m$ is the Fourier transform on $\mathbb{R}^m$.

Suppose $\mathbb{X}=\mathbb{R}^m$. It follows from [4, Theorem 3.4.1] and (11.13) that

$$\boldsymbol{\Lambda}^s\in\mathcal{L}\mathrm{is}(B_\infty^{t+s},B_\infty^t),\quad (\boldsymbol{\Lambda}^s)^{-1}=\boldsymbol{\Lambda}^{-s},\qquad t,\ s+t>0. \tag{11.14}$$

We set $A:=-\boldsymbol{\Lambda}^{s_1-s_0}$, considered as a linear operator in $B_\infty^{s_0}$ with domain $B_\infty^{s_1}$. Then [4, Proposition 1.5.2 and Theorem 3.4.2] guarantee the existence of $\varphi\in(\pi/2,\pi)$ such that the sector $S_\varphi:=\{\,z\in\mathbb{C}\,;\ |\arg z|\le\varphi\,\}\cup\{0\}$ belongs to the resolvent set of A and $\|(\lambda-A)^{-1}\|\le c/|\lambda|$ for $\lambda\in S_\varphi$. Furthermore, by [4, Proposition 1.5.4 and Theorem 3.4.2] we find that $A^z\in\mathcal{L}(B_\infty^{s_0})$ and there exists $\gamma>0$ such that $\|A^z\|\le ce^{\gamma\,|\,\mathrm{Im}\,z|}$ for $\mathrm{Re}\,z\le 0$. Now Seeley's theorem, more precisely: the proof in R. Seeley [44], and (11.14) imply $[B_\infty^{s_0},B_\infty^{s_1}]_\theta\doteq B_\infty^{s_\theta}$. This proves the second part of (iii) if $\mathbb{X}=\mathbb{R}^m$. The case $\mathbb{X}=\mathbb{H}^m$ is then covered by [4, Theorem 4.4.1].

(3) By [4, Theorems 3.3.2, 3.5.2, and 4.4.1] we get $B_{\infty,\infty}^s\hookrightarrow BUC$. Using this and the arguments of the proof of [4, Theorem 4.4.3(i)] we infer from [4, Theorem 3.6.1] that $\|\cdot\|_{B_{\infty,\infty}^s}\sim\|\cdot\|_{s,\infty}^*$. By appealing to [50, Theorem 1.13.1] in the proof of [4, Theorem 3.6.1] we obtain similarly $\|\cdot\|_{B_{\infty,\infty}^s}\sim\|\cdot\|_{s,\infty}^{**}$, making also use of (11.12) in the usual extension-restriction argument. Due to (11.13) this proves (i). □

Corollary 11.2

(i) $B^s_\infty \doteq BC^s$ *for* $s \in \mathbb{R}^+ \backslash \mathbb{N}$.

(ii) $BUC^k \hookrightarrow B^k_\infty$ *and* $BUC^k \neq B^k_\infty$.

P r o o f. (i) is implied by part (i) of the theorem.

(ii) The first claim is a consequence of [4, Theorem 3.5.2]. It follows from Example IV.4.3.1 in E. Stein [48] that the 'Zygmund space' B^1_∞ contains functions which are not uniformly Lipschitz continuous. This proves the second statement. □

By (11.12) we see that

$$BC^{s_1} \hookrightarrow BC^{s_0}, \qquad 0 \leq s_0 < s_1.$$

However, these embeddings are not dense. Since dense embeddings are of great importance in the theory of elliptic and parabolic differential equations we introduce the smaller subscale of 'little' Hölder spaces which enjoy the desired property.

Suppose $s \in \mathbb{R}^+$. The **little Hölder space**

$$bc^s = bc^s(\mathbb{X}, \mathcal{X}) \text{ is the closure of } BC^\infty \text{ in } BC^s.$$

Similarly, the **little Besov-Hölder space scale** $[\, b_\infty \ ; \ s > 0\,]$ is defined by

$$b^s_\infty \text{ is the closure of } BC^\infty \text{ in } B^s_\infty. \tag{11.15}$$

These spaces possess intrinsic characterizations.

Theorem 11.3

(i) $bc^k = BUC^k$.

(ii) $b^s_\infty \doteq bc^s$ *for* $s \in \mathbb{R}^+ \backslash \mathbb{N}$.

(iii) *Suppose* $k < s \leq k+1$. *Then* $u \in B^s_\infty$ *belongs to* b^s_∞ *iff*

$$\lim_{\delta \to 0} [\partial^\alpha_x u]^\delta_{s-k,\infty} = 0, \qquad |\alpha| = k. \tag{11.16}$$

(iv) $BC^s \overset{d}{\hookrightarrow} b^{s_0}_\infty$ *for* $0 < s_0 < s$.

P r o o f. (1) Assertion (i) is a consequence of (11.5). Statement (ii) follows from Corollary 11.2(i).

(2) Suppose $k < s \leq k+1$. We denote by $\widetilde{b}^s_\infty$ the linear subspace of B^s_∞ of all u satisfying (11.16). Then we infer from (11.9) that

$$BC^\infty \hookrightarrow BUC^{k+1} \hookrightarrow \widetilde{b}^s_\infty. \tag{11.17}$$

Let $u \in b^s_\infty$ and $\varepsilon > 0$. Then (11.12) implies the existence of $v \in BUC^{k+2}$ with $\|u - v\|^{**}_{s,\infty} < \varepsilon/2$. By (11.17) we can find $\delta_\varepsilon > 0$ such that $[\partial^\alpha_x v]^{\delta_\varepsilon}_{s-k,\infty} \leq \varepsilon/2$ for $|\alpha| = k$ and $0 < \delta \leq \delta_\varepsilon$. Hence

$$[\partial^\alpha_x u]^\delta_{s-k,\infty} \leq \big[\partial^\alpha_x (u - v)\big]_{s-k,\infty} + [\partial^\alpha_x v]^{\delta_\varepsilon}_{s-k,\infty} \leq \|u - v\|^{**}_{s,\infty} + \varepsilon/2 < \varepsilon$$

for $|\alpha| = k$ and $\delta \leq \delta_\varepsilon$. This proves $b^s_\infty \subset \widetilde{b}^s_\infty$.

(3) Suppose $\mathbb{X} = \mathbb{R}^m$ and $u \in \widetilde{b}^s_\infty$. We claim that $w_\eta * u$ converges in B^s_∞ towards u as $\eta \to 0$. Using (11.4) and $\partial^\alpha_x (w_\eta * u) = w_\eta * \partial^\alpha_x u$ we can assume $0 < s \leq 1$ and then have to show

$$[w_\eta * u - u]_{s,\infty} \to 0 \quad \text{as } \eta \to 0. \tag{11.18}$$

Note

$$w_\eta * u(x) - u(x) = \int \big(u(x-y) - u(x)\big) w_\eta(y)\, dy.$$

From this we infer

$$[w_\eta * u - u]^\delta_{s,\infty} \le 2\,[u]^\delta_{s,\infty}, \qquad \delta > 0. \tag{11.19}$$

Fix $\delta_\varepsilon > 0$ such that $[u]^\delta_{s,\infty} < \varepsilon/4$. Then we get from (11.6) and (11.19) that there exists $\eta_\varepsilon > 0$ such that

$$[w_\eta * u - u]_{s,\infty} \le \varepsilon/2 + 4\delta_\varepsilon^{-s}\,\|w_\eta * u - u\|_\infty \le \varepsilon$$

for $\eta \le \eta_\varepsilon$, due to $B^s_\infty \hookrightarrow BUC$ and (11.4). This proves (11.18). Thus $\widetilde{b}^s_\infty \subset b^s_\infty$.

(4) If $\mathbb{X} = \mathbb{H}^m$, then we get $\widetilde{b}^s_\infty \subset b^s_\infty$ from (3) and a standard extension and restriction argument based on the extension operator (4.1.7) of [4]. Together with the result of step (2) this proves claim (iii). The last assertion follows from (11.12) and (11.7). □

It should be remarked that assertion (iii) is basically known (see, for example, Proposition 0.2.1 in A. Lunardi [32], where the case $m = 1$ is considered). The proof is included here for further reference.

Little Besov-Hölder spaces can be characterized by interpolation as well. For this we recall that, given Banach spaces $\mathcal{X}_1 \overset{d}{\hookrightarrow} \mathcal{X}_0$, the *continuous interpolation space* $(\mathcal{X}_0, \mathcal{X}_1)^0_{\theta,\infty}$ of exponent $\theta \in (0,1)$ is the closure of $\mathcal{X}_1$ in $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,\infty}$. This defines an interpolation functor of exponent θ in the category of densely injected Banach couples, the *continuous interpolation functor*. It possesses the reiteration property (cf. [2, Section I.2] for more details and, in particular, G. Dore and A.Favini [13]).

Theorem 11.4

(i) *Suppose* $k_0 < s < k_1$ *with* $s \notin \mathbb{N}$. *Then* $(bc^{k_0}, bc^{k_1})^0_{(s-k_0)/(k_1-k_0),\infty} \doteq b^s_\infty$.

(ii) *If* $0 < s_0 < s_1$ *and* $0 < \theta < 1$, *then* $(b^{s_0}_\infty, b^{s_1}_\infty)^0_{\theta,\infty} \doteq b^{s_\theta}_\infty \doteq [b^{s_0}_\infty, b^{s_1}_\infty]_\theta$.

Proof. (1) The validity of (i) and the first part of (ii) follow from Theorem 11.1(ii) and (iii) and Theorem 11.3(i).

(2) We deduce from (11.2), (11.12), and Corollary 11.2 that $BC^\infty = \bigcap_{s>0} B^s_\infty$. From this and (11.14) we infer $\boldsymbol{\Lambda}^s \in \mathcal{L}\mathrm{aut}(BC^\infty)$. Hence, using the definition of the little Besov-Hölder spaces and once more (11.14) and Corollary 11.2, we find

$$\boldsymbol{\Lambda}^s \in \mathcal{L}\mathrm{is}(b^{t+s}_\infty, b^t_\infty), \quad (\boldsymbol{\Lambda}^s)^{-1} = \boldsymbol{\Lambda}^{-s}, \qquad t,\ t+s > 0.$$

Thus the relevant arguments of part (2) of the proof of Theorem 11.1 apply literally to give the second part of (ii). This is due to the fact that the Fourier multiplier Theorem [4, Theorem 3.4.2] holds for b^s_∞ also (see [3, Theorem 6.2]). □

Now we turn to anisotropic spaces. We set

$$BC^{kr/\vec{r}} := \big(\big\{\, u \in C(\mathbb{X} \times J, \mathcal{X}) \ ;\ \partial_x^\alpha \partial^j u \in BC(\mathbb{X} \times J, \mathcal{X}),\ |\alpha| + jr \le kr \,\big\},\ \|\cdot\|_{kr/\vec{r}}\big),$$

where

$$\|u\|_{kr/\vec{r}} := \max_{|\alpha|+jr \le kr} \|\partial_x^\alpha \partial^j u\|_\infty.$$

This space is complete and contains

$$BUC^{kr/\vec{r}} := \big\{\, u \in BC^{kr/\vec{r}} \ ;\ \partial_x^\alpha \partial^j u \in BUC(\mathbb{X} \times J, \mathcal{X}),\ |\alpha| + jr \le kr \,\big\}$$

as a closed linear subspace.

Proposition 11.5 $BUC^{kr/\vec{r}} = \bigcap_{j=0}^k BUC^j(J, BUC^{(k-j)r})$.

P r o o f. (1) Due to $u(x,t)-u(y,s)=u(x,t)-u(y,t)+u(y,t)-u(y,s)$ for $(x,t),(y,s)\in\mathbb{X}\times J$, the claim is immediate for $k=0$.

(2) Suppose $k\in\mathbb{N}^\times$ and $u\in BUC^{kr/\vec{r}}$. Suppose also $0\le j\le k-1$ and $|\alpha|\le(k-j)r$. Then, by the mean value theorem,

$$\partial_x^\alpha\partial^j u(x,t+h)-\partial_x^\alpha\partial^j u(x,t)-h\partial_x^\alpha\partial^{j+1}u(x,t)=h\int_0^1\big(\partial_x^\alpha\partial^{j+1}u(x,t+\tau h)-\partial_x^\alpha\partial^{j+1}u(x,t)\big)\,d\tau$$

for $x\in\mathbb{X}$ and $t,h\in J$. Thus, given $\varepsilon>0$, the uniform continuity of $\partial_x^\alpha\partial^{j+1}u$ implies the existence of $\delta>0$ such that

$$\begin{aligned}&\big\|h^{-1}\big(\partial_x^\alpha\partial^j u(\cdot,t+h)-\partial_x^\alpha\partial^j u(\cdot,t)\big)-\partial_x^\alpha\partial^{j+1}u(\cdot,t)\big\|_{\infty;\mathcal{X}}\\&\quad\le\max_{0\le\tau\le1}\|\partial_x^\alpha\partial^{j+1}u(\cdot,t+\tau h)-\partial_x^\alpha\partial^{j+1}u(\cdot,t)\|_\infty\le\varepsilon\end{aligned}$$

for $h\in J\backslash\{0\}$ with $|h|\le\delta$. Hence the map $\big(t\mapsto\partial_x^\alpha\partial^j u(\cdot,t)\big):\ J\to B(\mathbb{X},\mathcal{X})$ is differentiable and its derivative equals $t\mapsto\partial_x^\alpha\partial^{j+1}u(\cdot,t)$. From this and step (1) we infer $u\in BUC^j(J,BUC^{(k-j)r})$ for $0\le j\le k$. This implies $BUC^{kr/\vec{r}}\hookrightarrow\bigcap_{j=0}^k BUC^j(J,BUC^{(k-j)r})$. The converse embedding is an obvious consequence of step (1). □

It is an immediate consequence of this lemma that

$$BUC^{kr/\vec{r}}\hookrightarrow BUC(J,BUC^{kr})\cap BUC^k(J,BUC).$$

It follows from Remark 1.13.4.2 in [50], for instance, that $BUC^{kr/\vec{r}}$ is a proper subspace of the intersection space on the right hand side.

We infer from (11.1) that $BC^{(k+1)r/\vec{r}}\hookrightarrow BUC^{kr/\vec{r}}\hookrightarrow BC^{kr/\vec{r}}$. Consequently,

$$BC^{\infty/\vec{r}}:=\bigcap_k BC^{kr/\vec{r}}=\bigcap_k BUC^{kr/\vec{r}}=BC^\infty(\mathbb{X}\times J,\mathcal{X}).\tag{11.20}$$

For $s>0$ we set

$$\begin{aligned}\|u\|_{s/\vec{r},\infty}^*&:=\sup_t\|u(\cdot,t)\|_{s,\infty}^*+\sup_x\big[u(x,\cdot)\big]_{s/r,\infty}\\&=\|u\|_\infty+\sup_t\big[u(\cdot,t)\big]_{s,\infty}+\sup_x\big[u(x,\cdot)\big]_{s/r,\infty}.\end{aligned}\tag{11.21}$$

Suppose $0<s\le r$. Then

$$\|u\|_{s/\vec{r},\infty}^{**}:=\sup_t\|u(\cdot,t)\|_{s,\infty}^{**}+\sup_x\big[u(x,\cdot)\big]_{s/r,\infty}.$$

If $kr<s\le(k+1)r$ with $k\in\mathbb{N}^\times$, then

$$\|u\|_{s/\vec{r},\infty}^{**}:=\max_{|\alpha|+jr\le kr}\|\partial_x^\alpha\partial^j u\|_{(s-kr)/\vec{r},\infty}^{**}.\tag{11.22}$$

The **anisotropic Besov-Hölder space scale** $[\,B_\infty^{s/\vec{r}}\ ;\ s>0\,]$ is defined by

$$B_\infty^{s/\vec{r}}:=\begin{cases}(BUC^{kr/\vec{r}},BUC^{(k+1)r/\vec{r}})_{(s-kr)/r,\infty}, & kr<s<(k+1)r,\\(BUC^{kr/\vec{r}},BUC^{(k+2)r/\vec{r}})_{1/2,\infty}, & s=(k+1)r.\end{cases}$$

The next theorem is the anisotropic analogue of Theorem 11.1.

Theorem 11.6

(i) $\|\cdot\|_{s/\vec{r},\infty}^*$ *and* $\|\cdot\|_{s/\vec{r},\infty}^{**}$ *are norms for* $B_\infty^{s/\vec{r}}$.

(ii) *Suppose* $k_0r<s<k_1r$. *Then* $(BUC^{k_0r/\vec{r}},BUC^{k_1r/\vec{r}})_{(s-k_0r)/(k_1-k_0)r,\infty}\doteq B_\infty^{s/\vec{r}}$.

(iii) *If* $0 < s_0 < s_1$ *and* $0 < \theta < 1$, *then* $(B_\infty^{s_0/\vec{r}}, B_\infty^{s_1/\vec{r}})_{\theta,\infty} \doteq B_\infty^{s_\theta/\vec{r}} \doteq [B_\infty^{s_0/\vec{r}}, B_\infty^{s_1/\vec{r}}]_\theta$.

(iv) $\partial_x^\alpha \partial^j \in \mathcal{L}(B_\infty^{(s+|\alpha|+jr)/\vec{r}}, B_\infty^{s/\vec{r}})$ *for* $\alpha \in \mathbb{N}^m$ *and* $j \in \mathbb{N}$.

P r o o f. (1) We infer from [4, (3.3.12), (3.5.2), and Theorem 4.4.1] that

$$B_\infty^{s/\vec{r}} = B_{\infty,\infty}^{s/\boldsymbol{\nu}} \tag{11.23}$$

and that (ii) is true.

(2) The first part of (iii) follows from (ii) by reiteration.

(3) For $(\xi,\tau) \in \mathbb{R}^m \times \mathbb{R}$ we set $\widetilde{\Lambda}(\xi,\tau) := (1 + |\xi|^{2r} + \tau^2)^{1/2r}$. Then $\widetilde{\boldsymbol{\Lambda}}^s := \mathcal{F}_{m+1}^{-1} \widetilde{\Lambda}^s \mathcal{F}_{m+1}$ for $s \in \mathbb{R}$. From [4, Theorem 3.4.1] and (11.23) we get

$$\widetilde{\boldsymbol{\Lambda}}^s \in \mathcal{L}\mathrm{is}(B_\infty^{(t+s)/\vec{r}}, B_\infty^{t/\vec{r}}), \quad (\widetilde{\boldsymbol{\Lambda}}^s)^{-1} = \widetilde{\boldsymbol{\Lambda}}^{-s}, \qquad t,\ t+s > 0,$$

provided $\mathbb{X} = \mathbb{R}^m$ and $J = \mathbb{R}$. Now we obtain the second part of (iii) by obvious modifications of the relevant sections of part (2) of the proof of Theorem 11.1.

(4) Taking [4, Section 4.4] into account, we get from Theorems 3.3.2 and 3.5.2 therein that $B_\infty^{s/\vec{r}} \hookrightarrow BUC$. Suppose $kr < s \le (k+1)r$. By [4, Theorem 3.6.1]

$$\|u\|_{B_\infty^{s/\vec{r}}} \sim \|u\|_\infty + \sup_{h \in (0,\infty)^m} \frac{\|\triangle_{(h,0)}^{[s]+1} u\|_\infty}{|h|^s} + \sup_{h>0} \frac{\|\triangle_{(0,h)}^{[s/r]+1} u\|_\infty}{h^{s/r}},$$

where $[t]$ is the largest integer less than or equal to $t \in \mathbb{R}$. Since $u \in BC$ it follows

$$\|\triangle_{(h,0)}^{[s]+1} u\|_\infty = \sup_t \|\triangle_h^{[s]+1} u(\cdot,t)\|_\infty, \quad \|\triangle_{(0,h)}^{[s/r]+1} u\|_\infty = \sup_x \|\triangle_h^{[s/r]+1} u(x,\cdot)\|_\infty.$$

Thus $\|\cdot\|_{B_\infty^{s/\vec{r}}} \sim \|\cdot\|_{s/\vec{r},\infty}^*$.

(5) Suppose $\mathbb{X} = \mathbb{R}^m$ and $J = \mathbb{R}$. Then (iv) follows by straightforward modifications of the proof of [4, Lemma 2.3.7] by invoking the Fourier multiplier Theorem 3.4.2 therein. Similarly as in the proof of [4, Theorem 2.3.8], we see that, given $0 < s \le r$ and $k \in \mathbb{N}$,

$$\|\cdot\|_{B_\infty}^{(s+kr)/\vec{r}} \sim \max_{|\alpha|+jr \le kr} \|\partial_x^\alpha \partial^j \cdot\|_{B_\infty^{s/\vec{r}}} \tag{11.24}$$

(cf. [4, Corollary 2.3.4]). In the general case we now obtain the validity of (iv) and (11.24) by extension and restriction, taking $B_\infty^{s/\vec{r}} \hookrightarrow BUC$ into account.

(6) Suppose $0 < s \le r$. Then $\|\cdot\|_{s/\vec{r},\infty}^* \sim \|\cdot\|_{s/\vec{r},\infty}^{**}$ follows from Theorem 11.1(i). By combining this with (11.24) we see that the latter equivalence holds for every $k \in \mathbb{N}$. This proves the theorem. □

Corollary 11.7

(i) $B_\infty^{s/\vec{r}} \doteq B(J, B_\infty^s) \cap B_\infty^{s/r}(J,B)$.

(ii) *Set*

$$\|u\|_{s/\vec{r},\infty}^\sim := \sup_t \|u(\cdot,t)\|_{s,\infty}^{**} + \sup_x \|u(x,\cdot)\|_{s/r}^{**}, \qquad s > 0.$$

Then $\|\cdot\|_{s/\vec{r},\infty}^\sim$ *is a norm for* B_∞^s.

P r o o f. (i) is implied by Theorem 11.6(i), $B_\infty^s \hookrightarrow BUC^k$ if $k < s \le k+1$, and Proposition 11.5. (ii) follows from (i) and Theorem 11.1(i). □

We define **anisotropic Hölder spaces** by $BC^{s/\vec{r}} := B_\infty^{s/\vec{r}}$ for $s \in \mathbb{R}^+ \backslash r\mathbb{N}$. By means of the mean value theorem and using the norm $\|\cdot\|_{s/\vec{r},\infty}^\sim$, for example, we find, similarly as in the isotropic case, that

$$BC^{s/\vec{r}} \hookrightarrow BC^{s_0/\vec{r}}, \qquad 0 \le s_0 < s.$$

In order to obtain scales of spaces enjoying dense embeddings we define **anisotropic little Hölder spaces** by

$$bc^{s/\vec{r}} \text{ is the closure of } BC^{\infty/\vec{r}} \text{ in } BC^{s/\vec{r}}, \qquad s \in \mathbb{R}^+. \tag{11.25}$$

Similarly, the **anisotropic little Besov-Hölder space**

$$b_\infty^{s/\vec{r}} \text{ is the closure of } BC^{\infty/\vec{r}} \text{ in } B_\infty^{s/\vec{r}}, \qquad s > 0.$$

These spaces possess intrinsic characterizations as well. To allow for a simple formulation we denote by $[s]_-$ the largest integer strictly less than s.

Theorem 11.8

(i) $bc^{kr/\vec{r}} = BUC^{kr/\vec{r}}$.

(ii) $bc^{s/\vec{r}} = b_\infty^{s/\vec{r}}$ *if* $s \in \mathbb{R}^+ \backslash \mathbb{N}$.

(iii) $u \in b_\infty^{s/\vec{r}}$ *iff* $u \in B_\infty^{s/\vec{r}}$ *and*

$$\sup_t \max_{|\alpha|=[s]_-} \big[\partial_x^\alpha u(\cdot,t)\big]_{s-[s]_-,\infty}^\delta + \sup_x \big[\partial^{[s/r]_-} u(x,\cdot)\big]_{s/r-[s/r]_-,\infty}^\delta \to 0 \tag{11.26}$$

as $\delta \to 0$.

(iv) $BC^{s/\vec{r}} \overset{d}{\hookrightarrow} b_\infty^{s_0/\vec{r}}$ *for* $0 < s_0 < s$.

P r o o f. As in previous proofs it suffices to consider the case $\mathbb{X} = \mathbb{R}^m$ and $J = \mathbb{R}$.

(1) We know from (11.20) that $BC^{\infty/\vec{r}} \hookrightarrow BUC^{kr/\vec{r}}$. Let $\{\, w_\eta \,;\, \eta > 0 \,\}$ be a mollifier on $\mathbb{R}^{m+1}$. If u belongs to $BUC^{kr/\vec{r}}$, then it follows from (11.4) and $\partial_x^\alpha \partial^j (w_\eta * u) = w_\eta * (\partial_x^\alpha \partial^j u)$ that $w_\eta * u \to u$ in $BC^{kr/\vec{r}}$ as $\eta \to 0$. This proves assertion (i). Claim (ii) is trivial.

(2) Let $kr \le i < s \le i+1 \le (k+1)r$ with $i \in \mathbb{N}$. Suppose $u \in b_\infty^{s/\vec{r}}$ and $\varepsilon > 0$. Then we can find v belonging to $\in BUC^{(k+2)r/\vec{r}} \hookrightarrow B_\infty^{s/\vec{r}}$ such that $\|u - v\|_{s/\vec{r},\infty}^{**} < \varepsilon/2$. By Proposition 11.5 we know

$$BUC^{(k+2)r/\vec{r}} \hookrightarrow BUC(J, BUC^{(k+2)r}) \cap BUC^{k+2}(J, BUC).$$

Hence it follows from (11.9) that

$$\sup_t \big[\partial_x^\alpha v(\cdot,t)\big]_{s-i,\infty}^\delta \le c\delta \, \|v\|_{(k+2)r/\vec{r},\infty}, \qquad 0 < \delta \le 1, \quad |\alpha| = i.$$

Similarly,

$$\sup_x \big[\partial^k v(x,\cdot)\big]_{s/r-k,\infty}^\delta \le c\delta \, \|v\|_{(k+2)r/\vec{r},\infty}, \qquad 0 < \delta \le 1.$$

Thus we find $\delta_\varepsilon > 0$ such that

$$\sup_t \max_{|\alpha|=i} \big[\partial_x^\alpha v(\cdot,t)\big]_{s-i,\infty}^\delta + \sup_x \big[\partial^k v(x,\cdot)\big]_{s/r-k,\infty}^\delta < \varepsilon/2, \qquad 0 < \delta \le \delta_\varepsilon.$$

Consequently,

$$\sup_t \max_{|\alpha|=i} \big[\partial_x^\alpha u(\cdot,t)\big]_{s-i,\infty}^\delta \le \sup_t \max_{|\alpha|=i} \big[\partial_x^\alpha (u-v)(\cdot,t)\big]_{s-i,\infty} + \sup_t \max_{|\alpha|=i} \big[\partial_x^\alpha v(\cdot,t)\big]_{s-i,\infty}^\delta \le \varepsilon$$

for $0 < \delta \le \delta_\varepsilon$. This shows that the first term in (11.26) converges to zero. Analogously, we see that this is true for the second summand.

(3) Suppose $0 < s \le 1$ and $u \in B_\infty^{s/\vec r}$ satisfies (11.26). By (11.3) it suffices to show that

$$\|w_\eta * u - u\|_{s/\vec r,\infty}^{\sim} \to 0 \quad \text{as } \eta \to 0. \tag{11.27}$$

It follows from $\triangle_{(h,0)}^{[s]+1}(w_\eta * u) = w_\eta * (\triangle_{(h,0)}^{[s]+1} u)$ that

$$\|\triangle_{(h,0)}^{[s]+1}(w_\eta * u)(x,t)\| \le \sup_t \|\triangle_h^{[s]+1} u(\cdot,t)\|_\infty, \qquad (x,t) \in \mathbb{X} \times J.$$

Consequently,

$$\sup_t \bigl[w_\eta * u(\cdot,t)\bigr]_{s,\infty}^{\delta} \le \sup_t \bigl[u(\cdot,t)\bigr]_{s,\infty}^{\delta}, \qquad 0 < \delta < \infty.$$

Let $\varepsilon > 0$ and fix $\delta_\varepsilon > 0$ with $\sup_t \bigl[u(\cdot,t)\bigr]_{s,\infty}^{\delta_\varepsilon} < \varepsilon/4$. Then

$$\sup_t \bigl[(w_\eta * u - u)(\cdot,t)\bigr]_{s,\infty}^{\delta_\varepsilon} \le 2 \sup_t \bigl[u(\cdot,t)\bigr]_{s,\infty}^{\delta_\varepsilon} < \varepsilon/2.$$

Thus we infer from (11.6) that

$$\sup_t \bigl[(w_\eta * u - u)(\cdot,t)\bigr]_{s,\infty} \le \varepsilon/2 + 4\delta_\varepsilon^{-s} \sup_t \|(w_\eta * u - u)(\cdot,t)\|_\infty.$$

Since $u \in BUC(\mathbb{X} \times J, \mathcal{X})$ it follows from (11.4) that

$$\sup_t \|(w_\eta * u - u)(t)\|_\infty = \|w_\eta * u - u\|_{B(\mathbb{X}\times J,\mathcal{X})} \to 0 \quad \text{as } \eta \to 0.$$

Hence

$$\sup_t \bigl[(w_\eta * u - u)(\cdot,t)\bigr]_{s,\infty} \to 0 \quad \text{as } \eta \to 0.$$

Similarly,

$$\sup_x \bigl[(w_\eta * u - u)(x,\cdot)\bigr]_{s/r,\infty} \to 0 \quad \text{as } \eta \to 0.$$

This proves (11.27), thus, due to step (2), assertion (iii) for $0 < s \le 1$.

(4) To prove (iv) assume $kr \le i < s \le i+1 \le (k+1)r$ and $u \in B_\infty^{s/\vec r}$ satisfies (11.26). Then it follows from $\partial_x^\alpha \partial^j (w_\eta * u) = w_\eta * (\partial_x^\alpha \partial^j u)$ for $|\alpha| + jr < s$ and step (3) that $w_\eta * u \to u$ in $B_\infty^{s/\vec r}$ as $\eta \to 0$. Hence $u \in b_\infty^{s/\vec r}$, which shows that claim (iii) is always true.

(5) The proof of (iv) is obtained by employing (11.7), (11.9), and Corollary 11.7(ii). □

Anisotropic little Hölder spaces can be characterized by interpolation, similarly as their isotropic relatives.

Theorem 11.9

(i) $b_\infty^{s/\vec r} \doteq (bc^{k_0 r/\vec r}, bc^{k_1 r/\vec r})^0_{(s/r-k_0)/(k_1-k_0),\infty}$ *for* $k_0 r < s < k_1 r$.

(ii) *If* $0 < s_0 < s_1$ *and* $0 < \theta < 1$, *then* $(b_\infty^{s_0/\vec r}, b_\infty^{s_1/\vec r})^0_{\theta,\infty} \doteq b_\infty^{s_\theta/\vec r} \doteq [b_\infty^{s_0/\vec r}, b_\infty^{s_1/\vec r}]_\theta$.

(iii) $\partial_x^\alpha \partial^j \in \mathcal{L}(b_\infty^{(s+|\alpha|+jr)/\vec r}, b_\infty^{s/\vec r})$ *for* $\alpha \in \mathbb{N}^m$ *and* $j \in \mathbb{N}$.

Proof. (1) The first assertion as well as the first part of (ii) follow from part (i) of Theorem 11.8. Part two of (ii) and the first claim are implied by part (iii) of Theorem 11.6.

(2) The last part of statement (ii) is obtained by replacing BC^∞ and $\mathbf{\Lambda}^s$ in step (2) of the proof of Theorem 11.4 by $BC^{\infty/\vec r}$ and $\widetilde{\mathbf{\Lambda}}^s$, respectively.

(3) Theorem 11.6(iv) implies $\partial_x^\alpha \partial^j \in \mathcal{L}(BC^{\infty/\vec r})$. Thus, using the definition of $b_\infty^{s/\vec r}$ and once more the latter theorem, we obtain (iii). □

In the next section we need to employ Hölder spaces with a particular choice of $\mathcal{X}$ which we discuss now. For this we remind the reader of the notations and conventions introduced at the beginning of Section 7.

Let $\{ F_\beta \ ; \ \beta \in \mathsf{B} \}$ be a countable family of Banach spaces. Then it is obvious that

$$\boldsymbol{f} : \boldsymbol{F}^{\mathbb{X}} \to \prod_\beta F_\beta^{\mathbb{X}}, \quad u \mapsto \boldsymbol{f} u := (\mathrm{pr}_\beta \circ u) \tag{11.28}$$

is a linear bijection. Since $\boldsymbol{F}$ carries the product topology $u \in \boldsymbol{F}^{\mathbb{X}}$ is continuously differentiable iff

$$u_\beta := \mathrm{pr}_\beta \circ u \in C^1(\mathbb{X}, F_\beta), \qquad \beta \in \mathsf{B}.$$

Then $\partial_j u = (\partial_j u_\beta)$, that is,

$$\boldsymbol{f} \circ \partial_x^\alpha = \partial_x^\alpha \circ \boldsymbol{f}, \qquad \alpha \in \mathbb{N}^m. \tag{11.29}$$

Setting $\boldsymbol{C}^k(\mathbb{X}, \mathbb{F}) := \prod_\beta C^k(\mathbb{X}, F_\beta)$ etc., it follows

$$\boldsymbol{f} \in \mathcal{L}\mathrm{is}\big(C^k(\mathbb{X}, \boldsymbol{F}), \boldsymbol{C}^k(\mathbb{X}, \mathbb{F})\big). \tag{11.30}$$

Furthermore,

$$\boldsymbol{f} \in \mathcal{L}\big(BC^k\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big), \ell_\infty\big(\boldsymbol{BC}^k(\mathbb{X}, \mathbb{F})\big)\big). \tag{11.31}$$

Suppose $u \in BC^1\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big)$. Then, given $x \in \mathbb{X}$,

$$\sup_{\beta \in \mathsf{B}} \big\| t^{-1}\big(u_\beta(x + te_j) - u_\beta(x)\big) - \partial_j u_\beta(x) \big\|_{F_\beta} = \big\| t^{-1}\big(u(x + te_j) - u(x)\big) - \partial_j u(x) \big\|_{\ell_\infty(\boldsymbol{F})} \to 0$$

as $t \to 0$, with $t > 0$ if $\mathbb{X} = \mathbb{H}^m$ and $j = 1$. From this we see that $\boldsymbol{f}$ maps $u \in BC^k\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big)$ into the linear subspace of $\ell_\infty\big(\boldsymbol{BC}^k(\mathbb{X}, \mathbb{F})\big)$ consisting of all $\boldsymbol{v} = (v_\beta)$ for which v_β is k-times continuously differentiable, uniformly with respect to $\beta \in \mathsf{B}$. Thus (11.31) is not surjective if $k \geq 1$.

We denote by

$$\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}, \mathbb{F})\big)$$

the linear subspace of $\ell_\infty\big(\boldsymbol{BC}^k(\mathbb{X}, \mathbb{F})\big)$ of all $\boldsymbol{v} = (v_\beta)$ such that $\partial^\alpha v_\beta$ is uniformly continuous on $\mathbb{X}$ for $|\alpha| \leq k$, uniformly with respect to $\beta \in \mathsf{B}$.

Lemma 11.10 *$\boldsymbol{f}$ is an isomorphism*

$$\textit{from } bc^k\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big) \textit{ onto } \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}, \mathbb{F})\big) \tag{11.32}$$

and

$$\textit{from } B_\infty^s\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big) \textit{ onto } \ell_\infty\big(\boldsymbol{B}_\infty^s(\mathbb{X}, \mathbb{F})\big), \qquad s > 0. \tag{11.33}$$

Proof. (1) Suppose $u \in bc^k\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big)$. Then, by the above, it is obvious that $\boldsymbol{f} u \in \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}, \mathbb{F})\big)$. Conversely, assume $\boldsymbol{u} = (u_\beta) \in \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}, \mathbb{F})\big)$. Set $u := \boldsymbol{f}^{-1}\boldsymbol{u}$, which is defined due to (11.30). Then

$$\|u(x)\|_{\ell_\infty(\boldsymbol{F})} = \sup_\beta \|u_\beta(x)\|_{F_\beta}, \qquad x \in \mathbb{X},$$

and

$$\|u(x) - u(y)\|_{\ell_\infty(\boldsymbol{F})} = \sup_\beta \|u_\beta(x) - u_\beta(y)\|_{F_\beta}, \qquad x, y \in \mathbb{X},$$

show $u \in bc\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big)$. Hence we infer from (11.29) that $\partial_x^\alpha u \in bc\big(\mathbb{X}, \ell_\infty(\boldsymbol{F})\big)$ for $|\alpha| \leq k$.

(2) Let $k \geq 1$ and $1 \leq j \leq m$. Then, by the mean value theorem,

$$t^{-1}\big(u_\beta(x + te_j) - u_\beta(x)\big) - \partial_j u_\beta(x) = \int_0^1 \big(\partial_j u_\beta(x + ste_j) - \partial_j u_\beta(x)\big)\, ds,$$

where $t > 0$ if $j = 1$ and $\mathbb{X} = \mathbb{H}^m$. Hence

$$\begin{aligned}\big\|t^{-1}\big(u_\beta(x+te_j)-u_\beta(x)\big)-\partial_j u_\beta(x)\big\|_{F_\beta} &\le \sup_{|t|\le\delta}\sup_{x\in\mathbb{X}}\|\partial_j u_\beta(x+te_j)-\partial_j u_\beta(x)\|_{F_\beta}\\ &\le \sup_{|t|\le\delta}\|\partial_j \boldsymbol{u}(\cdot+te_j)-\partial_j\boldsymbol{u}\|_{\ell_\infty(\boldsymbol{BC}(\mathbb{X},\mathbb{F}))}\end{aligned}$$

for $|t|\le\delta,\ x\in\mathbb{X}$, and $\beta\in\mathsf{B}$. Thus

$$\big\|t^{-1}\big(u(\cdot+te_j)-u\big)-\partial_j u\big\|_{B(\mathbb{X},\ell_\infty(\boldsymbol{F}))}\le \sup_{|t|\le\delta}\|\partial_j \boldsymbol{u}(\cdot+te_j)-\partial_j\boldsymbol{u}\|_{\ell_\infty(\boldsymbol{BC}(\mathbb{X},\mathbb{F}))}$$

for $|t|\le\delta$. This implies that u is differentiable in the topology of $BC\big(\mathbb{X},\ell_\infty(\boldsymbol{F})\big)$. From this, step (1), and by induction we infer

$$\boldsymbol{f}^{-1}\in\mathcal{L}\big(\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X},\mathbb{F})\big),bc^k\big(\mathbb{X},\ell_\infty(\boldsymbol{F})\big)\big).$$

This proves (11.32).

(3) Suppose $0<s\le 1$ and set $i:=[s]_-$. It is convenient to write $h\gg 0$ iff $h\in(0,\infty)^m$. Given u belonging to $B^s_\infty\big(\mathbb{X},\ell_\infty(\boldsymbol{F})\big)$, we deduce from $\triangle_h u_\beta=\mathrm{pr}_\beta(\triangle_h u)$ that

$$\begin{aligned}\sup_\beta\big[\mathrm{pr}_\beta(\boldsymbol{f}u)\big]_{s,\infty;F_\beta}=\sup_\beta[u_\beta]_{s,\infty;F_\beta}&=\sup_\beta\sup_{h\gg0}\sup_x\frac{\|\triangle_h^{i+1}u_\beta(x)\|_{F_\beta}}{|h|^s}\\ &=\sup_{h\gg0}\sup_x\frac{\|\triangle_h^{i+1}u(x)\|_{\ell_\infty(\boldsymbol{F})}}{|h|^s}=\sup_{h\gg0}\frac{\|\triangle_h^{i+1}u\|_{\infty;\ell_\infty(\boldsymbol{F})}}{|h|^s}\\ &=[u]_{s,\infty;\ell_\infty(\boldsymbol{F})}.\end{aligned}\tag{11.34}$$

From (11.31) and (11.34) we infer

$$\boldsymbol{f}\in\mathcal{L}\big(B^s_\infty\big(\mathbb{X},\ell_\infty(\boldsymbol{F})\big),\ell_\infty\big(\boldsymbol{B}^s_\infty(\mathbb{X},\mathbb{F})\big)\big).\tag{11.35}$$

Now it follows from (11.29) that (11.35) holds for any $s>0$.

It is obvious from (11.11) and (11.29) that, given $k<s\le k+1$,

$$\ell_\infty\big(\boldsymbol{B}^s_\infty(\mathbb{X},\mathbb{F})\big)\hookrightarrow\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X},\mathbb{F})\big).$$

From this, (11.34), and (11.32) we get that $\boldsymbol{f}$ is onto $\ell_\infty\big(\boldsymbol{B}^s_\infty(\mathbb{X},\mathbb{F})\big)$. Due to (11.30) this proves (11.33). □

We denote for $k<s\le k+1$ by

$$\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{b}^s_\infty(\mathbb{X},\mathbb{F})\big)$$

the linear subspace of $\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X},\mathbb{F})\big)$ of all $\boldsymbol{v}=(v_\beta)$ such that $\lim_{\delta\to0}\max_{|\alpha|=k}[\partial_x^\alpha v_\beta]^\delta_{s-k,\infty;F_\beta}=0$, uniformly with respect to $\beta\in\mathsf{B}$.

Lemma 11.11 $\boldsymbol{f}\in\mathcal{L}\mathrm{is}\big(b^s_\infty\big(\mathbb{X},\ell_\infty(\boldsymbol{F})\big),\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{b}^s_\infty(\mathbb{X},\mathbb{F})\big)\big)$.

P r o o f. The proof of (11.34) shows that, given $k<s\le k+1$,

$$\sup_\beta\big[\mathrm{pr}_\beta(\boldsymbol{f}u)\big]^\delta_{s,\infty;F_\beta}=[u]^\delta_{s,\infty;\ell_\infty(\boldsymbol{F})},\qquad\delta>0.$$

Thus the claim follows by the arguments of step (2) of the proof of Lemma 11.10 and from Theorem 11.3. □

Now we extend $\boldsymbol{f}$ point-wise over J:

$$\widetilde{\boldsymbol{f}}:\ \boldsymbol{F}^{\mathbb{X}\times J}\to\prod_\beta F_\beta^{\mathbb{X}\times J},\quad u\mapsto\widetilde{\boldsymbol{f}}u:=\big(t\mapsto\boldsymbol{f}u(\cdot,t)\big).$$

As above, $\boldsymbol{B}^{s/\vec{r}}_\infty(\mathbb{X}\times J,\mathbb{F}):=\prod_\beta B^{s/\vec{r}}_\infty(\mathbb{X}\times J,F_\beta)$ for $s>0$. Analogous definitions apply to $\boldsymbol{b}^{s/\vec{r}}_\infty(\mathbb{X}\times J,\mathbb{F})$.

Clearly,

$$\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^{kr/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big)$$

is the closed subspace of $\ell_\infty\big(\boldsymbol{BC}^{kr/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big)$ of all $\boldsymbol{u}=(u_\beta)$ for which $\partial_x^\alpha\partial^j u_\beta\in BUC(\mathbb{X}\times J,F_\beta)$ for $|\alpha|+jr\le kr$, uniformly with respect to $\beta\in\mathsf{B}$.

Suppose $kr<s\le(k+1)r$. We denote by

$$\ell_{\infty,\mathrm{unif}}\big(\boldsymbol{b}_\infty^{s/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big)$$

the set of all $\boldsymbol{u}=(u_\beta)\in\ell_\infty\big(\boldsymbol{B}_\infty^{s/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big)$ satisfying

$$\sup_\beta\sup_t\max_{|\alpha|=[s]_-}\big[\partial_x^\alpha u_\beta(\cdot,t)\big]^\delta_{s-[s]_-,\infty;F_\beta}+\sup_\beta\sup_x\big[\partial^{[s/r]_-}u_\beta(x,\cdot)\big]^\delta_{s/r-[s/r]_-,\infty;F_\beta}\to 0$$

as $\delta\to 0$.

Now we can prove the following anisotropic analogue of Lemmas 11.10 and 11.11.

Lemma 11.12 $\widetilde{\boldsymbol{f}}$ *is an isomorphism*

$$\textit{from } bc^{kr/\vec{r}}\big(\mathbb{X}\times J,\ell_\infty(\boldsymbol{F})\big)\textit{ onto } \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^{kr/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big)$$

and

$$\textit{from } B_\infty^{s/\vec{r}}\big(\mathbb{X}\times J,\ell_\infty(\boldsymbol{F})\big)\textit{ onto } \ell_\infty\big(\boldsymbol{B}_\infty^{s/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big)$$

as well as

$$\textit{from } b_\infty^{s/\vec{r}}\big(\mathbb{X}\times J,\ell_\infty(\boldsymbol{F})\big)\textit{ onto } \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{b}_\infty^{s/\vec{r}}(\mathbb{X}\times J,\mathbb{F})\big).$$

Proof. Note $\partial^j\circ\widetilde{\boldsymbol{f}}=\widetilde{\boldsymbol{f}}\circ\partial^j$. Hence the first assertion follows from (11.32). The remaining statements are verified by obvious modifications of the relevant parts of the proofs of Lemmas 11.10 and 11.11, taking Corollary 11.7(ii) and Theorem 11.8 into account. □

12 Weighted Hölder Spaces

Having investigated Hölder spaces on $\mathbb{R}^m$ and $\mathbb{H}^m$ in the preceding section we now return to the setting of singular manifolds. First we introduce isotropic weighted Hölder spaces and study some of their properties. Afterwards we study to anisotropic Hölder spaces of time-dependent W-valued (σ,τ)-tensor fields on M. Making use of the results of Section 11 we can give coordinate-free invariant definitions of these spaces.

By $B^{0,\lambda}=B^{0,\lambda}(V)$ we mean the weighted Banach space of all sections u of V satisfying

$$\|u\|_{\infty;\lambda}=\|u\|_{0,\infty;\lambda}:=\big\|\rho^{\lambda+\tau-\sigma}\,|u|_h\,\big\|_\infty<\infty,$$

endowed with the norm $\|\cdot\|_{\infty;\lambda}$, and $B:=B^{0,0}$.

For $k\in\mathbb{N}$

$$BC^{k,\lambda}=BC^{k,\lambda}(V):=\big(\big\{\,u\in C^k(M,V)\,;\,\|u\|_{k,\infty;\lambda}<\infty\,\big\},\ \|\cdot\|_{k,\infty;\lambda}\big),$$

where

$$\|u\|_{k,\infty;\lambda}:=\max_{0\le i\le k}\big\|\rho^{\lambda+\tau-\sigma+i}\,|\nabla^i u|_h\big\|_\infty.$$

The topologies of $B^{0,\lambda}$ and $BC^{k,\lambda}$ are independent of the particular choice of $\rho\in\mathfrak{T}(M)$. Consequently, this is also true for all other spaces of this section as follows from their definition which involves the topology of $BC^{k,\lambda}$ for $k\in\mathbb{N}$ only. It is a consequence of Theorem 12.1 below that $BC^{k,\lambda}$ is a Banach space.

We set

$$BC^{\infty,\lambda}=BC^{\infty,\lambda}(V):=\textstyle\bigcap_k BC^{k,\lambda},$$

endowed with the obvious projective topology. Then

$$bc^{k,\lambda} = bc^{k,\lambda}(V) \text{ is the closure of } BC^{\infty,\lambda} \text{ in } BC^{k,\lambda}, \qquad k \in \mathbb{N}.$$

The **weighted Besov-Hölder space scale** $[\, B_\infty^{s,\lambda} \,;\, s > 0\,]$ is defined by

$$B_\infty^{s,\lambda} = B_\infty^{s,\lambda}(V) := \begin{cases} (bc^{k,\lambda}, bc^{k+1,\lambda})_{s-k,\infty}, & k < s < k+1, \\ (bc^{k,\lambda}, bc^{k+2,\lambda})_{1/2,\infty}, & s = k+1. \end{cases} \tag{12.1}$$

It is a scale of Banach spaces.

The following fundamental retraction theorem allows to characterize Besov-Hölder spaces locally.

Theorem 12.1 *Suppose $k \in \mathbb{N}$ and $s > 0$. Then ψ_∞^λ is a retraction from $\ell_\infty(\boldsymbol{BC}^k)$ onto $BC^{k,\lambda}$ and from $\ell_\infty(\boldsymbol{B}_\infty^s)$ onto $B_\infty^{s,\lambda}$, and φ_∞^λ is a coretraction.*

Proof. (1) The first claim is settled by Theorem 6.3 of [5].

(2) Suppose $k \in \mathbb{N}$. It is obvious by the definition of $\boldsymbol{bc}^k$, step (1), (4.1), and (7.3) that

$$\psi_\infty^\lambda \text{ is a retraction from } \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k) \text{ onto } bc^{k,\lambda}, \text{ and } \varphi_\infty^\lambda \text{ is a coretraction.} \tag{12.2}$$

(3) If $\partial M = \emptyset$, then we put $\mathbb{M} := \mathbb{R}^m$, $\mathsf{B} := \mathfrak{K}$, and $F_\kappa := E_\kappa := E$ for $\kappa \in \mathfrak{K}$. Then, defining $\boldsymbol{f}$ by (11.28) with this choice of F_β and $\mathbb{X} := \mathbb{R}^m$, Lemma 11.10 implies

$$\boldsymbol{f} \in \mathcal{L}\mathrm{is}\big(bc^k\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big), \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k)\big). \tag{12.3}$$

(4) Suppose $\partial M \neq \emptyset$. Then we set $\mathfrak{K}_0 := \mathfrak{K} \backslash \mathfrak{K}_{\partial M}$ and $\mathfrak{K}_1 := \mathfrak{K}_{\partial M}$. With $E_\kappa := E$ for $\kappa \in \mathfrak{K}$ we put $\boldsymbol{E}_i := \prod_{\kappa \in \mathfrak{K}_i} E_\kappa$ and define $\boldsymbol{f}_i$ by setting $\mathsf{B} = \mathfrak{K}_i$ and $F_\kappa = E_\kappa$. Then, letting $\mathbb{X}_0 := \mathbb{R}^m$ and $\mathbb{X}_1 := \mathbb{H}^m$, we infer from Lemma 11.10

$$\boldsymbol{f}_i \in \mathcal{L}\mathrm{is}\big(bc^k\big(\mathbb{X}_i, \ell_\infty(\boldsymbol{E}_i)\big), \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}_i, \mathbb{E}_i)\big)\big), \tag{12.4}$$

with $\boldsymbol{bc}^k(\mathbb{X}_i, \mathbb{E}_i) := \prod_{\kappa \in \mathfrak{K}_i} bc^k(\mathbb{X}_i, E_\kappa)$.

For $\boldsymbol{bc}^k = \prod_{\kappa \in \mathfrak{K}} bc_\kappa^k$ we use the natural identification $\boldsymbol{bc}^k = \boldsymbol{bc}^k(\mathbb{X}_0, \mathbb{E}_0) \oplus \boldsymbol{bc}^k(\mathbb{X}_1, \mathbb{E}_1)$. It induces a topological direct sum decomposition

$$\ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k) = \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}_0, \mathbb{E}_0)\big) \oplus \ell_{\infty,\mathrm{unif}}\big(\boldsymbol{bc}^k(\mathbb{X}_1, \mathbb{E}_1)\big), \tag{12.5}$$

where on the right side we use the maximum of the norms of the two summands.

Denoting by $\sqcup$ the disjoint union, we set $\mathbb{M} := \mathbb{R}^m \sqcup \mathbb{H}^m$ and

$$bc^k\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big) := bc^k\big(\mathbb{X}_0, \ell_\infty(\boldsymbol{E}_0)\big) \oplus bc^k\big(\mathbb{X}_1, \ell_\infty(\boldsymbol{E}_1)\big).$$

It follows from (12.4) and (12.5) that

$$\boldsymbol{f} := f_0 \circ \mathrm{pr}_0 + f_1 \circ \mathrm{pr}_1 \in \mathcal{L}\mathrm{is}\big(bc^k\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big), \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k)\big). \tag{12.6}$$

(5) Returning to the general case, where ∂M may or may not be empty, we set

$$\Phi_\infty^\lambda := \boldsymbol{f}^{-1} \circ \varphi_\infty^\lambda, \quad \Psi_\infty^\lambda := \psi_\infty^\lambda \circ \boldsymbol{f}.$$

We deduce from (12.2), (12.3), and (12.6) that

$$\Psi_\infty^\lambda \text{ is a retraction from } bc^k\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big) \text{ onto } bc^{k,\lambda}, \text{ and } \Phi_\infty^\lambda \text{ is a coretraction.} \tag{12.7}$$

As a consequence of this, Theorem 11.1(ii), definition (12.1), and general properties of interpolation functors (cf. [2], Proposition I.2.3.3) we find

$$\Psi_\infty^\lambda \text{ is a retraction from } B_\infty^s\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big) \text{ onto } B_\infty^{s,\lambda}, \text{ and } \Phi_\infty^\lambda \text{ is a coretraction.} \tag{12.8}$$

Since

$$\psi_\infty^\lambda = \Psi_\infty^\lambda \circ \boldsymbol{f}^{-1}, \quad \varphi_\infty^\lambda = \boldsymbol{f} \circ \Phi_\infty^\lambda \tag{12.9}$$

we get the second assertion from (12.8) and Lemma 11.10. □

Corollary 12.2

(i) $u \mapsto |||u|||_{k,\infty;\lambda} := \sup_{\kappa \ltimes \varphi} \rho_\kappa^\lambda \, \|(\kappa \ltimes \varphi)_*(\pi_\kappa u)\|_{k,\infty;E}$ *is a norm for* $BC^{k,\lambda}$.

(ii) *Suppose* $s > 0$. *Then*

$$u \mapsto |||u|||^*_{s,\infty;\lambda} := \sup_{\kappa \ltimes \varphi} \rho_\kappa^\lambda \, \|(\kappa \ltimes \varphi)_*(\pi_\kappa u)\|^*_{s,\infty;E}$$

and

$$u \mapsto |||u|||^{**}_{s,\infty;\lambda} := \sup_{\kappa \ltimes \varphi} \rho_\kappa^\lambda \, \|(\kappa \ltimes \varphi)_*(\pi_\kappa u)\|^{**}_{s,\infty;E}$$

are norms for $B_\infty^{s,\lambda}$.

(iii) *Assume* $k_0 < s < k_1$ *with* $k_0, k_1 \in \mathbb{N}$. *Then* $(bc^{k_0,\lambda}, bc^{k_1,\lambda})_{(s-k_0)/(k_1-k_0),\infty} \doteq B_\infty^{s,\lambda}$.

(iv) *If* $0 < s_0 < s_1$ *and* $0 < \theta < 1$, *then* $(B_\infty^{s_0,\lambda}, B_\infty^{s_1,\lambda})_{\theta,\infty} \doteq B_\infty^{s_\theta,\lambda} \doteq [B_\infty^{s_0,\lambda}, B_\infty^{s_1,\lambda}]_\theta$.

Proof. (i) and (ii) are implied by (7.9) and Theorem 11.1(i). Assertions (iii) and (iv) follow from (12.7) and (12.8) and parts (ii) and (iii) of Theorem 11.1, respectively, and (12.9) and Lemma 11.10. □

Weighted Hölder spaces are defined by $BC^{s,\lambda} := B_\infty^{s,\lambda}$ for $s \in \mathbb{R}^+ \backslash \mathbb{N}$. This is in agreement with Theorem 11.1(ii).

Parts (i) and (ii) of Corollary 12.2 show that the present definition of weighted Hölder spaces is equivalent to the one used in [5]. It should be noted that Corollary 12.2(iii) gives a positive answer to the conjecture of Remark 8.2 of [5], provided $BC^{k,\lambda}$ and $BC^{k+1,\lambda}$ are replaced by $bc^{k,\lambda}$ and $bc^{k+1,\lambda}$, respectively.

We define **weighted little Hölder spaces** by

$$bc^{s,\lambda} \text{ is the closure of } BC^{\infty,\lambda} \text{ in } BC^{s,\lambda}, \qquad s \geq 0.$$

Similarly, the **weighted little Besov-Hölder space scale** $[\, b_\infty^{s,\lambda} \,;\, s > 0 \,]$ is obtained by

$$b_\infty^{s,\lambda} \text{ is the closure of } BC^{\infty,\lambda} \text{ in } B_\infty^{s,\lambda}. \tag{12.10}$$

Theorem 12.3 ψ_∞^λ *is a retraction from* $\ell_{\infty,\mathrm{unif}}(\boldsymbol{b}_\infty^s)$ *onto* $b_\infty^{s,\lambda}$, *and* φ_∞^λ *is a coretraction.*

Proof. We infer from (11.20) that $BC^\infty\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big) = \bigcap_k bc^k\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big)$. Hence we get from (12.7) that Ψ_∞^λ is a retraction from $BC^\infty\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big)$ onto $BC^{\infty,\lambda}$, and Φ_∞^λ is a coretraction. Due to this and definitions (11.15) and (12.10) we deduce from (12.7) and (12.8) that

$$\Psi_\infty^\lambda \text{ is a retraction from } b_\infty^s\big(\mathbb{M}, \ell_\infty(\boldsymbol{E})\big) \text{ onto } b_\infty^{s,\lambda}, \text{ and } \Phi_\infty^\lambda \text{ is a coretraction.} \tag{12.11}$$

Now the assertion follows from Lemma 11.11 and (12.9). □

Corollary 12.4

(i) *Suppose* $k_0 < s < k_1$ *with* $k_0, k_1 \in \mathbb{N}$. *Then* $(bc^{k_0,\lambda}, bc^{k_1,\lambda})^0_{(s-k_0)/(k_1-k_0),\infty} \doteq b_\infty^{s,\lambda}$.

(ii) *If* $0 < s_0 < s_1$ *and* $0 < \theta < 1$, *then* $(b_\infty^{s_0,\lambda}, b_\infty^{s_1,\lambda})^0_{\theta,\infty} \doteq b_\infty^{s_\theta,\lambda} \doteq [b_\infty^{s_0,\lambda}, b_\infty^{s_1,\lambda}]_\theta$.

Proof. These predications are derived from (12.11) and Theorem 11.4. □

Now we turn to weighted anisotropic spaces. We set

$$BC^{0/\vec{r},\vec{\omega}} = BC^{0/\vec{r},(\lambda,0)} := \big(\big\{\, u \in C\big(J, C(V)\big) \,;\, \|u\|_{\infty;B^{0,\lambda}} < \infty \,\big\},\ \|\cdot\|_{\infty;B^{0,\lambda}}\big) \tag{12.12}$$

and, for $k \in \mathbb{N}^\times$,

$$BC^{kr/\vec{r},\vec{\omega}} := \big\{\, u \in C\big(J, C(V)\big) \,;\, \nabla^i \partial^j u \in BC^{0/\vec{r},(\lambda+i+j\mu,0)},\ i + jr \leq kr \,\big\}, \tag{12.13}$$

endowed with the norm

$$u \mapsto \|u\|_{kr/\vec{r},\infty;\vec{\omega}} := \max_{i+jr\le kr} \|\nabla^i \partial^j u\|_{\infty;\lambda+i+j\mu}. \tag{12.14}$$

It is a consequence of Theorem 12.6 below that $BC^{kr/\vec{r},\vec{\omega}}$ is a Banach space.

Similarly as in the isotropic case, anisotropic Hölder spaces can be characterized by means of local coordinates. For this we prepare the following analogue of Lemma 7.2.

Lemma 12.5 *Suppose $k \in \mathbb{N}$ and $s > 0$. Then*

$$S_{\widetilde{\kappa}\kappa} \in \mathcal{L}(BC^k_{\widetilde{\kappa}}, BC^k_\kappa) \cap \mathcal{L}(bc^k_{\widetilde{\kappa}}, bc^k_\kappa) \cap \mathcal{L}(B^s_{\infty,\widetilde{\kappa}}, B^s_{\infty,\kappa}) \cap \mathcal{L}(b^s_{\infty,\widetilde{\kappa}}, b^s_{\infty,\kappa})$$

and $\|S_{\widetilde{\kappa}\kappa}\| \le c$ for $\widetilde{\kappa} \in \mathfrak{N}(\kappa)$ and $\kappa \in \mathfrak{K}$.

Proof. As in the proof of Lemma 7.2 we see that the statement applies for the spaces BC^k and bc^k. Now we get the remaining assertions by interpolation, due to Theorems 11.1(ii) and 11.4(i). □

Theorem 12.6 *$\psi^{\vec{\omega}}_\infty$ is a retraction from $\ell_\infty(\boldsymbol{BC}^{kr/\vec{r}})$ onto $BC^{kr/\vec{r},\vec{\omega}}$, and $\varphi^{\vec{\omega}}_\infty$ is a coretraction.*

Proof. (1) From (9.6) and Theorem 12.1 we get

$$\|\varphi^{\vec{\omega}}_{\infty,\kappa} u\|_{\infty;BC^{kr}_\kappa} = \|\varphi^{\lambda}_{\infty,\kappa} u\|_{\infty;BC^{kr}_\kappa} \le c\,\|u\|_{\infty;BC^{kr,\lambda}}, \qquad \kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi.$$

Similarly, by invoking (9.5) as well,

$$\|\partial^j \varphi^{\vec{\omega}}_{\infty,\kappa} u\|_{\infty;BC^{(k-j)r}_\kappa} = \|\varphi^{\lambda+jr}_{\infty,\kappa} \partial^j u\|_{\infty;BC^{(k-j)r}_\kappa} \le c\,\|\partial^j u\|_{\infty;BC^{(k-j)r,\lambda+jr}}$$

for $0 \le j \le k$ and $\kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi$. From this and definition (12.14) we infer

$$\|\varphi^{\vec{\omega}}_\infty u\|_{\ell_\infty(\boldsymbol{BC}^{kr/\vec{r}})} \le c\,\|u\|_{kr/\vec{r},\infty;\vec{\omega}}.$$

(2) Given $\kappa \in \mathfrak{K}$ and $\widetilde{\kappa} \in \mathfrak{N}(\kappa)$,

$$\varphi^\lambda_{\infty,\kappa} \circ \psi^\lambda_{\infty,\widetilde{\kappa}} = a_{\widetilde{\kappa}\kappa} S_{\widetilde{\kappa}\kappa} \tag{12.15}$$

with

$$a_{\widetilde{\kappa}\kappa} := (\rho_\kappa/\rho_{\widetilde{\kappa}})^\lambda (\kappa_*\pi_\kappa) S_{\widetilde{\kappa}\kappa}(\widetilde{\kappa}_*\pi_{\widetilde{\kappa}}).$$

It is obvious that the scalar-valued BC^k-spaces form continuous multiplication algebras. Hence (4.3), (7.3), and Lemma 12.5 imply

$$\|a_{\widetilde{\kappa}\kappa}\|_{BC^{kr}} \le c, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \qquad \kappa \in \mathfrak{K}. \tag{12.16}$$

Thus we deduce from (12.15), (12.16), and Lemma 12.5 that

$$\|\varphi^\lambda_{\infty,\kappa} \circ \psi^{\vec{\omega}}_{\infty,\widetilde{\kappa}} v_{\widetilde{\kappa}}\|_{\infty;BC^{kr}_\kappa} = \|\varphi^\lambda_{\infty,\kappa} \circ \psi^\lambda_{\infty,\widetilde{\kappa}} v_{\widetilde{\kappa}}\|_{\infty;BC^{kr}_\kappa} \le c\,\|v_{\widetilde{\kappa}}\|_{\infty;BC^{kr}_{\widetilde{\kappa}}}$$

for $\widetilde{\kappa} \in \mathfrak{N}(\kappa)$ and $\kappa\ltimes\varphi,\ \widetilde{\kappa}\ltimes\widetilde{\varphi} \in \mathfrak{K}\ltimes\Phi$. By this and the finite multiplicity of $\mathfrak{K}$ we obtain

$$\begin{aligned}\|\varphi^\lambda_{\infty,\kappa} \circ \psi^{\vec{\omega}}_\infty \boldsymbol{v}\|_{\infty;BC^{kr}_\kappa} &= \Big\| \sum_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)} \varphi^\lambda_{\infty,\kappa} \circ \psi^{\vec{\omega}}_{\infty,\widetilde{\kappa}} v_{\widetilde{\kappa}} \Big\|_{\infty;BC^{kr}_\kappa} \\ &\le c \max_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)} \|\varphi^\lambda_{\infty,\kappa} \circ \psi^{\vec{\omega}}_{\infty,\widetilde{\kappa}} v_{\widetilde{\kappa}}\|_{\infty;BC^{kr}_\kappa} \le c\,\|\boldsymbol{v}\|_{\ell_\infty(B(J,\boldsymbol{BC}^{kr}))}\end{aligned}$$

for $\kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi$.

Note

$$\|\varphi^{\lambda+j\mu}_{\infty,\kappa} \circ \partial^j \circ \psi^{\vec{\omega}}_{\infty,\widetilde{\kappa}} v_{\widetilde{\kappa}}\|_{\infty;BC^{(k-j)r}_\kappa} = \|\varphi^{\lambda+j\mu}_{\infty,\kappa} \circ \psi^{\lambda+j\mu}_{\infty,\widetilde{\kappa}} (\partial^j v_{\widetilde{\kappa}})\|_{\infty;BC^{(k-j)r}_\kappa}$$

for $0 \le j \le k$ and $\kappa\ltimes\varphi,\ \widetilde{\kappa}\ltimes\widetilde{\varphi} \in \mathfrak{K}\ltimes\Phi$. Thus, as above,

$$\|\varphi^{\lambda+j\mu}_\infty \circ \partial^j \circ \psi^{\vec{\omega}}_\infty \boldsymbol{v}\|_{\ell_\infty(\boldsymbol{BC}^{(k-j)r})} \le c\,\|\partial^j \boldsymbol{v}\|_{\ell_\infty(B(J,\boldsymbol{BC}^{(k-j)r}))}, \qquad 0 \le j \le k.$$

Now we deduce from Corollary 12.2(i)

$$\|\psi^{\vec{\omega}}_\infty \boldsymbol{v}\|_{kr/\vec{r},\infty;\vec{\omega}} \le c\,\|\boldsymbol{v}\|_{\ell_\infty(\boldsymbol{BC}^{kr/\vec{r}})}.$$

Since $\varphi^{\vec{\omega}}_\infty$ is a right inverse for $\psi^{\vec{\omega}}_\infty$ the theorem is proved. □

Next we introduce a linear subspace of $BC^{kr/\vec r,\vec\omega}$ by

$$bc^{kr/\vec r,\vec\omega} \text{ is the set of all } u \text{ in } BC^{kr/\vec r,\vec\omega} \text{ with } \varphi_\infty^{\vec\omega} u \in \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r}).$$

Due to the fact that $\ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})$ is a closed linear subspace of $\ell_\infty(\boldsymbol{BC}^{kr/\vec r})$ it follows from the continuity of $\varphi_\infty^{\vec\omega}$ that $bc^{kr/\vec r,\vec\omega}$ is a closed linear subspace of $BC^{kr/\vec r,\vec\omega}$.

The next theorem shows, in particular, that $bc^{kr/\vec r,\vec\omega}$ is independent of the particular choice of $\mathfrak{K}\ltimes\Phi$ and the localization system used in the preceding definition. For this we set

$$BC^{\infty/\vec r,\vec\omega} := \textstyle\bigcap_k BC^{kr/\vec r,\vec\omega},$$

equipped with the natural projective topology.

Theorem 12.7

(i) *$\psi_\infty^{\vec\omega}$ is a retraction from $\ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})$ onto $bc^{kr/\vec r,\vec\omega}$, and $\varphi_\infty^{\vec\omega}$ is a coretraction.*

(ii) *$bc^{kr/\vec r,\vec\omega}$ is the closure of $BC^{\infty/\vec r,\vec\omega}$ in $BC^{kr/\vec r,\vec\omega}$.*

Proof. (1) Suppose $\varphi_\infty^{\vec\omega} u = 0$ for some $u \in BC^{kr/\vec r,\vec\omega}$. Then it follows from (9.3) that $(\kappa\ltimes\varphi)_*(\pi_\kappa u) = 0$ for $\kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi$. Hence $\pi_\kappa u = 0$ for $\kappa \in \mathfrak{K}$, and consequently $\pi_\kappa^2 u = 0$ for $\kappa \in \mathfrak{K}$. This implies $u = \sum_\kappa \pi_\kappa^2 u = 0$. Thus $\varphi_\infty^{\vec\omega}$ is injective.

(2) We denote by $\mathcal{Y}$ the image space of $BC^{kr/\vec r,\vec\omega}$ under $\varphi_\infty^{\vec\omega}$. Theorem 12.6 and [4, Lemma 4.1.5] imply

$$\ell_\infty(\boldsymbol{BC}^{kr/\vec r}) = \mathcal{Y} \oplus \ker(\psi_\infty^{\vec\omega}), \qquad \psi_\infty^{\vec\omega} \in \mathcal{L}\mathrm{is}(\mathcal{Y}, BC^{kr/\vec r,\vec\omega}). \tag{12.17}$$

Thus, by step (1) (see Remarks 2.2.1 of [4]),

$$\varphi_\infty^{\vec\omega} \in \mathcal{L}\mathrm{is}(BC^{kr/\vec r,\vec\omega}, \mathcal{Y}), \quad (\varphi_\infty^{\vec\omega})^{-1} = \psi_\infty^{\vec\omega} \,|\, \mathcal{Y}.$$

Since $\mathcal{X} := \mathcal{Y} \cap \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})$ is a closed linear subspace of $\mathcal{Y}$ we thus get

$$\varphi_\infty^{\vec\omega} \in \mathcal{L}\mathrm{is}(bc^{kr/\vec r,\vec\omega}, \mathcal{X}), \quad (\varphi_\infty^{\vec\omega} \,|\, bc^{kr/\vec r,\vec\omega})^{-1} = \psi_\infty^{\vec\omega} \,|\, \mathcal{X}. \tag{12.18}$$

Due to (12.17) we can write $\boldsymbol{w} \in \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})$ in the form $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ with $\boldsymbol{u} \in \mathcal{X}$ and $\boldsymbol{v} \in \ker(\psi_\infty^{\vec\omega})$. From this and (12.18) it follows $\psi_\infty^{\vec\omega}\big(\ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})\big) \subset bc^{kr/\vec r,\vec\omega}$. Hence $\psi_\infty^{\vec\omega} \in \mathcal{L}\big(\ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r}), bc^{kr/\vec r,\vec\omega}\big)$ and $\psi_\infty^{\vec\omega} \circ \varphi_\infty^{\vec\omega} u = u$ for $u \in bc^{kr/\vec r,\vec\omega}$. This proves (i).

(3) Using obvious adaptions of the notations of the proof of Theorem 12.1 we deduce from Lemma 11.12

$$\widetilde{\boldsymbol{f}} \in \mathcal{L}\mathrm{is}\big(bc^{kr/\vec r}\big(\mathbb{M} \times J, \ell_\infty(\boldsymbol{E})\big), \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})\big). \tag{12.19}$$

We set

$$\Phi_\infty^{\vec\omega} := \widetilde{\boldsymbol{f}}^{-1} \circ \varphi_\infty^{\vec\omega}, \quad \Psi_\infty^{\vec\omega} := \psi_\infty^{\vec\omega} \circ \widetilde{\boldsymbol{f}}.$$

Then we infer from (i) and (12.19) that

$$\Psi_\infty^{\vec\omega} \text{ is a retraction from } bc^{kr/\vec r}\big(\mathbb{M} \times J, \ell_\infty(\boldsymbol{E})\big) \text{ onto } bc^{kr/\vec r,\vec\omega}, \text{ and } \Phi_\infty^{\vec\omega} \text{ is a coretraction.} \tag{12.20}$$

Definition (11.25) guarantees

$$BC^\infty\big(\mathbb{M} \times J, \ell_\infty(\boldsymbol{E})\big) \overset{d}{\hookrightarrow} bc^{kr/\vec r}\big(\mathbb{M} \times J, \ell_\infty(\boldsymbol{E})\big).$$

It is an easy consequence of the mean value theorem that $\ell_\infty(\boldsymbol{BC}^{(k+1)r/\vec r}) \hookrightarrow \ell_{\infty,\mathrm{unif}}(\boldsymbol{bc}^{kr/\vec r})$. From these embeddings, Theorem 12.6, and (i) we infer that the first of the injections

$$BC^{(k+1)r/\vec r,\vec\omega} \hookrightarrow bc^{kr/\vec r,\vec\omega} \hookrightarrow BC^{kr/\vec r,\vec\omega}$$

is valid. Thus

$$BC^{\infty/\vec{r},\vec{\omega}} = \bigcap_k BC^{kr/\vec{r},\vec{\omega}} = \bigcap_k bc^{kr/\vec{r},\vec{\omega}}.$$

Now it follows from (11.20) and (12.20) that

$$\Psi^{\vec{\omega}}_{\infty} \text{ is a retraction from } BC^{\infty}\big(\mathbb{M}\times J, \ell_{\infty}(\boldsymbol{E})\big) \text{ onto } BC^{\infty/\vec{r},\vec{\omega}}.$$

Assertion (ii) is implied by (12.20) and [4, Lemma 4.1.6]. □

We define the **weighted anisotropic Besov-Hölder space scale** $[\, B^{s/\vec{r},\vec{\omega}}_{\infty}\ ;\ s>0\,]$ by

$$B^{s/\vec{r},\vec{\omega}}_{\infty} = B^{s/\vec{r},\vec{\omega}}_{\infty}(J,V) := \begin{cases} (bc^{kr/\vec{r},\vec{\omega}}, bc^{(k+1)r/\vec{r},\vec{\omega}})_{(s-kr)/r,\infty}, & kr < s < (k+1)r, \\ (bc^{kr/\vec{r},\vec{\omega}}, bc^{(k+2)r/\vec{r},\vec{\omega}})_{1/2,\infty}, & s = (k+1)r. \end{cases} \tag{12.21}$$

These spaces allow for a retraction-coretraction theorem as well which provides representations via local coordinates.

Theorem 12.8 *$\psi^{\vec{\omega}}_{\infty}$ is a retraction from $\ell_{\infty}(\boldsymbol{B}^{s/\vec{r}}_{\infty})$ onto $B^{s/\vec{r},\vec{\omega}}_{\infty}$, and $\varphi^{\vec{\omega}}_{\infty}$ is a coretraction.*

Proof. We infer from (12.20), Theorem 11.6(ii), and definition (12.21) that

$$\Psi^{\vec{\omega}}_{\infty} \text{ is a retraction from } B^{s/\vec{r}}_{\infty}\big(\mathbb{M}\times J, \ell_{\infty}(\boldsymbol{E})\big) \text{ onto } B^{s/\vec{r},\vec{\omega}}_{\infty}, \text{ and } \Phi^{\vec{\omega}}_{\infty} \text{ is a coretraction.} \tag{12.22}$$

Thus the assertion follows from

$$\varphi^{\vec{\omega}}_{\infty} = \widetilde{\boldsymbol{f}} \circ \Phi^{\vec{\omega}}_{\infty}, \quad \psi^{\vec{\omega}}_{\infty} = \Psi^{\vec{\omega}}_{\infty} \circ \widetilde{\boldsymbol{f}}^{-1}, \tag{12.23}$$

and Lemma 11.12. □

Corollary 12.9

(i) *Suppose $k_0 r < s < k_1 r$ with $k_0, k_1 \in \mathbb{N}$. Then $(bc^{k_0 r/\vec{r},\vec{\omega}}, bc^{k_1 r/\vec{r},\vec{\omega}})_{(s-k_0)/(k_1-k_0),\infty} \doteq B^{s/\vec{r},\vec{\omega}}_{\infty}$.*

(ii) *If $0 < s_0 < s_1$ and $0 < \theta < 1$, then $(B^{s_0/\vec{r},\vec{\omega}}_{\infty}, B^{s_1/\vec{r},\vec{\omega}}_{\infty})_{\theta,\infty} \doteq B^{s_\theta/\vec{r},\vec{\omega}}_{\infty} \doteq [B^{s_0/\vec{r},\vec{\omega}}_{\infty}, B^{s_1/\vec{r},\vec{\omega}}_{\infty}]_{\theta}$.*

Proof. This is implied by (12.20), (12.22), and Theorem 11.6. □

Weighted anisotropic Hölder spaces are defined by setting $BC^{s/\vec{r},\vec{\omega}} := B^{s/\vec{r},\vec{\omega}}_{\infty}$ for $s \in \mathbb{R}\backslash\mathbb{N}$. Then we introduce **weighted anisotropic little Hölder spaces** by

$$bc^{s/\vec{r},\vec{\omega}} = bc^{s/\vec{r},\vec{\omega}}(J,V) \text{ is the closure of } BC^{\infty/\vec{r},\vec{\omega}} \text{ in } BC^{s/\vec{r},\vec{\omega}}$$

for $s \geq 0$. Note that this is consistent with Theorem 12.7(ii).

Lastly, we get the **weighted anisotropic little Besov-Hölder space scale** $[\, b^{s/\vec{r},\vec{\omega}}_{\infty}\ ;\ s>0\,]$ by

$$b^{s/\vec{r},\vec{\omega}}_{\infty} \text{ is the closure of } BC^{\infty/\vec{r},\vec{\omega}} \text{ in } B^{s/\vec{r},\vec{\omega}}_{\infty}. \tag{12.24}$$

Theorem 12.10 (i) *$\psi^{\vec{\omega}}_{\infty}$ is a retraction from $\ell_{\infty,\mathrm{unif}}(\boldsymbol{b}^{s/\vec{r}}_{\infty})$ onto $b^{s/\vec{r},\vec{\omega}}_{\infty}$, and $\varphi^{\vec{\omega}}_{\infty}$ is a coretraction.*
(ii) *Suppose $k_0 r < s < k_1 r$ with $k_0, k_1 \in \mathbb{N}$. Then*

$$(bc^{k_0 r/\vec{r},\vec{\omega}}, bc^{k_1 r/\vec{r},\vec{\omega}})^0_{(s/r-k_0)/(k_1-k_0),\infty} \doteq bc^{s/\vec{r},\vec{\omega}}.$$

(iii) *If $0 < s_0 < s_1$ and $0 < \theta < 1$, then*

$$(bc^{s_0/\vec{r},\vec{\omega}}, bc^{s_1/\vec{r},\vec{\omega}})^0_{\theta,\infty} \doteq bc^{s_\theta/\vec{r},\vec{\omega}} \doteq [bc^{s_0/\vec{r},\vec{\omega}}, bc^{s_1/\vec{r},\vec{\omega}}]_{\theta}.$$

Proof. Assertion (ii) and the first part of (iii) follow from Corollary 12.9(i) and definition (12.24). From (ii), Theorem 11.9(i), and (12.20) it follows that

$$\Psi^{\vec{\omega}}_{\infty} \text{ is a retraction from } bc^{s/\vec{r}}\big(\mathbb{M}\times J, \ell_{\infty}(\boldsymbol{E})\big) \text{ onto } bc^{s/\vec{r},\vec{\omega}}, \text{ and } \Phi^{\vec{\omega}}_{\infty} \text{ is a coretraction.} \tag{12.25}$$

Due to this the second part of (iii) is now implied by Theorem 11.9(ii). Statement (i) is a consequence of (12.25), (12.23), and Lemma 11.12. □

13 Point-Wise Multipliers

In connection with differential and pseudodifferential operators there occur naturally 'products' of tensor fields possessing different regularity of the factors, so called 'point-wise products' or 'multiplications'. Although there is no problem in establishing mapping properties of differential operators say, if the coefficients are smooth, this is a much more difficult task if one is interested in operators with little regularity of the coefficients. Since such low regularity coefficients are of great importance in practice we derive in this and the next section point-wise multiplier theorems which are (almost) optimal.

Let $\mathcal{X}_j,\ j=0,1,2$, be Banach spaces. A *multiplication* $\mathcal{X}_0\times\mathcal{X}_1\to\mathcal{X}_2$ *from* $\mathcal{X}_0\times\mathcal{X}_1$ *into* $\mathcal{X}_2$ is an element of $\mathcal{L}(\mathcal{X}_0,\mathcal{X}_1;\mathcal{X}_2)$, the Banach space of continuous bilinear maps from $\mathcal{X}_0\times\mathcal{X}_1$ into $\mathcal{X}_2$.

Before considering multiplications in tensor bundles we first investigate point-wise products in Euclidean settings. Let $E_i=(E_i,|\cdot|_i),\ i=0,1,2$, be finite-dimensional Banach spaces, $\mathbb{X}\in\{\mathbb{R}^m,\mathbb{H}^m\}$, and $\mathbb{Y}:=\mathbb{X}\times J$.

Theorem 13.1 *Suppose* $\mathsf{b}\in\mathcal{L}(E_0,E_1;E_2)$ *and*

$$\mathsf{m}:\ E_0^{\mathbb{Y}}\times E_1^{\mathbb{Y}}\to E_2^{\mathbb{Y}},\quad (u_0,u_1)\mapsto\mathsf{b}(u_0,u_1)$$

is its point-wise extension. Then

(i) $\mathsf{m}\in\mathcal{L}\big(\mathcal{B}^{s/\vec r}(\mathbb{Y},E_0),\mathcal{B}^{s/\vec r}(\mathbb{Y},E_1);\mathcal{B}^{s/\vec r}(\mathbb{Y},E_2)\big)$ *if either* $s\in r\mathbb{N}$ *and* $\mathcal{B}\in\{BC,bc\}$, *or* $s>0$ *and* $\mathcal{B}\in\{B_\infty,b_\infty\}$.

(ii) $\mathsf{m}\in\mathcal{L}\big(BC^{s/\vec r}(\mathbb{Y},E_0),W_p^{s/\vec r}(\mathbb{Y},E_1);W_p^{s/\vec r}(\mathbb{Y},E_2)\big),\ s\in r\mathbb{N}$.

(iii) $\mathsf{m}\in\mathcal{L}\big(B_\infty^{s_0/\vec r}(\mathbb{Y},E_0),\mathfrak{F}_p^{s/\vec r}(\mathbb{Y},E_1);\mathfrak{F}_p^{s/\vec r}(\mathbb{Y},E_2)\big),\ 0<s<s_0$.

In either case the map $\mathsf{b}\mapsto\mathsf{m}$ *is linear and continuous.*

Proof. (1) Assertion (i) for $s\in r\mathbb{N}$ and $\mathcal{B}\in\{BC,bc\}$ as well as assertion (ii) follow from the product rule.

(2) Suppose $u_i\in E_i^{\mathbb{Y}},\ i=0,1$, and $0<\theta<1$. Then

$$\triangle_\xi\big(\mathsf{m}(u_0,u_1)\big)=\mathsf{m}\big(\triangle_\xi u_0,u_1(\cdot+\xi)\big)+\mathsf{m}(u_0,\triangle_\xi u_1),\qquad \xi\in\mathbb{Y}. \tag{13.1}$$

From this we infer, letting $\xi=(h,0)$ with $h\in(0,\delta)^m$,

$$\sup_t\big[\mathsf{m}(u_0,u_1)(\cdot,t)\big]_{\theta,\infty}^{\delta}\le c\big(\sup_t\big[u_0(\cdot,t)\big]_{\theta,\infty}^{\delta}\|u_1\|_\infty+\sup_t\big[u_1(\cdot,t)\big]_{\theta,\infty}^{\delta}\|u_0\|_\infty\big)$$

for $0<\delta\le\infty$. Similarly,

$$\sup_x\big[\mathsf{m}(u_0,u_1)(x,\cdot)\big]_{\theta,\infty}^{\delta}\le c\big(\sup_x\big[u_0(x,\cdot)\big]_{\theta,\infty}^{\delta}\|u_1\|_\infty+\sup_x\big[u_1(x,\cdot)\big]_{\theta,\infty}^{\delta}\|u_0\|_\infty\big).$$

By step (1), (11.21), and (11.22) we infer that (i) is true if $s\in\mathbb{R}^+\backslash\mathbb{N}$. Now we fill in the gaps $s\in\mathbb{N}$ by means of Theorems 11.6(iii) and 11.9(ii) and bilinear complex interpolation (cf. J. Bergh and J. Löfström [8, Theorem 4.4.1]). This proves (i) for $s>0$ and $\mathcal{B}\in\{B_\infty,b_\infty\}$.

(3) Assume $s\in r\mathbb{N}$ and $s_0>s$. By Theorem 11.8 $B_\infty^{s_0/\vec r}(\mathbb{Y},E_0)\hookrightarrow b_\infty^{s/\vec r}(\mathbb{Y},E_0)\hookrightarrow BC_\infty^{s/\vec r}(\mathbb{Y},E_0)$. Hence we deduce from (ii)

$$\mathsf{m}\in\mathcal{L}\big(B_\infty^{s_0/\vec r}(\mathbb{Y},E_0),W_p^{s/\vec r}(\mathbb{Y},E_1);W_p^{s/\vec r}(\mathbb{Y},E_2)\big),\qquad s\in r\mathbb{N},\quad s<s_0.$$

Using this, Theorem 8.2(iv), and once more bilinear complex interpolation we obtain

$$\mathsf{m}\in\mathcal{L}\big(B_\infty^{s_0/\vec r}(\mathbb{Y},E_0),H_p^{s/\vec r}(\mathbb{Y},E_1);H_p^{s/\vec r}(\mathbb{Y},E_2)\big),\qquad 0<s<s_0.$$

(4) We assume $kr<s<(k+1)r$ with $k\in\mathbb{N}$. It is well-known that

$$B_\infty^{s_0}(\mathbb{X},E_0)\times B_p^{s}(\mathbb{X},E_1)\to B_p^{s}(\mathbb{X},E_2),\quad (v_0,v_1)\mapsto b(v_0,v_1)$$

is a multiplication (see Remark 4.2(b) in H. Amann [1], where $B_\infty^{s_0}$ is denoted by BUC^{s_0}, Th. Runst and W. Sickel [39, Theorem 4.7.1], or V.G. Maz'ya and T.O. Shaposhnikova [33], and H. Triebel [51]), depending linearly and continuously on b. From this we infer

$$\|\mathsf{m}(u_0,u_1)\|_{p;B_p^s(\mathbb{X},E_2)} \le c\,\|u_0\|_{\infty;B_\infty^{s_0}(\mathbb{X},E_0)}\,\|u_1\|_{p;B_p^s(\mathbb{X},E_1)}. \tag{13.2}$$

By the product rule and (ii)

$$\begin{aligned}\big\|\partial^\ell\big(\mathsf{m}(u_0,u_1)\big)\big\|_{p;L_p(\mathbb{X},E_2)} &\le c\sum_{j=0}^{\ell}\|\partial^j u_0\|_{\infty;B(\mathbb{X},E_0)}\,\|\partial^{\ell-j}u_1\|_{p;L_p(\mathbb{X},E_1)}\\ &\le c\,\|u_0\|_{k,\infty;B(\mathbb{X},E_0)}\,\|u_1\|_{k,p;L_p(\mathbb{X},E_1)}\\ &\le c\,\|u_0\|^{**}_{s_0/r,\infty;B(\mathbb{X},E_0)}\,\|u_1\|_{s/r,p;L_p(\mathbb{X},E_1)}\end{aligned} \tag{13.3}$$

for $0\le\ell\le k$.

We deduce from (13.1) that, given $\theta\in(0,1)$,

$$\big[\mathsf{m}(u_0,u_1)\big]_{\theta,p;L_p(\mathbb{X},E_2)} \le c\big([u_0]_{\theta,\infty;B(\mathbb{X},E_0)}\,\|u_1\|_{p;L_p(\mathbb{X},E_1)} + \|u_0\|_{\infty;B(\mathbb{X},E_0)}\,[u_1]_{\theta,p;L_p(\mathbb{X},E_1)}\big).$$

Hence

$$\begin{aligned}\big[\mathsf{m}(\partial^j u_0,\partial^{k-j}u_1)\big]_{(s-kr)/r,p;L_p(\mathbb{X},E_2)} &\le c\,\|\partial^j u_0\|^{*}_{(s-kr)/r,\infty;B(\mathbb{X},E_0)}\,\|\partial^{k-j}u_1\|^{*}_{(s-kr)/r,p;L_p(\mathbb{X},E_2)}\\ &\le c\,\|u_0\|^{**}_{s_0/r,\infty;B(\mathbb{X},E_0)}\,\|u_1\|^{**}_{s/r,p;L_p(\mathbb{X},E_2)},\end{aligned}$$

where we used $B_\infty^{s_0/r}\big(J,B(\mathbb{X},E_0)\big)\hookrightarrow B_\infty^{s/r}\big(J,B(\mathbb{X},E_0)\big)$ in the last estimate. Thus

$$\big[\partial^k\big(\mathsf{m}(u_0,u_1)\big]_{(s-kr)/r,p;L_p(\mathbb{X},E_2)} \le c\,\|u_0\|^{**}_{s_0/r,\infty;B(\mathbb{X},E_0)}\,\|u_1\|^{**}_{s/r,p;L_p(\mathbb{X},E_1)}. \tag{13.4}$$

By Corollary 10.2

$$B_p^{s/\vec r}(\mathbb{Y},E_2)\doteq L_p\big(J,B_p^s(\mathbb{X},E_2)\big)\cap B_p^{s/r}\big(J,L_p(\mathbb{X},E_2)\big).$$

Thus we infer from (10.1) and (13.2)–(13.4) that

$$\mathsf{m}\in\mathcal{L}\big(B_\infty^{s_0/\vec r}(\mathbb{Y},E_0),B_p^{s/\vec r}(\mathbb{Y},E_1);B_p^{s/\vec r}(\mathbb{Y},E_2)\big),\qquad s\notin r\mathbb{N}.$$

Now we fill in the gaps at $s\in r\mathbb{N}$ once more by bilinear complex interpolation, which is possible due to Theorems 8.2(iv) and 11.6(iii).

Since the last part of the statement is obvious from the above considerations, the theorem is proved. □

It should be remarked that J. Johnson [20] has undertaken a detailed study of point-wise multiplication in anisotropic Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$. However, it does not seem to be possible to derive Theorem 13.1 from his results.

Next we extend the preceding theorem to (x,t)-dependent bilinear operators.

Theorem 13.2 *Suppose* $\mathsf{b}\in\mathcal{L}(E_0,E_1;E_2)^{\mathbb{Y}}$ *and set*

$$\mathsf{m}:\ E_0^{\mathbb{Y}}\times E_1^{\mathbb{Y}}\to E_2^{\mathbb{Y}},\quad (u_0,u_1)\mapsto\big((x,t)\mapsto\mathsf{b}(x,t)\big(u_0(x,t),u_1(x,t)\big)\big).$$

Then assertions (i)–(iii) *of the preceding theorem are valid in this case also, provided* b *possesses the same regularity as* u_0.

P r o o f. Consider the multiplication

$$\mathsf{b}_0:\ \mathcal{L}(E_0,E_1;E_2)\times E_0\to\mathcal{L}(E_1,E_2),\quad (\mathsf{b},e_0)\mapsto\mathsf{b}(e_0,\cdot)$$

and let m_0 be its point-wise extension. By applying Theorem 13.1(i) we obtain

$$\mathsf{m}_0 \in \mathcal{L}\big(\mathcal{B}^{s/\vec{r}}\big(\mathbb{Y}, \mathcal{L}(E_0, E_1; E_2)\big), \mathcal{B}^{s/\vec{r}}(\mathbb{Y}, E_0); \mathcal{B}^{s/\vec{r}}\big(\mathbb{Y}, \mathcal{L}(E_1, E_2)\big)\big), \tag{13.5}$$

where either $s \in r\mathbb{N}$ and $\mathcal{B} \in \{BC, bc\}$, or $s > 0$ and $\mathcal{B} \in \{B_\infty, b_\infty\}$.

Next we introduce the multiplication

$$\mathcal{L}(E_1, E_2) \times E_1 \to E_2, \quad (A, e_1) \mapsto Ae_1$$

and its point-wise extension m_1. Then we infer from Theorem 13.1

$$\mathsf{m}_1 \in \mathcal{L}\big(\mathcal{B}^{s/\vec{r}}\big(\mathbb{Y}, \mathcal{L}(E_1, E_2)\big), \mathcal{B}^{s/\vec{r}}(\mathbb{Y}, E_1); \mathcal{B}^{s/\vec{r}}(\mathbb{Y}, E_2)\big), \tag{13.6}$$

if either $s \in r\mathbb{N}$ and $\mathcal{B} \in \{BC, bc\}$, or $s > 0$ and $\mathcal{B} \in \{B_\infty, b_\infty\}$,

$$\mathsf{m}_1 \in \mathcal{L}\big(BC^{s/\vec{r}}\big(\mathbb{Y}, \mathcal{L}(E_1, E_2)\big), W_p^{s/\vec{r}}(\mathbb{Y}, E_1); W_p^{s/\vec{r}}(\mathbb{Y}, E_2)\big), \qquad s \in r\mathbb{N}, \tag{13.7}$$

and

$$\mathsf{m}_1 \in \mathcal{L}\big(B_\infty^{s_0/\vec{r}}\big(\mathbb{Y}, \mathcal{L}(E_1, E_2)\big), \mathfrak{F}_p^{s/\vec{r}}(\mathbb{Y}, E_1); \mathfrak{F}_p^{s/\vec{r}}(\mathbb{Y}, E_2)\big), \qquad 0 < s < s_0. \tag{13.8}$$

Note

$$\mathsf{m}(u_0, u_1) = \mathsf{m}_1\big(\mathsf{m}_0(\mathsf{b}, u_0), u_1\big), \qquad (u_0, u_1) \in E_0^{\mathbb{Y}} \times E_1^{\mathbb{Y}}.$$

Thus the statement is a consequence of (13.5)–(13.8). □

In order to study point-wise multiplications on manifolds we prepare a technical lemma which is a relative of Lemma 12.5. For this we set

$$T_{\widetilde{\kappa}\kappa} u(t) := u\big((\rho_\kappa/\rho_{\widetilde{\kappa}})^\mu t\big), \quad t \in J, \qquad R_{\widetilde{\kappa}\kappa} := T_{\widetilde{\kappa}\kappa} \circ S_{\widetilde{\kappa}\kappa}, \quad \kappa, \widetilde{\kappa} \in \mathfrak{K}. \tag{13.9}$$

Note

$$\Theta_{q,\kappa}^\mu = (\rho_\kappa/\rho_{\widetilde{\kappa}})^{\mu/q} T_{\widetilde{\kappa}\kappa} \Theta_{q,\widetilde{\kappa}}^\mu, \qquad \kappa, \widetilde{\kappa} \in \mathfrak{K}, \quad 1 \le q \le \infty. \tag{13.10}$$

We also put

$$\widehat{\varphi}_{q,\kappa}^{\vec{\omega}} := \rho_\kappa^{\lambda + m/q} \Theta_{q,\kappa}^\mu (\kappa \ltimes \varphi)_* (\chi_\kappa \cdot), \qquad \kappa \ltimes \varphi \in \mathfrak{K} \ltimes \Phi.$$

Then, using $u = \sum_{\widetilde{\kappa}} \pi_{\widetilde{\kappa}}^2 u$,

$$\widehat{\varphi}_{q,\kappa}^{\vec{\omega}} = \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} a_{\widetilde{\kappa}\kappa} R_{\widetilde{\kappa}\kappa} \varphi_{q,\widetilde{\kappa}}^{\vec{\omega}}, \tag{13.11}$$

where

$$a_{\widetilde{\kappa}\kappa} := (\rho_\kappa/\rho_{\widetilde{\kappa}})^{\lambda + (m+\mu)/q} \chi S_{\widetilde{\kappa}\kappa}(\widetilde{\kappa}_* \pi_{\widetilde{\kappa}}). \tag{13.12}$$

Hence, given $q \in [1, \infty]$, we deduce from (4.3), Lemma 12.5, and (7.3)(iii) that

$$a_{\widetilde{\kappa}\kappa} \in BC^k(\mathbb{X}_\kappa), \quad \|a_{\widetilde{\kappa}\kappa}\|_{k,\infty} \le c(k), \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}. \tag{13.13}$$

Lemma 13.3 *Suppose $k \in \mathbb{N}$ and $s > 0$. Let $\mathfrak{G}_\kappa \in \{W_{p,\kappa}^{kr/\vec{r}}, BC_\kappa^{kr/\vec{r}}, bc_\kappa^{kr/\vec{r}}, \mathfrak{F}_{p,\kappa}^{s/\vec{r}}, B_{\infty,\kappa}^{s/\vec{r}}, b_{\infty,\kappa}^{s/\vec{r}}\}$. Then*

$$R_{\widetilde{\kappa}\kappa} \in \mathcal{L}(\mathfrak{G}_{\widetilde{\kappa}}, \mathfrak{G}_\kappa), \quad \|R_{\widetilde{\kappa}\kappa}\| \le c, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa \ltimes \varphi,\ \widetilde{\kappa} \ltimes \widetilde{\varphi} \in \mathfrak{K} \ltimes \Phi. \tag{13.14}$$

Proof. It is immediate from (9.5), (9.6), (4.3), and Lemma 12.5 that

$$R_{\widetilde{\kappa}\kappa} \in \mathcal{L}(W_{p,\widetilde{\kappa}}^{kr/\vec{r}}, W_{p,\kappa}^{kr/\vec{r}}) \cap \mathcal{L}(BC_{\widetilde{\kappa}}^{kr/\vec{r}}, BC_\kappa^{kr/\vec{r}}) \cap \mathcal{L}(bc_{\widetilde{\kappa}}^{kr/\vec{r}}, bc_\kappa^{kr/\vec{r}})$$

and that the uniform estimates of (13.14) are satisfied. Now the remaining statements follow by interpolation. □

Assume $V_j = (V_j, h_j)$, $j = 0, 1, 2$, are metric vector bundles. By a *bundle multiplication* from $V_0 \oplus V_1$ into V_2, denoted by

$$\mathsf{m} : V_0 \oplus V_1 \to V_2, \quad (v_0, v_1) \mapsto \mathsf{m}(v_0, v_1),$$

we mean a smooth section $\mathfrak{m}$ of $\mathrm{Hom}(V_0 \otimes V_1, V_2)$ such that $\mathsf{m}(v_0, v_1) := \mathfrak{m}(v_0 \otimes v_1)$ and

$$|\mathsf{m}(v_0, v_1)|_{h_2} \le c\, |v_0|_{h_0}\, |v_1|_{h_1}, \qquad v_i \in \Gamma(M, V_i), \quad i = 0, 1.$$

Examples 13.4 **(a)** The duality pairing

$$\langle \cdot, \cdot \rangle_{V_1} : V_1^* \oplus V_1 \to M \times \mathbb{K}, \quad (v^*, v) \mapsto \langle v^*, v \rangle_{V_1}$$

is a bundle multiplication.

(b) Assume $\sigma_i, \tau_i \in \mathbb{N}$ for $i = 0, 1$. Then the tensor product

$$\otimes : T_{\tau_0}^{\sigma_0} M \oplus T_{\tau_1}^{\sigma_1} M \to T_{\tau_0+\tau_1}^{\sigma_0+\sigma_1} M, \quad (a, b) \mapsto a \otimes b$$

is a bundle multiplication where $(X^{\otimes\sigma_0} \otimes X^{*\otimes\tau_0}) \otimes (X^{\otimes\sigma_1} \otimes X^{*\otimes\tau_1}) := X^{\otimes(\sigma_0+\sigma_1)} \otimes X^{*(\tau_0+\tau_1)}$, where we set $X^{\otimes\sigma} = X^1 \otimes \cdots \otimes X^\sigma$ etc.

(c) Suppose $1 \le i \le \sigma$ and $1 \le j \le \tau$. We denote by $\mathsf{C}_j^i : T_\tau^\sigma M \to T_{\tau-1}^{\sigma-1} M$ the contraction with respect to positions i and j, defined by

$$\mathsf{C}_j^i \Big(\bigotimes_{k=1}^{\sigma} X^k \otimes \bigotimes_{\ell=1}^{\tau} X_\ell^* \Big) := \langle X_j^*, X^i \rangle \bigotimes_{\substack{k=1\\ k\neq i}}^{\sigma} X^k \otimes \bigotimes_{\substack{\ell=1\\ \ell\neq j}}^{\tau} X_\ell^*, \qquad X^k \in \Gamma(M, TM), \quad X_\ell^* \in \Gamma(M, T^*M).$$

It follows from (a) and (b) that

$$\mathsf{C}_j^i : T_{\tau_1}^{\sigma_1} M \oplus T_{\tau_2}^{\sigma_2} M \to T_{\tau_1+\tau_2-1}^{\sigma_1+\sigma_2-1} M, \qquad (a, b) \mapsto \mathsf{C}_j^i (a \otimes b)$$

is a bundle multiplication, where $1 \le i \le \sigma_1 + \sigma_2$ and $1 \le j \le \tau_1 + \tau_2$.

(d) Let $W_j = (W_j, h_{W_j})$, $j = 0, 1, 2$, be metric vector bundles and $\sigma_j, \tau_j \in \mathbb{N}$. Suppose

$$\mathsf{w} : W_0 \oplus W_1 \to W_2, \quad \mathsf{t} : T_{\tau_0}^{\sigma_0} M \oplus T_{\tau_1}^{\sigma_1} M \to T_{\tau_2}^{\sigma_2} M$$

are bundle multiplications. Set $T_{\tau_j}^{\sigma_j}(M, W_j) := (T_{\tau_j}^{\sigma_j} M \otimes W_j, h_j)$ with $h_j := (\cdot, \cdot)_{\sigma_j}^{\tau_j} \otimes h_{W_j}$. Then

$$\mathsf{t} \otimes \mathsf{w} : T_{\tau_0}^{\sigma_0}(M, W_0) \oplus T_{\tau_1}^{\sigma_1}(M, W_1) \to T_{\tau_2}^{\sigma_2}(M, W_2),$$

defined by $\mathsf{t} \otimes \mathsf{w}(a_0 \otimes u_0,\ a_1 \otimes u_1) := \mathsf{t}(a_0, a_1) \otimes \mathsf{w}(u_0, u_1)$, is a bundle multiplication. □

Let m be a bundle multiplication from $V_0 \oplus V_1$ into V_2. Then

$$\Gamma(M, V_0 \oplus V_1)^J \to \Gamma(M, V_2)^J, \quad \big(v_0(t), v_1(t)\big) \mapsto \mathsf{m}\big(v_0(t), v_1(t)\big), \qquad t \in J,$$

is the *point-wise extension* of m, denoted by m also.

After these preparations we can prove the following point-wise multiplier theorem which is the basis of the more specific results of the next section.

Theorem 13.5 *Let $W_j = (W_j, h_{W_j}, D_j)$, $j = 0, 1, 2$, be fully uniformly regular vector bundles over M. Assume $\sigma_j, \tau_j \in \mathbb{N}$ satisfy*

$$\sigma_2 - \tau_2 = \sigma_0 + \sigma_1 - \tau_0 - \tau_1. \tag{13.15}$$

Set

$$V_j = (V_j, h_j, \nabla_j) := \big(T_{\tau_j}^{\sigma_j}(M, W_j), (\cdot, \cdot)_{\sigma_j}^{\tau_j} \otimes h_{W_j}, \nabla(\nabla_g, D_j) \big)$$

and suppose $\mathsf{m} : V_0 \oplus V_1 \to V_2$ is a bundle multiplication, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_2 := \lambda_0 + \lambda_1$, and $\vec{\omega}_j := (\lambda_j, \mu)$. Then

(i) $\mathsf{m} \in \mathcal{L}\big(\mathcal{B}^{s/\vec{r}, \vec{\omega}_0}(J, V_0), \mathcal{B}^{s/\vec{r}, \vec{\omega}_1}(J, V_1); \mathcal{B}^{s/\vec{r}, \vec{\omega}_2}(J, V_2) \big)$, *where either $s \in r\mathbb{N}$ and $\mathcal{B} \in \{BC, bc\}$, or $s > 0$ and $\mathcal{B} \in \{B_\infty, b_\infty\}$.*

(ii) $\mathsf{m} \in \mathcal{L}\bigl(BC^{s/\vec{r},\vec{\omega}_0}(J,V_0), W_p^{s/\vec{r},\vec{\omega}_1}(J,V_1); W_p^{s/\vec{r},\vec{\omega}_2}(J,V_2)\bigr)$, $s \in r\mathbb{N}$.

(iii) $\mathsf{m} \in \mathcal{L}\bigl(B_\infty^{s_0/\vec{r},\vec{\omega}_0}(J,V_0), \mathfrak{F}_p^{s/\vec{r},\vec{\omega}_1}(J,V_1); \mathfrak{F}_p^{s/\vec{r},\vec{\omega}_2}(J,V_2)\bigr)$, $0 < s < s_0$.

Proof. (1) Suppose

$$M = \mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}, \quad g = g_m, \quad \rho \sim \mathbf{1}, \quad W_j = \bigl(M \times F_j, (\cdot,\cdot)_{F_j}, d_{W_j}\bigr),$$

where d_{W_j} is the F_j-valued differential. Set

$$E_j := \bigl(E_{\tau_j}^{\sigma_j}(F_j), (\cdot|\cdot)_{HS}\bigr), \quad V_j = \bigl(\mathbb{X} \times E_j, (\cdot,\cdot)_j, d_{E_j}\bigr),$$

where $(\cdot|\cdot)_j := (\cdot|\cdot)_{E_j}$.

Introducing bases, we define isomorphisms $E_j \simeq \mathbb{K}^{N_j}$. By means of them m is transported onto an element of $\mathcal{L}(\mathbb{K}^{N_0}, \mathbb{K}^{N_1}; \mathbb{K}^{N_2})^{\mathbb{X}}$ which has the 'matrix representation'

$$\mathbb{K}^{N_0} \times \mathbb{K}^{N_1} \ni (\xi,\eta) \mapsto (\mathsf{m}_{\nu_0\nu_1}^{\nu_2}(x)\xi^{\nu_0}\eta^{\nu_1})_{1 \le \nu_2 \le N_2} \in \mathbb{K}^{N_2}.$$

Assume $\mathsf{m} \in BC^\infty\bigl(\mathbb{X}, \mathcal{L}(E_0,E_1;E_2)\bigr)$. Then the assertion follows from Theorem 13.2.

(2) Now we consider the general case. We choose uniformly regular atlases $\mathfrak{K} \ltimes \Phi_j$ for W_j over $\mathfrak{K}$ with model fiber F_j. Given $\kappa \ltimes \varphi_j \in \mathfrak{K} \ltimes \Phi_j$ we define, recalling (5.3), $\mathsf{m}_\kappa \in \mathcal{D}\bigl(\mathbb{X}_\kappa, \mathcal{L}(E_0,E_1;E_2)\bigr)$ by

$$\mathsf{m}_\kappa(\eta_0,\eta_1) := (\kappa \ltimes \varphi_2)_*\bigl(\chi_\kappa \mathsf{m}\bigl((\kappa \ltimes \varphi_0)^*\eta_0, (\kappa \ltimes \varphi_1)^*\eta_1\bigr)\bigr)$$

for $\eta_j \in E_j^{\mathbb{X}_\kappa}$. It follows from (5.11) and the fact that m is a bundle multiplication that

$$|\mathsf{m}_\kappa(\eta_0,\eta_1)|_2 \le c\, \rho_\kappa^{\tau_2-\sigma_2} \rho_\kappa^{\sigma_0-\tau_0} \rho_\kappa^{\sigma_1-\tau_1}\, |\eta_0|_0\, |\eta_1|_1, \qquad \eta_j \in E_j^{\mathbb{X}_\kappa}.$$

Hence we infer from (13.15)

$$\mathsf{m}_\kappa \in BC^k\bigl(\mathbb{X}_\kappa, \mathcal{L}(E_0,E_1;E_2)\bigr), \quad \|\mathsf{m}_\kappa\|_{k,\infty} \le c(k), \qquad \kappa \ltimes \varphi_j \in \mathfrak{K} \ltimes \Phi_j, \quad k \in \mathbb{N}.$$

(3) In the following, it is understood that $\varphi_q^{\vec{\omega}_j}$ is defined by means of $\mathfrak{K} \ltimes \Phi_j$ for $1 \le q \le \infty$. Then, given $v_j \in \Gamma(M,V_j)^J$,

$$\varphi_{q,\kappa}^{\vec{\omega}_2}\bigl(\mathsf{m}(v_0,v_1)\bigr) = \rho_\kappa^{\lambda_0} \rho_\kappa^{\lambda_1+m/q} \Theta_{q,\kappa}^\mu (\kappa \ltimes \varphi_2)_*\bigl(\pi_\kappa \mathsf{m}(v_0,v_1)\bigr) = \mathsf{m}_\kappa(\varphi_{\infty,\kappa}^{\vec{\omega}_0} v_0, \widehat{\varphi}_{q,\kappa}^{\vec{\omega}_1} v_1).$$

Consequently, we get from (13.11)

$$\varphi_{q,\kappa}^{\vec{\omega}_2}\bigl(\mathsf{m}(v_0,v_1)\bigr) = \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} a_{\widetilde{\kappa}\kappa} \mathsf{m}_\kappa(\varphi_{\infty,\kappa}^{\vec{\omega}_0} v_0, R_{\widetilde{\kappa}\kappa} \varphi_{q,\kappa}^{\vec{\omega}_1} v_1). \tag{13.16}$$

(4) Suppose either $s \in \mathbb{N}$ and $\mathcal{B} \in \{BC, bc\}$, or $s > 0$ and $\mathcal{B} \in \{B_\infty, b_\infty\}$. Then we infer from (13.13), Lemma 13.3, Theorem 13.2, and steps (1) and (3) that

$$\|a_{\widetilde{\kappa}\kappa} \mathsf{m}_\kappa(\eta_0,\eta_1)\|_{\mathcal{B}^{s/\vec{r}}(\mathbb{Y}_\kappa,E_2)} \le c\, \|\eta_0\|_{\mathcal{B}^{s/\vec{r}}(\mathbb{Y}_\kappa,E_0)}\, \|\eta_1\|_{\mathcal{B}^{s/\vec{r}}(\mathbb{Y}_\kappa,E_1)},$$

uniformly with respect to $\kappa \ltimes \varphi$, $\widetilde{\kappa} \ltimes \widetilde{\varphi} \in \mathfrak{K} \ltimes \Phi_2$. Hence we get from (13.16) and the finite multiplicity of $\mathfrak{K}$

$$\bigl\|\varphi_\infty^{\vec{\omega}_2}\bigl(\mathsf{m}(v_0,v_1)\bigr)\bigr\|_{\ell_\infty(\boldsymbol{\mathcal{B}}^{s/\vec{r}}(\mathbb{Y},E_2))} \le c \|v_0\|_{\ell_\infty(\boldsymbol{\mathcal{B}}^{s/\vec{r}}(\mathbb{Y},E_0))}\, \|v_1\|_{\ell_\infty(\boldsymbol{\mathcal{B}}^{s/\vec{r}}(\mathbb{Y},E_1))}. \tag{13.17}$$

Thus Theorems 12.6 and 12.8 imply, due to (7.8),

$$\|\mathsf{m}(v_0,v_1)\|_{\mathcal{B}^{s/\vec{r},\vec{\omega}_2}(J,V_2)} \le c\, \|v_0\|_{\mathcal{B}^{s/\vec{r},\vec{\omega}_0}(J,V_0)}\, \|v_1\|_{\mathcal{B}^{s/\vec{r},\vec{\omega}_1}(J,V_1)},$$

provided either $s \in r\mathbb{N}$ and $\mathcal{B} = BC$, or $s > 0$ and $\mathcal{B} = B_\infty$.

If $s \in r\mathbb{N}$ and $\mathcal{B} = bc$, or $s > 0$ and $\mathcal{B} = b_\infty$, then (13.17) holds with ℓ_∞ replaced by $\ell_{\infty,\mathrm{unif}}$ everywhere. Thanks to Theorems 12.7(i) and 12.10(i) this proves assertion (i). The proofs for (ii) and (iii) are similar. □

It is clear that obvious analogues of the results of this section hold in the case of time-independent isotropic spaces. This generalizes and improves [5, Theorem 9.2].

14 Contractions

In practice, most pointwise multiplications in tensor bundles occur through contractions of tensor fields. For this reason we specialize in this section the general multiplier Theorem 13.5 to this setting and study the problem of right invertibility of multiplier operators induced by contraction.

Let $V_i = (V_i, h_i),\ i = 1, 2$, be uniformly regular metric vector bundles of rank n_i over M with model fiber E_i. Set $V_0 = (V_0, h_0) := \big(\mathrm{Hom}(V_1, V_2), h_{12}\big)$. By Example 3.1(f), V_0 is a uniformly regular vector bundle of rank $n_1 n_2$ over M with model fiber $\mathcal{L}(E_1, E_2)$. The *evaluation map*

$$\mathrm{ev}:\ \Gamma(M, V_0 \otimes V_1) \to \Gamma(M, V_2), \quad (a, v) \mapsto av$$

is defined by $av(p) := a(p)v(p)$ for $p \in M$.

Lemma 14.1 *The evaluation map is a bundle multiplication.*

P r o o f. We fix uniformly regular atlases $\mathfrak{K}\ltimes\Phi_i,\ i = 1, 2$, for V_i over $\mathfrak{K}$. Then, using the notation of Section 2, it follows from (2.15)

$$(a^* a)^{\nu_1}_{\widetilde{\nu}_1} = h_1^{*\nu_1 \widehat{\nu}_1}\ \overline{a^{\widehat{\nu}_2}_{\widehat{\nu}_1}}\ \overline{h_{2,\widehat{\nu}_2 \widetilde{\nu}_2}}\, a^{\widetilde{\nu}_2}_{\widetilde{\nu}_1}.$$

Hence we infer from (3.5)

$$\kappa_*(|a|^2_{h_0}) = \kappa_*\big(\mathfrak{tr}(a^* a)\big) = \kappa_* h_1^{*\nu_1 \widehat{\nu}_1}\ \kappa_* a^{\widetilde{\nu}_2}_{\nu_1}\ \kappa_* h_{2,\widetilde{\nu}_2 \widehat{\nu}_2}\ \overline{\kappa_* a^{\widehat{\nu}_2}_{\widehat{\nu}_1}} \sim \sum_{\nu_1,\nu_2} |\kappa_* a^{\nu_2}_{\nu_1}|^2 = \mathfrak{tr}\big([\kappa_* a]^* [\kappa_* a]\big),$$

uniformly with respect to $\kappa \in \mathfrak{K}$. Furthermore, (2.12) and (3.5) imply

$$\begin{aligned} \kappa_*(|au|_{h_2}) &= \big|\big((\kappa\ltimes\varphi_{12})_* a\big)(\kappa\ltimes\varphi_1)_* u\big|_{(\kappa\ltimes\varphi_2)_* h_2} \\ &\sim \big|\big((\kappa\ltimes\varphi_{12})_* a\big)(\kappa\ltimes\varphi_1)_* u\big|_{E_2} \le |(\kappa\ltimes\varphi_{12})_* a|_{\mathcal{L}(E_1,E_2)}\, |(\kappa\ltimes\varphi_1)_* u|_{E_1} \end{aligned} \tag{14.1}$$

for $u \in \Gamma(M, V_1)$ and $\kappa\ltimes\varphi_i \in \mathfrak{K}\ltimes\Phi_i$. Since $\mathcal{L}(E_1, E_2)$ is finite-dimensional the operator norm $|\cdot|_{\mathcal{L}(E_1,E_2)}$ is equivalent to the trace norm. Hence, using $\mathcal{L}(E_1, E_2) \simeq \mathbb{K}^{n_2 \times n_1}$ and (2.12) and (3.5) once more, we deduce from (14.1) that $\kappa_*(|au|_{h_2}) \le c\kappa_*(|a|_{h_0})\kappa_*(|u|_{h_1})$ for $\kappa \in \mathfrak{K}$. Consequently,

$$|au|_{h_2} \le c\, |a|_{h_0}\, |u|_{h_1}, \qquad (a, u) \in \Gamma(M, V_0 \oplus V_1).$$

This proves the lemma. □

Suppose $\sigma, \sigma_i, \tau, \tau_i \in \mathbb{N}$ for $i = 1, 2$ with $\sigma + \tau > 0$. We define the *center contraction* of order $\sigma + \tau$,

$$\mathsf{C} = \mathsf{C}^{[\sigma]}_{[\tau]}:\ \Gamma(M, T^{\sigma_2+\tau}_{\tau_2+\sigma} M \oplus T^{\sigma+\sigma_1}_{\tau+\tau_1} M) \to \Gamma(M, T^{\sigma_1+\sigma_2}_{\tau_1+\tau_2} M), \tag{14.2}$$

as follows: Given $(i_k) \in \mathbb{J}_{\sigma_k},\ (j_k) \in \mathbb{J}_{\tau_k}$ for $k = 1, 2$, and $\sigma \in \mathbb{J}_\sigma,\ \tau \in \mathbb{J}_\tau$ we set

$$(i_2; j) := (i_{2,1}, \dots, i_{2,\sigma_2}, j_1, \dots, j_\tau) \in \mathbb{J}_{\sigma_2+\tau}$$

etc. Assume $a \in \Gamma(M, T^{\sigma_2+\tau}_{\tau_2+\sigma} M)$ is locally represented on U_κ by

$$a = a^{(i_2;j)}_{(j_2;i)}\, \frac{\partial}{\partial x^{(i_2)}} \otimes \frac{\partial}{\partial x^{(j)}} \otimes dx^{(j_2)} \otimes dx^{(i)}$$

and b has a corresponding representation. Then the local representation of $\mathsf{C}(a, b)$ on U_κ is given by

$$a^{(i_2;j)}_{(j_2;i)}\, b^{(i;i_1)}_{(j;j_1)}\, \frac{\partial}{\partial x^{(i_2)}} \otimes \frac{\partial}{\partial x^{(i_1)}} \otimes dx^{(j_2)} \otimes dx^{(j_1)}.$$

A center contraction (14.2) is a *complete contraction* (on the right) if $\sigma_1 = \tau_1 = 0$. If C is a complete contraction, then we usually simply write $a \cdot u$ for $\mathsf{C}(a, u)$.

Lemma 14.2 *The* center contraction *associated with the evaluation map* ev*,*

$$\mathsf{C}\otimes \mathrm{ev}:\ \Gamma\big(M,T^{\sigma_2+\tau}_{\tau_2+\sigma}(M,V_0)\oplus T^{\sigma+\sigma_1}_{\tau+\tau_1}(M,V_1)\big)\to \Gamma\big(M,T^{\sigma_1+\sigma_2}_{\tau_1+\tau_2}(M,V_2)\big),$$

is a bundle multiplication.

P r o o f. Note that C is a composition of $\sigma+\tau$ simple contractions of type C^i_j. Hence the assertion follows from Lemma 14.1 and Examples 13.4(c) and (d). □

Henceforth, we write again C for $\mathsf{C}\otimes\mathrm{ev}$, if no confusion seems likely. Furthermore, we use the same symbol for point-wise extensions to time-dependent tensor fields. In addition, we do not indicate notationally the tensor bundles on which C is operating. This will always be clear from the context.

Throughout the rest of this section we presuppose

- $W_i=(W_i,h_i,D_i)$, $i=1,2,3,$ are fully uniformly regular vector bundles of rank n_i over M with model fiber F_i.

For $i,j\in\{1,2,3\}$ we set

$$W_{ij}=(W_{ij},h_{W_{ij}},D_{ij}):=\big(\mathrm{Hom}(W_i,W_j),(\cdot|\cdot)_{HS},\nabla(D_i,D_j)\big).$$

Example 3.1(f) guarantees that W_{ij} is a fully uniformly regular vector bundle over M.

We also assume for $i,j\in\{1,2,3\}$

- $\sigma_i,\tau_i,\sigma_{ij},\tau_{ij}\in\mathbb{N}$;
- $V_i=(V_i,h_i,\nabla_i):=\big(T^{\sigma_i}_{\tau_i}(M,W_i),(\cdot|\cdot)^{\tau_i}_{\sigma_i}\otimes h_{W_i},\nabla(\nabla_g,D_i)\big)$;
- $V_{ij}=(V_{ij},h_{ij},\nabla_{ij}):=\big(T^{\sigma_{ij}}_{\tau_{ij}}(M,W_{ij}),(\cdot|\cdot)^{\tau_{ij}}_{\sigma_{ij}}\otimes h_{W_{ij}},\nabla(\nabla_g,D_{ij})\big)$;
- $\lambda_i,\lambda_{ij}\in\mathbb{R}$, $\vec{\omega}_i=(\lambda_i,\mu)$, $\vec{\omega}_{ij}=(\lambda_{ij},\mu)$.

Due to Lemma 14.2 we can apply Theorem 13.5 and its corollary with $\mathsf{m}=\mathsf{C}$. For simplicity and for their importance in the theory of differential and pseudodifferential operators, *we restrict ourselves in the following to complete contractions*. It should be observed that condition (14.4) below is void if $\partial M=\emptyset$ and $J=\mathbb{R}$.

Theorem 14.3

(i) *Suppose*

$$\lambda_2=\lambda_{12}+\lambda_1,\quad \sigma_2=\sigma_{12}-\tau_1,\quad \tau_2=\tau_{12}-\sigma_1, \tag{14.3}$$

and

$$s>\begin{cases} -1+1/p & \text{if } \partial M\neq\emptyset,\\ r(-1+1/p) & \text{if } \partial M=\emptyset \text{ and } J=\mathbb{R}^+.\end{cases} \tag{14.4}$$

Let one of the following additional conditions be satisfied:

(α) $s=t\in r\mathbb{N}$, $q:=\infty$, $\mathcal{B}=\mathfrak{G}\in\{BC,bc\}$;
(β) $s=t\in r\mathbb{N}$, $q:=p$, $\mathcal{B}=BC$, $\mathfrak{G}=W$;
(γ) $s=t>0$, $q:=\infty$, $\mathcal{B}=\mathfrak{G}\in\{B_\infty,bc_\infty\}$;
(δ) $|s|<t$, $q:=p$, $\mathcal{B}=B_\infty$, $\mathfrak{G}=\mathfrak{F}$.

Assume $a\in\mathcal{B}^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$. *Then*

$$A:=(u\mapsto a\cdot u)\in\mathcal{L}\big(\mathfrak{G}^{s/\vec{r},\vec{\omega}_1}_q(J,V_1),\mathfrak{G}^{s/\vec{r},\vec{\omega}_2}_q(J,V_2)\big),$$

where $BC_\infty:=BC$ *and* $bc_\infty:=bc$ *if* (α) *applies. The map* $a\mapsto A$ *is linear and continuous.*

(ii) *Assume, in addition,*

$$\lambda_3=\lambda_{23}+\lambda_2,\quad \sigma_3=\sigma_{23}-\tau_2,\quad \tau_3=\tau_{23}-\sigma_2$$

and $b\in\mathcal{B}^{t/\vec{r},\vec{\omega}_{23}}(J,V_{23})$. *Set* $B:=(v\mapsto b\cdot v)$. *Then*

$$BA=\Big(u\mapsto \mathsf{C}^{[\sigma_2]}_{[\tau_2]}(b,a)\cdot u\Big)\in\mathcal{L}\big(\mathfrak{G}^{s/\vec{r},\vec{\omega}_1}_q(J,V_1),\mathfrak{G}^{s/\vec{r},\vec{\omega}_3}_q(J,V_3)\big).$$

Proof. (1) Suppose $s\ge 0$ with $s>0$ if $\mathfrak{F}=B$. Then, due to Lemma 14.2, assertion (i) is immediate from Theorem 13.5.

(2) Choose uniformly regular atlases $\mathfrak{K}\ltimes\Phi_i,\ i=1,2$, for W_i over $\mathfrak{K}$. Let

$$a=a^{(i_{12}),\nu_2}_{(j_{12}),\nu_1}(t)\,\frac{\partial}{\partial x^{(i_{12})}}\otimes dx^{(j_{12})}\otimes b^2_{\nu_2}\otimes\beta^{\nu_1}_1,\qquad t\in J,\tag{14.5}$$

be the local representation of a in the local coordinate frame for V_{12} over U_κ associated with $\kappa\ltimes\varphi_{12}\in\mathfrak{K}\ltimes\Phi_{12}$, where $(b^i_1,\dots,b^i_{n_i})$ is the local coordinate frame for W_i over U_κ associated with $\kappa\ltimes\Phi_i$, and $(\beta^1_i,\dots,\beta^{n_i}_i)$ is its dual frame (cf. Example 2.1(b) and (5.5)). Write $(i_{k\ell})=(i_\ell;j_k)\in\mathbb{J}_{\sigma_{k\ell}}$ and $(j_{k\ell})=(j_\ell;i_k)\in\mathbb{J}_{\tau_{k\ell}}$ for $k,\ell\in\{1,2\}$ with $k\ne\ell$, where $(i_k)\in\mathbb{J}_{\sigma_k}$ and $(j_k)\in\mathbb{J}_{\tau_k}$.

We define $a'\in\Gamma\big(M,T^{\sigma_{12}}_{\tau_{12}}\big(M,\mathrm{Hom}(W_2',W_1')\big)\big)^J$ by

$${a'}^{(j_1;i_2),\nu_2}_{(i_1;j_2),\nu_1}(t)\,\frac{\partial}{\partial x^{(j_1)}}\otimes\frac{\partial}{\partial x^{(i_2)}}\otimes dx^{(i_1)}\otimes dx^{(j_2)}\otimes\beta^{\nu_1}_1\otimes b^2_{\nu_2},\qquad t\in J,$$

where ${a'}^{(j_1;i_2),\nu_2}_{(i_1;j_2),\nu_1}:=a^{(i_2;j_1),\nu_2}_{(j_2;i_1),\nu_1}$.

It is obvious that

$$a'\in\mathcal{B}^{t/\vec{r},\vec{\omega}_{12}}\big(J,T^{\sigma_{12}}_{\tau_{12}}\big(M,\mathrm{Hom}(W_2',W_1')\big)\big),\tag{14.6}$$

the map $a\mapsto a'$ is linear and continuous, and $(a')'=a$. Furthermore, since $V_i'=T^{\tau_i}_{\sigma_i}(M,W_i')$,

$$\langle v,a\cdot u\rangle_{V_2}=\langle a'\cdot v,u\rangle_{V_1},\qquad (v,u)\in\Gamma(M,V_2'\oplus V_1)^J.\tag{14.7}$$

(3) Suppose condition (δ) is satisfied and $s<0$. It follows from (14.3), step (1), and (14.6)

$$\mathsf{C}(a'):=(v\mapsto a'\cdot v)\in\mathcal{L}\big(\mathfrak{F}^{-s/\vec{r},-\vec{\omega}_2}_{p'}(J,V_2'),\mathfrak{F}^{-s/\vec{r},-\vec{\omega}_1}_{p'}(J,V_1')\big).\tag{14.8}$$

From Theorem 8.3(ii) and assumption (14.4) we infer

$$\mathfrak{F}^{-s/\vec{r},-\vec{\omega}_i}_{p'}(J,V_i')=\mathring{\mathfrak{F}}^{-s/\vec{r},-\vec{\omega}_i}_{p'}(J,V_i'),\qquad i=1,2.$$

Thus we deduce from (8.5), (14.7), and (14.8) that $\mathsf{C}(a')=\mathsf{C}(a)'$. Hence, using $(a')'=a$, we get the remaining part of assertion (i), provided $s\ne 0$ if $\mathfrak{F}=B$. Now this gap is closed by interpolation.

(4) It is clear that $\mathsf{C}(b)\mathsf{C}(a):\ T^{\sigma_1}_{\tau_1}(M,V_1)\to T^{\sigma_3}_{\tau_3}(M,V_3)$ is given by

$$v\mapsto\mathsf{C}\big(b,\mathsf{C}(a,v)\big)=\mathsf{C}\Big(\mathsf{C}^{[\sigma_2]}_{[\tau_2]}(b,a),v\Big)=\Big(v\mapsto\mathsf{C}^{[\sigma_2]}_{[\tau_2]}(b,a)v\Big).\tag{14.9}$$

Set $\mathsf{m}=\mathsf{C}^{[\sigma_2]}_{[\tau_2]}$ in Theorem 13.5. Also set $V_0:=W_{23},\ \ V_1:=W_{12}$, and $V_2:=W_{13}$ in Lemma 14.2. Then it follows from that lemma and Theorem 13.5 that

$$\mathsf{C}^{[\sigma_2]}_{[\tau_2]}(b,a)\in\mathcal{B}^{t/\vec{r},(\lambda_3-\lambda_1,\mu)}\big(J,T^{\sigma_3+\tau_1}_{\tau_3+\sigma_1}(M,W_{13})\big).$$

Thus claim (ii) is a consequence of (14.9) and assertion (i). □

Next we study the invertibility of the linear map A. We introduce the following definition: Suppose $t > 0$ and

$$a \in B_\infty^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11}), \quad \sigma_{11} = \tau_{11} = \sigma_1 + \tau_1. \tag{14.10}$$

Then a ia said to be λ_{11}-*uniformly contraction invertible* if there exists $a^{-1} \in \Gamma(M, V_{11})^J$ satisfying

$$a^{-1} \cdot (a \cdot u) = u, \quad a \cdot (a^{-1} \cdot u) = u, \qquad u \in \Gamma(M, V_1)^J, \tag{14.11}$$

and

$$\rho^{-\lambda_{11}} \, |a^{-1}(t)|_{h_{11}} \le c, \qquad t \in J. \tag{14.12}$$

Note that the second part of (14.10) guarantees that the complete contractions in (14.11) are well-defined. Also note that there exists at most one a^{-1} satisfying (14.11), the *contraction inverse* of a. For abbreviation, we put

$$B_{\infty,\mathrm{inv}}^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11}) := \big\{ \, a \in B_\infty^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11}) \; ; \; a \text{ is } \lambda_{11}\text{-uniformly contraction invertible} \big\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let U be open in $\mathcal{X}$. Then $f : U \to \mathcal{Y}$ is *analytic* if each $x_0 \in U$ has a neighborhood in which f can be represented by a convergent series of continuous monomials. If f is analytic, then f is smooth and it can be locally represented by its Taylor series. If $\mathbb{K} = \mathbb{C}$, and f is (Fréchet) differentiable, then it is analytic. For this and further details we refer to E. Hille and R.S. Phillips [19].

To simplify the presentation we restrict ourselves now to the most important cases in which $\mathcal{B} = B_\infty$. We leave it to the reader to carry out the obvious modifications in the following considerations needed to cover the remaining instances as well.

Proposition 14.4 *Suppose* $\sigma_{11} = \tau_{11} = \sigma_1 + \tau_1$. *Then* $B_{\infty,\mathrm{inv}}^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11})$ *is open in* $B_\infty^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11})$. *If* $a \in B_{\infty,\mathrm{inv}}^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11})$, *then* $a^{-1} \in B_{\infty,\mathrm{inv}}^{t/\vec{r},(-\lambda_{11},\mu)}(J, V_{11})$. *The map*

$$B_{\infty,\mathrm{inv}}^{t/\vec{r},\vec{\omega}_{11}}(J, V_{11}) \to B_\infty^{t/\vec{r},(-\lambda_{11},\mu)}(J, V_{11}), \quad a \mapsto a^{-1}$$

is analytic.

P r o o f. (1) Without loss of generality we let $F_1 = \mathbb{K}^n$ and set $\sigma := \sigma_{11}$. Note that $E := \mathcal{L}(\mathbb{K}^n)^{m^\sigma \times m^\sigma}$ is a Banach algebra with unit of dimension $N^2 := (nm^\sigma)^2$. It is obvious that we can fix an algebra isomorphism from E onto $\mathbb{K}^{N \times N}$ by which we identify E with $\mathbb{K}^{N \times N}$.

For $b \in \mathbb{K}^{N \times N}$ we denote by $b^\natural$ the $(N \times N)$-matrix of cofactors of b. Thus $b^\natural = [b^\natural_{ij}]$ with

$$b^\natural_{ij} := \det[b_1, \ldots, b_{i-1}, e_j, b_{i+1}, \ldots, b_N], \tag{14.13}$$

where $b_1, \ldots, b_N$ are the columns of b and e_j is the j-th standard basis vector of $\mathbb{K}^N$. Then, if b is invertible,

$$b^{-1} = \big(\det(b)\big)^{-1} b^\natural. \tag{14.14}$$

(2) Suppose either $X := (Q^m, g_m)$ or $X := (Q^m \cap \mathbb{H}^m, g_m)$, and $Y = X \times J$. Set

$$\mathcal{X}^{t/\vec{r}}(Y, E) := B\big(J, B_\infty^t(X, E)\big) \cap B_\infty^{t/r}\big(J, B(X, E)\big), \tag{14.15}$$

where $B_\infty^t(X, E)$ is obtained from $B_\infty^t(\mathbb{R}^m, E)$ by restriction, of course. Note

$$\mathcal{X}^{t/\vec{r}}(Y, E) \hookrightarrow B_\infty(Y, E). \tag{14.16}$$

It follows from Theorem 13.5 that $\mathcal{X}^{t/\vec{r}}(Y, E)$ is a Banach algebra with respect to the point-wise extension of the (matrix) product of E.

Assume $b \in \mathcal{X}^{t/\vec{r}}(Y, E)$ and $b(y)$ is invertible for $y \in Y$ such that

$$|b^{-1}(y)|_E \le c_0, \qquad y \in Y. \tag{14.17}$$

Then the spectrum $\sigma\big(b(y)\big)$ of $b(y)$ is bounded and has a positive distance from $0\in\mathbb{C}$, uniformly with respect to $y\in Y$. Hence

$$1/c(c_0)\le \big|\det\big(b(y)\big)\big|\le c(c_0),\qquad y\in Y, \tag{14.18}$$

due to the fact that $\det\big(b(y)\big)$ can be represented as the product of the eigenvalues of $b(y)$, counted with multiplicities.

Since $\det\big(b(y)\big)$ is a polynomial in the entries of $b(y)$ and $\mathcal{X}^{t/\vec r}(Y):=\mathcal{X}^{t/\vec r}(Y,\mathbb{K})$ is a multiplication algebra we infer

$$\det(b)\in \mathcal{X}^{t/\vec r}(Y). \tag{14.19}$$

Using the chain rule if $t\ge 1$ (cf. Lemma 1.4.2 of [4]), we get $\big(\det(b)\big)^{-1}\in\mathcal{X}^{t/\vec r}(Y)$ from (14.18) and (14.19). Now we deduce from (14.13), (14.14), and the fact that $\mathcal{X}^{t/\vec r}(Y)$ is a multiplication algebra, that

$$b^{-1}\in\mathcal{X}^{t/\vec r}(Y,E),\quad \|b^{-1}\|_{\mathcal{X}^{t/\vec r}(Y,E)}\le c(c_0),$$

whenever $b\in\mathcal{X}^{t/\vec r}(Y,E)$ satisfies (14.17).

By (14.16) it is obvious that the set of all invertible elements of $\mathcal{X}^{t/\vec r}(Y,E)$ satisfying (14.17) for some $c_0=c_0(b)\ge 1$ is open in $\mathcal{X}^{t/\vec r}(Y,E)$.

(3) Assume $\mathfrak{K}\ltimes\Phi_1$ is a uniformly regular atlas for W_1 over $\mathfrak{K}$. Given $\kappa\ltimes\varphi_1\in\mathfrak{K}\ltimes\Phi_1$, put

$$\chi_\kappa^{\vec\omega_1}v:=\rho_\kappa^{\lambda_1}\Theta_{\infty,\kappa}^{\mu}(\kappa\ltimes\varphi_1)_*v,\quad \chi_\kappa^{\vec\omega_{11}}a:=\rho_\kappa^{\lambda_{11}}\Theta_{\infty,\kappa}^{\mu}(\kappa\ltimes\varphi_{11})_*a$$

for $v\in\Gamma(M,V_1)^J$ and $a\in\Gamma(M,V_{11})^J$, respectively, and $Y_\kappa:=Q_\kappa^m\times J$.

Suppose $a\in B_{\infty,\mathrm{inv}}^{t/\vec r,\vec\omega_{11}}(J,V_{11})$. Then we deduce from (14.11) (see Example 3.1(f)) and

$$\chi_\kappa^{\vec\omega_1}v=\chi_\kappa^{\vec\omega_1}\big(a^{-1}\cdot(a\cdot v)\big)=(\chi_\kappa^{(-\lambda_{11},\mu)}a^{-1})(\chi_\kappa^{\vec\omega_{11}}a)\chi_\kappa^{\vec\omega_1}v \tag{14.20}$$

for $\kappa\ltimes\varphi_1\in\mathfrak{K}\ltimes\Phi_1$ and $v\in\Gamma(U_\kappa,V_1)^J$. Note that $\chi_\kappa^{\vec\omega_1}$ is a bijection from $\Gamma(U_\kappa,V_1)^J$ onto $(E_{\tau_1}^{\sigma_1})^{Y_\kappa}$. Thus it follows from (14.20) that $\chi_\kappa^{(-\lambda_{11},\mu)}a^{-1}$ is a left inverse for $\chi_\kappa^{\vec\omega_{11}}a$ in $B_\infty(Y_\kappa,E)$. Similarly, we see that it is also a right inverse. Hence $b_\kappa:=\chi_\kappa^{\vec\omega_{11}}a$ is invertible in $B_\infty(Y_\kappa,E)$ and

$$b_\kappa^{-1}=\chi_\kappa^{(-\lambda_{11},\mu)}a^{-1}. \tag{14.21}$$

We infer from (4.1)(iv), (5.11), (3.5), (3.10), (14.10), and (14.12) that

$$|b_\kappa^{-1}|_E\le c\,\Theta_{\infty,\kappa}^{\mu}\kappa_*(\rho^{-\lambda_{11}}\,|a^{-1}|_{h_{11}})\le c,\qquad \kappa\ltimes\varphi_1\in\mathfrak{K}\ltimes\Phi_1. \tag{14.22}$$

Recalling (4.3), (7.10), and (13.10) we find

$$b_\kappa=\chi_\kappa^{\vec\omega_{11}}\Big(\sum_{\widetilde\kappa\in\mathfrak{N}(\kappa)}\pi_{\widetilde\kappa}^2a\Big)=\sum_{\widetilde\kappa\in\mathfrak{N}(\kappa)}S_{\widetilde\kappa\kappa}(\widetilde\kappa_*\pi_{\widetilde\kappa})R_{\widetilde\kappa\kappa}\varphi_{\infty,\widetilde\kappa}^{\vec\omega_{11}}a. \tag{14.23}$$

Since $a\in B_\infty^{t/\vec r,\vec\omega_{11}}(J,V_{11})$ implies $\varphi_\infty^{\vec\omega_{11}}a\in\ell_\infty(\boldsymbol{B}_\infty^{t/\vec r})$ we deduce from (14.23), (7.3)(iii), Lemmas 12.5 and 13.3, Theorem 13.5, and definition (14.15)

$$\|b_\kappa\|_{\mathcal{X}^{t/\vec r}(Y_\kappa,E)}\le c\,\|a\|_{t/\vec r,\infty;\vec\omega_{11}},\qquad \kappa\ltimes\varphi_1\in\mathfrak{K}\ltimes\Phi_1. \tag{14.24}$$

Set $a_\kappa:=\rho_\kappa^{-\lambda_{11}}b_\kappa$. Then it follows from (14.22) and (14.24) that

$$\rho_\kappa^{-\lambda_{11}}a_\kappa^{-1}\in\mathcal{X}^{t/\vec r}(Y_\kappa,E),\quad \|\rho_\kappa^{-\lambda_{11}}a_\kappa^{-1}\|_{\mathcal{X}^{t/\vec r}(Y_\kappa,E)}\le c,\qquad \kappa\ltimes\varphi_1\in\mathfrak{K}\ltimes\Phi_1. \tag{14.25}$$

Employing (7.3)(iii) and Theorem 13.5 once more we derive from (14.25)

$$\varphi_{\infty,\kappa}^{(-\lambda_{11},\mu)}a^{-1}=\chi_\kappa^{(-\lambda_{11},\mu)}(\pi_\kappa a^{-1})=(\kappa_*\pi_\kappa)b_\kappa^{-1}\in B_\infty^{t/\vec r}(Y_\kappa,E)=B_{\infty,\kappa}^{t/\vec r} \tag{14.26}$$

and $\varphi_\infty^{(-\lambda_{11},\mu)} a^{-1} \in \ell_\infty(\boldsymbol{B}_\infty^{t/\vec r})$. Hence Theorem 12.8 implies

$$a^{-1} = \psi_\infty^{(-\lambda_{11},\mu)}(\varphi_\infty^{(-\lambda_{11},\mu)} a^{-1}) \in B_\infty^{t/\vec r,(-\lambda_{11},\mu)}(J, V_{11}). \tag{14.27}$$

(4) Let $\mathcal{X}$ be a Banach algebra with unit e. Denote by $\mathcal{G}$ the group of invertible elements of $\mathcal{X}$. For $b_0 \in \mathcal{X}$ and $\delta > 0$ let $\mathcal{X}(b_0, \delta)$ be the open ball in $\mathcal{X}$ of radius δ, centered at b_0. Suppose $b_0 \in \mathcal{G}$. Then $b = b_0 - (b_0 - b) = \big(e - (b_0 - b)b_0^{-1}\big)b_0$ and

$$\|(b_0 - b)b_0^{-1}\| \le \|b_0 - b\|\,\|b_0^{-1}\| < 1/2, \qquad b \in \mathcal{X}(b_0, \|b_0^{-1}\|/2),$$

imply that $b \in \mathcal{X}(b_0, \|b_0^{-1}\|/2)$ is invertible and

$$b^{-1} = b_0^{-1}\big(e - (b_0 - b)b_0^{-1}\big)^{-1} = b_0^{-1} \sum_{i=0}^{\infty} \big((b_0 - b)b_0^{-1}\big)^i. \tag{14.28}$$

In fact, this Neumann series has the convergent majorant $\sum_i 2^{-i}$. Note that $p_i(x) := (-1)^i b_0^{-1}(x b_0^{-1})^i$ is a continuous homogenous polynomial in $x \in \mathcal{X}$. Hence it follows from (14.28)

$$b^{-1} = \sum_{i=0}^{\infty} p_i(b - b_0), \qquad b \in \mathcal{X}(b_0, \|b_0^{-1}\|/2),$$

and this series converges uniformly on $\mathcal{X}(b_0, \|b_0^{-1}\|/2)$. Thus $\mathcal{G}$ is open and the inversion map $\mathrm{inv} : \mathcal{G} \to \mathcal{X}$, $b \mapsto b^{-1}$ is analytic.

(5) We set $\mathcal{X} := B\big(J, B(M, V_{11})\big)$ and define a multiplication by $(a, b) \mapsto \mathsf{C}_{[\tau_1]}^{[\sigma_1]}(a, b)$. Then $\mathcal{X}$ is a Banach algebra with unit $e := \big((p, t) \mapsto \mathrm{id}_{\mathcal{L}((V_1)_p)}\big)$.

Consider the continuous linear map

$$f : B_\infty^{t/\vec r,\vec\omega_{11}}(J, V_{11}) \to \mathcal{X}, \quad a \mapsto \rho^{\lambda_{11}} a.$$

Then $G := f^{-1}(\mathcal{G})$ is open in $B_\infty^{t/\vec r,\vec\omega_{11}}(J, V_{11})$. Consequently,

$$f_0 := \mathrm{inv} \circ (f\,|\,G) : G \to \mathcal{X}, \quad a \mapsto (\rho^{\lambda_{11}} a)^{-1}$$

is continuous (in fact, analytic) by step (4). Note that $a^{-1} = \rho^{\lambda_{11}} f_0(a)$ is the contraction inverse of a. Furthermore, $f_0(a) \in \mathcal{X}$ implies

$$\rho^{-\lambda_{11}}\, |a^{-1}(t)|_{h_{11}} = |f_0(a)(t)|_{h_{11}} \le c, \qquad t \in J.$$

Hence each $a \in G$ is λ_{11}-uniformly contraction invertible. Conversely, if $a \in B_\infty^{t/\vec r,\vec\omega_{11}}(J, V_{11})$ is λ_{11}-uniformly contraction invertible, then a belongs to G. Thus $G = B_{\infty,\mathrm{inv}}^{t/\vec r,\vec\omega_{11}}(J, V_{11})$ which shows that $B_{\infty,\mathrm{inv}}^{t/\vec r,\vec\omega_{11}}(J, V_{11})$ is open.

(6) We denote by $\mathcal{G}_\kappa$ the group of invertible elements of $B_{\infty,\kappa}^{t/\vec r}$. Suppose $a_0 \in G$. Then step (3) (see (14.24) and (14.25)) guarantees that $b_{0,\kappa} := \chi_\kappa^{\vec\omega_{11}} a_0 \in \mathcal{G}_\kappa$ and

$$\|b_{0,\kappa}\|_{B_{\infty,\kappa}^{t/\vec r}} + \|b_{0,\kappa}^{-1}\|_{B_{\infty,\kappa}^{t/\vec r}} \le c, \qquad \kappa \ltimes \varphi_1 \in \mathfrak{K} \ltimes \Phi_1.$$

Hence we infer from step (4) that there exists $\delta > 0$ such that the open ball $B_{\infty,\kappa}^{t/\vec r}(b_{0,\kappa}, \delta)$ belongs to $\mathcal{G}_\kappa$ for $\kappa \in \mathfrak{K}$ and the inversion map $\mathrm{inv}_\kappa : \mathcal{G}_\kappa \to B_{\infty,\kappa}^{t/\vec r}$ is analytic on $B_{\infty,\kappa}^{t/\vec r}(b_{0,\kappa}, \delta)$, uniformly with respect to $\kappa \in \mathfrak{K}$ in the sense that the series

$$\sum_i b_{0,\kappa}^{-1}\big((b_{0,\kappa} - b_\kappa)b_{0,\kappa}^{-1}\big)^i$$

converges in $B_{\infty,\kappa}^{t/\vec r}$, uniformly with respect to $b_\kappa \in B_{\infty,\kappa}^{t/\vec r}(b_{0,\kappa}, \delta)$ and $\kappa \in \mathfrak{K}$.

Note that

$$\boldsymbol{B}_\infty^{t/\vec r}(\boldsymbol{b}_0,\delta) := \prod_\kappa B_{\infty,\kappa}^{t/\vec r}(b_{0,\kappa},\delta)$$

is open in $\ell_\infty(\boldsymbol{B}_\infty^{t/\vec r})$. The above considerations show that

$$\mathbf{inv} : \boldsymbol{B}_\infty^{t/\vec r}(\boldsymbol{b}_0,\delta) \to \ell_\infty(\boldsymbol{B}_\infty^{t/\vec r}), \quad \boldsymbol{b} \mapsto \bigl(\mathrm{inv}_\kappa(b_\kappa)\bigr) \tag{14.29}$$

is analytic. It follows from (14.24) that the linear map

$$\chi^{\vec\omega_{11}} : B_\infty^{t/\vec r,\vec\omega_{11}}(J,V_{11}) \to \ell_\infty(\boldsymbol{B}_\infty^{t/\vec r}), \quad v \mapsto (\chi_\kappa^{\vec\omega_{11}} v) \tag{14.30}$$

is continuous. Hence $G_0 := (\chi^{\vec\omega_{11}})^{-1}\bigl(\boldsymbol{B}_\infty^{t/\vec r}(\boldsymbol{b}_0,\delta)\bigr) \cap G$ is an open neighborhood of a_0 in G. It is a consequence of (14.29) and (14.30) that $\mathbf{inv} \circ \chi^{\vec\omega_{11}}$ is an analytic map from G_0 into $\ell_\infty(\boldsymbol{B}_\infty^{t/\vec r})$.

Consider the point-wise multiplication operator

$$\boldsymbol{\pi} : \ell_\infty(\boldsymbol{B}_\infty^{t/\vec r}) \to \ell_\infty(\boldsymbol{B}_\infty^{t/\vec r}), \quad \boldsymbol{b} \mapsto \bigl((\kappa_*\pi_\kappa)b_\kappa\bigr).$$

It follows from (7.3) and Theorem 13.5 that it is a well-defined continuous linear map.

If $a \in G_0$, then we know from (14.21) and (14.24) that

$$\mathbf{inv} \circ \chi^{\vec\omega_{11}}(a) = \bigl(\chi_\kappa^{(-\lambda_{11},\mu)} a^{-1}\bigr) \in \ell_\infty(\boldsymbol{B}_\infty^{t/\vec r}).$$

Hence we see by (14.26) and (14.27) that $a^{-1} = \psi_\infty^{(-\lambda_{11},\mu)} \circ \boldsymbol{\pi} \circ \mathbf{inv} \circ \chi^{\vec\omega_{11}} a$. Thus

$$(a \mapsto a^{-1}) = \psi_\infty^{(-\lambda_{11},\mu)} \circ \boldsymbol{\pi} \circ \mathbf{inv} \circ \chi^{\vec\omega_{11}} : G_0 \to B_\infty^{t/\vec r,(-\lambda_{11},\mu)}(J,V_{11})$$

is analytic, being a composition of analytic maps. This proves the proposition. □

Henceforth, we set $\mathfrak{F}_\infty := B_\infty$ so that $\mathfrak{F}_q$ is defined for $1 < q \le \infty$.

Theorem 14.5 *Suppose* $1 < q \le \infty$ *and*

$$t > 0 \text{ and } s \text{ satisfies (14.4) with } |s| < t \text{ if } q = p, \text{ and } s = t \text{ if } q = \infty.$$

Assume $\sigma_{11} = \tau_{11} = \sigma_1 + \tau_1$ *and* $\lambda_2 = \lambda_{11} + \lambda_1$. *If* $a \in B_{\infty,\mathrm{inv}}^{t/\vec r,\vec\omega_{11}}(J,V_{11})$, *then*

$$A = \mathsf{C}(a) \in \mathcal{L}\mathrm{is}\bigl(\mathfrak{F}_q^{s/\vec r,\vec\omega_1}(J,V_1), \mathfrak{F}_q^{s/\vec r,\vec\omega_2}(J,V_1)\bigr) \tag{14.31}$$

and $A^{-1} = \mathsf{C}(a^{-1})$. *The map* $a \mapsto A^{-1}$ *is analytic.*

P r o o f. It follows from Theorem 14.3(i) and Proposition 14.4 that (14.31) applies and $a \mapsto \mathsf{C}(a^{-1})$ is analytic. Part (ii) of that theorem implies $A^{-1} = \mathsf{C}(a^{-1})$. □

Next we study the problem of the right invertibility of the operator A of Theorem 14.3. This is of particular importance in connection with boundary value problems. First we need some preparation.

We assume

$$\sigma_{12} = \sigma_2 + \tau_1, \quad \tau_{12} = \tau_2 + \sigma_1, \quad \sigma_{21} = \tau_{12}, \quad \tau_{21} = \sigma_{12}. \tag{14.32}$$

Then, given $a \in \Gamma(M,V_{12})^J$, there exists a unique $a^* \in \Gamma(M,V_{21})^J$, the *complete contraction adjoint of* a, such that

$$h_2(a \cdot u, v) = h_1(u, a^* \cdot v), \qquad (u,v) \in \Gamma(M, V_1 \oplus V_2)^J. \tag{14.33}$$

Indeed, recalling (14.5) set

$$(a^*)^{(i_{21}),\nu_1}_{(j_{21}),\nu_2} := g^{(i_1)(\widetilde\imath_1)}_{(j_1)(\widetilde\jmath_1)}\, h^{*\nu_1\widetilde\nu_1}_{W_1}\, \overline{a^{(\widetilde\imath_2;\widetilde\jmath_1),\widetilde\nu_2}_{(\widetilde\jmath_2;\widetilde\imath_1),\widetilde\nu_1}}\, g^{(\widetilde\jmath_2)(j_2)}_{(\widetilde\imath_2)(i_2)}\, \overline{h_{W_2,\widetilde\nu_2\nu_2}}. \tag{14.34}$$

Then it follows from (2.15) and $h_{k\ell} = (\cdot|\cdot)_{\sigma_{k\ell}}^{\tau_{k\ell}} \otimes h_{W_{k\ell}}$ that

$$(a^*)_{(j_{21}),\nu_2}^{(i_{21}),\nu_1} \frac{\partial}{\partial x^{(i_{21})}} \otimes dx^{(j_{21})} \otimes b_{\nu_1}^1 \otimes \beta_2^{\nu_2} \tag{14.35}$$

is the local representation of a^* over U_κ with respect to the coordinate frame for V_{21} over U_κ associated with $\kappa \ltimes \varphi_{21}$,

We set

$$\lambda_{21}^* := \lambda_{12} + \sigma_{21} - \tau_{21}, \quad \vec{\omega}_{21}^* := (\lambda_{21}^*, \mu) \tag{14.36}$$

and suppose $a \in B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12})$. Then it is a consequence of (3.5), (5.9), (5.10), (14.34), and (14.35) that

$$\|\varphi_{\infty,\kappa}^{\vec{\omega}_{21}^*} a^*\|_{B_\infty^{t/\vec{r}}(\mathbb{Y}_\kappa, E_{\tau_{21}}^{\sigma_{21}})} \sim \|\varphi_{\infty,\kappa}^{\vec{\omega}_{12}} a\|_{B_\infty^{t/\vec{r}}(\mathbb{Y}_\kappa, E_{\tau_{12}}^{\sigma_{12}})}, \qquad \kappa \ltimes \varphi_i \in \mathfrak{K} \ltimes \Phi_i, \quad i = 1, 2.$$

From this, Theorem 12.8, (7.8), and (14.34) we infer

$$(a \mapsto a^*) \in \mathcal{L}\big(B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12}), B_\infty^{t/\vec{r},\vec{\omega}_{21}^*}(J, V_{21})\big). \tag{14.37}$$

Assume $a^*(p,t) \in \mathcal{L}\big((V_2)_p, (V_1)_p\big)$ is injective for $(p,t) \in M \times J$. Then $a(p,t) \in \mathcal{L}\big((V_1)_p, (V_2)_p\big)$ is surjective. This motivates the following definition:

$$\begin{aligned} &a \in B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12}) \text{ is } \lambda_{12}\textit{-uniformly contraction surjective} \text{ if} \\ &\rho^{\lambda_{12}+(\tau_{12}-\sigma_{12})/2} \, |a^*(t) \cdot u|_{h_1} \geq |u|_{h_2}/c, \qquad u \in \Gamma(M, V_2), \quad t \in J. \end{aligned} \tag{14.38}$$

The reason for the specific choice of the exponent of ρ will become apparent below. We set

$$B_{\infty,\mathrm{surj}}^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12}) := \big\{\, a \in B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12}) \;;\; a \text{ is } \lambda_{12}\text{-uniformly contraction surjective} \,\big\}.$$

For abbreviation, we put

$$a \odot a^* := \mathsf{C}_{[\tau_1]}^{[\sigma_1]}(a, a^*), \quad \sigma_{22} := \tau_{22} := \sigma_2 + \tau_2, \quad \lambda_{22} := 2\lambda_{12} + \tau_{12} - \sigma_{12}.$$

It follows from (14.37) and Theorem 13.5 that

$$B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12}) \to B_\infty^{t/\vec{r},\vec{\omega}_{22}}(J, V_{22}), \quad a \mapsto a \odot a^* \tag{14.39}$$

is a well-defined continuous quadratic map. Hence it is analytic.

Lemma 14.6 $a \in B_{\infty,\mathrm{surj}}^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12})$ *iff* $a \odot a^* \in B_{\infty,\mathrm{inv}}^{t/\vec{r},\vec{\omega}_{22}}(J, V_{22})$.

P r o o f. It follows from (14.33) that

$$h_2\big((a \odot a^*) \cdot u, v\big) = h_2\big(a \cdot (a^* \cdot u), v\big) = h_1(a^* \cdot u, a^* \cdot v), \qquad (u,v) \in \Gamma(M, V_2 \oplus V_2)^J. \tag{14.40}$$

Hence $\mathsf{C}(a \odot a^*)$ is symmetric and positive semi-definite. We see from (14.40) that (14.38) is equivalent to

$$\rho^{\lambda_{22}} h_2\big((a \odot a^*)(t) \cdot u, u\big) \geq |u|_{h_2}^2/c, \qquad u \in \Gamma(M, V_2), \quad t \in J.$$

By symmetry this inequality is equivalent to the λ_{22}-uniform contraction invertibility of $a \odot a^*$. □

In the next proposition we give a local criterion for checking λ_{12}-uniform surjectivity.

Proposition 14.7 *Suppose* $a \in B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J, V_{12})$. *Let* $\mathfrak{K} \ltimes \Phi_i$, $i = 1, 2$, *be uniformly regular atlases for* V_i *over* $\mathfrak{K}$. *Set*

$$\mathsf{a}_\kappa(t)(\zeta, \zeta) := \sum_{\substack{(i_1) \in \mathbb{J}_{\sigma_1},\, (j_1) \in \mathbb{J}_{\tau_1} \\ 1 \leq \nu_1 \leq n_1}} \Big| \kappa_* a_{(j_2;i_1),\nu_1}^{(i_2;j_1),\nu_2}(t) \, \zeta_{(i_2),\nu_2}^{(j_2)} \Big|^2$$

for $\zeta \in E_{\sigma_2}^{\tau_2}(F_2^*)^{Q_\kappa^m}$ *and* $t \in J$. *Then* a *is* λ_{12}*-uniformly contraction surjective iff*

$$\rho_\kappa^{2\lambda_{12}} \mathsf{a}_\kappa(t)(\zeta, \zeta) \sim |\zeta|^2, \qquad \zeta \in E_{\sigma_2}^{\tau_2}(F_2^*)^{Q_\kappa^m}, \quad \kappa \in \mathfrak{K}, \quad t \in J.$$

P r o o f. Assume $v\in\Gamma(M,V_2)^J$ and put $w:=(h_2)_\flat v\in\Gamma(M,V_2')^J$, where $V_2'=T_{\sigma_2}^{\tau_2}(M,W_2')$. Then, locally on U_κ,

$$v=v_{(j_2)}^{(i_2),\nu_2}\,\frac{\partial}{\partial x^{(i_2)}}\otimes dx^{(j_2)}\otimes b_{\nu_2}^2,\qquad w=w_{(i_2),\nu_2}^{(j_2)}\,\frac{\partial}{\partial x^{(j_2)}}\otimes dx^{(i_2)}\otimes\beta_2^{\nu_2},$$

where

$$w_{(i_2),\nu_2}^{(j_2)}=g_{(i_2)(\widetilde{\imath}_2)}^{(j_2)(\widetilde{\jmath}_2)}\,h_{W_2,\nu_2\widetilde{\nu}_2}\,\overline{v_{(\widetilde{\jmath}_2)}^{(\widetilde{\imath}_2),\widetilde{\nu}_2}},$$

due to $h_2=(\cdot|\cdot)_{\sigma_2}^{\tau_2}\otimes h_{W_2}$. Thus it follows from (14.34) that, locally on U_κ,

$$\bigl((a\odot a^*)\cdot v\bigr)_{(j_2)}^{(i_2),\nu_2}=a_{(j_2;i_1),\nu_1}^{(i_2;j_1),\nu_2}\,g_{(j_1)(\widetilde{\jmath}_1)}^{(i_1)(\widetilde{\imath}_1)}\,h_{W_1}^{*\nu_1\widetilde{\nu}_1}\,\overline{a_{(\widetilde{\jmath}_2;\widetilde{\imath}_1),\widetilde{\nu}_1}^{(\widetilde{\imath}_2;\widetilde{\jmath}_1),\widetilde{\nu}_2}}\,\overline{w_{(\widetilde{\imath}_2),\widetilde{\nu}_2}^{(\widetilde{\jmath}_2)}}.$$

Hence

$$\begin{aligned}h_2\bigl((a\odot a^*)\cdot v,v\bigr)&=g_{(i_2)(\widehat{\imath}_2)}^{(j_2)(\widehat{\jmath}_2)}\,h_{W_2,\nu_2\widehat{\nu}_2}\bigl((a\odot a^*)\cdot v\bigr)_{(j_2)}^{(i_2),\nu_2}\,\overline{v_{(\widehat{\jmath}_2)}^{(\widehat{\imath}_2),\widehat{\nu}_2}}\\&=\bigl((a\odot a^*)\cdot v\bigr)_{(j_2)}^{(i_2),\nu_2}\,w_{(i_2),\nu_2}^{(j_2)}\\&=a_{(j_2;i_1),\nu_1}^{(i_2;j_1),\nu_2}\,w_{(i_2),\nu_2}^{(j_2)}\,g_{(j_1)(\widetilde{\jmath}_1)}^{(i_1)(\widetilde{\imath}_1)}\,h_{W_1}^{*\nu_1\widetilde{\nu}_1}\,\overline{a_{(\widetilde{\jmath}_2;\widetilde{\imath}_1),\widetilde{\nu}_1}^{(\widetilde{\imath}_2;\widetilde{\jmath}_1),\widetilde{\nu}_2}}\,\overline{w_{(\widetilde{\imath}_2),\widetilde{\nu}_2}^{(\widetilde{\jmath}_2)}}.\end{aligned}$$

Thus we deduce from (3.5), (4.3), (5.8), (5.9) (applied to W_i)

$$\kappa_*\bigl(\rho^{\lambda_{22}}h_2\bigl((a\odot a^*)\cdot v,v\bigr)\bigr)\sim\rho_\kappa^{\lambda_{22}+2(\tau_1-\sigma_1)}\mathsf{a}_\kappa(\zeta,\zeta)\tag{14.41}$$

for $\kappa\ltimes\varphi_i\in\mathfrak{K}\ltimes\Phi_i$, $i=1,2$, and $v\in\Gamma(M,V_2)^J$, where

$$\zeta:=(\kappa\ltimes\varphi_2)_*\bigl((h_2)_\flat v\bigr)\in\bigl(E_{\sigma_2}^{\tau_2}(F_2^*)\bigr)^{Q_\kappa^m\times J}.\tag{14.42}$$

Since $(h_2)_\flat$ is an isometry and h_2^* is the bundle metric of V_2^* we get from (5.11)

$$\kappa_*(|v|_{h_2}^2)=\kappa_*(|w|_{h_2^*}^2)\sim\rho_\kappa^{2(\tau_2-\sigma_2)}\,|\zeta|_{E_{\sigma_2}^{\tau_2}(F_2^*)}^2,\qquad\kappa\in\mathfrak{K},\tag{14.43}$$

with v and ζ being related by (14.42). Now the assertion follows from (14.32), (14.36), (14.41), and (14.43). □

Suppose $a\in B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$ and $a^c\in B_\infty^{t/\vec{r},(-\lambda_{12},\mu)}(J,V_{21})$ are such that $a\cdot(a^c\cdot v)=v$ for v belonging to $\Gamma(M,V_2)^J$. Then a^c is a *right contraction inverse of* a.

Proposition 14.8 *Let conditions* (14.32) *be satisfied. Then* $B_{\infty,\mathrm{surj}}^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$ *is open in* $B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$ *and there exists an analytic map*

$$I^c:\ B_{\infty,\mathrm{surj}}^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})\to B_\infty^{t/\vec{r},(-\lambda_{12},\mu)}(J,V_{21})$$

such that $I^c(a)$ *is a right contraction inverse for* a.

P r o o f. It follows from (14.39), Proposition 14.4, and Lemma 14.6 that $S:=B_{\infty,\mathrm{surj}}^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$ is open in $B_\infty^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$. Set

$$I^c(a):=\mathsf{C}_{[\tau_2]}^{[\sigma_2]}\bigl(a^*,(a\odot a^*)^{-1}\bigr),\qquad a\in S,$$

where $(a\odot a^*)^{-1}$ is the contraction inverse of $a\odot a^*\in B_\infty^{t/\vec{r},\vec{\omega}_{22}}(J,V_{22})$. Then (14.37), (14.39), and Theorem 13.5 imply that I^c is an analytic map from S into $B_\infty^{t/\vec{r},(-\lambda_{12},\mu)}(J,V_{21})$. Since

$$a\cdot\bigl(I^c(a)\cdot v\bigr)=a\cdot\bigl(a^*\cdot\bigl((a\odot a^*)^{-1}\cdot v\bigr)\bigr)=(a\odot a^*)\cdot\bigl((a\odot a^*)^{-1}\cdot v\bigr)=v,\qquad v\in\Gamma(M,V_2),$$

the assertion follows. □

After these preparations it is easy to prove the second main theorem of this section. For this it should be noted that definition (14.38) applies equally well if $a\in\mathcal{B}^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$ where either $\mathcal{B}=b_\infty$, or $t\in r\mathbb{N}$ and $\mathcal{B}\in\{BC,bc\}$. Hence $\mathcal{B}_{\mathrm{surj}}^{t/\vec{r},\vec{\omega}_{12}}(J,V_{12})$ is defined in these cases also.

Theorem 14.9 *Let assumptions* (14.3) *and* (14.4) *be satisfied and* $1 < q \le \infty$.

(i) *Assume* $|s| < t$ *if* $q = p$, *and* $s = t > 0$ *if* $q = \infty$. *Then there exists an analytic map*

$$A^c : B^{t/\vec{r},\vec{\omega}_{12}}_{\infty,\mathrm{surj}}(J, V_{12}) \to \mathcal{L}\big(\mathfrak{F}^{s/\vec{r},\vec{\omega}_2}_q(J, V_2), \mathfrak{F}^{s/\vec{r},\vec{\omega}_1}_q(J, V_1)\big)$$

such that $A^c(a)$ *is a right inverse for* $A(a) = (v \mapsto a \cdot v)$.

(ii) *There exists an analytic map*

$$A^c : \mathcal{B}^{s/\vec{r},\vec{\omega}_{12}}_{\mathrm{surj}}(J, V_{12}) \to \mathcal{L}\big(\mathcal{B}^{s/\vec{r},\vec{\omega}_2}(J, V_2), \mathcal{B}^{s/\vec{r},\vec{\omega}_1}(J, V_1)\big)$$

such that $A^c(a)$ *is a right inverse for* $A(a)$ *if either* $s \in r\mathbb{N}$ *and* $\mathcal{B} \in \{BC, bc\}$, *or* $s > 0$ *and* $\mathcal{B} = b_\infty$.

Proof. The first assertion is an obvious consequence of Theorem 14.3 and Proposition 14.8. The second claim is obtained by modifying the above arguments in the apparent way. □

As in the preceding section, the above results possess obvious analogues applying in the isotropic case.

15 Embeddings

Now we complement the embedding theorems of Section 8 by establishing further inclusions between anisotropic weighted spaces.

Theorem 15.1 *Suppose* $\lambda_0 < \lambda_1$ *and put* $\vec{\omega}_i := (\lambda_i, \mu)$ *for* $i = 0, 1$. *Then* $\mathfrak{F}^{s/\vec{r},\vec{\omega}_0}_p \overset{d}{\hookrightarrow} \mathfrak{F}^{s/\vec{r},\vec{\omega}_1}_p$ *if* $\rho \le 1$, *whereas* $\rho \ge 1$ *implies* $\mathfrak{F}^{s/\vec{r},\vec{\omega}_1}_p \overset{d}{\hookrightarrow} \mathfrak{F}^{s/\vec{r},\vec{\omega}_0}_p$ *for* $s \in \mathbb{R}$.

Similarly, $\mathcal{B}^{s/\vec{r},\vec{\omega}_0} \hookrightarrow \mathcal{B}^{s/\vec{r},\vec{\omega}_1}$ *if* $\rho \le 1$, *and* $\mathcal{B}^{s/\vec{r},\vec{\omega}_1} \hookrightarrow \mathcal{B}^{s/\vec{r},\vec{\omega}_0}$ *for* $\rho \ge 1$, *if either* $s > 0$ *and* $\mathcal{B} \in \{B_\infty, b_\infty\}$, *or* $s \in r\mathbb{N}$ *and* $\mathcal{B} \in \{BC, bc\}$.

Proof. If $\rho \le 1$, then it is obvious that

$$W^{kr/\vec{r},\vec{\omega}_0}_p \overset{d}{\hookrightarrow} W^{kr/\vec{r},\vec{\omega}_1}_p, \quad \mathring{W}^{kr/\vec{r},\vec{\omega}_0}_p \overset{d}{\hookrightarrow} \mathring{W}^{kr/\vec{r},\vec{\omega}_1}_p, \quad BC^{kr/\vec{r},\vec{\omega}_0} \hookrightarrow BC^{kr/\vec{r},\vec{\omega}_1}$$

for $k \in \mathbb{N}$. Thus, by duality,

$$W^{kr/\vec{r},\vec{\omega}_0}_p \overset{d}{\hookrightarrow} W^{kr/\vec{r},\vec{\omega}_1}_p, \qquad k \in -\mathbb{N}^\times.$$

By interpolation $\mathfrak{F}^{s/\vec{r},\vec{\omega}_0}_p \overset{d}{\hookrightarrow} \mathfrak{F}^{s/\vec{r},\vec{\omega}_1}_p$ follows. The proof of the other embeddings is similar. □

The next theorem contains Sobolev-type embedding results. In the anisotropic case they involve the weight exponents as well as the regularity parameters.

Theorem 15.2

(i) *Suppose* $s_0 < s_1$ *and* $p_0, p_1 \in (1, \infty)$ *satisfy*

$$s_1 - (m + r)/p_1 = s_0 - (m + r)/p_0. \tag{15.1}$$

Set $\vec{\omega}_0 := \big(\lambda + (m + \mu)(1/p_1 - 1/p_0), \mu\big)$. *Then* $\mathfrak{F}^{s_1/\vec{r},\vec{\omega}}_{p_1} \overset{d}{\hookrightarrow} \mathfrak{F}^{s_0/\vec{r},\vec{\omega}_0}_{p_0}$.

(ii) *Assume* $t > 0$ *and* $s \ge t + (m + r)/p$. *Set* $\vec{\omega}_\infty := \big(\lambda + (m + \mu)/p, \mu\big)$. *Then* $\mathfrak{F}^{s/\vec{r},\vec{\omega}}_p \hookrightarrow b^{t/\vec{r},\vec{\omega}_\infty}_\infty$.

P r o o f. (1) Note that $s_1 > s_0$ and (15.1) imply $p_0 > p_1$. Hence it follows from (15.1) and Theorems 3.3.2, 3.7.5, and 4.4.1 of [4] that

$$\ell_{p_1}(\mathfrak{F}_{p_1}^{s_1/\vec{r}}) \overset{d}{\hookrightarrow} \ell_{p_1}(\mathfrak{F}_{p_0}^{s_0/\vec{r}}) \overset{d}{\hookrightarrow} \ell_{p_0}(\mathfrak{F}_{p_0}^{s_0/\vec{r}}).$$

Also note that we get $\psi_{p_1}^{\vec{\omega}} = \psi_{p_0}^{\vec{\omega}_0}$ from (15.1). Thus we infer from Theorem 9.3 that

$$\begin{array}{ccc}
\mathfrak{F}_{p_1}^{s_1/\vec{r},\vec{\omega}} & \xrightarrow{\ \varphi_{p_1}^{\vec{\omega}}\ } & \ell_{p_1}(\mathfrak{F}_{p_1}^{s_1/\vec{r}}) \\
\big\downarrow & & \big\downarrow d \\
\mathfrak{F}_{p_0}^{s_0/\vec{r},\vec{\omega}_0} & \xleftarrow{\ \psi_{p_0}^{\vec{\omega}_0}\ } & \ell_{p_0}(\mathfrak{F}_{p_0}^{s_0/\vec{r}})
\end{array}$$

is commuting. From this we obtain assertion (i).

(2) We infer from Lemma 9.2 and [4, Theorem 3.3.2] that

$$B_{p,\kappa}^{s/\vec{r}} = B_{p,\kappa}^{s/\boldsymbol{\nu}} \hookrightarrow B_{p,\infty,\kappa}^{s/\boldsymbol{\nu}} \hookrightarrow B_{\infty,\infty,\kappa}^{t/\boldsymbol{\nu}} = B_{\infty,\kappa}^{t/\vec{r}}$$

and from [4, Theorem 3.7.1] that $H_{p,\kappa}^{s/\vec{r}} \hookrightarrow B_{p,\infty,\kappa}^{s/\boldsymbol{\nu}}$. Consequently, $\mathfrak{F}_{p,\kappa}^{s/\vec{r}} \hookrightarrow B_{\infty,\kappa}^{t/\vec{r}}$. From this and the density of $\mathcal{D}(\mathbb{Y}_\kappa, E)$ in $\mathfrak{F}_{p,\kappa}^{s/\vec{r}}$ it follows, due to $\mathcal{D}(\mathbb{Y}_\kappa, E) \hookrightarrow b_{\infty,\kappa}^{t/\vec{r}}$, that $\mathfrak{F}_{p,\kappa}^{s/\vec{r}} \hookrightarrow b_{\infty,\kappa}^{t/\vec{r}}$. Thus, by (7.1),

$$\ell_p(\mathfrak{F}_p^{s/\vec{r}}) \hookrightarrow \ell_\infty(\boldsymbol{b}_\infty^{t/\vec{r}}). \tag{15.2}$$

It is obvious that $\boldsymbol{\mathcal{D}}(\mathbb{Y}, E) \hookrightarrow \ell_{\infty,\mathrm{unif}}(\boldsymbol{b}_\infty^{t/\vec{r}})$. By Theorem 9.3 we know that $\boldsymbol{\mathcal{D}}(\mathbb{Y}, E)$ is dense in $\ell_p(\mathfrak{F}_p^{s/\vec{r}})$. From this and (15.2) we deduce $\ell_p(\mathfrak{F}_p^{s/\vec{r}}) \hookrightarrow \ell_{\infty,\mathrm{unif}}(\boldsymbol{b}_\infty^{t/\vec{r}})$. Observing $\psi_p^{\vec{\omega}} = \psi_\infty^{\vec{\omega}_\infty}$, we infer from Theorems 9.3 and 12.10 that the diagram

$$\begin{array}{ccc}
\mathfrak{F}_p^{s/\vec{r},\vec{\omega}} & \xrightarrow{\ \varphi_p^{\vec{\omega}}\ } & \ell_p(\mathfrak{F}_p^{s/\vec{r}}) \\
\big\downarrow & & \big\downarrow \\
b_\infty^{t/\vec{r},\vec{\omega}_\infty} & \xleftarrow{\ \psi_\infty^{\vec{\omega}_\infty}\ } & \ell_{\infty,\mathrm{unif}}(\boldsymbol{b}_\infty^{t/\vec{r}})
\end{array}$$

is commuting. This proves (ii). □

Remark 15.3 Define the *anisotropic small Hölder space* $C_0^{s/\vec{r},\vec{\omega}} = C_0^{s/\vec{r},\vec{\omega}}(J, V)$ to be the closure of $\mathcal{D}(J, \mathcal{D})$ in $B_\infty^{s/\vec{r},\vec{\omega}}$ for $s > 0$. Then the above proof shows $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}} \hookrightarrow C_0^{t/\vec{r},\vec{\omega}_\infty}$ if the hypotheses of (ii) are satisfied. □

16 Differential Operators

First we establish the mapping properties of ∇ and ∂ in anisotropic weighted Bessel potential and Besov spaces. They are, of course, of fundamental importance for the theory of differential equations.

Theorem 16.1 *Suppose either $s \geq 0$ and $\mathfrak{G} = \mathfrak{F}_p$, or $s > 0$ and $\mathfrak{G} \in \{B_\infty, b_\infty\}$. Then*

$$\nabla \in \mathcal{L}\big(\mathfrak{G}^{s+1,\lambda}, \mathfrak{G}^{s,\lambda}(V_{\tau+1}^\sigma)\big) \cap \mathcal{L}\big(\mathfrak{G}^{(s+1)/\vec{r},\vec{\omega}}, \mathfrak{G}^{s/\vec{r},\vec{\omega}}(J, V_{\tau+1}^\sigma)\big)$$

and $\partial \in \mathcal{L}\big(\mathfrak{G}^{(s+r)/\vec{r},\vec{\omega}}, \mathfrak{G}^{s/\vec{r},(\lambda+\mu,\mu)}\big)$.

P r o o f. We consider the time-dependent case. The proof in the stationary setting is similar.

(1) From (5.15) and (5.16) we know that

$$(\kappa \ltimes \varphi)_* \nabla v = \partial_x v + a_\kappa v, \qquad v \in C\big(J, C^1(\mathbb{X}_\kappa, E)\big),$$

where $a_\kappa \in C^\infty\big(Q_\kappa^m, \mathcal{L}(E_\tau^\sigma, E_{\tau+1}^\sigma)\big)$ satisfies $\|a_\kappa\| \le c(k)$ for $\kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi$. Hence it follows from Theorem 13.1 and $\mathfrak{G}_\kappa^{(s+1)/\vec{r}} \hookrightarrow \mathfrak{G}_\kappa^{s/\vec{r}}$ that

$$A_\kappa := (v \mapsto a_\kappa \chi v) \in \mathcal{L}\big(\mathfrak{G}_\kappa^{(s+1)/\vec{r}}, \mathfrak{G}^{s/\vec{r}}(\mathbb{Y}_\kappa, E_{\tau+1}^\sigma)\big), \qquad \|A_\kappa\| \le c, \quad \kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi.$$

By [4, Theorem 4.4.2] and Theorems 11.6 and 11.9 we get

$$\partial_x \in \mathcal{L}\big(\mathfrak{G}_\kappa^{(s+1)/\vec{r}}, \mathfrak{G}^{s/\vec{r}}(\mathbb{Y}_\kappa, E_{\tau+1}^\sigma)\big), \quad \partial \in \mathcal{L}(\mathfrak{G}_\kappa^{(s+r)/\vec{r}}, \mathfrak{G}_\kappa^{s/\vec{r}}). \tag{16.1}$$

(2) Set $q := p$ if $\mathfrak{G} = \mathfrak{F}_p$, and $q := \infty$ otherwise. Then, given $u \in \mathfrak{G}_\kappa^{(s+1)/\vec{r},\vec{\omega}}$,

$$\varphi_{q,\kappa}^{\vec{\omega}}(\nabla u) = \rho_\kappa^{\lambda+m/q} \Theta_{q,\kappa}^\mu (\kappa\ltimes\varphi)_*(\pi_\kappa \nabla u) = (\kappa_* \pi_\kappa)\big((\kappa\ltimes\varphi)_*\nabla\big)(\widehat{\varphi}_{q,\kappa}^{\vec{\omega}} u).$$

Hence we get from (13.11)

$$\varphi_{q,\kappa}^{\vec{\omega}}(\nabla u) = \sum_{\widetilde{\kappa}\in\mathfrak{K}} b_{\widetilde{\kappa}\kappa} (\kappa\ltimes\varphi)_* \nabla(R_{\widetilde{\kappa}\kappa} \varphi_{q,\widetilde{\kappa}}^{\vec{\omega}} u),$$

where $b_{\widetilde{\kappa}\kappa} = (\kappa_*\pi_\kappa) a_{\widetilde{\kappa}\kappa}$ and $a_{\widetilde{\kappa}\kappa}$ is defined by (13.12). From this, (7.3), (13.13), Lemma 13.3, step (1), Theorem 13.5, and the finite multiplicity of $\mathfrak{K}$ we infer

$$\|\varphi_q^{\vec{\omega}}(\nabla u)\|_{\ell_q(\boldsymbol{\mathfrak{G}}^{s/\vec{r}}(\mathbb{Y}, E_{\tau+1}^\sigma))} \le c\,\|\varphi_q^{\vec{\omega}} u\|_{\ell_q(\boldsymbol{\mathfrak{G}}^{(s+1)/\vec{r}})}$$

for $u \in \mathfrak{G}^{(s+1)/\vec{r},\vec{\omega}}$. Using Theorems 9.3, 12.8, and 12.10 we thus obtain

$$\|\varphi_q^{\vec{\omega}}(\nabla u)\|_{\ell_q(\boldsymbol{\mathfrak{G}}^{s/\vec{r}}(\mathbb{Y}, E_{\tau+1}^\sigma))} \le c\,\|u\|_{\mathfrak{G}^{(s+1)/\vec{r},\vec{\omega}}}, \qquad u \in \mathfrak{G}^{(s+1)/\vec{r},\vec{\omega}}.$$

Thus the first assertion follows from $\nabla u = \psi_q^{\vec{\omega}}\big(\varphi_q^{\vec{\omega}}(\nabla u)\big)$ by invoking these theorems once more.

(3) Since (see (9.5))

$$\varphi_{q,\kappa}^{(\lambda+\mu,\mu)}(\partial u) = \rho_\kappa^\mu \varphi_{q,\kappa}^{\vec{\omega}} \partial u = \partial(\varphi_{q,\kappa}^{\vec{\omega}} u),$$

the second assertion is implied by the second part of (16.1) and the arguments of step (2). □

By combining this result with Theorem 14.3 and embedding theorems of the preceding section we can derive mapping properties of differential operators. To be more precise, for $k \in \mathbb{N}^\times$ we consider operators of the form

$$\mathcal{A} = \sum_{i+jr\le kr} a_{ij} \cdot \nabla^i \partial^j$$

where a_{ij} are suitably regular time-dependent vector-bundle-valued tensor field homomorphisms and $a_{ij} \cdot \nabla^i \partial^j$ equals $\big(u \mapsto a_{ij} \cdot (\nabla^i \partial^j u)\big)$, of course. Recall that $\mathfrak{F}_\infty = B_\infty$.

Theorem 16.2 *Let $\bar{W} = (\bar{W}, h_{\bar{W}}, D_{\bar{W}})$ be a fully uniformly regular vector bundle over M. Suppose k, $\bar{\sigma}$, $\bar{\tau}$ belong to $\mathbb{N}$ and $\bar{\lambda} \in \mathbb{R}$. For $0 \le i \le k$ set*

$$\sigma_i := \bar{\sigma} + \tau + i, \quad \tau_i := \bar{\tau} + \sigma, \quad \vec{\bar{\omega}} := (\bar{\lambda}, \mu).$$

(i) *Given $i, j \in \mathbb{N}$ with $i + jr \le k$, put*

$$\lambda_{ij} := \bar{\lambda} - \lambda - j\mu, \quad \vec{\omega}_{ij} := (\lambda_{ij}, \mu).$$

Let condition (14.4) *be satisfied. Suppose $\widehat{s} > |s|$ if $q = p$, and $\widehat{s} = |s| > 0$ if $q = \infty$, and*

$$a_{ij} \in B_\infty^{\widehat{s}/\vec{r},\vec{\omega}_{ij}}\big(J, T_{\tau_i}^{\sigma_i}\big(M, \mathrm{Hom}(W, \bar{W})\big)\big), \qquad i + jr \le k. \tag{16.2}$$

Then

$$\mathcal{A} \in \mathcal{L}\big(\mathfrak{F}_q^{(s+kr)/\vec{r},\vec{\omega}}, \mathfrak{F}_q^{s/\vec{r},\vec{\bar{\omega}}}\big(J, V_{\bar{\tau}}^{\bar{\sigma}}(\bar{W})\big)\big), \qquad 1 < q \le \infty.$$

If $B_\infty^{\widehat{s}/\vec{r},\vec{\omega}_{ij}}$ *in* (16.2) *is replaced by* $b_\infty^{\widehat{s}/\vec{r},\vec{\omega}_{ij}}$, *then*

$$\mathcal{A}\in\mathcal{L}\big(b_\infty^{(s+kr)/\vec{r},\vec{\omega}},b_\infty^{s/\vec{r},\vec{\omega}}\big(J,V_{\bar{\tau}}^{\bar{\sigma}}(\bar{W})\big)\big).$$

(ii) *Fix*

$$p_{ij}\begin{cases}=(m+r)/(kr-i-jr), & i+jr>kr-(m+r)/p,\\ >p, & i+jr=kr-(m+r)/p,\\ =p, & i+jr<kr-(m+r)/p,\end{cases}$$

and set

$$\lambda_{ij}:=\bar{\lambda}-\lambda-j\mu-(m+\mu)/p_{ij},\quad \vec{\omega}_{ij}:=(\lambda_{ij},\mu)$$

for $i+jr\le kr$. *Suppose*

$$a_{ij}\in L_{p_{ij}}\big(J,L_{p_{ij}}^{\lambda_{ij}}\big(T_{\tau_i}^{\sigma_i}(M,\mathrm{Hom}(W,\bar{W}))\big)\big).$$

Then

$$\mathcal{A}\in\mathcal{L}\big(W_p^{kr/\vec{r},\vec{\omega}},L_p\big(J,L_p^{\bar{\lambda}}(V_{\bar{\tau}}^{\bar{\sigma}}(\bar{W}))\big)\big).$$

(iii) *In either case the map* $(a_{ij}\mapsto\mathcal{A})$ *is linear and continuous.*

P r o o f. (1) Theorem 16.1 implies

$$\nabla^i\partial^j\in\mathcal{L}\big(\mathfrak{F}_q^{(s+kr)/\vec{r},\vec{\omega}},\mathfrak{F}_q^{(s-i+(k-j)r)/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})\big)\tag{16.3}$$

and this is also true if $\mathfrak{F}_\infty$ is replaced by b_∞. Since

$$\mathfrak{F}_q^{(s-i+(k-j)r)/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})\hookrightarrow\mathfrak{F}_q^{s/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})$$

assertion (i) follows from Theorem 14.3.

(2) If $i+jr>kr-(m+r)/p$, then we get from Theorem 15.2(i)

$$H_p^{(kr-i-jr)/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})\hookrightarrow L_{q_{ij}}\big(J,L_{q_{ij}}^{\bar{\lambda}-\lambda_{ij}}(V_{\tau+i}^{\sigma})\big),$$

where $1/q_{ij}:=1/p-1/p_{ij}$.

Suppose $i+jr=kr-(m+r)/p$. Then $p_{ij}>p$ implies $s:=i+jr+(m+r)/p_{ij}<kr$. Thus, invoking Theorem 15.2(i) once more,

$$H_p^{(kr-i-jr)/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})\hookrightarrow H_p^{(s-i-jr)/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})\hookrightarrow L_{q_{ij}}\big(J,L_{q_{ij}}^{\bar{\lambda}-\lambda_{ij}}(V_{\tau+i}^{\sigma})\big).$$

If $i+jr<kr-(m+r)/p$, then we deduce from Theorem 15.2(i)

$$H_p^{(kr-i-jr)/\vec{r},(\lambda+j\mu,\mu)}(J,V_{\tau+i}^{\sigma})\hookrightarrow L_\infty\big(J,L_\infty^{\bar{\lambda}-\lambda_{ij}}(V_{\tau+i}^{\sigma})\big).$$

Since $q_{ij}=\infty$ if $p_{ij}=p$ we get in either case from (16.3)

$$\nabla^i\partial^j u\in L_{q_{ij}}\big(J,L_{q_{ij}}^{\bar{\lambda}-\lambda_{ij}}(V_{\tau+i}^{\sigma})\big)=:L_{q_{ij}}(J,X_{ij}),\qquad u\in H_p^{kr/\vec{r},\vec{\omega}}.$$

Note $\bar{\lambda}+\bar{\tau}-\bar{\sigma}=\lambda_{ij}+\tau_i-\sigma_i+\bar{\lambda}-\lambda_{ij}+\tau+i-\sigma$ implies, due to Lemma 14.2,

$$\rho^{\bar{\lambda}+\bar{\tau}-\bar{\sigma}}\,|a_{ij}\cdot\nabla^i\partial^j u|_{\bar{h}}\le c\rho^{\lambda_{ij}+\tau_i-\sigma_i}\,|a_{ij}|_{h_{ij}}\,\rho^{\bar{\lambda}-\lambda_{ij}+\tau+i-\sigma}\,|\nabla^i\partial^j u|_{h_i},$$

where $\bar{h}:=(\cdot,\cdot)_{\bar{\sigma}}^{\bar{\tau}}\otimes h_{\bar{W}}$, $h_{ij}:=(\cdot,\cdot)_{\sigma_i}^{\tau_i}\otimes h_{W\bar{W}}$, and $h_i:=(\cdot,\cdot)_{\sigma}^{\tau+i}\otimes h_W$. Hence, by Hölder's inequality,

$$\|a_{ij}\cdot\nabla^i\partial^j u\|_{L_p(J,L_p^{\bar{\lambda}}(V_{\bar{\tau}}^{\bar{\sigma}}(\bar{W})))}\le\|a_{ij}\|_{L_{p_{ij}}(J,Y_{ij})}\,\|\nabla^i\partial^j u\|_{L_{q_{ij}}(J,X_{ij})},$$

where $Y_{ij}:=L_{p_{ij}}^{\lambda_{ij}}\big(T_{\tau_i}^{\sigma_i}\big(M,\mathrm{Hom}(W,\bar{W})\big)\big)$. By combining this with (16.3) and using $W_p^{kr/\vec{r},\vec{\omega}}\doteq\mathfrak{F}_p^{kr/\vec{r},\vec{\omega}}$ we get assertion (ii).

(3) The last claim is obvious. □

It is clear which changes have to be made to get analogous results for 'stationary' differential operators in the time-independent isotropic case. Details are left to the reader.

17 Extensions and Restrictions

In many situations it is easier to consider anisotropic function spaces on the whole line rather than on the half-line. Therefore we investigate in this section the possibility of extending half-line spaces to spaces on all of $\mathbb{R}$.

We fix $h \in C^\infty\big((0,\infty),\mathbb{R}\big)$ satisfying

$$\int_0^\infty t^s\,|h(t)|\,dt < \infty, \quad s \in \mathbb{R}, \qquad (-1)^k \int_0^\infty t^k h(t)\,dt = 1, \quad k \in \mathbb{Z}, \tag{17.1}$$

and $h(1/t) = -th(t)$ for $t > 0$. Lemma 4.1.1 of [4], which is taken from [18], guarantees the existence of such a function.

Let $\mathcal{X}$ be a locally convex space. Then the *point-wise restriction*,

$$r^+ : C(\mathbb{R},\mathcal{X}) \to C(\mathbb{R}^+,\mathcal{X}), \quad u \mapsto u\,|\,\mathbb{R}^+, \tag{17.2}$$

is a continuous linear map. For $v \in C(\mathbb{R}^+,\mathcal{X})$ we set

$$\varepsilon v(t) := \int_0^\infty h(s)v(-st)\,ds, \qquad t < 0, \tag{17.3}$$

and

$$e^+ v := \begin{cases} v & \text{on } \mathbb{R}^+, \\ \varepsilon v & \text{on } (-\infty,0). \end{cases} \tag{17.4}$$

It follows from (17.1) that e^+ is a continuous linear map from $C(\mathbb{R}^+,\mathcal{X})$ into $C(\mathbb{R},\mathcal{X})$, and $r^+e^+ = \mathrm{id}$. Thus point-wise restriction (17.2) is a retraction, and e^+ is a coretraction.

By replacing $\mathbb{R}^+$ in (17.2) by $-\mathbb{R}^+$ and using obvious modifications we get the point-wise restriction r^- 'to the negative half-line' and a corresponding extension operator e^-. The *trivial extension operator*

$$e_0^+ : C_{(0)}(\mathbb{R}^+,\mathcal{X}) := \big\{ u \in C(\mathbb{R}^+,\mathcal{X})\,;\, u(0) = 0 \big\} \to C(\mathbb{R},\mathcal{X})$$

is defined by $e_0^+ v := v$ on $\mathbb{R}^+$ and $e_0^+ v := 0$ on $(-\infty,0)$. Then

$$r_0^+ := r^+(1 - e^- r^-) : C(\mathbb{R},\mathcal{X}) \to C_{(0)}(\mathbb{R}^+,\mathcal{X}) \tag{17.5}$$

is a retraction, and e_0^+ is a coretraction.

We define:

$$\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty),V\big) \text{ is the closure of } \mathcal{D}\big((0,\infty),\mathcal{D}\big) \text{ in } \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V).$$

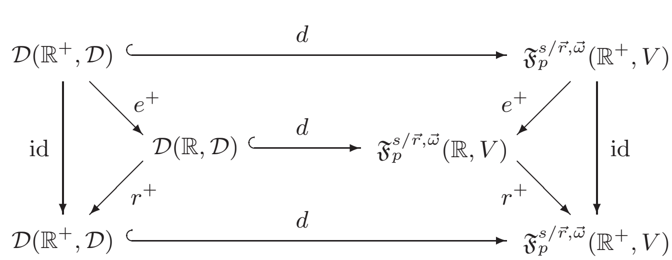
Thus

$$\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V) \hookrightarrow \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty),V\big) \hookrightarrow \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V).$$

Now we can prove an extension theorem 'from the half-cylinder $M \times \mathbb{R}^+$ to the full cylinder $M \times \mathbb{R}$.'

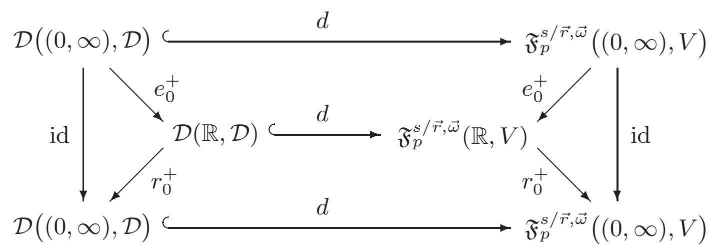
Theorem 17.1

(i) *Suppose* $s \in \mathbb{R}$ *where* $s > -1 + 1/p$ *if* $\partial M \neq \emptyset$. *Then the diagram*



is commuting.

(ii) *If* $s>0$, *then*



is a commuting diagram as well.

P r o o f. (1) Suppose

$$M=(\mathbb{X},g_m) \text{ with } \mathbb{X}\in\{\mathbb{R}^m,\mathbb{H}^m\},\quad \rho=\mathbf{1},\quad W=\mathbb{X}\times F,\quad D=d_F. \tag{17.6}$$

If $k\in\mathbb{N}$, then it is not difficult to see that r^+ is a retraction from $W_p^{kr/\vec{r}}(\mathbb{X}\times\mathbb{R},E)$ onto $W_p^{kr/\vec{r}}(\mathbb{X}\times\mathbb{R}^+,E)$, and e^+ is a coretraction. (cf. steps (1) and (2) of the proof of Theorem 4.4.3 of [4]). Thus, if $s>0$, the first assertion follows by interpolation.

(2) Let (17.6) be satisfied. Suppose $s>0$ and $J=\mathbb{R}^+$. It is an easy consequence of

$$\mathfrak{F}_p^{s/\vec{r}}(J,V)=L_p\big(J,\mathfrak{F}_p^s(V)\big)\cap\mathfrak{F}_p^{s/r}\big(J,L_p(V)\big) \tag{17.7}$$

that

$$\mathfrak{F}_p^{s/\vec{r}}(\mathring{J},V)=L_p\big(J,\mathfrak{F}_p^s(V)\big)\cap\mathfrak{F}_p^{s/r}\big(\mathring{J},L_p(V)\big). \tag{17.8}$$

From this it is obvious that

$$e_0^+\in\mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r}}(\mathring{J},V),\mathfrak{F}_p^{s/\vec{r}}(\mathbb{R},V)\big).$$

Note that $L_p(V)=L_p(\mathbb{X},E)$ is a UMD space (e.g.; [2, Theorem III.4.5.2]). Hence [4, Lemma 4.1.4], definition (17.5), and the arguments of step (1) show

$$r_0^+\in\mathcal{L}\big(\mathfrak{F}_p^{s/r}\big(\mathbb{R},L_p(V)\big),\mathfrak{F}_p^{s/r}\big(\mathring{J},L_p(V)\big)\big).$$

From this, (17.7), and (17.8) we deduce assertion (ii) in this setting.

(3) Assume (17.6) and $s<0$ with $s>-1+1/p$ if $\mathbb{X}=\mathbb{H}^m$. Then $\mathfrak{F}_{p'}^{-s}(V')=\mathring{\mathfrak{F}}_{p'}^{-s}(V')$ by Theorem 4.7.1(ii) of [4]. Hence

$$\begin{aligned}\mathfrak{F}_{p'}^{-s/\vec{r}}(\mathring{J},V')&=L_{p'}\big(J,\mathfrak{F}_{p'}^{-s}(V')\big)\cap\mathfrak{F}_{p'}^{-s/r}\big(\mathring{J},L_{p'}(V')\big)=L_{p'}\big(J,\mathring{\mathfrak{F}}_{p'}^{-s}(V')\big)\cap\mathfrak{F}_{p'}^{-s/r}\big(\mathring{J},L_{p'}(V')\big)\\&=\mathring{\mathfrak{F}}_{p'}^{s/\vec{r}}(J,V').\end{aligned}$$

Thus, by (8.5),

$$\mathfrak{F}_p^{s/\vec{r}}(J,V)\doteq\big(\mathfrak{F}_{p'}^{-s/\vec{r}}(\mathring{J},V')\big)'.$$

The results of Section 4.2 of [4] imply r^+, respectively e^+, is the dual of e_0^+, respectively r_0^+. From this and step (2) it follows (see [4, (4.2.3)] that assertion (i) holds in the present setting if $s<0$, provided $s>-1+1/p$ if $\mathbb{X}=\mathbb{H}^m$.

(4) It follows from (9.2) and (17.2)–(17.4) that

$$r^+\circ\Theta_{q,\kappa}^\mu=\Theta_{q,\kappa}^\mu\circ r^+,\quad e^+\circ\Theta_{q,\kappa}^\mu=\Theta_{q,\kappa}^\mu\circ e^+$$

for $1\le q\le\infty$. Hence

$$r_0^+\circ\Theta_{q,\kappa}^\mu=\Theta_{q,\kappa}^\mu\circ r_0^+,\quad e_0^+\circ\Theta_{q,\kappa}^\mu=\Theta_{q,\kappa}^\mu\circ e_0^+. \tag{17.9}$$

Thus

$$\varphi_{q,\kappa}^{\vec{\omega}}(r^+u)=\rho_\kappa^{\lambda+m/q}\Theta_{q,\kappa}^\mu(\kappa\ltimes\varphi)_*(\pi_\kappa r^+u)=r^+(\varphi_{q,\kappa}^{\vec{\omega}}u)$$

and, similarly,

$$\psi_{q,\kappa}^{\vec{\omega}}(r^+ v_\kappa) = r^+(\psi_{q,\kappa}^{\vec{\omega}} v_\kappa).$$

This implies that $\varphi_q^{\vec{\omega}}$ and $\psi_q^{\vec{\omega}}$ commute with r^+, r_0^+, e^+, and e_0^+. Hence the statements follow from steps (1)–(3) and Theorem 9.3. □

The next theorem concerns the extension of Besov-Hölder spaces from half- to full cylinders.

Theorem 17.2 *Suppose either $s \in r\mathbb{N}$ and $\mathcal{B} \in \{BC, bc\}$, or $s > 0$ and $\mathcal{B} \in \{B_\infty, b_\infty\}$. Then r^+ is a retraction from $\mathcal{B}^{s/\vec{r},\vec{\omega}}(\mathbb{R}, V)$ onto $\mathcal{B}^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$, and e^+ is a coretraction.*

P r o o f. (1) Let $k \in \mathbb{N}$ and $\mathcal{B} \in \{BC, bc\}$. It is obvious that

$$r^+ \in \mathcal{L}\big(\mathcal{B}^{kr/\vec{r},\vec{\omega}}(\mathbb{R}, V), \mathcal{B}^{kr/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)\big).$$

It follows from (17.1) that $\varepsilon \in \mathcal{L}\big(\mathcal{B}^{kr/\vec{r},\vec{\omega}}(\mathbb{R}^+, V), \mathcal{B}^{kr/\vec{r},\vec{\omega}}(-\mathbb{R}^+, V)\big)$. Thus, by the second part of (17.1) and (17.4),

$$e^+ \in \mathcal{L}\big(\mathcal{B}^{kr/\vec{r},\vec{\omega}}(\mathbb{R}^+, V), \mathcal{B}^{kr/\vec{r},\vec{\omega}}(\mathbb{R}, V)\big).$$

From this we get the assertion in this case.

(2) If $s > 0$ and $\mathcal{B} \in \{B_\infty, b_\infty\}$, then, due to Corollary 12.9, we obtain the statement by interpolation from the results of step (1). □

Lastly, we consider little Besov-Hölder spaces 'with vanishing initial values'. They are defined as follows: If $k \in \mathbb{N}$, then

$$\begin{aligned} &u \in bc^{kr/\vec{r},\vec{\omega}}\big((0,\infty), V\big) \text{ iff} \\ &u \in bc^{kr/\vec{r},\vec{\omega}}(\mathbb{R}^+, V) \text{ and } \partial^j u(0) = 0 \text{ for } 0 \le j \le k. \end{aligned} \tag{17.10}$$

Furthermore, $b_\infty^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big)$ is defined by

$$\begin{cases} \big(bc^{kr/\vec{r},\vec{\omega}}((0,\infty), V), bc^{(k+1)r/\vec{r},\vec{\omega}}((0,\infty), V)\big)^0_{(s-kr)/r,\infty}, & kr < s < (k+1)r, \\ \big(bc^{kr/\vec{r},\vec{\omega}}((0,\infty), V), bc^{(k+2)r/\vec{r},\vec{\omega}}((0,\infty), V)\big)^0_{1/2,\infty}, & s = (k+1)r, \end{cases} \tag{17.11}$$

where $k \in \mathbb{N}$.

Theorem 17.3 *Let $k \in \mathbb{N}$ and $s > 0$. Then r_0^+ is a retraction from $bc^{kr/\vec{r},\vec{\omega}}(\mathbb{R}, V)$ onto $bc^{kr/\vec{r},\vec{\omega}}\big((0,\infty), V\big)$ and from $b_\infty^{s/\vec{r},\vec{\omega}}(\mathbb{R}, V)$ onto $b_\infty^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big)$, and e_0^+ is a coretraction.*

P r o o f. It is easily seen by (17.5) and the preceding theorem that the assertion is true for $bc^{kr/\vec{r},\vec{\omega}}$ spaces. The stated results in the remaining cases now follow by interpolation. □

18 Trace Theorems

Suppose Γ is a union of connected components of ∂M. We denote by $\mathring{\iota} \colon \Gamma \hookrightarrow M$ the natural injection and endow Γ with the induced Riemannian metric $\mathring{g} := \mathring{\iota}^* g$. Let $(\rho, \mathfrak{K})$ be a singularity datum for M. For $\kappa \in \mathfrak{K}_\Gamma$ we put $U_{\mathring{\kappa}} := \partial U_\kappa = U_\kappa \cap \Gamma$ and $\mathring{\kappa} := \iota_0 \circ (\mathring{\iota}^* \kappa) : U_{\mathring{\kappa}} \to \mathbb{R}^{m-1}$, where $\iota_0 : \{0\} \times \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$, $(0, x') \mapsto x'$. Then $\mathring{\mathfrak{K}} := \{\, \mathring{\kappa} \,;\, \kappa \in \mathfrak{K}_\Gamma \,\}$ is a normalized atlas for Γ, the one induced by $\mathfrak{K}$. We set $\mathring{\rho} := \mathring{\iota}^* \rho = \rho | \Gamma$. It follows that $(\mathring{\rho}, \mathring{\mathfrak{K}})$ is a singularity datum for Γ, so that Γ is singular of type $[\![\mathring{\rho}]\!]$. Henceforth, it is understood that Γ is given this singularity structure *induced by* $\mathfrak{T}(M)$.

We denote by $\mathring{W} = W_\Gamma$ the restriction of W to Γ and by $h_{\mathring{W}} := \mathring{\iota}^* h_W$ the bundle metric on Γ induced by h_W. Furthermore, the connection $D_{\mathring{W}}$ for $\mathring{W}$, induced by D, is defined by restricting

$$D : \mathcal{T}M \times C^\infty(M, W) \to C^\infty(M, W) \quad \text{to} \quad \mathcal{T}\Gamma \times C^\infty(\Gamma, \mathring{W}),$$

considered as a map into $C^\infty(\Gamma,\mathring{W})$. Then $\mathring{W}=(\mathring{W},h_{\mathring{W}},D_{\mathring{W}})$ is a fully uniformly regular vector bundle over Γ.

We set $\mathring{V}:=T_\tau^\sigma(\Gamma,\mathring{W})$ and endow it with the bundle metric $\mathring{h}:=(\cdot|\cdot)_{T_\tau^\sigma\Gamma}\otimes h_{\mathring{W}}$, where $(\cdot|\cdot)_{T_\tau^\sigma\Gamma}$ is the bundle metric on $T_\tau^\sigma\Gamma$ induced by $\mathring{g}$. Then we equip $\mathring{V}$ with the metric connection $\mathring{\nabla}:=\nabla(\nabla_{\mathring{g}},D_{\mathring{W}})$. Hence $\mathring{V}=(\mathring{V},\mathring{h},\mathring{\nabla})$. It follows that $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,\mathring{V})$ is a well-defined anisotropic weighted space with respect to the boundary weight function $\mathring{\rho}$.

We write $\boldsymbol{n}=\boldsymbol{n}(\Gamma)$ for the inward pointing unit normal on Γ. In local coordinates, $\kappa=(x^1,\ldots,x^m)$,

$$\boldsymbol{n}=\big(\sqrt{g_{11}\,|\,\partial U_\kappa}\,\big)^{-1}\frac{\partial}{\partial x^1}. \tag{18.1}$$

Let $u\in\mathcal{D}=\mathcal{D}(M,V)$ and $k\in\mathbb{N}$. The *trace of order k of u on Γ*, $\partial_{\boldsymbol{n}}^k u=\partial_{\boldsymbol{n}(\Gamma)}^k u\in\mathcal{D}(\Gamma,\mathring{V})$, is defined by

$$\langle\partial_{\boldsymbol{n}}^k u,a\rangle_{\mathring{V}^*}:=\big\langle\nabla^k u\,|\,\Gamma,a\otimes\boldsymbol{n}^{\otimes k}\big\rangle_{\mathring{V}^*},\qquad a\in\mathcal{D}(\Gamma,\mathring{V}^*). \tag{18.2}$$

We also set $\gamma_\Gamma:=\partial_{\boldsymbol{n}(\Gamma)}^0$ and call it *trace operator on* Γ. We write again $\partial_{\boldsymbol{n}}^k=\partial_{\boldsymbol{n}(\Gamma)}^k$ for the point-wise extension of $\partial_{\boldsymbol{n}(\Gamma)}^k$ over J, that is, $(\partial_{\boldsymbol{n}}^k u)(t):=\partial_{\boldsymbol{n}}^k\big(u(t)\big)$ for $t\in J$ and $u\in\mathcal{D}(J,\mathcal{D})$, and call it *lateral trace operator of order k on* $\Gamma\times J$. Correspondingly, the *lateral trace operator on* $\Gamma\times J$ is the point-wise extension of γ_Γ, denoted by γ_Γ as well. Moreover,

$$\partial_{\boldsymbol{n},0}^k:\ \mathcal{D}\big((0,\infty),\mathcal{D}\big)\to\mathcal{D}\big((0,\infty),\mathcal{D}(\Gamma,\mathring{V})\big),\quad u\mapsto\partial_{\boldsymbol{n}}^k u$$

is the restriction of $\partial_{\boldsymbol{n}}^k$ to $\mathcal{D}\big((0,\infty),\mathcal{D}\big)$.

Assume $J=\mathbb{R}^+$. Then $M_0:=M\times\{0\}$ is the *initial boundary* of the space-time (half-)cylinder $M\times\mathbb{R}^+$. The *initial trace operator* is the linear map

$$\gamma_{M_0}:\ \mathcal{D}(\mathbb{R}^+,\mathcal{D})\to\mathcal{D},\quad u\mapsto u(0),$$

where M_0 is identified with M. Furthermore,

$$\partial_{t=0}^k:=\gamma_{M_0}\circ\partial^k:\ \mathcal{D}(\mathbb{R}^+,\mathcal{D})\to\mathcal{D},\quad u\mapsto(\partial^k u)(0)$$

is the *initial trace operator of order k*.

Suppose $s_0>1/p$. The following theorem shows, in particular, that there exists a unique

$$(\gamma_\Gamma)_{s_0}\in\mathcal{L}\big(\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}},B_p^{(s_0-1/p)/\vec{r},(\lambda+1/p,\mu)}(J,\mathring{V})\big)$$

extending γ_Γ and being a retraction. Furthermore, there exists a coretraction $(\gamma_\Gamma^c)_{s_0}$ such that, for each $s\in\mathbb{R}$, there is

$$(\gamma_\Gamma^c)_s\in\mathcal{L}\big(B_p^{(s-1/p)/\vec{r},(\lambda+1/p,\mu)}(J,\mathring{V}),\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big)$$

such that

$$\begin{aligned}&\text{(i)}\quad(\gamma_\Gamma^c)_s\,\big|\,\mathcal{D}\big(J,\mathcal{D}(\Gamma,\mathring{V})\big)=(\gamma_\Gamma^c)_{s_0}\,\big|\,\mathcal{D}\big(J,\mathcal{D}(\Gamma,\mathring{V})\big),\\&\text{(ii)}\quad(\gamma_\Gamma^c)_s\ \text{is for each } s>1/p \text{ a coretraction for } (\gamma_\Gamma)_s.\end{aligned} \tag{18.3}$$

Thus $(\gamma_\Gamma)_{s_0}$ is for each $s_0>1/p$ uniquely determined by γ_Γ and $(\gamma_\Gamma^c)_s$ can be obtained for any $s\in\mathbb{R}$ by unique continuous extension or restriction of $(\gamma_\Gamma^c)_{s_0}$ for any $s_0>1/p$. Hence we simply write γ_Γ and γ_Γ^c for $(\gamma_\Gamma)_s$ and $(\gamma_\Gamma^c)_s$, respectively, without fearing confusion. So we can say γ_γ^c is a *universal coretraction* for the retraction

$$\gamma_\Gamma\in\mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}},B_p^{(s-1/p)/\vec{r},(\lambda+1/p,\mu)}(J,\mathring{V})\big),\qquad s>1/p,$$

herewith expressing properties (18.3). Similar conventions hold for higher order trace operators and traces occurring below.

Theorem 18.1 *Suppose $k \in \mathbb{N}$.*

(i) *Assume $\Gamma \neq \emptyset$ and $s > k + 1/p$. Then $\partial_{\boldsymbol{n}}^k$ is a retraction*

$$\text{from } \mathfrak{F}_p^{s/\vec{r},(\lambda,\mu)}(J,V) \text{ onto } B_p^{(s-k-1/p)/\vec{r},(\lambda+k+1/p,\mu)}(J,\mathring{V}).$$

It possesses a universal coretraction $(\gamma_{\boldsymbol{n}}^k)^c$ satisfying $\partial_{\boldsymbol{n}}^i \circ (\gamma_{\boldsymbol{n}}^k)^c = 0$ for $0 \leq i \leq k-1$.

(ii) *Suppose $s > r(k+1/p)$. Then $\partial_{t=0}^k$ is a retraction*

$$\text{from } \mathfrak{F}_p^{s/\vec{r},(\lambda,\mu)}(\mathbb{R}^+,V) \text{ onto } B_p^{s-r(k+1/p),\lambda+\mu(k+1/p)}(V).$$

There exists a universal coretraction $(\gamma_{t=0}^k)^c$ such that $\partial_{t=0}^i \circ (\gamma_{t=0}^k)^c = 0$ for $0 \leq i \leq k-1$.

(iii) *Let $\Gamma \neq \emptyset$ and $s > k + 1/p$. Then $\partial_{\boldsymbol{n},0}^k$ is a retraction*

$$\text{from } \mathfrak{F}_p^{s/\vec{r},(\lambda,\mu)}\big((0,\infty),V\big) \text{ onto } B_p^{(s-k-1/p)/\vec{r},(\lambda+k+1/p,\mu)}\big((0,\infty),\mathring{V}\big). \tag{18.4}$$

The restriction of $(\gamma_{\boldsymbol{n}}^k)^c$ to the space on the right side of (18.4) *is a universal coretraction.*

P r o o f. (1) Suppose $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$, $M = (\mathbb{X}, g_m)$, $\rho = \mathbf{1}$, $W = \mathbb{X} \times F$, and $D = d_F$ so that $V = \mathbb{X} \times E$. Put $\mathbb{Y} := \mathbb{X} \times J$. Assume either $\Gamma \neq \emptyset$ or $J = \mathbb{R}^+$. If $J = \mathbb{R}$, then $M \times J = \mathbb{Y} = \mathbb{H}^{m+1}$. If $J = \mathbb{R}^+$ and $\Gamma = \emptyset$, then $M \times J = \mathbb{R}^m \times \mathbb{R}^+ \simeq \mathbb{H}^{m+1}$. Finally, if $J = \mathbb{R}^+$ and $\Gamma \neq \emptyset$, then

$$M \times J = \mathbb{H}^m \times \mathbb{R}^+ \simeq \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{m-1},$$

that is, $M \times J$ is a closed 2-corner in the sense of Section 4.3 of [4]. In each case $\simeq$ is simply a permutation diffeomorphism.

If either $J = \mathbb{R}$ or $\Gamma = \emptyset$, then assertions (i) and (ii) follow from Theorem 4.6.3 of [4]. If $J = \mathbb{R}^+$ and $\Gamma \neq \emptyset$, then assertion (i) follows from Theorem 4.6.3 and the definition of the trace operator for a face of $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{m-1}$, that is, formula (4.10.12) of [4]. Claim (iii) is a consequence of Theorem 4.10.3 of [4] (choose any κ therein with $\kappa > s+1$).

(2) Now we consider the general case. Suppose $\Gamma \neq \emptyset$. For $t > 1/p$ we set

$$\mathring{B}_{p,\kappa}^{(t-1/p)/\vec{r}} := \begin{cases} B_p^{(t-1/p)/\vec{r}}(\partial\mathbb{Y}_\kappa, E) & \text{if } \kappa \in \mathfrak{K}_\Gamma, \\ \{0\} & \text{otherwise.} \end{cases}$$

Let γ_κ be the trace operator on $\partial\mathbb{Y}_\kappa = \{0\} \times \mathbb{R}^{m-1} \times J$ if $\kappa \in \mathfrak{K}_\Gamma$, and $\gamma_\kappa := 0$ otherwise. Set

$$\gamma_{k,\kappa} := \rho_\kappa^k \big(\sqrt{\gamma_\kappa(\kappa_* g_{11})}\,\big)^{-k} \gamma_\kappa \circ \partial_1^k, \qquad \kappa \in \mathfrak{K}.$$

It follows from step (1), (18.1), and (18.2) that $\gamma_\kappa \circ \partial_1^k$ is a retraction from $\mathfrak{F}_{p,\kappa}^{s/\vec{r}}$ onto $\mathring{B}_{p,\kappa}^{(s-1/p)/\vec{r}}$ and that there exists a universal coretraction $\widetilde{\gamma}_{k,\kappa}^c$ satisfying

$$(\gamma_\kappa \circ \partial_1^i) \circ \widetilde{\gamma}_{k,\kappa}^c = 0, \qquad 0 \leq i \leq k-1, \tag{18.5}$$

(setting $\widetilde{\gamma}_{k,\kappa}^c := 0$ if $\kappa \in \mathfrak{K} \backslash \mathfrak{K}_\Gamma$). We put

$$\gamma_{k,\kappa}^c := \rho_\kappa^{-k} \big(\sqrt{\gamma_\kappa(\kappa_* g_{11})}\,\big)^k \, \widetilde{\gamma}_{k,\kappa}^c, \qquad \kappa \in \mathfrak{K}.$$

Then (3.7) and (4.1) imply

$$\gamma_{k,\kappa} \in \mathcal{L}(\mathfrak{F}_{p,\kappa}^{s/\vec{r}}, \mathring{B}_{p,\kappa}^{(s-k-1/p)/\vec{r}}), \quad \gamma_{k,\kappa}^c \in \mathcal{L}(\mathring{B}_{p,\kappa}^{(s-k-1/p)/\vec{r}}, \mathfrak{F}_{p,\kappa}^{s/\vec{r}})$$

and

$$\|\gamma_{k,\kappa}\| + \|\gamma^c_{k,\kappa}\| \le c, \qquad \kappa \in \mathfrak{K}.$$

From (18.5) and Leibniz' rule we thus infer

$$\gamma_{i,\kappa} \circ \gamma^c_{k,\kappa} = \delta_{ik}\mathrm{id}, \qquad 0 \le i \le k. \tag{18.6}$$

(3) We set $(\mathring{\pi}_{\mathring{\kappa}}, \mathring{\chi}_{\mathring{\kappa}}) := (\pi_\kappa, \chi_\kappa)\,|\,U_{\mathring{\kappa}}$ for $\mathring{\kappa} \in \mathring{\mathfrak{K}}$. Then it is verified that $\big\{ (\mathring{\pi}_{\mathring{\kappa}}, \mathring{\chi}_{\mathring{\kappa}}) \ ; \ \mathring{\kappa} \in \mathring{\mathfrak{K}} \big\}$ is a localization system subordinate to $\mathring{\mathfrak{K}}$. We denote by

$$\mathring{\psi}^{\vec{\omega}}_p : \ \ell_p(\mathring{\boldsymbol{B}}^{(s-k-1/p)/\vec{r}}_p) \to B^{(s-k-1/p)/\vec{r},\vec{\omega}}_p(J, \mathring{V})$$

the 'boundary retraction' defined analogously to $\psi^{\vec{\omega}}_p$. Correspondingly, $\mathring{\varphi}^{\vec{\omega}}_p$ is the 'boundary coretraction'.

We write $\mathring{\kappa}\ltimes\mathring{\varphi}$ for the restriction of $\kappa\ltimes\varphi \in \mathfrak{K}\ltimes\Phi$ to Γ and put

$$C_{k,\kappa} := \mathring{\rho}^{\,k}_{\mathring{\kappa}}(\mathring{\kappa}\ltimes\mathring{\varphi})_* \circ \partial^k_{\boldsymbol{n}} \circ (\kappa\ltimes\varphi)^*, \qquad \kappa\ltimes\varphi \in \mathfrak{K}_\Gamma\ltimes\Phi,$$

and $C_{k,\kappa} := 0$ otherwise. Note $\mathring{\rho}_{\mathring{\kappa}} = \rho_\kappa$ for $\kappa \in \mathfrak{K}_\Gamma$. It follows from (5.15), (18.1), and (18.2) that

$$C_{k,\kappa}v = \gamma_{k,\kappa}v + \sum_{\ell=0}^{k-1} a_{\ell,\kappa}\gamma_{\ell,\kappa}v, \qquad v \in \mathcal{D}\big(J, \mathcal{D}(\partial\mathbb{X}_\kappa, E)\big), \tag{18.7}$$

and (5.16) implies $\|a_{\ell,k}\|_{k-1,\infty} \le c$ for $0 \le \ell \le k-1$ and $\kappa \in \mathfrak{K}$. Hence, using $\mathfrak{F}^{s/\vec{r}}_{p,\kappa} \hookrightarrow \mathfrak{F}^{(s-k+\ell)/\vec{r}}_{p,\kappa}$ and Theorem 13.5, we find

$$C_{k,\kappa} \in \mathcal{L}(\mathfrak{F}^{s/\vec{r}}_{p,\kappa}, \mathring{B}^{(s-k-1/p)/\vec{r}}_{p,\kappa}), \quad \|C_{k,\kappa}\| \le c, \qquad \kappa \in \mathfrak{K}. \tag{18.8}$$

(4) For $u \in \mathcal{D}(J, \mathcal{D})$

$$\mathring{\pi}_{\mathring{\kappa}}\partial^k_{\boldsymbol{n}}u = \partial^k_{\boldsymbol{n}}(\pi_\kappa u) - \sum_{j=0}^{k-1}\binom{k}{j}(\partial^{k-j}_{\boldsymbol{n}}\pi_\kappa)\partial^j_{\boldsymbol{n}}(\chi_\kappa u), \qquad \kappa \in \mathfrak{K}, \tag{18.9}$$

setting $\partial^k_{\boldsymbol{n}}v := 0$ if $\mathrm{supp}(v) \cap \Gamma = \emptyset$. Note

$$\begin{aligned} &\mathring{\rho}^{\,\lambda+k+1/p+(m-1)/p}_{\mathring{\kappa}}\Theta^\mu_{p,\mathring{\kappa}}(\mathring{\kappa}\ltimes\mathring{\varphi})_*\big(\partial^k_{\boldsymbol{n}}(\pi_\kappa u)\big) \\ &\qquad = \mathring{\rho}^{\,k}_\kappa(\mathring{\kappa}\ltimes\mathring{\varphi})_* \circ \partial^k_{\boldsymbol{n}} \circ (\kappa\ltimes\varphi)^*\big(\rho^{\lambda+m/p}_\kappa\Theta^\mu_{p,\kappa}(\kappa\ltimes\varphi)_*(\pi_\kappa u)\big) = C_{k,\kappa}(\varphi^{\vec{\omega}}_{p,\kappa}u), \end{aligned} \tag{18.10}$$

since $\Theta^\mu_{p,\mathring{\kappa}} = \Theta^\mu_{p,\kappa}$ for $\kappa \in \mathfrak{K}_\Gamma$. Similarly, using (13.11) and (13.12) also,

$$\begin{aligned} &\mathring{\rho}^{\,\lambda+k+1/p+(m-1)/p}_{\mathring{\kappa}}\Theta^\mu_{p,\mathring{\kappa}}(\mathring{\kappa}\ltimes\mathring{\varphi})_*\big((\partial^{k-j}_{\boldsymbol{n}}\pi_\kappa)\partial^j_{\boldsymbol{n}}(\chi_\kappa u)\big) \\ &\qquad = C_{k-j,\kappa}(\kappa_*\pi_\kappa)C_{j,\kappa}(\widehat{\varphi}^{\vec{\omega}}_{p,\kappa}u) = \sum_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)} C_{k-j,\kappa}(\kappa_*\pi_\kappa)C_{j,\kappa}(a_{\widetilde{\kappa}\kappa}R_{\widetilde{\kappa}\kappa}\varphi^{\vec{\omega}}_{p,\widetilde{\kappa}}u). \end{aligned}$$

From this, (18.9), and (18.10) we get

$$\mathring{\varphi}^{(\lambda+k+1/p,\mu)}_{p,\mathring{\kappa}}(\partial^k_{\boldsymbol{n}}u) = C_{k,\kappa}(\varphi^{\vec{\omega}}_{p,\kappa}u) + \sum_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)} A_{k-1,\widetilde{\kappa}\kappa}(\varphi^{\vec{\omega}}_{p,\widetilde{\kappa}}u), \tag{18.11}$$

where

$$A_{k-1,\widetilde{\kappa}\kappa} := \sum_{i=0}^{k-1} b_{i,\widetilde{\kappa}\kappa}C_{i,\kappa} \circ R_{\widetilde{\kappa}\kappa}, \quad b_{i,\widetilde{\kappa}\kappa} := -\sum_{j=i}^{k-1}\binom{k}{j}\binom{j}{i}C_{k-j,\kappa}(\kappa_*\pi_\kappa)C_{j-i}a_{\widetilde{\kappa}\kappa}.$$

It is obvious that

$$C_{\ell,\kappa} \in \mathcal{L}\big(BC_\kappa^{n+\ell}, BC^n(\partial\mathbb{X}_\kappa, E)\big), \quad \|C_{\ell,\kappa}\| \le c(n), \qquad \kappa \in \mathfrak{K}, \quad 0 \le \ell \le k, \quad n \in \mathbb{N}.$$

From this, (7.3), (13.13), and Theorem 13.5 we obtain

$$b_{i,\widetilde{\kappa}\kappa} \in BC^\infty(\partial\mathbb{X}_\kappa, E), \quad \|b_{i,\widetilde{\kappa}\kappa}\|_{n,\infty} \in c, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa \in \mathfrak{K}, \quad 0 \le i \le k, \quad n \in \mathbb{N}.$$

Hence, using Theorem 13.5 once more, we get from (18.8) and Lemma 13.3

$$A_{k-1,\widetilde{\kappa}\kappa} \in \mathcal{L}(\mathfrak{F}_{p,\widetilde{\kappa}}^{s/\vec{r}}, \overset{\bullet}{B}{}_{p,\kappa}^{(s-k-1/p)/\vec{r}}), \quad \|A_{k-1,\widetilde{\kappa}\kappa}\| \le c, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa \in \mathfrak{K}. \tag{18.12}$$

(5) We define $\boldsymbol{C}_k$ by

$$\boldsymbol{C}_k \boldsymbol{v} := \Big(C_{k,\kappa} v_\kappa + \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} A_{k-1,\widetilde{\kappa}\kappa} v_{\widetilde{\kappa}}\Big)_{\kappa \in \mathfrak{K}}, \qquad \boldsymbol{v} = (v_\kappa).$$

Then we deduce from (18.8), (18.12), and the finite multiplicity of $\mathfrak{K}$

$$\boldsymbol{C}_k \in \mathcal{L}\big(\ell_p(\boldsymbol{\mathfrak{F}}_p^{s/\vec{r}}), \ell_p(\overset{\bullet}{\boldsymbol{B}}{}_p^{(s-k-1/p)/\vec{r}})\big). \tag{18.13}$$

Employing (18.6) and (18.7) we infer $C_{k,\kappa} \circ \gamma_{k,\kappa}^c = \mathrm{id}$. Furthermore, recalling (13.9) and using $\overset{\bullet}{\rho}_\kappa = \rho_\kappa$ for $\kappa \in \mathfrak{K}_\Gamma$,

$$\begin{aligned} C_{i,\kappa} \circ R_{\widetilde{\kappa}\kappa} &= \overset{\bullet}{\rho}{}_\kappa^k (\overset{\bullet}{\kappa} \ltimes \overset{\bullet}{\varphi})_* \circ \partial_{\boldsymbol{n}}^i \circ (\kappa \ltimes \varphi)^* \circ T_{\widetilde{\kappa}\kappa} \circ (\kappa \ltimes \varphi)_* (\widetilde{\kappa} \ltimes \widetilde{\varphi})^* (\chi \cdot) \\ &= (\rho_\kappa / \rho_{\widetilde{\kappa}})^k T_{\overset{\bullet}{\widetilde{\kappa}}\overset{\bullet}{\kappa}} S_{\overset{\bullet}{\widetilde{\kappa}}\overset{\bullet}{\kappa}} C_{i,\widetilde{\kappa}} = (\rho_\kappa / \rho_{\widetilde{\kappa}})^k R_{\overset{\bullet}{\widetilde{\kappa}}\overset{\bullet}{\kappa}} C_{i,\widetilde{\kappa}}. \end{aligned}$$

By this, (18.6), and (18.7) it follows $C_{i,\kappa} \circ R_{\widetilde{\kappa}\kappa} \circ \gamma_{k,\widetilde{\kappa}}^c = 0$ for $0 \le i \le k-1$. Thus, setting $\boldsymbol{\gamma}_k^c \boldsymbol{v} := (\gamma_{k,\kappa}^c v_\kappa)$,

$$\boldsymbol{\gamma}_k^c \in \mathcal{L}\big(\ell_p(\overset{\bullet}{\boldsymbol{B}}{}_p^{(s-k-1/p)/\vec{r}}), \ell_p(\boldsymbol{\mathfrak{F}}_p^{s/\vec{r}})\big), \quad \boldsymbol{C}_i \circ \boldsymbol{\gamma}_k^c = \delta_{ik} \mathrm{id}, \qquad 0 \le i \le k. \tag{18.14}$$

From (18.11), (18.13), and the first claim of Theorem 9.3 we get

$$\partial_{\boldsymbol{n}}^k = \overset{\bullet}{\psi}{}_p^{(\lambda+k+1/p,\mu)} \circ \boldsymbol{C}_k \circ \varphi_p^{\vec{\omega}} \in \mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}, B_p^{(s-k-1/p)/\vec{r},(\lambda+k+1/p,\mu)}(J, \overset{\bullet}{V})\big).$$

(6) Given $\boldsymbol{v} \in \overset{\bullet}{\boldsymbol{B}}{}_p^{(s-k-1/p)/\vec{r}}$,

$$\begin{aligned} \partial_{\boldsymbol{n}}^i (\psi_p^{\vec{\omega}} \boldsymbol{v}) &= \sum_\kappa \rho_\kappa^{-(\lambda+m/p)} \partial_{\boldsymbol{n}}^i \big(\Theta_{p,\kappa}^{-\mu} \pi_\kappa (\kappa \ltimes \varphi)^* v_\kappa\big) \\ &= \sum_{\overset{\bullet}{\kappa}} \overset{\bullet}{\rho}{}_{\overset{\bullet}{\kappa}}^{-(\lambda+i+m/p)} \Theta_{p,\overset{\bullet}{\kappa}}^{-\mu} \Big(\overset{\bullet}{\pi}_{\overset{\bullet}{\kappa}} (\overset{\bullet}{\kappa} \ltimes \overset{\bullet}{\varphi})^* C_{i,\kappa} v_\kappa + \overset{\bullet}{\rho}{}_{\overset{\bullet}{\kappa}}^i \sum_{j=0}^{i-1} \binom{i}{j} (\partial_{\boldsymbol{n}}^{i-j} \pi_\kappa) \partial_{\boldsymbol{n}}^j \big((\kappa \ltimes \varphi)^* v_\kappa\big)\Big) \\ &= \overset{\bullet}{\psi}{}_p^{(\lambda+i+1/p,\mu)} C_{i,\kappa} v_\kappa + \sum_\kappa \overset{\bullet}{\rho}{}_{\overset{\bullet}{\kappa}}^{-(\lambda+i+m/p)} \Theta_{p,\overset{\bullet}{\kappa}}^{-\mu} (\overset{\bullet}{\kappa} \ltimes \overset{\bullet}{\varphi})^* \sum_{j=0}^{i-1} \binom{i}{j} C_{i-j,\kappa}(\pi_\kappa) C_{j,\kappa} v_\kappa. \end{aligned}$$

Thus we infer from (18.6), (18.7), and (18.14)

$$\partial_{\boldsymbol{n}}^i (\psi_p^{\vec{\omega}} \boldsymbol{\gamma}_k^c \boldsymbol{w}) = \delta_{ik} \overset{\bullet}{\psi}{}_p^{(\lambda+i+1/p,\mu)} \boldsymbol{w}, \qquad \boldsymbol{w} \in \overset{\bullet}{\boldsymbol{B}}{}_p^{(s-k-1/p)/\vec{r}}, \quad 0 \le i \le k.$$

Now (18.14) and the first part of Theorem 9.3 imply

$$(\gamma_{\boldsymbol{n}}^k)^c := \psi_p^{\vec{\omega}} \circ \boldsymbol{\gamma}_k^c \circ \overset{\bullet}{\varphi}{}_p^{(\lambda+k+1/p,\mu)} \in \mathcal{L}\big(B_p^{(s-k-1/p)/\vec{r},(\lambda+k+1/p,\mu)}(J, \overset{\bullet}{V}), \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big)$$

and $\partial_{\boldsymbol{n}}^i (\gamma_{\boldsymbol{n}}^k)^c = \delta_{ik} \mathrm{id}$ for $0 \le i \le k$. This proves assertion (i).

(7) By invoking in the preceding argumentation the second statement of Theorem 9.3 we see that assertion (iii) is true.

(8) We denote by $\partial^k_{t=0,\kappa}$ the initial trace operator of order k for $\mathbb{Y}_\kappa = \mathbb{X}_\kappa \times \mathbb{R}^+$. It follows from step (1) that

$$\boldsymbol{\partial}^k_{t=0} : \ell_p(\boldsymbol{\mathfrak{F}}^{s/\vec r}_p) \to \ell_p(\boldsymbol{B}^{s-r(k+1/p)}_p), \quad \boldsymbol{v} \mapsto (\partial^k_{t=0,\kappa} v_\kappa)$$

is a retraction and there exists a universal coretraction

$$(\boldsymbol{\partial}^k_{t=0})^c : \ell_p(\boldsymbol{B}^{s-r(k+1/p)}_p) \to \ell_p(\boldsymbol{\mathfrak{F}}^{s/\vec r}_p), \quad \boldsymbol{w} \mapsto \big((\partial^k_{t=0,\kappa})^c w_\kappa\big)$$

such that

$$\boldsymbol{\partial}^j_{t=0} \circ (\boldsymbol{\partial}^k_{t=0})^c = \delta_{jk}\mathrm{id}, \qquad 0 \le j \le k. \tag{18.15}$$

(9) We deduce from (9.5) and step (1)

$$\partial^k_{t=0,\kappa} \circ \varphi^{\vec\omega}_{p,\kappa} = \varphi^{\lambda+\mu(k+1/p)}_{p,\kappa} \circ \partial^k_{t=0}, \qquad \kappa \in \mathfrak{K}. \tag{18.16}$$

Hence

$$\boldsymbol{\partial}^k_{t=0} \circ \varphi^{\vec\omega}_p = \varphi^{\lambda+\mu(k+1/p)}_p \circ \partial^k_{t=0}.$$

From this and Theorems 7.1 and 9.3 we infer

$$\partial^k_{t=0} = \psi^{\lambda+\mu(k+1/p)}_p \circ \boldsymbol{\partial}^k_{t=0} \circ \varphi^{\vec\omega}_p \in \mathcal{L}\big(\mathfrak{F}^{s/\vec r,\vec\omega}_p(\mathbb{R}^+, V), B^{s-r(k+1/p),\lambda+\mu(k+1/p)}_p(V)\big).$$

(10) Set

$$(\gamma^k_{t=0})^c := \psi^{\vec\omega}_p \circ (\boldsymbol{\partial}^k_{t=0})^c \circ \varphi^{\lambda+\mu(k+1/p)}_p.$$

Then, similarly as above,

$$(\gamma^k_{t=0})^c \in \mathcal{L}\big(B^{s-r(k+1/p),\lambda+\mu(k+1/p)}_p(V), \mathfrak{F}^{s/\vec r,\vec\omega}_p(\mathbb{R}^+, V)\big).$$

For $0 \le j \le k$ we get from (9.5) and (18.15)

$$\begin{aligned} \partial^j_{t=0}(\gamma^k_{t=0})^c w &= \partial^j_{t=0}\Big(\sum_\kappa \psi^{\vec\omega}_{p,\kappa} \circ (\partial^k_{t=0,\kappa})^c \circ \varphi^{\lambda+\mu(k+1/p)}_{p,\kappa} w\Big) \\ &= \sum_\kappa \psi^{\lambda+\mu(j+1/p)}_{p,\kappa} \circ \partial^j_{t=0,\kappa} \circ (\partial^k_{t=0,\kappa})^c \circ \varphi^{\lambda+\mu(k+1/p)}_{p,\kappa} w \\ &= \delta_{jk} \psi^{\lambda+\mu(j+1/p)}_p \circ \varphi^{\lambda+\mu(k+1/p)}_{p,\kappa} w = \delta_{jk} w \end{aligned}$$

for $w \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $B^{s-r(k+1/p),\lambda+\mu(k+1/p)}_p$, assertion (ii) follows. □

Suppose M is a compact m-dimensional submanifold of $\mathbb{R}^m$. In this setting and if $s = r \in 2\mathbb{N}^\times$ assertions (i) and (ii) reduce to the trace theorems for anisotropic Sobolev spaces due to P. Grisvard [14]; also see O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'ceva [30] and R. Denk, M. Hieber, and J. Prüss [11]. (In the latter paper the authors consider vector-valued spaces.) The much simpler Hilbertian case $p = 2$ has been presented by J.-L. Lions and E. Magenes in [31, Chapter 4, Section 2] following the approach by P. Grisvard [15].

19 Spaces With Vanishing Traces

In this section we characterize $\mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_p$ and $\mathfrak{F}^{s/\vec r,\vec\omega}_p(\mathring{J}, V)$ by the vanishing of certain traces. In fact, we need to characterize those subspaces of $\mathfrak{F}^{s/\vec r,\vec\omega}_p(J, V)$ whose traces vanish on Γ even if $\Gamma \ne \partial M$. More precisely, we denote by

$$\mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_{p,\Gamma} = \mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_{p,\Gamma}(J, V) \text{ the closure of } \mathcal{D}\big(\mathring{J}, \mathcal{D}(M\backslash\Gamma, V)\big) \text{ in } \mathfrak{F}^{s/\vec r,\vec\omega}_p(J, V). \tag{19.1}$$

Note that $\mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_{p,\partial M} = \mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_p$. By Theorem 8.3(ii) we know already

$$\mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_p = \mathfrak{F}^{s/\vec r,\vec\omega}_p, \qquad s < 1/p, \tag{19.2}$$

and, trivially, $\mathring{\mathfrak{F}}^{s/\vec r,\vec\omega}_p = \mathfrak{F}^{s/\vec r,\vec\omega}_p$ if $\partial M = \emptyset$ and $J = \mathbb{R}$. The following theorem concerns the case $s > 1/p$ and $(\Gamma, J) \ne (\emptyset, \mathbb{R})$.

Theorem 19.1

(i) *If* $\Gamma \neq \emptyset$ *and* $k + 1/p < s < k + 1 + 1/p$ *with* $k \in \mathbb{N}$*, then*

$$\mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}} = \{\, u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}} \,;\, \partial_{\boldsymbol{n}}^i u = 0,\ i \leq k \,\}. \tag{19.3}$$

(ii) *Assume* $r(\ell + 1/p) < s < r(\ell + 1 + 1/p)$ *with* $\ell \in \mathbb{N}$*. Then*

$$\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big) = \big\{\, u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V) \,;\, \partial_{t=0}^j u = 0,\ j \leq \ell \,\big\}. \tag{19.4}$$

Suppose $s < r/p$ *with* $s > r(-1 + 1/p)$ *if* $\Gamma \neq \emptyset$*. Then* $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big) = \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$.

P r o o f. (1) Let the assumptions of (i) be satisfied. Since $\partial_{\boldsymbol{n}}^i$ is continuous and vanishes on the dense subset $\mathcal{D}\big(\mathring{J}, \mathcal{D}(M \setminus \Gamma)\big)$ of $\mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}$ it follows that the latter space is contained in the one on the right side of (19.3).

Conversely, let $u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$ satisfy $\partial_{\boldsymbol{n}}^i u = 0$ for $i \leq k$. Suppose $\alpha \in \mathcal{D}\big(\mathring{M} \cup \Gamma, [0,1]\big)$ and $\alpha = 1$ in a neighborhood of Γ. Then $v := \alpha u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$ and $\partial_{\boldsymbol{n}}^i v = 0$ for $i \leq k$. We infer from (18.7), (18.9), and (18.10) that $\gamma_\kappa \circ \partial_1^i(\varphi_{p,\kappa}^{\vec{\omega}} v) = 0$ for $i \leq k$ and $\kappa \ltimes \varphi \in \mathfrak{K} \ltimes \Phi$. Since $\gamma_\kappa = 0$ for $\kappa \notin \mathfrak{K}_\Gamma$ it follows from [4, Theorem 4.7.1] that $\varphi_{p,\kappa}^{\vec{\omega}} v \in \mathring{\mathfrak{F}}_{p,\kappa}^{s/\vec{r}}$ for $\kappa \ltimes \varphi \in \mathfrak{K}_\Gamma \ltimes \Phi$. If $\kappa \in \mathfrak{K} \setminus \mathfrak{K}_\Gamma$, then $\varphi_{p,\kappa}^{\vec{\omega}} v$ belongs to $\mathring{\mathfrak{F}}_{p,\kappa}^{s/\vec{r}}$ as well. Moreover, v vanishes near $\partial M \setminus \Gamma$ and $\mathring{\mathfrak{F}}_{p,\kappa}^{s/\vec{r}} = \mathfrak{F}_{p,\kappa}^{s/\vec{r}}$ for $\kappa \in \mathfrak{K} \setminus \mathfrak{K}_{\partial M}$. Hence we deduce from Theorem 9.3 that $\varphi_p^{\vec{\omega}} v \in \ell_p(\mathring{\boldsymbol{\mathfrak{F}}}_p^{s/\vec{r}})$. Now part (ii) of that theorem guarantees $v = \psi_p^{\vec{\omega}}(\varphi_p^{\vec{\omega}} v) \in \mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}$. Consequently, $u \in \mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}$. This proves claim (i).

(2) Assume $J = \mathbb{R}^+$ and $r(\ell + 1/p) < s < r(\ell + 1 + 1/p)$. As above, we see that $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big)$ is contained in the space on the right side of (19.4).

Let $u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$ satisfy $\partial_{t=0}^j u = 0$ for $0 \leq j \leq \ell$. We get from (18.16) that $\partial_{t=0,\kappa}^j(\varphi_{p,\kappa}^{\vec{\omega}} u) = 0$ for $j \leq \ell$ and $\kappa \ltimes \varphi \in \mathfrak{K} \ltimes \Phi$.

Suppose $\kappa \in \mathfrak{K} \setminus \mathfrak{K}_{\partial M}$. Then [4, Theorem 4.7.1] implies $\varphi_{p,\kappa}^{\vec{\omega}} u \in \mathfrak{F}_p^{s/\vec{r}}\big(\mathbb{X}_\kappa \times (0,\infty), E\big)$. If $\kappa \in \mathfrak{K}_{\partial M}$, then we obtain the latter result by extending $v_\kappa := \varphi_{p,\kappa}^{\vec{\omega}} u$ first from $\mathbb{H}^m \times \mathbb{R}^+$ to $\mathbb{R}^m \times \mathbb{R}^+$ (as in Section 4.1 of [4]), then applying [4, Theorem 4.7.1], and restricting afterwards to $\mathbb{H}^m \times \mathbb{R}^+$. From this and Theorems 9.3 and 17.1 we obtain

$$e_0^+(\varphi_p^{\vec{\omega}} u) \in \ell_p\big(\boldsymbol{\mathfrak{F}}^{s/\vec{r}}(\mathbb{X} \times \mathbb{R}, E)\big). \tag{19.5}$$

Thus, using these theorems once more and the fact that, by (17.9), e_0^+ commutes with $\psi_p^{\vec{\omega}}$, we find

$$u = r_0^+ \circ e_0^+ \circ \psi_p^{\vec{\omega}} \circ \varphi_p^{\vec{\omega}} u = r_0^+ \circ \psi_p^{\vec{\omega}} \circ e_0^+ \circ \varphi_p^{\vec{\omega}} u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big). \tag{19.6}$$

This implies the first part of claim (ii).

Assume $s < r/p$. If $\partial M = \emptyset$, then $\mathcal{D}\big((0,\infty), M\big) = \mathcal{D}(\mathring{J}, \mathring{M})$. Hence $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big) = \mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$. Thus, by (19.2), $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big) = \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$ for $s < 1/p$ and $\partial M = \emptyset$. This shows that in either case $s > r(-1 + 1/p)$. Consequently, as above, we deduce $\mathfrak{F}_p^{s/\vec{r}}(\mathbb{X}_\kappa \times \mathring{J}, E) = \mathfrak{F}_{p,\kappa}^{s/\vec{r}}$ from [4, Theorem 4.7.1(ii)]. Now the second part of assertion (ii) is implied by (19.5) and (19.6). □

20 Boundary Operators

Throughout this section we suppose $\Gamma \neq \emptyset$.

For $k \in \mathbb{N}$ we consider differential operators on Γ of the form

$$\sum_{i=0}^{k} b_i(\dot{\nabla}) \circ \partial_{\boldsymbol{n}}^i, \quad b_i(\dot{\nabla}) := \sum_{j=0}^{k-i} b_{ij} \cdot \dot{\nabla}^j,$$

where $b_{ij} \cdot \dot{\nabla}^j := (u \mapsto b_{ij} \cdot \dot{\nabla}^j u)$, of course. Thus $b_i(\dot{\nabla})$ is a tangential differential operator of order at most $k - i$.

In fact, we consider systems of such operators. Thus we assume

$$\begin{aligned} &\bullet\quad k, r_i \in \mathbb{N} \text{ with } r_0 < \cdots < r_k, \\ &\bullet\quad \sigma_i, \tau_i \in \mathbb{N} \text{ and } \lambda_i \in \mathbb{R}, \\ &\bullet\quad G_i = (G_i, h_{G_i}, D_{G_i}) \text{ is a fully uniformly regular vector bundle over } \Gamma \end{aligned} \tag{20.1}$$

for $0 \le i \le k$. For abbreviation,

$$\nu_i := (r_i, \sigma_i, \tau_i, \lambda_i), \qquad 0 \le i \le k, \quad \boldsymbol{\nu}_k := (\nu_0, \ldots, \nu_k).$$

Then we define *boundary operators* on Γ of order at most r_i by

$$\mathcal{B}_i(\boldsymbol{b}_i) := \sum_{j=0}^{r_i} \mathcal{B}_{ij}(\boldsymbol{b}_{ij}) \circ \partial_{\boldsymbol{n}}^j, \quad \mathcal{B}_{ij}(\boldsymbol{b}_{ij}) := \sum_{\ell=0}^{r_i-j} b_{ij,\ell} \cdot \dot{\nabla}^\ell,$$

where $\boldsymbol{b}_i := (\boldsymbol{b}_{i0}, \ldots, \boldsymbol{b}_{ir_i})$ and $\boldsymbol{b}_{ij} := (\boldsymbol{b}_{ij,0}, \ldots, \boldsymbol{b}_{ij,r_i-j})$ with $b_{ij,\ell}$ being time-dependent $\text{Hom}(\dot{W}, G_i)$-valued tensor fields on Γ. To be more precise, we introduce data spaces for $s > r_i$ by

$$\mathfrak{B}_{ij}^s(\Gamma, G_i) = \mathfrak{B}_{ij}^s(\Gamma, G_i, \nu_i, \mu) := \prod_{\ell=0}^{r_i-j} B_\infty^{(s-r_i)/\vec{r}, (\lambda_i+r_i-j,\mu)}\big(\mathbb{R}, T_{\tau_i+\sigma}^{\sigma_i+\tau+\ell}\big(\Gamma, \text{Hom}(\dot{W}, G_i)\big)\big)$$

with general point $(\boldsymbol{b}_{ij})$, and

$$\mathfrak{B}_i^s(\Gamma, G_i) = \mathfrak{B}_i^s(\Gamma, G_i, \nu_i, \mu) := \prod_{j=0}^{r_i} \mathfrak{B}_{ij}^s(\Gamma, G_i)$$

whose general point is $\boldsymbol{b}_i$.

Remarks 20.1 **(a)** For the ease of writing we assume that these data spaces are defined on the whole line $\mathbb{R}$. In the following treatment, when studying function spaces on $\mathbb{R}^+$ or $(0, \infty)$ it suffices, of course, to consider data defined on $\mathbb{R}^+$ only. It follows from Theorem 17.2 that this is no restriction of generality to assume that the data are given on all of $\mathbb{R}$.

(b) It should be observed that everything which follows below remains valid if we replace the data space $B_\infty^{(s-r_i)/\vec{r},(\lambda_i+r_i-j,\mu)}$ by $B_\infty^{\bar{s}_i/\vec{r},(\lambda_i+r_i-j,\mu)}$ with $\bar{s} > s - r_i - 1/p$. The selected choice has the advantage that $\mathfrak{B}_i^s(\Gamma, G_i)$ is independent of p. □

Henceforth, $I \in \{ J, (0,\infty) \}$. Given $\boldsymbol{b}_i \in \mathfrak{B}_i^s(\Gamma, G_i)$, it follows from Theorem 16.2, by taking also Theorem 19.1(ii) into consideration if $I = (0, \infty)$, that

$$\mathcal{B}_{ij}(\boldsymbol{b}_{ij}) \in \mathcal{L}\big(B_p^{(s-j-1/p)/\vec{r},(\lambda+j+1/p,\mu)}(I, \dot{V}), B_p^{(s-r_i-1/p)/\vec{r},(\lambda+\lambda_i+r_i+1/p,\mu)}\big(I, T_{\tau_i}^{\sigma_i}(\Gamma, G_i)\big)\big). \tag{20.2}$$

Hence, by Theorem 18.1,

$$\mathcal{B}_i(\boldsymbol{b}_i) \in \mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(I, V), B_p^{(s-r_i-1/p)/\vec{r},(\lambda+\lambda_i+r_i+1/p,\mu)}\big(I, T_{\tau_i}^{\sigma_i}(\Gamma, G_i)\big)\big). \tag{20.3}$$

Finally, we set $G := G_0 \oplus \cdots \oplus G_k$,

$$\mathfrak{B}^s(\Gamma, G) = \mathfrak{B}^s(\Gamma, G, \boldsymbol{\nu}, \mu) := \prod_{i=0}^{k} \mathfrak{B}_i^s(\Gamma, G_i)$$

and

$$\mathcal{B}(\boldsymbol{b}) := \big(\mathcal{B}_0(\boldsymbol{b}_0), \ldots, \mathcal{B}_k(\boldsymbol{b}_k)\big), \quad \boldsymbol{b} := (\boldsymbol{b}_0, \ldots, \boldsymbol{b}_k) \in \mathfrak{B}^s(\Gamma, G).$$

The boundary operator $\mathcal{B}_i(\boldsymbol{b}_i)$ is *normal* if $b_{ir_i} := b_{ir_i,0}$ is λ_i-uniformly contraction surjective, and $\mathcal{B}(\boldsymbol{b})$ is *normal* if each $\mathcal{B}_i(\boldsymbol{b}_i)$, $0 \le i \le k$, has this property. Then

$$\mathfrak{B}_{\text{norm}}^s(\Gamma, G) := \big\{ \boldsymbol{b} \in \mathfrak{B}^s(\Gamma, G) \ ;\ \mathcal{B}(\boldsymbol{b}) \text{ is normal} \big\}.$$

It should be observed that $\Gamma \neq \partial M$, in general. This will allow us to consider boundary value problems where the order of the boundary operators may be different on different parts of ∂M.

Lastly, we introduce the 'boundary space'

$$\partial_{\Gamma\times I}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G) = \partial_{\Gamma\times I}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G,\boldsymbol{\nu},\mu) := \prod_{i=0}^{k} B_p^{(s-r_i-1/p)/\vec{r},(\lambda+\lambda_i+r_i+1/p,\mu)}\big(I, T_{\tau_i}^{\sigma_i}(\Gamma, G_i)\big).$$

The following lemma shows that it is an image space for the boundary operators under consideration.

Lemma 20.2 *If $s > r_k + 1/p$ and $\boldsymbol{b} \in \mathfrak{B}^s(\Gamma, G)$, then*

$$\mathcal{B}(\boldsymbol{b}) \in \mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(I,V), \partial_{\Gamma\times I}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)\big).$$

The map $\mathcal{B}(\cdot) = \big(\boldsymbol{b} \mapsto \mathcal{B}(\boldsymbol{b})\big)$ is linear and continuous, and $\mathfrak{B}^s_{\rm norm}(\Gamma, G)$ is open in $\mathfrak{B}^s(\Gamma, G)$.

Proof. The first assertion is immediate from (20.3). The second one is obvious, and the last one is a consequence of Proposition 14.8. □

Theorem 20.3 *Suppose assumption* (20.1) *applies. Let $s > r_k + 1/p$ and $\boldsymbol{b} \in \mathfrak{B}^s_{\rm norm}(\Gamma, G)$. Then $\mathcal{B}(\boldsymbol{b})$ is a retraction from $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V)$ onto $\partial_{\Gamma\times J}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$. There exists an analytic map*

$$\mathcal{B}^c(\cdot): \ \mathfrak{B}^s_{\rm norm}(\Gamma, G) \to \mathcal{L}\big(\partial_{\Gamma\times J}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G), \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V)\big)$$

such that

(i) *$\mathcal{B}^c(\boldsymbol{b})$ is a coretraction for $\mathcal{B}(\boldsymbol{b})$,*

(ii) *$\partial_{\boldsymbol{n}}^j \circ \mathcal{B}^c(\boldsymbol{b}) = 0$ for $0 \le j < s - 1/p$ with $j \notin \{r_0, \dots, r_k\}$.*

If $J = \mathbb{R}^+$, then $\mathcal{B}^c(\boldsymbol{b})g \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\big((0,\infty), V\big)$ whenever $g \in \partial_{\Gamma\times(0,\infty)}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$.

Proof. (1) We deduce from Theorem 14.9 for $0 \le i \le k$ the existence of an analytic map $A_i^c(\cdot)$ from

$$B_{\infty,\rm surj}^{(s-r_i)/\vec{r},(\lambda_i,\mu)}\big(J, T_{\tau_i+\sigma}^{\sigma_i+\tau}\big(\Gamma, {\rm Hom}(\mathring{W}, G_i)\big)\big)$$

into

$$\mathcal{L}\big(B_p^{(s-r_i-1/p)/\vec{r},(\lambda+\lambda_i+r_i+1/p,\mu)}\big(J, T_{\tau_i}^{\sigma_i}(\Gamma, G_i)\big), B_p^{(s-r_i-1/p)/\vec{r},(\lambda+r_i+1/p,\mu)}(J, \mathring{V})\big)$$

such that $A_i^c(a)$ is a right inverse for $A_i(a) := (u \mapsto a\cdot u)$.

(2) Suppose $\boldsymbol{b} \in \mathfrak{B}^s_{\rm norm}(\Gamma, G)$. For $0 \le i \le k$ we set

$$\mathcal{C}_{r_i}(\boldsymbol{b}_i) := -\sum_{j=0}^{r_i-1} A_i^c(b_{ir_i})\mathcal{B}_{ij}(\boldsymbol{b}_{ij}) \circ \partial_{\boldsymbol{n}}^j.$$

It follows from (20.2), step (1), and Theorem 18.1 that

$$\mathcal{C}_{r_i}(\boldsymbol{b}_i) \in \mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V), B_p^{(s-j-1/p)/\vec{r},(\lambda+r_i+1/p,\mu)}(J,\mathring{V})\big) \tag{20.4}$$

and the map $\boldsymbol{b}_i \to \mathcal{C}_{r_i}(\boldsymbol{b}_i)$ is analytic.

Let $N := [s - 1/p]_-$ and define

$$\mathcal{C} = (\mathcal{C}_0, \dots, \mathcal{C}_N) \in \mathcal{L}\Big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V), \prod_{\ell=0}^{N} B_p^{(s-\ell-1/p)/\vec{r},(\lambda+\ell+1/p,\mu)}(J,\mathring{V})\Big)$$

by setting $\mathcal{C}_\ell := 0$ for $0 \le \ell \le N$ with $\ell \notin \{r_0, \dots, r_k\}$.

(3) Assume $g=(g_0,\dots,g_k)\in\partial_{\Gamma\times J}\mathfrak{F}_p^{s/\vec{r}}(G)$. Define

$$h=(h_0,\dots,h_N)\in\prod_{\ell=0}^N B_p^{(s-\ell-1/p)/\vec{r},(\lambda+\ell+1/p,\mu)}(J,\mathring{V})$$

by $h_{r_i}:=A_i^c(b_{ir_i})g_i$ for $0\le i\le k$, and $h_\ell:=0$ otherwise.

By Theorem 18.1 there exists for $j\in\{0,\dots,N\}$ a universal coretraction $(\gamma_{\boldsymbol{n}}^j)^c$ for $\partial_{\boldsymbol{n}}^j$ satisfying

$$\partial_{\boldsymbol{n}}^\ell\circ(\gamma_{\boldsymbol{n}}^j)^c=\delta^{\ell j}\mathrm{id},\qquad 0\le\ell\le j. \tag{20.5}$$

We put $u_0:=(\gamma_{\boldsymbol{n}}^0)^c h_0\in\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V)$. Suppose $1\le j\le N$ and $u_0,u_1,\dots,u_{j-1}$ have already been defined. Set

$$u_j:=u_{j-1}+(\gamma_{\boldsymbol{n}}^j)^c\,(h_j+\mathcal{C}_j u_{j-1}-\partial_{\boldsymbol{n}}^j u_{j-1}). \tag{20.6}$$

This defines $u_0,u_1,\dots,u_N\in\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V)$. It follows from (20.5) and (20.6)

$$\partial_{\boldsymbol{n}}^j u_j=h_j+\mathcal{C}_j u_{j-1},\qquad 0\le j\le N, \tag{20.7}$$

and

$$\partial_{\boldsymbol{n}}^\ell u_j=\partial_{\boldsymbol{n}}^\ell u_{j-1},\qquad 0\le\ell\le j-1,\quad 1\le j\le N.$$

The latter relation implies

$$\partial_{\boldsymbol{n}}^\ell u_j=\partial_{\boldsymbol{n}}^\ell u_n,\qquad 0\le\ell<j<n\le N.$$

Hence, since $\mathcal{C}_j$ involves $\partial_{\boldsymbol{n}}^0,\dots,\partial_{\boldsymbol{n}}^{j-1}$ only, we deduce from (20.7)

$$\partial_{\boldsymbol{n}}^j u_n=h_j+\mathcal{C}_j u_n,\qquad 0\le j\le n\le N.$$

If $j=r_i$, then we apply $A_i(b_{ir_i})$ to this equation to find

$$\mathcal{B}_i u_n=g_i,\qquad r_i\le n\le N. \tag{20.8}$$

For $0\le i\le k$ we set $G^i:=G_0\oplus\cdots\oplus G_i$ and $\boldsymbol{\nu}^i:=(\nu_1,\dots,\nu_i)$ as well as $\boldsymbol{b}^i:=(\boldsymbol{b}_0,\dots,\boldsymbol{b}_i)$. Then it follows from (20.3) that

$$\mathcal{B}^i(\boldsymbol{b}^i):=\big(\mathcal{B}_0(\boldsymbol{b}_0),\dots,\mathcal{B}_i(\boldsymbol{b}_i)\big)\in\mathcal{L}\big(\mathfrak{F}_p^{t/\vec{r},\vec{\omega}}(J,V),\partial_{\Gamma\times J}\mathfrak{F}_p^{t/\vec{r},\vec{\omega}}(G^i,\boldsymbol{\nu}^i,\mu)\big) \tag{20.9}$$

for $r_i+1/p<t\le s$. We define $\mathcal{B}^{ic}(\boldsymbol{b}^i)$ by

$$\mathcal{B}^{ic}(\boldsymbol{b}^i)\,(g_0,\dots,g_i):=u_{r_i}.$$

It follows from (20.4), (20.6), and Theorem 18.1 that

$$\mathcal{B}^{ic}(\boldsymbol{b}^i)\in\mathcal{L}\big(\partial_{\Gamma\times J}\mathfrak{F}_p^{t/\vec{r},\vec{\omega}}(G^i,\boldsymbol{\nu}^i,\mu),\mathfrak{F}_p^{t/\vec{r},\vec{\omega}}(J,V)\big),\qquad r_i+1/p<t\le s. \tag{20.10}$$

Furthermore, (20.8) and the definition of h imply

$$\mathcal{B}_\alpha(\boldsymbol{b}_\alpha)\mathcal{B}^{ic}(\boldsymbol{b}^i)(g_0,\dots,g_i)=\mathcal{B}_\alpha(\boldsymbol{b}_\alpha)\mathcal{B}^{jc}(\boldsymbol{b}^j)(g_0,\dots,g_j)=g_\alpha,\qquad 0\le\alpha\le i\le j\le k, \tag{20.11}$$

and $\partial_{\boldsymbol{n}}^j\mathcal{B}^{ic}(\boldsymbol{b}^i)=0$ for $0\le j<t-1/p$ with $j\notin\{r_0,\dots,r_k\}$.

Now we set $\mathcal{B}^c(\boldsymbol{b}):=\mathcal{B}^{kc}(\boldsymbol{b})$. Then (20.9) and (20.11) show that it is a right inverse for $\mathcal{B}(\boldsymbol{b})$. It is a consequence of step (2) and (20.6) that $\mathcal{B}^c(\cdot)$ is analytic. Due to Theorem 18.1 it is easy to see that the last assertion applies as well. □

There is a similar, though much simpler result concerning the ‘extension of initial conditions’.

Theorem 20.4 *Suppose* $0 \le j_0 < \cdots < j_\ell$ *and* $s > r(j_\ell + 1/p)$. *Set* $\mathcal{C} := (\partial_{t=0}^{j_0}, \dots, \partial_{t=0}^{j_\ell})$ *and*

$$B_p^{s-r(\boldsymbol{j}_\ell+1/p),\lambda+\mu(\boldsymbol{j}_\ell+1/p)}(V) := \prod_{i=0}^{\ell} B_p^{s-r(j_i+1/p),\lambda+\mu(j_i+1/p)}(V). \tag{20.12}$$

Then $\mathcal{C}$ *is a retraction from* $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$ *onto* $B_p^{s-\vec{r}(\boldsymbol{j}_\ell+1/p),\lambda+\mu(\boldsymbol{j}_\ell+1/p)}(V)$, *and there exists a coretraction* $\mathcal{C}^c$ *satisfying* $\partial_{t=0}^{j} \circ \mathcal{C}^c = 0$ *for* $0 \le j < s/r - 1/p$ *with* $j \notin \{j_0, \dots, j_\ell\}$.

P r o o f. Theorem 18.1(ii) guarantees that $\mathcal{C}$ is a continuous linear map from $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$ into (20.12). Due to that theorem the assertion follows from step (3) of the proof of Theorem 20.3 using the following modifications: $h_{j_i} := g_i$ for $0 \le i \le \ell$ and

$$u_j := u_{j-1} + (\gamma_{t=0}^{j})^c (h_j - \partial_{t=0}^{j} u_{j-1})$$

with $u_{-1} := 0$. □

Now we suppose $\Gamma \ne \emptyset$ and $J = \mathbb{R}^+$. We write $\Sigma := \Gamma \times \mathbb{R}^+$ for the *lateral boundary over* Γ and recall that $M_0 := M \times \{0\}$ is the initial boundary. Then $\Sigma \cap M_0 = \Gamma \times \{0\} =: \Gamma_0$ is the *corner manifold over* Γ. We suppose

- assumption (20.1) is satisfied,
- $\ell \in \mathbb{N}$ and $s > \max\{r_k + 1/p,\ r(\ell + 1/p)\}$. (20.13)

We set $\mathcal{C} := \overrightarrow{\partial_{t=0}^{\ell}} := (\partial_{t=0}^{0}, \dots, \partial_{t=0}^{\ell})$. Then, by Theorem 20.4, $\mathcal{C}$ is a retraction from $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$ onto

$$B_p^{s-r(\boldsymbol{\ell}+1/p),\lambda+\mu(\boldsymbol{\ell}+1/p)}(V) := \prod_{j=0}^{\ell} B_p^{s-r(j+1/p),\lambda+\mu(j+1/p)}(V).$$

By Theorem 20.3 $\mathcal{B}(\boldsymbol{b})$ is for $\boldsymbol{b} \in \mathfrak{B}^s_{\mathrm{norm}}(\Gamma, G)$ a retraction from $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$ onto $\partial_\Sigma \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$. We put

$$\partial_{\Sigma\cup M_0} \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G) := \partial_\Sigma \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G) \times B_p^{s-r(\boldsymbol{\ell}+1/p),\lambda+\mu(\boldsymbol{\ell}+1/p)}(V)$$

and $\vec{\mathcal{B}}(\cdot) := \bigl(\mathcal{B}(\cdot), \mathcal{C}\bigr)$. Then

$$\vec{\mathcal{B}}(\cdot) :\ \mathfrak{B}^s_{\mathrm{norm}}(\Gamma, G) \to \mathcal{L}\bigl(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V), \partial_{\Sigma\cup M_0} \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)\bigr)$$

is the restriction of a continuous linear map to the open subset $\mathfrak{B}^s_{\mathrm{norm}}(\Gamma, G)$ of $\mathfrak{B}^s(\Gamma, G)$, hence analytic.

However, $\vec{\mathcal{B}}(\boldsymbol{b})$ is not surjective, in general. Indeed, suppose

$$0 \le i \le k, \quad 0 \le j \le \ell, \quad s > r_i + 1/p + r(j + 1/p) =: r_{ij}.$$

Then we deduce from (20.3) and Theorem 18.1(ii)

$$\partial_{t=0}^{j} \circ \mathcal{B}_i(\boldsymbol{b}) \in \mathcal{L}\bigl(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V), B_p^{s-r_{ij},\lambda+\lambda_i+r_{ij}}\bigl(T_{\tau_i}^{\sigma_i}(\Gamma_0, G_i)\bigr)\bigr).$$

Furthermore, $\partial^j \bigl(\mathcal{B}_i(\boldsymbol{b})u\bigr) = \mathcal{B}_i^{(j)}(\boldsymbol{b})u$, where

$$\mathcal{B}_i^{(j)}(\boldsymbol{b})u = \sum_{\alpha=0}^{j} \binom{j}{\alpha} \mathcal{B}_i(\partial^{j-\alpha}\boldsymbol{b}_i) \circ \partial^\alpha.$$

Theorem 16.1 implies

$$\partial^{j-\alpha}\boldsymbol{b}_i \in \mathfrak{B}_i^s\bigl(\Gamma, G_i, \bigl(r_i + r(j-\alpha), \sigma_i, \tau_i, \lambda_i + \mu(j-\alpha)\bigr), \mu\bigr).$$

From this and (20.3) we infer that $\mathcal{B}_i^{(j)}(\boldsymbol{b})$ possesses the same mapping properties as $\partial^j\circ\mathcal{B}_i(\boldsymbol{b})$. Set

$$\mathcal{B}_i^{(j)}(0)\vec{v}_j=\mathcal{B}_i^{(j)}(\boldsymbol{b},0)\vec{v}_j:=\sum_{\alpha=0}^{j}\binom{j}{\alpha}\mathcal{B}_i(\partial_{t=0}^{j-\alpha}\boldsymbol{b}_i)v_\alpha,\qquad \vec{v}_j:=(v_0,\dots,v_j)$$

with $v_\alpha\in B_p^{s-r(\alpha+1/p),\lambda+\mu(\alpha+1/p)}(V)$. Then $\mathcal{B}_i^{(j)}(0)$ is a continuous linear map

$$\prod_{\alpha=0}^{j}B_p^{s-r(\alpha+1/p),\lambda+\mu(\alpha+1/p)}(V)\to B_p^{s-r_{ij},\lambda+\lambda_j+r_{ij}}\bigl(T_{\tau_i}^{\sigma_i}(\Gamma_0,G_i)\bigr)$$

and $\boldsymbol{b}\mapsto\mathcal{B}_i^{(j)}(0)$ is the restriction of a linear and continuous map to $\mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)$. Furthermore,

$$\partial_{t=0}^j\bigl(\mathcal{B}_i(\boldsymbol{b})u\bigr)=\mathcal{B}_i^{(j)}(0)\overrightarrow{\partial_{t=0}^j}u,\qquad u\in\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V).\tag{20.14}$$

We denote for $\boldsymbol{b}\in\mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)$ by

$\partial_{\vec{\mathcal{B}}(\boldsymbol{b})}^{cc}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ the set of all $(g,h)\in\partial_{\Sigma\cup M_0}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ satisfying the **compatibility conditions**

$$\partial_{t=0}^j g_i=\mathcal{B}_i^{(j)}(0)\vec{h}_j$$

for $0\le i\le k$ and $0\le j\le\ell$ with $r_i+1/p+r(j+1/p)<s$.

The linearity and continuity of $\partial_{t=0}^j$ and $\mathcal{B}_i^{(j)}(0)$ guarantee that $\partial_{\vec{\mathcal{B}}(\boldsymbol{b})}^{cc}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ is a closed linear subspace of $\partial_{\Sigma\cup M_0}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$. By the preceding considerations it contains the range of $\vec{\mathcal{B}}(\boldsymbol{b})$. The following theorem shows that, in fact, $\partial_{\vec{\mathcal{B}}(\boldsymbol{b})}^{cc}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)=\mathrm{im}(\vec{\mathcal{B}})$, provided $\boldsymbol{b}\in\mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)$.

Theorem 20.5 *Let assumption* (20.13) *be satisfied and suppose*

$$s\notin\bigl\{\,r_i+1/p+r(j+1/p)\;;\;0\le i\le k,\;0\le j\le\ell\,\bigr\}.$$

Then $\vec{\mathcal{B}}(\boldsymbol{b})$ is for $\boldsymbol{b}\in\mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)$ a retraction from $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V)$ onto $\partial_{\vec{\mathcal{B}}(\boldsymbol{b})}^{cc}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$. There exists an analytic map

$$\vec{\mathcal{B}}^c(\cdot):\ \mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)\to\mathcal{L}\bigl(\partial_{\Sigma\cup M_0}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G),\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V)\bigr)\tag{20.15}$$

such that $\vec{\mathcal{B}}^c(\boldsymbol{b})\,|\,\partial_{\vec{\mathcal{B}}(\boldsymbol{b})}^{cc}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ is a coretraction for $\vec{\mathcal{B}}(\boldsymbol{b})$.

Proof. By the preceding remarks it suffices to construct $\vec{\mathcal{B}}^c(\cdot)$ satisfying (20.15) such that its restriction to $\partial_{\vec{\mathcal{B}}(\boldsymbol{b})}^{cc}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ is a right inverse for $\vec{\mathcal{B}}(\boldsymbol{b})$.

By Theorem 20.3 there exists an analytic map

$$\mathcal{B}^c(\cdot):\ \mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)\to\mathcal{L}\bigl(\partial_\Sigma\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G),\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+,V)\bigr)$$

such that

$$v\in\partial_{\Gamma\times(0,\infty)}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)\quad\Rightarrow\quad\mathcal{B}^c(\boldsymbol{b})v\in\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}\bigl((0,\infty),V\bigr)\tag{20.16}$$

for $\boldsymbol{b}\in\mathfrak{B}_{\mathrm{norm}}^s(\Gamma,G)$.

Let $\mathcal{C}^c$ be a coretraction for $\mathcal{C}$. Its existence is guaranteed by Theorem 20.4. Given $(g,h) \in \partial_{\Sigma\cup M_0}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ and $\boldsymbol{b} \in \mathfrak{B}^s_{\mathrm{norm}}(\Gamma, G)$, set

$$\vec{\mathcal{B}}^c(\boldsymbol{b})(g,h) := \mathcal{C}^c h + \mathcal{B}^c(\boldsymbol{b})\big(g - \mathcal{B}(\boldsymbol{b})\mathcal{C}^c h\big).$$

Then $\mathcal{B}^c(\cdot)$ satisfies (20.15) and is analytic. Furthermore,

$$\mathcal{B}(\boldsymbol{b})\big(\vec{\mathcal{B}}^c(\boldsymbol{b})(g,h)\big) = g. \tag{20.17}$$

We fix $\boldsymbol{b} \in \mathfrak{B}^s_{\mathrm{norm}}(\Gamma, G)$ and write $\mathcal{B} = \mathcal{B}(\boldsymbol{b})$ and $\mathcal{B}^c = \mathcal{B}^c(\boldsymbol{b})$. For $(g,h) \in \partial^{cc}_{\vec{\mathcal{B}}(\boldsymbol{b})}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$ we set

$$v := g - \mathcal{B}\mathcal{C}^c h \in \partial_\Sigma \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G).$$

Suppose $0 \le i \le k$. Then

$$v_i = g_i - \mathcal{B}_i\mathcal{C}^c h \in B_p^{(s-r_i-1/p)/\vec{r},(\lambda+\lambda_i+r_i+1/p,\mu)}(\mathbb{R}^+, V_i),$$

where $V_i := T^{\sigma_i}_{\tau_i}(G_i)$. Let $j_i \in \{0,\ldots,\ell\}$ be the largest integer satisfying $r_i + 1/p + r(j_i + 1/p) < s$. Then, by (20.14),

$$\partial^j_{t=0} v_i = \partial^j_{t=0} g_i - \partial^j_{t=0}(\mathcal{B}_i\mathcal{C}^c h) = \partial^j_{t=0} g_i - \mathcal{B}_i^{(j)}(0)\overrightarrow{\partial^j_{t=0}\mathcal{C}^c h} = \partial^j_{t=0} g_i - \mathcal{B}_i^{(j)}(0)\vec{h}_j = 0$$

for $0 \le j \le j_i$. Hence $r(j_i + 1/p) < s - r_i - 1/p < r(j_i + 1 + 1/p)$ and Theorem 19.1(ii) imply

$$v_i \in B_p^{(s-r_i-1/p)/\vec{r},(\lambda+\lambda_i+r_i+1/p,\mu)}\big((0,\infty), V_i\big). \tag{20.18}$$

If there is no such j_i, then $s - r_i - 1/p < r/p$. In this case that theorem guarantees (20.18) also. This shows that $v \in \partial_{\Gamma\times(0,\infty)}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$. Hence $\mathcal{C}\mathcal{B}^c v = 0$ by (20.16) and Theorem 19.1(ii) and since $s > r(j + 1/p)$ for $0 \le j \le \ell$. Consequently, $\mathcal{C}\big(\vec{\mathcal{B}}^c(\boldsymbol{b})(g,h)\big) = h$ for $(g,h) \in \partial^{cc}_{\vec{\mathcal{B}}(\boldsymbol{b})}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G)$. Together with (20.17) this proves the theorem. □

Remark 20.6 Let assumption (20.1) be satisfied. Suppose

$$r_0 + 1/p < s < r/p, \qquad s \notin \{\, r_i + 1/p\ ;\ 1 \le i \le k\,\}.$$

Then there is a lateral boundary operator $\mathcal{B}$ only, since there is no initial trace. Thus this case is covered by Theorem 20.3.

Assume

$$r/p < s < r_0 + 1/p, \qquad s \notin \big\{\, r(j + 1/p)\ ;\ 1 \le j \le \ell\,\big\}.$$

Then there is no lateral trace operator and we are in a situation to which Theorem 20.4 applies.

Lastly, if $-1 + 1/p < s < \min\{r_0 + 1/p,\ r/p\}$, then there is neither a lateral nor an initial trace operator and $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V) = \mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+, V)$. □

The theorems on the ‘extension of boundary values’ proved in this section are of great importance in the theory of nonhomogeneous time-dependent boundary value problems. The only results of this type available in the literature concern the case where M is an m-dimensional compact submanifold of $\mathbb{R}^m$. In this situation an anisotropic extension theorem involving compatibility conditions has been proved by P. Grisvard in [14] for the case where $s \in r\mathbb{N}^\times$, and in [15] if $p = 2$ (also see J.-L. Lions and E. Magenes [31, Chapter 4, Section 2] for the Hilbertian case) by means of functional analytical techniques. If $s = r = 2$, then corresponding results are derived in O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural’ceva [30] by studying heat potentials. In contrast to our work, in all these publications the exceptional values $r_i + 1/p + r(j + 1/p)$ for s are considered also.

21 Interpolation

In Section 8 the anisotropic spaces $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$ have been defined for $s>0$ by interpolating between anisotropic Sobolev spaces. From this we could derive some interpolation properties by means of reiteration theorems. However, such results would be restricted to spaces with one and the same value of λ. In this section we prove general interpolation theorems for anisotropic Bessel potential, Besov, and Besov-Hölder spaces involving different values of s and λ.

Reminding that $\xi_\theta=(1-\theta)\xi_0+\theta\xi_1$ for $\xi_0,\xi_1\in\mathbb{R}$ and $0\le\theta\le 1$, we set $\vec{\omega}_\theta:=(\lambda_\theta,\mu)$ for $\lambda_0,\lambda_1\in\mathbb{R}$. We also recall that $(\cdot,\cdot)_\theta=[\cdot,\cdot]_\theta$ if $\mathfrak{F}=H$, and $(\cdot,\cdot)_\theta=(\cdot,\cdot)_{\theta,p}$ if $\mathfrak{F}=B$.

Theorem 21.1 *Suppose* $-\infty<s_0<s_1<\infty,\ \lambda_0,\lambda_1\in\mathbb{R}$, *and* $0<\theta<1$.

(i) *Assume* $s_0>-1+1/p$ *if* $\partial M\ne\emptyset$. *Then*

$$(\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0},\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1})_\theta\doteq\mathfrak{F}_p^{s_\theta/\vec{r},\vec{\omega}_\theta}\doteq[\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0},\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1}]_\theta \tag{21.1}$$

and

$$(H_p^{s_0/\vec{r},\vec{\omega}_0},H_p^{s_1/\vec{r},\vec{\omega}_1})_{\theta,p}\doteq B_p^{s_\theta/\vec{r},\vec{\omega}_\theta}. \tag{21.2}$$

(ii) *If* $s_0>0$, *then*

$$[B_\infty^{s_0/\vec{r},\vec{\omega}_0},B_\infty^{s_1/\vec{r},\vec{\omega}_1}]_\theta\doteq B_\infty^{s_\theta/\vec{r},\vec{\omega}_\theta}$$

and

$$[b_\infty^{s_0/\vec{r},\vec{\omega}_0},b_\infty^{s_1/\vec{r},\vec{\omega}_1}]_\theta\doteq b_\infty^{s_\theta/\vec{r},\vec{\omega}_\theta}.$$

P r o o f. (1) Let X be a Banach space and $\delta>0$. Then $\delta X:=(X,\|\cdot\|_{\delta X})$, where $\|x\|_{\delta X}:=\|\delta^{-1}x\|_X$ for $x\in X$. Thus δX is the image space of X under the map $x\mapsto\delta x$ so that this function is an isometric isomorphism from X onto δX.

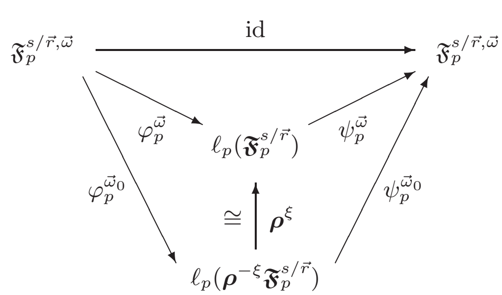
Assume X_β is a Banach space and $\delta_\beta>0$ for each β in a countable index set B. Then we set $\boldsymbol{\delta X}:=\prod_\beta\delta_\beta X_\beta$ and $\boldsymbol{\delta x}:=(\delta_\beta x_\beta)$. Hence $\boldsymbol{\delta}:=(\boldsymbol{x}\mapsto\boldsymbol{\delta x})\in\mathcal{L}\mathrm{is}(\boldsymbol{X},\boldsymbol{\delta X})$.

Let (X_0,X_1) be a pair of Banach spaces such that X_j is continuously injected in some locally convex space for $j=0,1$, that is, (X_0,X_1) is an interpolation couple. Suppose $\{\cdot,\cdot\}_\theta\in\bigl\{[\cdot,\cdot]_\theta,(\cdot,\cdot)_{\theta,p}\bigr\}$. Then interpolation theory guarantees

$$\{\delta_0X_0,\delta_1X_1\}_\theta=\delta_0^{1-\theta}\delta_1^\theta\{X_0,X_1\}_\theta,\qquad\delta_0,\delta_1>0, \tag{21.3}$$

(e.g., [50, formula (7) in Section 3.4.1]).

(2) Let $J=\mathbb{R}$ and $s>-1+1/p$ if $\partial M\ne\emptyset$. Put $\xi:=\lambda-\lambda_0$. Then $\varphi_{p,\kappa}^{\vec{\omega}}=\rho_\kappa^\xi\varphi_{p,\kappa}^{\vec{\omega}_0}$ and $\psi_{p,\kappa}^{\vec{\omega}}=\rho_\kappa^{-\xi}\psi_{p,\kappa}^{\vec{\omega}_0}$ imply, due to Theorem 9.3, that the diagram



is commuting. Hence $\psi_p^{\vec{\omega}_0}$ is for each s a retraction from $\ell_p(\boldsymbol{\rho}^{-\xi}\boldsymbol{\mathfrak{F}}_p^{s/\vec{r}})$ onto $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$, and $\varphi_p^{\vec{\omega}_0}$ is a coretraction.

(3) Let $\xi:=\lambda_1-\lambda_0$. By Theorem 9.3 and the preceding step each of the maps

$$\psi_p^{\vec{\omega}_0}\ :\ \ell_p(\boldsymbol{\mathfrak{F}}_p^{s_0/\vec{r}})\to\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0},\quad \psi_p^{\vec{\omega}_0}\ :\ \ell_p(\boldsymbol{\rho}^{-\xi}\boldsymbol{\mathfrak{F}}_p^{s_1/\vec{r}})\to\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1}$$

is a retraction, and $\varphi_p^{\vec\omega_0}$ is a coretraction. Thus, by interpolation,

$$\psi_p^{\vec\omega_0}:\ \big\{\ell_p(\boldsymbol{\mathfrak{F}}_p^{s_0/\vec r}),\ell_p(\boldsymbol{\rho}^{-\xi}\boldsymbol{\mathfrak{F}}_p^{s_1/\vec r})\big\}_\theta\to\big\{\mathfrak{F}_p^{s_0/\vec r,\vec\omega_0},\mathfrak{F}_p^{s_1/\vec r,\vec\omega_1}\big\}_\theta \tag{21.4}$$

is a retraction, and $\varphi_p^{\vec\omega_0}$ is a coretraction. From [50, Theorem 1.18.1] and (21.3) we infer

$$\big\{\ell_p(\boldsymbol{\mathfrak{F}}_p^{s_0/\vec r}),\ell_p(\boldsymbol{\rho}^{-\xi}\boldsymbol{\mathfrak{F}}_p^{s_1/\vec r})\big\}_\theta\doteq\ell_p\big(\boldsymbol{\rho}^{-\theta\xi}\{\boldsymbol{\mathfrak{F}}_p^{s_0/\vec r},\boldsymbol{\mathfrak{F}}_p^{s_1/\vec r}\}_\theta\big). \tag{21.5}$$

(Recall the definition of $(\boldsymbol{E},\boldsymbol{F})_\theta$ after (9.1).) Suppose $\partial M=\emptyset$. Then [4, formulas (3.3.12) and (3.4.1) and Theorem 3.7.1(iv)] imply

$$(\mathfrak{F}_{p,\kappa}^{s_0/\vec r},\mathfrak{F}_{p,\kappa}^{s_1/\vec r})_\theta\doteq\mathfrak{F}_{p,\kappa}^{s_\theta/\vec r},\quad (H_{p,\kappa}^{s_0/\vec r},H_{p,\kappa}^{s_1/\vec r})_{\theta,p}\doteq B_{p,\kappa}^{s_\theta/\vec r}\doteq[B_{p,\kappa}^{s_0/\vec r},B_{p,\kappa}^{s_1/\vec r}]_\theta. \tag{21.6}$$

This is due to the fact that, on account of [4, Corollary 3.3.4 and Theorem 3.7.1(i)], the definition of $\mathfrak{F}_{p,\kappa}^{s/\vec r}$ for $s<0$ used in that publication coincides with the definition by duality employed in this paper.

If $\partial M\neq\emptyset$, then it follows from $s>-1+1/p$ and Theorem 4.7.1(ii) of [4] by the same arguments that (21.6) holds in this case as well.

Thus, in either case, due to (21.4)–(21.6) $\psi_p^{\vec\omega_0}$ is a retraction from $\ell_p(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{\mathfrak{F}}_p^{s_\theta/\vec r})$ onto $(\mathfrak{F}_p^{s_0/\vec r,\vec\omega_0},\mathfrak{F}_p^{s_1/\vec r,\vec\omega_1})_\theta$ and onto $[\mathfrak{F}_p^{s_0/\vec r,\vec\omega_0},\mathfrak{F}_p^{s_1/\vec r,\vec\omega_1}]_\theta$, and $\varphi_p^{\vec\omega_0}$ is a coretraction. On the other hand, we infer from step (2), setting $\xi=\theta(\lambda_1-\lambda_0)$, that $\psi_p^{\vec\omega_0}$ is a retraction from $\ell_p(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{\mathfrak{F}}_p^{s_\theta/\vec r})$ onto $\mathfrak{F}_p^{s_\theta/\vec r,\vec\omega_\theta}$, and $\varphi_p^{\vec\omega_0}$ is a coretraction. This implies the validity of (21.1) if $J=\mathbb{R}$. The proof for (21.2) is similar.

(4) Assume $J=\mathbb{R}^+$. In this case we get assertion (i) by Theorem 17.1(i) in conjunction with what has just been proved.

(5) Set $\xi=\lambda_1-\lambda_0$. Then as above, we infer from Theorem 12.8 that $\psi_\infty^{\vec\omega_0}$ is a retraction from $\ell_\infty(\boldsymbol{B}_\infty^{s_0/\vec r})$ onto $B_\infty^{s_0/\vec r,\vec\omega_0}$ and from $\ell_\infty(\boldsymbol{\rho}^{-\xi}\boldsymbol{B}_\infty^{s_1/\vec r})$ onto $B_\infty^{s_1/\vec r,\vec\omega_0}$, and $\varphi_\infty^{\vec\omega_0}$ is a coretraction. Hence

$$\psi_\infty^{\vec\omega_0}:\ \big[\ell_\infty(\boldsymbol{B}_\infty^{s_0/\vec r}),\ell_\infty(\boldsymbol{\rho}^{-\xi}\boldsymbol{B}_\infty^{s_1/\vec r})\big]_\theta\to[B_\infty^{s_0/\vec r,\vec\omega_0},B_\infty^{s_1/\vec r,\vec\omega_1}]_\theta \tag{21.7}$$

is a retraction, and $\varphi_\infty^{\vec\omega_0}$ is a coretraction.

We use the notation of Sections 11 and 12. Then, setting $B_\infty^{s/\vec r}(\mathbb{M}\times J,\boldsymbol{\rho}^{-\xi}\boldsymbol{E}):=\prod_\kappa B_\infty^{s/\vec r}(\mathbb{M}\times J,\rho_\kappa^{-\xi}E)$, it is not difficult to verify (cf. Lemma 11.12) that

$$\widetilde{\boldsymbol{f}}\in\mathcal{L}\mathrm{is}\big(B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{\rho}^{-\xi}\boldsymbol{E})\big),\ell_\infty(\boldsymbol{\rho}^{-\xi}\boldsymbol{B}_\infty^{s/\vec r})\big) \tag{21.8}$$

for $s>0$. Hence we deduce from (21.7) that the map $\Psi_\infty^{\vec\omega_0}=\psi_\infty^{\vec\omega_0}\circ\widetilde{\boldsymbol{f}}$

$$\big[B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{E})\big),B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{\rho}^{-\xi}\boldsymbol{B}_\infty^{s/\vec r})\big)\big]_\theta\to[B_\infty^{s_0/\vec r,\vec\omega_0},B_\infty^{s_1/\vec r,\vec\omega_1}]_\theta \tag{21.9}$$

is a retraction, and $\Phi_\infty^{\vec\omega_0}$ is a coretraction.

(6) Set $\mathfrak{K}_0:=\{\,\kappa\in\mathfrak{K}\,;\ \rho_\kappa\le 1\,\}$ and $\mathfrak{K}_1:=\mathfrak{K}\backslash\mathfrak{K}_0$. Let X_κ be a Banach space for $\kappa\in\mathfrak{K}$ and set $\boldsymbol{X}:=\prod_\kappa X_\kappa$ and $\boldsymbol{X}_j:=\prod_{\kappa\in\mathfrak{K}_j}X_\kappa$ as well as $\ell_\infty^j(\boldsymbol{X}):=\ell_\infty(\boldsymbol{X}_j)$ for $j=0,1$. Then $\ell_\infty(\boldsymbol{X})\doteq\ell_\infty^0(\boldsymbol{X})\oplus\ell_\infty^1(\boldsymbol{X})$. Consequently,

$$B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{\rho}^{-\eta}\boldsymbol{E})\big)\doteq B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^0(\boldsymbol{\rho}^{-\eta}\boldsymbol{E})\big)\oplus B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^1(\boldsymbol{\rho}^{-\eta}\boldsymbol{E})\big) \tag{21.10}$$

for $\eta\in\{0,\xi\}$.

(7) Put $Y_0:=B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^0(\boldsymbol{\rho}^{-\xi}\boldsymbol{E})\big)$ and $Y_1:=B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^0(\boldsymbol{E})\big)$. It follows from $\rho_\kappa\le 1$ for $\kappa\in\mathfrak{K}_0$ that $Y_1\hookrightarrow Y_0$. Define a linear operator A_0 in Y_0 with domain Y_1 by $A_0u=\boldsymbol{\rho}^{-\xi}u$. Then A_0 is closed, $-A_0$ contains the sector $S_{\pi/4}$ in its resolvent set and satisfies $\|(\lambda+A_0)^{-1}\|_{\mathcal{L}(Y_0)}\le c/|\lambda|$ for $\lambda\in S_{\pi/4}$. Furthermore,

$$\|(-A_0)^z\|_{\mathcal{L}(Y_0)}\le\sup_{\kappa\in\mathfrak{K}_0}\rho_\kappa^{-\xi\,\mathrm{Re}\,z}\le 1,\qquad \mathrm{Re}\,z\le 0.$$

Hence Seeley's theorem, alluded to in the proof of Theorem 11.1, implies

$$\begin{aligned}\big[B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^0(\boldsymbol{E})\big),B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^0(\boldsymbol{\rho}^{-\xi}\boldsymbol{E})\big)\big]_\theta &= [Y_0,Y_1]_{1-\theta}\\ &\doteq B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^0(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{E})\big),\end{aligned} \tag{21.11}$$

due to the fact that the space on the right side equals, except for equivalent norms, $\mathrm{dom}(A_0^{1-\theta})$ equipped with the graph norm.

(8) Set $Z_0:=B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^1(\boldsymbol{E})\big)$ and $Z_1:=B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^1(\boldsymbol{\rho}^{-\xi}\boldsymbol{E})\big)$. Then $\rho_\kappa>1$ for $\kappa\in\mathfrak{K}_1$ implies $Z_1\hookrightarrow Z_0$. Define a linear map A_1 in Z_0 with domain Z_1 by $A_1u:=\boldsymbol{\rho}^\xi u$. Then A_1 is closed and satisfies $\|(\lambda+A_1)^{-1}\|_{\mathcal{L}(Z_0)}\le c/|\lambda|$ for $\lambda\in S_{\pi/4}$ as well as

$$\|(-A_1)^z\|_{\mathcal{L}(Z_0)}\le\sup_{\kappa\in\mathfrak{K}_0}\rho_\kappa^{\xi\,\mathrm{Re}\,z}\le 1,\qquad \mathrm{Re}\,z\le 0.$$

Thus, using Seeley's theorem once more,

$$\big[B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^1(\boldsymbol{E})\big),B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^1(\boldsymbol{\rho}^{-\xi}\boldsymbol{E})\big)\big]_\theta\doteq B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty^1(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{E})\big). \tag{21.12}$$

Now we deduce from (21.10)–(21.12) that

$$\big[B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{E})\big),B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{\rho}^{-\xi}\boldsymbol{E})\big)\big]_\theta\doteq B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{E})\big).$$

Thus (21.9) shows that

$$\Psi_\infty^{\vec\omega}:\ B_\infty^{s/\vec r}\big(\mathbb{M}\times J,\ell_\infty(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{E})\big)\to[B_\infty^{s_0/\vec r,\vec\omega_0},B_\infty^{s_1/\vec r,\vec\omega_1}]_\theta$$

is a retraction, and $\Phi_\infty^{\vec\omega}$ is a coretraction. Hence (12.23) and (21.8) imply that

$$\psi_\infty^{\vec\omega}:\ \ell_\infty(\boldsymbol{\rho}^{-\theta\xi}\boldsymbol{B}_\infty^{s/\vec r})\to[B_\infty^{s_0/\vec r},B_\infty^{s_1/\vec r}]_\theta$$

is a retraction, and $\varphi_\infty^{\vec\omega}$ is a coretraction. From this and the observation at the beginning of step (5) we derive that the first part of the second statement is true.

(9) By replacing ℓ_∞ in the preceding considerations by $\ell_{\infty,\mathrm{unif}}$ and invoking Theorem 12.10 instead of Theorem 12.8 we see that the second part of claim (ii) is also true. □

For completeness and complementing the results of [5] we include the following interpolation theorem for isotropic Besov-Hölder spaces.

Remark 21.2 Suppose $0<s_0<s_1$, $\lambda_0,\lambda_1\in\mathbb{R}$, and $0<\theta<1$. Then

$$[B_\infty^{s_0,\lambda_0},B_\infty^{s_1,\lambda_1}]_\theta\doteq B_\infty^{s_\theta,\lambda_\theta},\quad [b_\infty^{s_0,\lambda_0},b_\infty^{s_1,\lambda_1}]_\theta\doteq b_\infty^{s_\theta,\lambda_\theta}.$$

Proof. This follows from the above proof by relying on the corresponding isotropic results of Sections 11 and 12. □

Throughout the rest of this section we suppose

(21.13)

- $\Gamma\ne\emptyset$.
- assumption (20.1) is satisfied.
- $\bar s>r_k+1/p$ and $\boldsymbol{b}\in\mathfrak{B}_{\mathrm{norm}}^{\bar s}$.
- $\mathcal{B}=(\mathcal{B}_0,\dots,\mathcal{B}_k):=\mathcal{B}(\boldsymbol{b})$.

Let $I\in\{J,(0,\infty)\}$. For $-1+1/p<s\le\bar s$ with $s\notin\{\,k_i+1/p\,;\ 0\le i\le k\,\}$ we set

$$\mathfrak{F}_{p,\mathcal{B}}^{s/\vec r,\vec\omega}(I):=\big\{\,u\in\mathfrak{F}_p^{s/\vec r,\vec\omega}(I)=\mathfrak{F}_p^{s/\vec r,\vec\omega}(I,V)\,;\ \mathcal{B}_iu=0\text{ for }r_i<s-1/p\,\big\}.$$

Thus $\mathfrak{F}_{p,\mathcal{B}}^{s/\vec r,\vec\omega}(I)=\mathfrak{F}_p^{s/\vec r,\vec\omega}(I)$ if $s<k_0+1/p$.

Suppose $\boldsymbol{b}$ is independent of t. Then we can define stationary isotropic spaces with vanishing boundary conditions analogously, that is,

$$\mathfrak{F}_{p,\mathcal{B}}^{s,\lambda} := \{\, u \in \mathfrak{F}_p^{s,\lambda}(V)\ ;\ \mathcal{B}_i u = 0 \text{ for } r_i < s - 1/p \,\}.$$

Theorem 21.3 *Let* (21.13) *be satisfied. Suppose* $-1 + 1/p < s_0 < s_1 \le \bar{s}$ *and* $0 < \theta < 1$ *satisfy*

$$s_0, s_1, s_\theta \notin \{\, r_i + 1/p\ ;\ 0 \le i \le k \,\}$$

and $\lambda_0, \lambda_1 \in \mathbb{R}$. *Then*

$$\big(\mathfrak{F}_{p,\mathcal{B}}^{s_0/\vec{r},\vec{\omega}_0}(I), \mathfrak{F}_{p,\mathcal{B}}^{s_1/\vec{r},\vec{\omega}_1}(I)\big)_\theta \doteq \mathfrak{F}_{p,\mathcal{B}}^{s_\theta/\vec{r},\vec{\omega}_\theta}(I) \doteq \big[\mathfrak{F}_{p,\mathcal{B}}^{s_0/\vec{r},\vec{\omega}_0}(I), \mathfrak{F}_{p,\mathcal{B}}^{s_1/\vec{r},\vec{\omega}_1}(I)\big]_\theta \tag{21.14}$$

and

$$\big(H_{p,\mathcal{B}}^{s_0/\vec{r},\vec{\omega}_0}(I), H_{p,\mathcal{B}}^{s_1/\vec{r},\vec{\omega}_1}(I)\big)_{\theta,p} \doteq B_{p,\mathcal{B}}^{s_\theta/\vec{r},\vec{\omega}_\theta}(I).$$

If $\boldsymbol{b}$ *is independent of* $t \in \mathbb{R}$, *then*

$$(\mathfrak{F}_{p,\mathcal{B}}^{s_0,\lambda_0}, \mathfrak{F}_{p,\mathcal{B}}^{s_1,\lambda_1})_\theta \doteq \mathfrak{F}_{p,\mathcal{B}}^{s_\theta,\lambda_\theta} \doteq [\mathfrak{F}_{p,\mathcal{B}}^{s_0,\lambda_0}, \mathfrak{F}_{p,\mathcal{B}}^{s_1,\lambda_1}]_\theta$$

and

$$(H_{p,\mathcal{B}}^{s_0,\lambda_0}, H_{p,\mathcal{B}}^{s_1,\lambda_1})_{\theta,p} \doteq B_{p,\mathcal{B}}^{s_\theta,\lambda_\theta}.$$

Proof. Theorem 20.3 guarantees the existence of a coretraction

$$\mathcal{B}^c \in \mathcal{L}\big(\partial_{\Gamma\times I}\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(G), \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(I)\big), \qquad r_k + 1/p < s \le \bar{s}.$$

Hence $\mathcal{B}^c\mathcal{B} \in \mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(I)\big)$ is a projection. Note that $\mathcal{B}^c\mathcal{B}$ depends on $\boldsymbol{b}$ and the universal extension operators (20.5) only. Thus we do not need to indicate the parameters s, λ, and p with $r_k + 1/p < s \le \bar{s}$ which characterize the domain $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(I)$.

Taking this into account and using the notation of the proof of Theorem 20.3 we set $X_\ell := \mathfrak{F}_p^{s_\ell/\vec{r},\vec{\omega}_\ell}(I)$ for $\ell \in \{0,1,\theta\}$ and, putting $r_{k+1} := \infty$,

$$P_\ell := \begin{cases} \mathrm{id}_\ell, & s_\ell < r_0 + 1/p, \\ \mathrm{id}_\ell - \mathcal{B}^{ic}\mathcal{B}^i, & r_i + 1/p < s_\ell < r_{i+1} + 1/p. \end{cases} \tag{21.15}$$

Since $X_\ell \hookrightarrow \mathcal{D}(\mathring{J}, \mathring{\mathcal{D}})'$ the sum space $X_0 + X_1$ is well-defined, that is, (X_0, X_1) is an interpolation couple. It follows from (20.9)–(20.11) that $P_0 \in \mathcal{L}(X_0 + X_1)$ and $P_\ell \in \mathcal{L}(X_\ell)$ with $P_0 \,|\, X_\ell = P_\ell$ for $\ell \in \{0,1,\theta\}$.

Theorem 21.1 guarantees $(X_0, X_1)_\theta \doteq X_\theta$. Theorem 20.3, definition (21.15), and [2, Lemma I.2.3.1] (also see [4, Lemma 4.1.5]) imply that P_ℓ is a projection onto $X_{\ell,\mathcal{B}} := \mathfrak{F}_{p,\mathcal{B}}^{s_\ell/\vec{r},\vec{\omega}_\ell}(I)$ for $\ell \in \{0,1,\theta\}$. Thus it is a retraction from X_ℓ onto $X_{\ell,\mathcal{B}}$ possessing the natural injection $X_{\ell,\mathcal{B}} \hookrightarrow X_\ell$ as a coretraction. Consequently, P_θ is a retraction from $X_\theta \doteq (X_0, X_1)_\theta$ onto $(X_{0,\mathcal{B}}, X_{1,\mathcal{B}})_\theta$. From this we get $(X_{0,\mathcal{B}}, X_{1,\mathcal{B}})_\theta \doteq X_{\theta,\mathcal{B}}$. This proves the first equivalence of (21.14). The remaining statements for the anisotropic case follow analogously.

Due to the observation at the end of Section 14 it is clear that the above proof applies to the isotropic case as well. □

There is a similar result concerning interpolations of spaces with vanishing initial conditions. For this we assume

- $\ell, j_0, \dots, j_\ell \in \mathbb{N}$ with $j_0 < j_1 < \cdots < j_\ell$.
- $\mathcal{C} := (\partial_{t=0}^{j_0}, \dots, \partial_{t=0}^{j_\ell})$. (21.16)

Then, given $s > -1 + 1/p$, we put

$$\mathfrak{F}_{p,\mathcal{C}}^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+) := \{\, u \in \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(\mathbb{R}^+)\ ;\ \partial_{t=0}^{j_i} u = 0 \text{ if } r(j_i + 1/p) < s \,\}.$$

Theorem 21.4 *Let* (21.16) *be satisfied. Suppose* $-1+1/p<s_0<s_1$ *and* $\theta\in(0,1)$ *satisfy*

$$s_0,s_1,s_\theta\notin\{\,r(j_i+1/p)\,;\,0\le i\le\ell\,\}$$

and $\lambda_0,\lambda_1\in\mathbb{R}$. *Then*

$$\big(\mathfrak{F}_{p,\mathcal{C}}^{s_0/\vec r,\vec\omega_0}(\mathbb{R}^+),\mathfrak{F}_{p,\mathcal{C}}^{s_1/\vec r,\vec\omega_1}(\mathbb{R}^+)\big)_\theta\doteq\mathfrak{F}_{p,\mathcal{C}}^{s_\theta/\vec r,\vec\omega_\theta}(\mathbb{R}^+)\doteq\big[\mathfrak{F}_{p,\mathcal{C}}^{s_0/\vec r,\vec\omega_0}(\mathbb{R}^+),\mathfrak{F}_{p,\mathcal{C}}^{s_1/\vec r,\vec\omega_1}(\mathbb{R}^+)\big]_\theta$$

and

$$\big(H_{p,\mathcal{C}}^{s_0/\vec r,\vec\omega_0}(\mathbb{R}^+),H_{p,\mathcal{C}}^{s_1/\vec r,\vec\omega_1}(\mathbb{R}^+)\big)_{\theta,p}\doteq B_{p,\mathcal{C}}^{s_\theta/\vec r,\vec\omega_\theta}(\mathbb{R}^+).$$

Proof. This is shown by the preceding proof using Theorem 20.4 instead of Theorem 20.3. □

Now we suppose, in addition to (21.13), that $\ell\in\mathbb{N}$ and $\bar s>r(\ell+1/p)$. Then we set

$$\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s/\vec r,\vec\omega}:=\big\{\,u\in\mathfrak{F}_p^{s/\vec r,\vec\omega}(\mathbb{R}^+)\,;\,\mathcal{B}_iu=0,\ \partial_{t=0}^ju=0\text{ if }r_i+1/p+r(j+1/p)<s\,\big\}$$

if $\max\{r_0+1/p,\ r/p\}<s\le\bar s$ and $s\notin\big\{\,r_i+1/p+r(j+1/p)\,;\,0\le i\le k,\ 0\le j\le\ell\,\big\}$,

$$\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s/\vec r,\vec\omega}:=\mathfrak{F}_{p,\mathcal{B}}^{s/\vec r,\vec\omega}(\mathbb{R}^+)$$

if $r_0+1/p<s<r/p$ and $s\notin\{\,r_i+1/p\,;\,1\le i\le k\,\}$,

$$\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s/\vec r,\vec\omega}:=\mathfrak{F}_{p,\overrightarrow{\partial_{t=0}^{\ell}}}^{s/\vec r,\vec\omega}(\mathbb{R}^+)$$

if $r/p<s<r_0+1/p$ with $s\notin\big\{\,r(j+1/p)\,;\,1\le j\le\ell\,\big\}$, and

$$\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s/\vec r,\vec\omega}:=\mathfrak{F}_{p}^{s/\vec r,\vec\omega}(\mathbb{R}^+)$$

if $-1+1/p<s<\max\{r_0+1/p,\ r/p\}$.

The following theorem is analogue to Theorem 21.3. It describes the interpolation behavior of anisotropic function spaces with vanishing boundary and initial conditions.

Theorem 21.5 *Let assumption* (21.13) *be satisfied. Also assume* $\ell\in\mathbb{N}$ *and* $r(\ell+1/p)<\bar s$. *Suppose* $-1+1/p<s_0<s_1\le\bar s$ *and* $0<\theta<1$ *satisfy*

$$s_0,s_1,s_\theta\notin\{\,r_i+1/p+r(j+1/p),\ r_i+1/p,\ r(j+1/p)\,;\,0\le i\le k,\ 0\le j\le\ell\,\}$$

and $\lambda_0,\lambda_1\in\mathbb{R}$. *Then*

$$(\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s_0/\vec r,\vec\omega_0},\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s_1/\vec r,\vec\omega_1})_\theta\doteq\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s_\theta/\vec r,\vec\omega_\theta}\doteq[\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s_0/\vec r,\vec\omega_0},\mathfrak{F}_{p,\vec{\mathcal{B}}}^{s_1/\vec r,\vec\omega_1}]_\theta$$

and

$$(H_{p,\vec{\mathcal{B}}}^{s_0/\vec r,\vec\omega_0},H_{p,\vec{\mathcal{B}}}^{s_1/\vec r,\vec\omega_1})_{\theta,p}\doteq B_{p,\vec{\mathcal{B}}}^{s_\theta/\vec r,\vec\omega_\theta}.$$

Proof. This follows by the arguments of the proof of Theorem 21.3 by invoking Theorem 20.5 and Remark 20.6. □

The preceding interpolation theorems combined with the characterization statements of Section 19 lead to interpolation results for spaces with vanishing traces. For abbreviation, $\mathfrak{F}_{p,\Gamma}^{s/\vec r,\vec\omega}(I)=\mathfrak{F}_{p,\Gamma}^{s/\vec r,\vec\omega}(I,V)$, etc.

Theorem 21.6 *Suppose* $-1+1/p<s_0<s_1<\infty,\ 0<\theta<1$, *and* $\lambda_0,\lambda_1\in\mathbb{R}$.

(i) *If* $s_0,s_1,s_\theta\notin\mathbb{N}+1/p$, *then*

$$\big(\mathring{\mathfrak{F}}_{p,\Gamma}^{s_0/\vec r,\vec\omega_0}(J),\mathring{\mathfrak{F}}_{p,\Gamma}^{s_1/\vec r,\vec\omega_1}(J)\big)_\theta\doteq\mathring{\mathfrak{F}}_{p,\Gamma}^{s_\theta/\vec r,\vec\omega_\theta}(J)\doteq\big[\mathring{\mathfrak{F}}_{p,\Gamma}^{s_0/\vec r,\vec\omega_0}(J),\mathring{\mathfrak{F}}_{p,\Gamma}^{s_1/\vec r,\vec\omega_1}(J)\big]_\theta$$

and

$$\big(\mathring{H}_{p,\Gamma}^{s_0/\vec{r},\vec{\omega}_0}(J),\mathring{H}_{p,\Gamma}^{s_1/\vec{r},\vec{\omega}_1}(J)\big)_{\theta,p}\doteq\mathring{B}_{p,\Gamma}^{s_\theta/\vec{r},\vec{\omega}_\theta}(J).$$

(ii) *Assume* $s_0,s_1,s_\theta\notin r(\mathbb{N}+1/p)$. *Then*

$$\big(\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0}(0,\infty),\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1}(0,\infty)\big)_\theta\doteq\mathfrak{F}_p^{s_\theta/\vec{r},\vec{\omega}_\theta}(0,\infty)\doteq\big[\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0}(0,\infty),\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1}(0,\infty)\big]_\theta$$

and

$$\big(H_p^{s_0/\vec{r},\vec{\omega}_0}(0,\infty),H_p^{s_1/\vec{r},\vec{\omega}_1}(0,\infty)\big)_{\theta,p}\doteq B_p^{s_\theta/\vec{r},\vec{\omega}_\theta}(0,\infty).$$

(iii) *Suppose* $s_0,s_1,s_\theta\notin\mathbb{N}+1/p$ *with* $s_0,s_1,s_\theta\notin r(\mathbb{N}+1/p)$ *if* $J=\mathbb{R}^+$. *Then*

$$(\mathring{\mathfrak{F}}_p^{s_0/\vec{r},\vec{\omega}_0},\mathring{\mathfrak{F}}_p^{s_1/\vec{r},\vec{\omega}_1})_\theta\doteq\mathring{\mathfrak{F}}_p^{s_\theta/\vec{r},\vec{\omega}_\theta}\doteq[\mathring{\mathfrak{F}}_p^{s_0/\vec{r},\vec{\omega}_0},\mathring{\mathfrak{F}}_p^{s_1/\vec{r},\vec{\omega}_1}]_\theta$$

and

$$(\mathring{H}_p^{s_0/\vec{r},\vec{\omega}_0},\mathring{H}_p^{s_1/\vec{r},\vec{\omega}_1})_{\theta,p}\doteq\mathring{B}_p^{s_\theta/\vec{r},\vec{\omega}_\theta}.$$

Proof. To prove (i) we can assume $s_1>1/p$, due to Theorem 8.3(ii). Hence $k:=[s_1-1/p]_-\geq 0$. Set $\mathcal{B}:=(\partial_{\boldsymbol{n}}^0,\dots,\partial_{\boldsymbol{n}}^k)$ on $\Gamma\times J$. Then Theorem 19.1(i) guarantees $\mathring{\mathfrak{F}}_{p,\Gamma}^{s_j/\vec{r},\vec{\omega}_j}(J)=\mathfrak{F}_{p,\mathcal{B}}^{s_j/\vec{r},\vec{\omega}_j}(J)$ for $j\in\{0,1,\theta\}$. Hence assertion (i) is a consequence of Theorem 21.1. The proofs for claims (ii) and (iii) follow analogous lines. □

Since, in (8.5), the negative order spaces have been defined by duality we can now prove interpolation theorems for these spaces as well.

Theorem 21.7 *Suppose* $-\infty<s_0<s_1<1/p$, $0<\theta<1$, *and* $\lambda_0,\lambda_1\in\mathbb{R}$. *Assume* $s_0,s_1,s_\theta\notin-\mathbb{N}+1/p$ *and, if* $J=\mathbb{R}^+$, *also* $s_0,s_1,s_\theta\notin r(-\mathbb{N}+1/p)$. *Then*

$$(\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0},\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1})_\theta\doteq\mathfrak{F}_p^{s_\theta/\vec{r},\vec{\omega}_\theta}\doteq[\mathfrak{F}_p^{s_0/\vec{r},\vec{\omega}_0},\mathfrak{F}_p^{s_1/\vec{r},\vec{\omega}_1}]_\theta$$

and

$$(H_p^{s_0/\vec{r},\vec{\omega}_0},H_p^{s_1/\vec{r},\vec{\omega}_1})_{\theta,p}\doteq B_p^{s_\theta/\vec{r},\vec{\omega}_\theta}.$$

Proof. This follows easily from Theorem 21.6(iii), the duality properties of $(\cdot,\cdot)_\theta$, and Theorem 8.3(ii) and Corollary 8.4(ii). □

Suppose M is an m-dimensional compact submanifold of $\mathbb{R}^m$ with boundary and $W=M\times\mathbb{C}^n$. In this situation it has been shown by R. Seeley [44] that

$$[L_p,H_{p,\mathcal{B}}^s]_\theta=H_{p,\mathcal{B}}^{\theta s},\qquad s>0, \tag{21.17}$$

with $\mathcal{B}$ a normal system of boundary operators (with smooth coefficients). This generalizes the earlier result by P. Grisvard [15] who obtained (21.17) in the case $p=2$ and $n=1$. The latter author proved in [16] that $(L_p,W_{p,\mathcal{B}}^k)_{\theta,p}\doteq B_{p,\mathcal{B}}^{\theta k}$ and $(L_p,B_{p,\mathcal{B}}^s)_{\theta,p}\doteq B_{p,\mathcal{B}}^{\theta s}$ for $k\in\mathbb{N}^\times$ and $s>0$. An extension of these results to arbitrary Banach spaces is due to D. Guidetti [17]. In each of those papers the 'singular values' $\mathbb{N}+1/p$ are considered also. (If $s\in\mathbb{N}+1/p$, then $H_{p,\mathcal{B}}^s$ and $B_{p,\mathcal{B}}^s$ are no longer closed subspaces of H_p^s and B_p^s, respectively.)

Following the ideas of R. Seeley and D. Guidetti we have given in [4, Theorem 4.9.1] a proof of the anisotropic part of Theorem 21.3 in the special case where $M=\mathbb{H}^m$ and $J=\mathbb{R}$, respectively $M=\mathbb{R}^m$ and $J=\mathbb{R}^+$ (to remain in the setting of this paper), $W=M\times\mathbb{C}^n$, and B has constant coefficients. The proof given here, which is solely based on Theorem 20.3 and general properties of interpolation functors, is new even in this simple Euclidean setting.

22 Bounded Cylinders

So far we have developed the theory of weighted anisotropic function spaces on full and half-cylinders, making use of the dilation invariance of J. In this final section we now show that all preceding results not explicitly depending on this dilation invariance remain valid in the case of cylinders of finite height.

Throughout this section

$$\bullet \quad J = \mathbb{R}^+, \quad 0 < T < \infty, \quad J_T := [0,T].$$

Furthermore, $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J) = \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J,V)$ etc.

For $k \in \mathbb{N}$ we introduce $W_p^{kr/\vec{r}}(J_T)$ by replacing J in definition (8.1) by J_T. Then $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J_T)$ is defined for $s > 0$ analogously to (8.3). Similarly as in (19.1)

$$\mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}(J_T) \text{ is the closure of } \mathcal{D}\big((0,T], \mathcal{D}(M \setminus \Gamma, V)\big) \text{ in } \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J_T) \text{ for } s > 0.$$

Moreover,

$$\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(J_T) := \mathring{\mathfrak{F}}_{p,\partial M}^{s/\vec{r},\vec{\omega}}(J_T), \quad \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(0,T] := \mathring{\mathfrak{F}}_{p,\emptyset}^{s/\vec{r},\vec{\omega}}(J_T).$$

Note that we do not require that $u \in \mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}(J_T)$ approaches zero near T. To take care of this situation also we define:

$$\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(0,T) \text{ is the closure of } \mathcal{D}\big((0,T), \mathring{\mathcal{D}}\big) \text{ in } \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J_T).$$

Then

$$\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J_T) := \big(\mathring{\mathfrak{F}}_{p'}^{s/\vec{r},\vec{\omega}}\big((0,T), V'\big)\big)', \qquad s > 0,$$

and

$$H_p^{0/\vec{r},\vec{\omega}}(J_T) := L_p(J_T, L_p^\lambda), \quad B_p^{0/\vec{r},\vec{\omega}}(J_T) := \big(B_p^{-s(p)/\vec{r},\vec{\omega}}(J_T), B_p^{s(p)/\vec{r},\vec{\omega}}(J_T)\big)_{1/2,p}.$$

This defines the weighted anisotropic Bessel potential space scale $\big[\, H_p^{s/\vec{r},\vec{\omega}}(J_T)\,;\ s \in \mathbb{R}\,\big]$ and Besov space scale $\big[\, B_p^{s/\vec{r},\vec{\omega}}(J_T)\,;\ s \in \mathbb{R}\,\big]$ on J_T.

As for Hölder space scales, $BC^{kr/\vec{r},\vec{\omega}}(J_T)$ is obtained by replacing J in (12.12) and (12.13) by J_T. Then $bc^{kr/\vec{r},\vec{\omega}}(J_T)$ is the closure of

$$BC^{\infty/\vec{r},\vec{\omega}}(J_T) := \bigcap_{i \in \mathbb{N}} BC^{ir/\vec{r},\vec{\omega}}(J_T)$$

in $BC^{kr/\vec{r},\vec{\omega}}(J_T)$. Besov-Hölder spaces are defined for $s > 0$ by

$$B_\infty^{s/\vec{r},\vec{\omega}}(J_T) := \begin{cases} \big(bc^{kr/\vec{r},\vec{\omega}}(J_T), bc^{(k+1)r/\vec{r},\vec{\omega}}(J_T)\big)_{(s-k)/r,\infty}, & kr < s < (k+1)r, \\ \big(bc^{kr/\vec{r},\vec{\omega}}(J_T), bc^{(k+2)r/\vec{r},\vec{\omega}}(J_T)\big)_{1/2,\infty}, & s = (k+1)r. \end{cases} \tag{22.1}$$

Moreover, $b_\infty^{s/\vec{r},\vec{\omega}}(J_T)$ is the closure of $BC^{\infty/\vec{r},\vec{\omega}}(J_T)$ in $B_\infty^{s/\vec{r},\vec{\omega}}(J_T)$. Lastly, $b_\infty^{s/\vec{r},\vec{\omega}}(0,T]$ is obtained by substituting $(0,T]$ for $(0,\infty)$ in (17.10) and (17.11).

Given a locally convex space $\mathcal{X}$, the continuous linear map

$$r_T : C(J,\mathcal{X}) \to C(J_T,\mathcal{X}), \quad u \mapsto u \,|\, J_T$$

is the *point-wise restriction to* J_T. As usual, we use the same symbol for r_T and any of its restrictions or (unique) continuous extensions.

Theorem 22.1 *Let one of the following conditions be satisfied:*

(α) $s \in \mathbb{R}$ *and* $\mathfrak{G} = \mathfrak{F}_p^{s/\vec{r},\vec{\omega}}$;

(β) $s > 0$ *and* $\mathfrak{G} \in \{B_\infty^{s/\vec{r},\vec{\omega}}, b_\infty^{s/\vec{r},\vec{\omega}}\}$;

(γ) $k \in \mathbb{N}$ *and* $\mathfrak{G} \in \{BC^{kr/\vec{r},\vec{\omega}}, bc^{kr/\vec{r},\vec{\omega}}\}$;

(δ) $s > 0$ *and* $\mathfrak{G} = \mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}$.

Then r_T is a retraction from $\mathfrak{G}(J)$ onto $\mathfrak{G}(J_T)$ possessing a universal coretraction e_T. It is also a retraction from $\mathfrak{G}(0,\infty)$ onto $\mathfrak{G}(0,T]$ with coretraction e_T if either $s>0$ and $\mathfrak{G}=b_\infty^{s/\vec{r},\vec{\omega}}$ or $k\in\mathbb{N}$ and $\mathfrak{G}=bc^{kr/\vec{r},\vec{\omega}}$.

P r o o f. (1) Suppose $k\in\mathbb{N}$. It is obvious that

$$r^+\in\mathcal{L}\big(W_p^{kr/\vec{r},\vec{\omega}}(J),W_p^{kr/\vec{r},\vec{\omega}}(J_T)\big).$$

Thus we get $r^+\in\mathcal{L}\big(\mathfrak{G}(J),\mathfrak{G}(J_T)\big)$ if (α) is satisfied with $s>0$ by interpolation, due to the definition of $\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(I)$ for $I\in\{J,J_T\}$.

(2) It is also clear that $r^+\in\mathcal{L}\big(BC^{kr/\vec{r},\vec{\omega}}(J),BC^{kr/\vec{r},\vec{\omega}}(J_T)\big)$ for $k\in\mathbb{N}$. Hence

$$r^+\in\mathcal{L}\big(BC^{\infty/\vec{r},\vec{\omega}}(J),BC^{\infty/\vec{r},\vec{\omega}}(J_T)\big). \tag{22.2}$$

From this we obtain

$$r^+\in\mathcal{L}\big(\mathfrak{G}(J),\mathfrak{G}(J_T)\big) \tag{22.3}$$

if either (β) or (γ) is satisfied. In fact, this is obvious from (22.2) if (γ) applies. If $s>0$ and $\mathfrak{G}=B_\infty^{s/\vec{r},\vec{\omega}}$, then (22.3) is obtained by interpolation on account of (12.21), Corollary 12.9(ii), and (22.1). From this and (22.2) it follows that (22.3) is valid if $\mathfrak{G}=b_\infty^{s/\vec{r},\vec{\omega}}$, due to (12.24) and the definition of $bc^{s/\vec{r},\vec{\omega}}(J_T)$.

Clearly, r^+ maps $\mathcal{D}\big(\mathring{J},\mathcal{D}(M\setminus\Gamma,V)\big)$ into $\mathcal{D}\big((0,T],\mathcal{D}(M\setminus\Gamma,V)\big)$. From this and step (1) we infer that (22.3) is true if (δ) applies. It is equally clear that $r^+\in\mathcal{L}\big(\mathfrak{G}(0,\infty),\mathfrak{G}(0,T)\big)$ if either $s>0$ and $\mathfrak{G}=b_\infty^{s/\vec{r},\vec{\omega}}$ or $k\in\mathbb{N}$ and $\mathfrak{G}=bc^{kr/\vec{r},\vec{\omega}}$.

(3) We set $\delta_T(t):=t+T$ for $t\in\mathbb{R}$. We fix $\alpha\in\mathcal{D}\big((-T,0],\mathbb{R}\big)$ satisfying $\alpha(t)=1$ for $-T/2\le t\le 0$ and put $\beta u(t):=\alpha\delta_T^*u(t)$ for $t\le 0$ and $u:\ J_T\to C(V)$. It follows that $\beta\in\mathcal{L}\big(\mathfrak{G}(J_T),\mathfrak{G}(-\mathbb{R}^+)\big)$, provided $s>0$ if (α) holds. Indeed, this is easily verified if $\mathfrak{G}$ is one of the spaces $W_p^{kr/\vec{r},\vec{\omega}}$ and $BC^{kr/\vec{r},\vec{\omega}}$. From this we get the claim by interpolation, similarly as in steps (1) and (2).

(4) We recall from Section 17 the definition of the extension operator e^- associated with the point-wise restriction r^- to $-\mathbb{R}^+$. Then we define a linear map

$$\varepsilon_T:\ C\big(J_T,C(V)\big)\to C\big([T,\infty),C(V)\big),\quad u\mapsto\delta_{-T}^*(e^-\beta u).$$

Finally, we put $e_Tu(t):=u(t)$ for $t\in J_T$ and $e_Tu(t):=\varepsilon_Tu(t)$ for $T<t<\infty$. It follows from step (3) and Theorems 17.1 and 17.2 that

$$e_T\in\mathcal{L}\big(\mathfrak{G}(J_T),\mathfrak{G}(J)\big) \tag{22.4}$$

if one of conditions (α)–(γ) is satisfied, provided $s>0$ if (α) applies.

Since α is compactly supported it follows from (17.1) that ε_Tu is smooth and rapidly decreasing if u is smooth. By the density of $\mathcal{D}\big([T,\infty),\mathcal{D}(M\setminus\Gamma,V)\big)$ in the Schwartz space of smooth rapidly decreasing $\mathcal{D}(M\setminus\Gamma,V)$-valued functions on $[T,\infty)$ we get $e_Tu\in\mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}(J)$ if $u\in\mathring{\mathfrak{F}}_{p,\Gamma}^{s/\vec{r},\vec{\omega}}(J_T)$. From this we see that (22.4) holds if (δ) is satisfied. It is obvious that $r_Te_T=\mathrm{id}$. Thus the assertion is proved, provided $s>0$ if (α) is satisfied.

(5) As in (17.5) we introduce the trivial extension map $e_0^-:\ C_{(0)}(-\mathbb{R}^+,\mathcal{X})\to C(\mathbb{R},\mathcal{X})$ by $e_0^-u(t):=u(t)$ if $t\le 0$, and $e_0^-u(t):=0$ if $t>0$. Then

$$r_0^-:=r^-(1-e^+r^+):\ C(\mathbb{R},\mathcal{X})\to C_0(-\mathbb{R}^+,\mathcal{X})$$

is a retraction possessing e_0^- as coretraction. We also set $r_{0,T}:=\delta_{-T}^*r_0^-\delta_T^*e_0^+$. Then

$$r_{0,T}\big(\mathcal{D}\big((0,T),\mathring{\mathcal{D}}\big)\big)\subset\mathcal{D}\big((0,\infty),\mathring{\mathcal{D}}\big). \tag{22.5}$$

The mapping properties of e^+ and r^+ described in Theorems 17.1 and 17.2, and the analogous ones for r^-, imply, similarly as above, that

$$r_{0,T}\in\mathcal{L}\big(\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J),\mathfrak{F}_p^{s/\vec{r},\vec{\omega}}(J_T)\big),\qquad s>0.$$

Consequently, we get from (22.5)

$$r_{0,T} \in \mathcal{L}\big(\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(J), \mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(0,T)\big), \qquad s > 0.$$

We define $e_{0,T} : \mathcal{D}\big((0,T), \mathring{\mathcal{D}}\big) \to \mathcal{D}\big((0,\infty), \mathring{\mathcal{D}}\big)$ by $e_{0,T}u \,|\, J_T := u$ and $e_{0,T}u \,|\, [0,\infty) := 0$. Then $e_{0,T}$ extends to a continuous linear map from $\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(0,T)$ into $\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(J)$ for $s > 0$, the trivial extension. Moreover, $r_{0,T}e_{0,T} = \mathrm{id}$. Thus $r_{0,T}$ is a retraction possessing $e_{0,T}$ as coretraction.

(6) Let $s > 0$. For $u \in \mathcal{D}(J, \mathcal{D})$ and $\varphi \in \mathcal{D}\big((0,T), \mathcal{D}(\mathring{M}, V')\big)$ we get

$$\int_0^T \int_M \langle \varphi, r_T u\rangle_V \, dV_g \, dt = \int_0^\infty \int_M \langle e_{0,T}\varphi, u\rangle_V \, dV_g \, dt.$$

Hence, by step (5) and the definition of the negative order spaces,

$$|\langle \varphi, r_T u\rangle_{M\times J}| \le c \, \|\varphi\|_{\mathfrak{F}_{p'}^{s/\vec{r},\vec{\omega}}(\mathring{J}_T, V')} \, \|u\|_{\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J)}$$

for $u \in \mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J)$ and $\varphi \in \mathring{\mathfrak{F}}_{p'}^{s/\vec{r},\vec{\omega}}\big((0,T), V'\big)$. Thus

$$r_T \in \mathcal{L}\big(\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J), \mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J_T)\big).$$

(7) For $v \in C(-J, \mathcal{D})$ we set

$$\varepsilon^- v(t) := \int_0^\infty h(s) v(-st)\, ds, \qquad t \ge 0.$$

Then, given $\varphi \in \mathcal{D}\big((0,\infty), \mathcal{D}(\mathring{M}, V')\big)$, we obtain from $h(1/s) = -sh(s)$ for $s > 0$ and (17.3)

$$\begin{aligned}
\int_0^\infty \big\langle \varphi(t), \varepsilon^- v(t)\big\rangle_M \, dt &= \int_0^\infty \int_0^\infty \big\langle \varphi(t), h(s)v(-st)\big\rangle_M \, ds\, dt \\
&= \int_{-\infty}^0 \int_0^\infty \big\langle s^{-1}\varphi(-\tau/s)h(s), v(\tau)\big\rangle_M \, ds\, d\tau \\
&= \int_{-\infty}^0 \int_0^\infty \sigma^{-1}\big\langle \varphi(-\tau\sigma)h(1/\sigma), v(\tau)\big\rangle_M \, d\sigma\, d\tau \\
&= -\int_{-\infty}^0 \Big\langle \int_0^\infty h(\sigma)\varphi(-\sigma\tau)\, d\sigma, v(\tau)\Big\rangle_M \, d\tau = -\int_{-\infty}^0 \langle \varepsilon\varphi, v\rangle_M \, d\tau.
\end{aligned} \tag{22.6}$$

Thus, by the definition of e_T, given $u \in \mathcal{D}(J_T, \mathcal{D})$,

$$\int_0^\infty \langle \varphi, e_T u\rangle_M \, dt = \int_0^T \langle \varphi, u\rangle_M \, dt + \int_T^\infty \langle \varphi, \delta_{-T}^* e^- \alpha \delta_T^* u\rangle_M \, dt.$$

The last integral equals, due to $e^- w(t) = \varepsilon^- w(t)$ for $t \ge 0$ and (22.6),

$$\begin{aligned}
\int_0^\infty \langle \delta_T^*\varphi, \varepsilon^- \alpha\delta_T^* u\rangle_M \, dt &= -\int_{-\infty}^0 \langle \varepsilon\delta_T^*\varphi, \alpha\delta_T^* u\rangle_M \, dt \\
&= -\int_0^T \int_0^\infty \big\langle h(\sigma)\varphi\big(-\sigma(s-T)+T\big)\, d\sigma, \alpha(s-T)u(s)\big\rangle_M \, ds
\end{aligned}$$

since α is supported in $(-T, 0]$. From this we infer as in steps (3) and (4) that, given $s > 0$,

$$|\langle \varphi, e_T u\rangle_{M\times J}| \le c \, \|\varphi\|_{\mathfrak{F}_{p'}^{s/\vec{r},\vec{\omega}}(J, V')} \, \|u\|_{\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J_T)}$$

for $\varphi \in \mathcal{D}\big((0,\infty),\mathcal{D}(\mathring{M},V')\big)$ and $u \in \mathcal{D}(J_T,\mathcal{D})$. Thus

$$e_T \in \mathcal{L}\big(\mathfrak{F}_p^{-s/\vec{r},\vec{\omega}}(J_T),\mathfrak{F}_{p'}^{-s/\vec{r},\vec{\omega}}(J)\big)$$

for $s>0$. This and step (6) imply that the assertion holds if (α) is satisfied with $s<0$.

The case $s=0$ and $\mathfrak{F}=H$ is covered by step (1). If $s=0$ and $\mathfrak{F}=B$, we now obtain the claim by interpolation, due to the definition of $B_p^{0/\vec{r},\vec{\omega}}(I)$ for $I\in\{J,J_T\}$. □

Corollary 22.2 *Suppose $s>0$. There exists a universal retraction $r_{0,T}$ from $\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(J)$ onto $\mathring{\mathfrak{F}}_p^{s/\vec{r},\vec{\omega}}(\mathring{J}_T)$ such that the trivial extension is a coretraction for it.*

Proof. This has been shown in step (5). □

As a consequence of this retraction theorem we find that, modulo obvious adaptions, everything proved in the preceding sections remains valid for cylinders of finite height.

Theorem 22.3 *All embedding, interpolation, trace, and point-wise contraction multiplier theorems, as well as the theorems involving boundary conditions, remain valid if J is replaced by J_T. Furthermore, all retraction theorems for the anisotropic spaces stay in force, provided $\varphi_q^{\vec{\omega}}$ and $\psi_q^{\vec{\omega}}$ are replaced by $\varphi_q^{\vec{\omega}}\circ e_T$ and $r_T\circ\psi_q^{\vec{\omega}}$, respectively.*

Proof. This is an immediate consequence of Theorem 22.1 and the fact that all contraction multiplication and boundary operators are local ones. □

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