OPTIMAL CONTROL PROBLEMS WITH FINAL OBSERVATION GOVERNED BY EXPLOSIVE PARABOLIC EQUATIONS

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Abstract. We study optimal controls problems with final observation. The governing parabolic equations or systems involve superlinear nonlinearities and their solutions may blow up in finite time. Our proof of the existence, regularity and optimality conditions for an optimal pair is based on uniform a priori estimates for the approximating solutions. Our conditions on the growth of the nonlinearity are essentially optimal. In particular, we also solve a longstanding open problem of J.L. Lions concerning singular systems.

1. Introduction

In his book [21], J.L. Lions studied several optimal control problems governed by nonlinear parabolic equations of the form

$$\partial_t y - \Delta y = y^\lambda + u, \qquad x \in \Omega,\ t \in [0,T], \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\lambda \in \{2,3\}$, $u = u(x,t)$ is the control and $y = y(x,t)$ is the state variable. Equation (1.1) is complemented by suitable boundary and initial conditions, for example

$$y = 0 \ \text{ on } \partial\Omega \times (0,T), \qquad y(\cdot,0) = y_0, \tag{1.2}$$

where $y_0 \in L_\infty(\Omega)$. If u is regular enough then the state problem (1.1)-(1.2) possesses a unique strong solution $y = y(u)$ defined on the maximal existence interval J_u (see Section 2 for the definition of a strong solution). However, even for smooth controls u, the solution $y(u)$ need not be global – the interval J_u need not coincide with $[0,T]$. In this case, $y(u)$ *blows up* at the time $t(u) := \sup J_u$, i.e. it develops a singularity and leaves its natural regularity class. After blow-up, the solution either can be continued in a weak sense (the blow-up is *incomplete* [16]) or such continuation is not possible (the solution blows up *completely* [9]).

Let $\mathbb{U}_{\rm ad}$ denote the set of admissible controls,

$$\mathbb{U}^G_{\rm ad} := \{u \in \mathbb{U}_{\rm ad} : \text{the solution } y(u) \text{ is global}\},$$

and $\mathbb{J} = \mathbb{J}(y,u)$ be the cost functional. A standard way to solve the optimal control problem

$$\text{minimize } \mathbb{J}(y(u),u) \text{ over } u \in \mathbb{U}^G_{\rm ad} \tag{1.3}$$

is to consider controls u_k, $k = 1,2,\dots$, such that $(\mathbb{J}(y(u_k),u_k))$ is a minimizing sequence for (1.3) and to show that a suitable subsequence of $((y(u_k),u_k))$ converges to an optimal pair $(y(u),u)$. Assume, for example, that $\mathbb{U}_{\rm ad}$ is a (weakly closed) subset of a reflexive Banach space. If $\mathbb{J}$ is coercive with respect to u (or $\mathbb{U}_{\rm ad}$ is bounded) then the sequence (u_k) is bounded and we may assume that $u_k \to u$ in the weak topology. Similarly, if $\mathbb{J}$ is coercive with respect to y (in a suitable space of functions defined in $Q := \Omega \times [0, T]$), then the sequence $(y(u_k))$ is bounded and standard compactness results for the state problem enable us to pass to the limit in order to find a minimizer for (1.3).

If we consider problems with final observation (where $\mathbb{J}$ depends just on u and the final value $y(\cdot, T)$), then the coerciveness of $\mathbb{J}$ provides a priori estimates for u_k and final values of $y(u_k)$. However, such estimates are, in general, not sufficient for the uniform boundedness of solutions $y(u_k)$ on the whole interval $[0, T]$. Consequently, we have to find sufficient conditions on λ and/or other parameters of the problem which guarantee a priori bounds for global solutions y of (1.1)-(1.2) depending only on suitable norms of u and $y(\cdot, T)$.

Let us discuss the question of a priori bounds for problems with final observation in the particular setting of [21, Section I.10]. Fix $N > 0$, $q \geq 1$, $y_d \in L_q(\Omega)$ and set

$$\mathbb{J}(y, u) := \int_\Omega |y(x, T) - y_d(x)|^q \, dx + N \int_Q u^2(x, t) \, dx \, dt,$$

Assume also that $\mathbb{U}_{\mathrm{ad}} \subset L_2(Q)$ is closed and convex, and $\mathbb{U}_{\mathrm{ad}}^G \neq \emptyset$. If $\lambda = 2$, $q = 3$ and $n \leq 3$ then [21, Theorem I.10.1] and its proof guarantee the required bounds for the solutions $y(u_k)$, hence the existence of an optimal pair (y, u). If, in addition, $n \leq 2$, then optimality conditions for the optimal pair (y, u) were derived in [21, Theorem I.10.3]. On the other hand, the existence of an optimal pair in the case $\lambda = 3$, $q = 4$ (or $\lambda = 2$, $q < 3$) and the optimality conditions for $n = 3$ were left as open problems, see [21, Remarks I.10.1, I.10.2 and I.10.4]. Our results give, in particular, positive answers to all those open problems. In fact, we consider an arbitrary dimension n, exponents $q \geq 2$, $\lambda > 1$ (where either $y^\lambda := |y|^{\lambda-1} y$ or $y^\lambda := |y|^\lambda$) and controls $u \in L_r([0, T], L_2(\Omega))$, $r \geq 2$, and find sufficient conditions on q, λ and r that guarantee the existence of optimal controls and the optimality conditions (see Section 2 for precise statements of our results).

We also show that many of our conditions are essentially optimal. In particular, if $\mathbb{U}_{\mathrm{ad}} \subset L_\infty([0, T], L_2(\Omega))$ then our sufficient conditions on q and λ guaranteeing the existence of optimal controls have the form

$$\lambda < \frac{n+2}{(n-2)_+} \qquad \text{and} \qquad q \in \Big((\lambda - 1)\frac{n}{2}, \frac{2n}{(n-4)_+} \Big),$$

where $a_+ := \max(a, 0)$ and $a/b_+ := \infty$ if $a > 0$ and $b \leq 0$. The upper bound for q is required by the (low) regularity of controls u: it guarantees $y(u) \in C([0, T], L_q(\Omega))$ so that $\mathbb{J}(y(u), u)$ is defined. If $q < (\lambda - 1)n/2$ or $\lambda > (n+2)/(n-2)_+$ and $n \leq 10$ then we show that problem (1.3) need not be solvable even if the set $\mathbb{U}_{\mathrm{ad}}$ is a compact subset of $C^\infty(\bar{\Omega} \times [0, T])$ and $\mathbb{U}_{\mathrm{ad}}^G \neq \emptyset$, see Remark 3.4. This nonexistence result is due to the fact that the set $\mathbb{U}_{\mathrm{ad}}^G$ need not be closed in $\mathbb{U}_{\mathrm{ad}}$: if $u_k \in \mathbb{U}_{\mathrm{ad}}^G$, $u_k \to u \in \mathbb{U}_{\mathrm{ad}}$, then the limiting solution $y(u)$ may blow-up at $t(u) < T$. The conditions on q show the importance of a good choice of the cost functional in order to control the equation. On the other hand, if $\lambda > (n+2)/(n-2)_+$ then (a strong) solvability of our control problem cannot be guaranteed for any q.

The solvability of (1.3) with $\mathbb{U}_{\mathrm{ad}}$, $\mathbb{J}$ as above was proved by Imanuvilov [18, Theorem 2.1] and Fursikov [15, Theorem 4.3] for $r = q = 2$ and any $\lambda > 1$ but their function $y(u)$ corresponding to the optimal control u need not be a strong solution in our sense. In fact, the results of [18, 15] also apply to the example of Remark 3.4(i), where $y(u)$ blows up at $t(u) < T$ (but can be continued in a weak sense). This lack of regularity causes serious problems in establishing the optimality conditions. In order to obtain these conditions, Imanuvilov and Fursikov have to assume $q = 2 \geq (\lambda - 1)n/2$, see [15, Theorem 5.1]. Note also that the proofs in [18, 15] substantially use the choice $q = 2$ hence require $\lambda \leq 1 + 4/n$. In particular, if $n = 3$ then their method cannot be used in the case $\lambda = 3$, $q = 4$ mentioned above.

Our proof of a priori estimates is based on energy and perturbation arguments in [25, 27]. The same approach can be used for more general problems. For example, the case of general second-order elliptic operators and/or general nonlinearities with polynomial growth can be solved by adopting the proofs in [26]. Similarly, if one considers linear or nonlinear parabolic equations complemented by nonlinear Neumann boundary conditions of the form $\partial_\nu y = y^\lambda$ or $\partial_\nu y = y^\lambda + u$ then one can use estimates in [28] and [11].

In this paper we consider two modifications of the model problem (1.1)-(1.2): a problem with multiplicative control and a problem governed by a parabolic system.

In the case of multiplicative control we replace the state equation (1.1) by

$$\partial_t y - \Delta y = y^\lambda + uy, \qquad x \in \Omega,\ t \in [0,T], \tag{1.4}$$

and we prove the required a priori bounds by using the energy and perturbation arguments mentioned above. This study is motivated by the fact that multiplicative controls often appear in the literature.

In Section 6 we investigate the existence of optimal controls for problems governed by the system

$$\left.\begin{aligned} \partial_t y_1 - \Delta y_1 &= \kappa y_1 y_2 - b y_1 + u, \qquad && x \in \Omega,\ t \in [0,T], \\ \partial_t y_2 - d\Delta y_2 &= a y_1, && x \in \Omega,\ t \in [0,T], \end{aligned}\right\} \tag{1.5}$$

which is complemented by suitable boundary and (nonnegative) initial conditions. Here $d \geq 0$, $\kappa, a > 0$ $b \in \mathbb{R}$ and u is a nonnegative control. System (1.5) (with $d = 0$ and $u = 0$) was derived in [19] as a model for the dynamics of a nuclear reactor close to a stationary state. The state variables y_1 and y_2 correspond to the neutron flux and the temperature, respectively, and the constant κ represents the temperature feedback (cf. also [29]). Since this system (with $d \geq 0$, $\kappa > 0$ and $u = 0$) possesses an interesting dynamics with possible blow-up in finite time, it became the object of study of many mathematical papers (see [10], [17], [23], [24], [31], [32] and the references therein). We consider the case $d = \kappa = 1$ and study the corresponding optimal control problem with final observation. Since the energy arguments used in the case of equations (1.1) or (1.4) cannot be applied, we use a different approach to the proof of a priori bounds.

This paper is organized as follows. In Section 2 we formulate our main results (Theorems 2.3, 2.6, 2.8 and 2.10). Sections 3 and 4 are devoted to the proof of existence of optimal controls and optimality conditions, respectively, for the problem governed by the model equation (1.1). Problems governed by (1.4) and (1.5) are studied in Sections 5 and 6, respectively. In the Appendix we recall, for the reader's convenience, from [6] the basic existence, uniqueness and stability results for semilinear parabolic equations which are the fundament for our investigations.

2. Main Results

First we introduce some notation which will be used throughout this paper. If $a, b \in \mathbb{R}$ then we denote $a \vee b := \max(a,b)$ and $a \wedge b := \min(a,b)$. If $p \in (1,\infty)$

then p' is the dual exponent defined by $1/p + 1/p' = 1$. For $X \subset \mathbb{R}^n$ we write $\mathcal{D}(X)$ for the space of smooth functions with compact support in X. The symbols w and w^* are used to denote the weak and weak-star topology, respectively. By Ω we mean an open bounded subset of $\mathbb{R}^n$ having a smooth boundary Γ. We also set $Q := \Omega \times J$ and $\Sigma := \Gamma \times J$, where $J := [0, T]$ with a fixed $T > 0$. By $\mathcal{B}$ we denote one of the boundary operators γ, ∂_ν, where γ is the trace operator and ∂_ν the derivative with respect to ν, the outer unit normal on Γ.

Let $s \in [-2, 2]$ and $1 < q < \infty$. We write $W_q^s := W_q^s(\Omega)$ for the usual Sobolev-Slobodeckii spaces; hence $W_q^0 = L_q(\Omega)$. If $\mathcal{B} = \gamma$ then we set

$$W_{q,\mathcal{B}}^s := \begin{cases} \{ u \in W_q^s \ ; \ \mathcal{B}u = 0 \}, & 1/q < s \le 2, \\ W_q^s, & 0 \le s < 1/q, \\ (W_{q',\mathcal{B}}^{-s})', & -2 \le s < 0, \quad s \ne -1 + 1/q, \end{cases}$$

where X' denotes the dual space to X. If $\mathcal{B} = \partial_\nu$ then

$$W_{q,\mathcal{B}}^s := \begin{cases} \{ u \in W_q^s \ ; \ \mathcal{B}u = 0 \}, & 1 + 1/q < s \le 2, \\ W_q^s, & 0 \le s < 1 + 1/q, \\ (W_{q',\mathcal{B}}^{-s})', & -2 \le s < 0, \quad s \ne -2 + 1/q. \end{cases}$$

In either case the dual spaces are determined by means of the standard L_q-duality pairing. We also set $S_q := \{-2 + 1/q, -1 + 1/q, 1/q, 1 + 1/q\}$.

Weak and Strong Solutions

Consider the problem

$$\left.\begin{aligned} \partial_t y - \Delta y &= f && \text{in } Q, \\ \mathcal{B}y &= 0 && \text{on } \Sigma, \\ y(\cdot, 0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{2.1}$$

where $y^0 \in L_1(\Omega)$ and $f \in L_1(Q)$.

Definition 2.1. Assume that $s \in [0, 2] \setminus S_q$ and $1 < p, q < \infty$. A **weak $L_p(W_q^s)$-solution** of (2.1) **on** $[0, t]$, $0 < t \le T$, is a function $y \in L_{p,\mathrm{loc}}([0, t), W_{q,\mathcal{B}}^s)$ such that

$$\int_0^t \int_\Omega (-\partial_t \varphi - \Delta \varphi) y \, dx \, d\tau = \int_0^t \int_\Omega \varphi f \, dx \, d\tau + \int_\Omega \varphi(0) y^0 \, dx$$

for any $\varphi \in \mathcal{D}(\overline{\Omega} \times [0, t))$ satisfying $\mathcal{B}\varphi = 0$ on $\Gamma \times [0, t]$. It is **global** if $t = T$ and $y \in L_p((0, T), W_{q,\mathcal{B}}^s)$.

The differential operator $\mathcal{C} := 1 - \Delta$ defines an isomorphism between $W_{q,\mathcal{B}}^2$ and $L_q(\Omega)$ and this isomorphism admits a unique extension to an isomorphism $C = C_s$ between $W_{q,\mathcal{B}}^s$ and $W_{q,\mathcal{B}}^{s-2}$ for any $s \in [0, 2] \setminus S_q$ (see [1]). Moreover, $-A := 1 - C$ generates a strongly continuous analytic semigroup $\{e^{-tA} \ ; \ t \ge 0\}$ on $W_{q,\mathcal{B}}^r$ for $r \in [-2, s] \setminus S_q$, and

$$(t \mapsto e^{-tA} x) \in C\big([0, T], W_{q,\mathcal{B}}^r\big) \cap C\big((0, T], W_{q,\mathcal{B}}^s\big) \tag{2.2}$$

with

$$\|e^{-tA} x\|_{W_{q,\mathcal{B}}^s} \le c t^{(r-s)/2} \, \|x\|_{W_{q,\mathcal{B}}^r}, \qquad 0 < t \le T, \tag{2.3}$$

for $x \in W^r_{q,\mathcal{B}}$ (cf. [1, Theorem 5.2] and [2, Theorem V.2.1.3]). Then, provided $1 < q < n/(n-2)$ and $0 \le s < 2 - n/q'$, (the weak form of) problem (2.1) is equivalent to the abstract evolution equation

$$\dot{y} + Ay = f \text{ in } [0,T], \qquad y(0) = y^0, \tag{2.4}$$

(see [6] for details).

Definition 2.2. A weak $L_p(W^s_q)$-solution y of (2.1) on $[0,t]$ is a **strong $L_p(W^s_q)$-solution** if

$$y \in W^1_{r,\mathrm{loc}}([0,t), W^{s-2}_{q,\mathcal{B}}) \cap L_{r,\mathrm{loc}}([0,t), W^s_{q,\mathcal{B}})$$

for some $r > 1$ and (2.4) is satisfied a.e. in $[0,t]$. If, in addition, $y \in C^\rho([0,t), W^s_{q,\mathcal{B}})$ for some $\rho \in [0,1)$ then y is called **strong $C^\rho(W^s_q)$-solution**.

A Model Problem

Now we are ready to formulate the main results of this paper. First consider the optimal control problem (1.3) for the model state equation

$$\left.\begin{aligned} \partial_t y - \Delta y &= |y|^{\lambda-1}y + u && \text{in } Q,\\ \mathcal{B}y &= 0 && \text{on } \Sigma,\\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{2.5}$$

where $\mathcal{B} \in \{\gamma, \partial_\nu\}$. As already announced in the introduction, instead of the operator $-\Delta$ and the model nonlinearity $|y|^{\lambda-1}y$ we could handle a general second-order elliptic operator $\mathcal{A}$ and a general superlinear function $f(x,y)$ satisfying suitable growth conditions (see [26] for details).

In the following theorem we consider cost functionals $\mathbb{J}$ which depend on the final value of y and which satisfy the coercivity condition

$$\mathbb{J}(y,u) \ge c_1 \|y(\cdot,T)\|_{L_q(\Omega)} - c_2, \tag{2.6}$$

with positive constants c_1 and c_2.

Theorem 2.3. *Let*

$$1 < \lambda < \frac{n+2}{(n-2)_+}, \tag{2.7}$$

$$q \in \Big((\lambda-1)\frac{n}{2}, \frac{2n}{(n-4)_+}\Big) \qquad \textit{and} \qquad q \ge 2. \tag{2.8}$$

Suppose that $r \ge 2$ satisfies

$$\frac{1}{r} < 1 - \frac{n}{4} + \frac{n}{2q} \tag{2.9}$$

and

$$r > \frac{\lambda+1}{\lambda}\,\frac{\lambda n - (n+4)}{n+2-\lambda(n-2)} - \frac{2}{\lambda}. \tag{2.10}$$

Assume that $y^0 \in W^2_{q,\mathcal{B}}$ and $\mathbb{U}_{\mathrm{ad}}$ is a weakly compact subset of $L_r(J, L_2(\Omega))$. If $u \in \mathbb{U}_{\mathrm{ad}}$ then (2.5) has a unique strong $L_{r\lambda}(L_{2\lambda})$-solution defined on the maximal existence interval J_u.

Assume $\mathbb{U}^G_{\mathrm{ad}} \ne \emptyset$. Let (2.6) be true and assume that $\mathbb{J}$ can be written in the form $\mathbb{J}(y,u) = \mathbb{J}_T(y(\cdot,T),u)$, where $\mathbb{J}_T : L_q(\Omega) \times (L_r(J, L_2(\Omega)), w) \to \mathbb{R}$ is lower semicontinuous. Then the optimal control problem (1.3) governed by (2.5) has a solution.

Remarks 2.4. (i) Theorem 2.3 remains true if we replace the nonlinearity $|y|^{\lambda-1}y$ with $|y|^{\lambda}$, see Remark 3.3.

(ii) Let λ, q, r satisfy (2.7)–(2.10), $y^0 \in W^2_{q,\mathcal{B}}$,

$$\mathbb{J}(y,u) := \int_\Omega |y(x,T) - y_d(x)|^q \, dx + N \int_0^T \Bigl(\int_\Omega u^2(x,t)\, dx \Bigr)^{r/2} dt, \tag{2.11}$$

where $y_d \in L_q(\Omega)$, $N \geq 0$, and let $\mathbb{U}_{\rm ad} \subset L_r(J, L_2(\Omega))$ be closed, convex and bounded. Then all assumptions of Theorem 2.3 are satisfied provided $\mathbb{U}^G_{\rm ad} \neq \emptyset$. In addition, if $N > 0$ then $\mathbb{U}_{\rm ad}$ need not be bounded (we can replace the set $\mathbb{U}_{\rm ad}$ in problem (1.3) with $\tilde{\mathbb{U}}_{\rm ad} := \mathbb{U}_{\rm ad} \cap \mathbb{B}_R$, where $\mathbb{B}_R$ is a large closed ball in $L_r(J, L_2(\Omega))$).

(iii) If $r = 2$ then conditions (2.7)–(2.10) in Theorem 2.3 read

$$1 < \lambda < \frac{3n+8}{(3n-4)_+}, \qquad q \in \Bigl((\lambda-1)\frac{n}{2}, \frac{2n}{(n-2)_+} \Bigr) \qquad \text{and} \qquad q \geq 2.$$

In particular, if $n \leq 3$ then we may choose $\lambda = 3$ and $q = 4$ (cf. the open problems of J.L. Lions mentioned above). □

Example 2.5. Let $\lambda, q, r, y^0, \mathbb{J}, \mathbb{U}_{\rm ad}$ be as in Remark 2.4(ii). Assume $|y^0| \leq C_0$ for some $C_0 \geq 0$ and $\{u \in L_\infty(Q)\,;\, |u| \leq C_0^\lambda\} \subset \mathbb{U}_{\rm ad}$. Then $\mathbb{U}^G_{\rm ad} \neq \emptyset$, hence the optimal control problem (1.3) has a solution. In fact, the solution $\tilde{y}$ of the linear problem

$$\left.\begin{aligned} \partial_t \tilde{y} - \Delta \tilde{y} &= 0 && \text{in } Q, \\ \mathcal{B}\tilde{y} &= 0 && \text{on } \Sigma, \\ \tilde{y}(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\}$$

satisfies $|\tilde{y}| \leq C_0$ by the maximum principle, thus $u := -|\tilde{y}|^{\lambda-1}\tilde{y} \in \mathbb{U}^G_{\rm ad}$ (the function $y := \tilde{y}$ is a global solution of (2.5)).

Optimality Conditions

In order to obtain the optimality conditions, we restrict ourselves to the case $r = 2$ and we also fix

$$\mathbb{J}(y,u) := \int_\Omega |y(x,T) - y_d(x)|^q \, dx + N \int_Q u^2(x,t)\, dx\, dt, \tag{2.12}$$

where $q > 1$, $y_d \in L_q(\Omega)$ and $N \geq 0$ are given. This particular choice of r and $\mathbb{J}$ corresponds to the setting of J.L. Lions in [21].

Theorem 2.6. *Let the assumptions of Theorem 2.3 be fulfilled and let, moreover,* $r = 2$, $\mathbb{U}_{\rm ad}$ *be convex and* $\mathbb{J}$ *be as in* (2.12). *If* (y,u) *is an optimal pair for problem* (1.3) *governed by* (2.5) *and* p *is the solution of*

$$\left.\begin{aligned} -\partial_t p - \Delta p &= \lambda |y|^{\lambda-1} p && \text{in } Q, \\ \mathcal{B}p &= 0 && \text{on } \Sigma, \\ p(\cdot,T) &= q|y(\cdot,T) - y_d|^{q-2}(y(\cdot,T) - y_d) && \text{in } \Omega, \end{aligned}\right\} \tag{2.13}$$

then

$$\int_Q (p + 2Nu)(v - u)\, dx\, dt \geq 0 \qquad \text{for all } v \in \mathbb{U}_{\rm ad}.$$

Remarks 2.7. (a) The existence of an optimal pair (y,u) in Theorem 2.6 is guaranteed by Theorem 2.3. The solvability of (2.13) follows from Lemma 4.1 and Remark 4.2 below.

(b) As in Remark 2.4(ii), in Theorem 2.6 we can allow $\mathbb{U}_{\rm ad}$ to be any closed convex subset of $L_2(Q)$ if $N>0$. □

Multiplicative Controls

Next we consider the optimal control problem (1.3) governed by the equation

$$\left.\begin{aligned} \partial_t y-\Delta y &= |y|^{\lambda-1}y+uy &&\text{in } Q,\\ \mathcal{B}y &= 0 &&\text{on } \Sigma,\\ y(\cdot,0) &= y^0 &&\text{in } \Omega, \end{aligned}\right\} \tag{2.14}$$

where $\mathcal{B}\in\{\gamma,\partial_\nu\}$.

Theorem 2.8. *Let* (2.6), (2.7) *and* (2.8) *be satisfied,* $y^0\in W^2_{q,\mathcal{B}}$, $\mathbb{U}_{\rm ad}\subset L_\infty(Q)$ *be* w^**-sequentially compact and* $\mathbb{U}^G_{\rm ad}\neq\emptyset$. *Assume that* $\mathbb{J}$ *can be written in the form* $\mathbb{J}(y,u)=\mathbb{J}_T(y(\cdot,T),u)$ *, where* $\mathbb{J}_T: L_q(\Omega)\times(L_\infty(Q),w^*)\to\mathbb{R}$ *is sequentially lower semicontinuous. Then problem* (1.3) *governed by* (2.14) *has a solution.*

Remark 2.9. Similarly as in Remark 2.4(ii) and Example 2.5, all assumptions of Theorem 2.8 concerning $\mathbb{U}_{\rm ad}$ and $\mathbb{J}$ are satisfied if, for example, $|y^0|\le C_0$, $D_1\ge C_0^{\lambda-1}$, $D_2\ge 0$, $N\ge 0$,

$$\mathbb{U}_{\rm ad}=\{u\in L_\infty(Q): -D_1\le u\le D_2\}$$

and

$$\mathbb{J}(y,u)=\int_\Omega |y(x,T)-y_d(x)|^q\,dx+N\|u\|_{L_\infty(Q)}.$$

Again, we may take $D_1=\infty$ and/or $D_2=\infty$ if $N>0$. □

Control of Systems

Finally, let us formulate our result concerning the parabolic system

$$\left.\begin{aligned} \partial_t y_1-\Delta y_1 &= y_1y_2-by_1+u &&\text{in } Q,\\ \partial_t y_2-\Delta y_2 &= ay_1 &&\text{in } Q,\\ \mathcal{B}y_1=\mathcal{B}y_2 &= 0 &&\text{on } \Sigma,\\ y_1(\cdot,0) &= y_1^0 &&\text{in } \Omega,\\ y_2(\cdot,0) &= y_2^0 &&\text{in } \Omega, \end{aligned}\right\} \tag{2.15}$$

where $a>0$, $b\in\mathbb{R}$, $\mathcal{B}\in\{\gamma,\partial_\nu\}$,

$$y_1^0,y_2^0\ge 0,\qquad y_1^0,y_2^0\in C^2(\bar\Omega),\qquad \mathcal{B}y_1^0=\mathcal{B}y_2^0=0, \tag{2.16}$$

$$u\in L_r(J,L_z^+(\Omega)),\qquad r,z>1,\ \frac{1}{r}+\frac{n}{2z}<1. \tag{2.17}$$

As usual, $L_r(J,L_z^+(\Omega))$ is the set of positive functions in $L_r(J,L_z(\Omega))$. The regularity assumption (2.17) guarantees that (2.15) possesses a unique strong solution (defined on the maximal existence interval J_u) and this solution is Hölder continuous both in x and t.

Theorem 2.10. *Consider problem* (2.15) *with* $a > 0$, $b \in \mathbb{R}$. *Let* (2.16), (2.17) *be satisfied, where either* $\mathcal{B} = \partial_\nu$ *and* $n \le 3$ *or* $\mathcal{B} = \gamma$ *and* $n \le 2$. *Assume that* $\mathbb{U}_{\rm ad}$ *is a compact set in* $L_r(J, L_z^+(\Omega))$, $\mathbb{U}_{\rm ad}^G \neq \emptyset$, *and* $\mathbb{J}$ *can be written in the form* $\mathbb{J}(y,u) = \mathbb{J}_T(y_1(T), u)$, *where*

$$\mathbb{J}_T : L_q(\Omega) \times L_r(J, L_z^+(\Omega)) \to \mathbb{R} \quad \textit{is lower semicontinuous,}$$

$q \in [1, \infty]$ *and* $\mathbb{J}(y,u) \ge c_1 \|y_1(T)\|_{L_q(\Omega)} - c_2$. *Then the optimal control problem* (1.3) *governed by* (2.15) *has a solution.*

Remark 2.11. As above, we can easily find examples of $\mathbb{U}_{\rm ad}$ and $\mathbb{J}$ satisfying the compactness and lower semicontinuity assumptions in Theorem 2.10. The assumption $\mathbb{U}_{\rm ad}^G \neq \emptyset$ is satisfied if, for example, $\mathcal{B} = \gamma$, $b \ge 0$, $0 \in \mathbb{U}_{\rm ad}$ and y_1^0, y_2^0 are small enough (e.g. in $L_\infty(\Omega)$). This is due to the fact that in this case, zero is an asymptotically stable equilibrium of (2.15) with $u = 0$. If $\mathcal{B} = \partial_\nu$, $y_1^0 = y_2^0 = 0$ and $0 \in \mathbb{U}_{\rm ad}$ then obviously $0 \in \mathbb{U}_{\rm ad}^G$. □

3. Solvability of the Model Problem

Proof of Theorem 2.3. Set $s := 0$, $q := 2\lambda$ and $p := r\lambda$. Since $r \ge 2$ and $1 < \lambda < \frac{n+2}{(n-2)_+}$, there exists $\sigma \notin S_q$ satisfying

$$\frac{2}{r\lambda'} < \sigma < \frac{2}{r} \wedge \Big(2 - \frac{n}{2\lambda'}\Big).$$

Now Theorem A.1 guarantees the existence of a unique $L_{r\lambda}(L_{2\lambda})$-solution y of (2.5) defined on the maximal existence interval J_u. Fixing $u \in \mathbb{U}_{\rm ad}^G$, this solution is global and $|y|^\lambda \in L_r(J, L_2(\Omega))$. The Sobolev maximal regularity for (2.5), [2, Theorem III.4.10.2] and interpolation theorems in [4] (also see [3, Theorem 3]) imply

$$y \in W_r^1(J, L_2(\Omega)) \cap L_r(J, W_{2,\mathcal{B}}^2) \hookrightarrow C(J, W_{2,\mathcal{B}}^1) \cap C(J, W_{q,\mathcal{B}}^z) \cap L_{r\lambda}(J, L_{2\lambda}(\Omega)) \quad (3.1)$$

for any

$$z < 2 - \frac{n}{2} + \frac{n}{q} - \frac{2}{r},$$

where the embedding into $C(J, W_{q,\mathcal{B}}^z) \cap L_{r\lambda}(J, L_{2\lambda}(\Omega))$ is compact.

Let (y_k, u_k) be a minimizing sequence for problem (1.3). We may assume $u_k \to u$ weakly in $L_r(J, L_2(\Omega))$ and $\|u_k\|_{L_r(J, L_2(\Omega))} \le C_r$. Part (a) of the proof of [6, Theorem 1.1] shows that there exists $t_0 > 0$ independent of k such that

$$y_k \text{ are uniformly bounded in } L_{r\lambda}([0, t_0], L_{2\lambda}(\Omega)). \quad (3.2)$$

Set $u_k(x,t) := 0$ for $t \in (T, 2T]$ and consider problem (2.5) with J replaced by $[0, 2T]$. This problem possesses a unique $L_{r\lambda}(L_{2\lambda})$-solution $\tilde{y}_k$ defined on the maximal existence interval $J_{\tilde{y}_k} \subset [0, 2T]$. The function $w_k(t) := \tilde{y}_k(T + t)$ is the $L_{r\lambda}(L_{2\lambda})$-solution of (2.5) with $u \equiv 0$, initial condition $w_k(0) = y_k(T)$ and the maximal existence interval $J_{w_k} \subset [0, T]$. The boundedness of $\mathbb{J}(y_k, u_k)$ implies a bound for $y_k(T)$ in $L_q(\Omega)$ and the well posedness of (2.5) in $L_q(\Omega)$, guaranteed by Lemma 3.1 below, shows the existence of $t_1 > 0$ such that $[0, t_1] \subset J_{w_k}$ for any k. Consequently, all solutions y_k can be continued on the interval $[T, T + t_1]$. Now Lemma 3.2 below implies $\|y_k(\tau)\|_{L_q(\Omega)} \le C_q$ for any $\tau \in [0, T]$.

Let $\tau^* = \tau^*(C_r, C_q)$ be from Lemma 3.1. Fixing $\delta \in (0, t_0 \wedge \tau^*)$ and using the last statement of Lemma 3.1 for $w_k(t) := y_k(\tau + t)$, $t \in [0, \tau^*]$, $\tau \in [t_0 - \delta, T - \tau^*]$, we get a uniform bound for y_k in $L_{r\lambda}([t_0,T],L_{2\lambda}(\Omega))$. This bound and (3.2) show the boundedness of $|y_k|^{\lambda-1}y_k$ in $L_r(J,L_2(\Omega))$. As in (3.1), we get that the sequence $(|y_k|^{\lambda-1}y_k)$ is compact in $L_r(J,L_2(\Omega))$ and $(y_k(T))$ is compact in $L_q(\Omega)$. Now it is easy to pass to the limit to get a solution of (1.3). □

Let λ, q be as in Theorem 2.3 and let $r\geq 2$ satisfy (2.9). These assumptions guarantee that there exists $s\notin S_q$ such that

$$0\vee\Big(\frac{n}{q}-\frac{n}{\lambda}\Big)\vee\Big[\frac{n}{q}-\frac{1}{\lambda}\Big(2+\frac{n}{q}\Big)\Big]<s<\frac{2}{\lambda}\wedge\Big(2+\frac{n}{q}-\frac{n}{2}-\frac{2}{r}\Big)\wedge\Big[\frac{1}{\lambda}\Big(2+\frac{n}{q}\Big)-\frac{2}{r}-\frac{n}{2}+2\Big]. \tag{3.3}$$

Lemma 3.1. *Let λ, q be as in Theorem 2.3 and let $r\geq 2$ satisfy (2.9). Assume $u\in L_r(J,L_2(\Omega))$. Then problem (2.5) is well posed in $L_q(\Omega)$. More precisely, if the norm of u in $L_r(J,L_2(\Omega))$ is bounded by a constant C_r, $y^0\in L_q(\Omega)$, $\|y^0\|_{L_q(\Omega)}\leq C_q$, and s satisfies (3.3), then there exists $\tau^*=\tau^*(C_r,C_q)>0$ and a unique solution*

$$y\in C([0,\tau^*],L_q(\Omega))\cap C((0,\tau^*],W^s_{q,\mathcal{B}}). \tag{3.4}$$

In addition, this solution satisfies

$$\|y(t)\|_{L_q(\Omega)}+t^{s/2}\|y(t)\|_{W^s_{q,\mathcal{B}}}\leq C,\qquad t\in(0,\tau^*], \tag{3.5}$$

where C depends only on s, C_r, C_q (and q, r, λ, Ω). If $\hat q\geq q$ satisfies

$$\hat q<\frac{2n}{(n-4)_+}\qquad\text{and}\qquad\frac{1}{r}<1-\frac{n}{4}+\frac{n}{2\hat q} \tag{3.6}$$

then

$$y\in C((0,\tau^*],L_{\hat q}(\Omega)) \tag{3.7}$$

and

$$\|y(t)\|_{L_{\hat q}(\Omega)}\leq C(\delta,\hat q,C_r,C_q),\qquad t\in[\delta,\tau^*],\quad\delta\in(0,\tau^*). \tag{3.8}$$

Finally,

$$y\in C([\delta,\tau^*],W^1_{2,\mathcal{B}}(\Omega))\cap L_{r\lambda}([\delta,\tau^*],L_{2\lambda}(\Omega)) \tag{3.9}$$

for any $\delta>0$ and the norm of y in this space can be bounded by $C(\delta,C_r,C_q)$.

Proof. The proof of the first part is an easy modification of [8, Theorem 4.1]. In fact, let X be the Banach space of all functions

$$y\in C([0,\tau^*],L_q(\Omega))\cap C((0,\tau^*],W^s_{q,\mathcal{B}})$$

for which

$$\|y\|_X:=\sup_{t\in(0,\tau^*]}(\|y(t)\|_{L_q(\Omega)}+t^{s/2}\|y(t)\|_{W^s_{q,\mathcal{B}}})<\infty.$$

Then it is sufficient to use the Banach fixed point theorem for the mapping

$$Ky(t)=e^{-At}y^0+\int_0^t e^{-A(t-\tau)}(|y(\tau)|^{\lambda-1}y(\tau)+u(\tau))\,d\tau$$

in a large closed ball $\mathbb{B}$ of X with radius R, where A is as in (2.4). For example, assume that $y\in\mathbb{B}$ and denote by $\|\cdot\|_s$ the norm in $W^s_{q,\mathcal{B}}$. Fixing s satisfying (3.3) there exists

$$z\in(1,q]\quad\text{such that}\qquad\lambda\Big(\frac{n}{q}-s\Big)\vee\lambda\Big(\frac{2}{r}+\frac{n}{2}-2\Big)<\frac{n}{z}<2+\frac{n}{q}-s\lambda. \tag{3.10}$$

Choose $\sigma_1 \in (s\lambda, 2+n/q-n/z)$ and $\sigma_2 \in (s+2/r, 2+n/q-n/2)$, $\sigma_1, \sigma_2 \notin S_q$. Then we have $L_z(\Omega) \hookrightarrow W_{q,\mathcal{B}}^{\sigma_1-2}$ and $L_2(\Omega) \hookrightarrow W_{q,\mathcal{B}}^{\sigma_2-2}$, hence it follows from (2.3) that

$$\begin{aligned} t^{s/2}\|Ky(t)\|_s &\le C(C_q) + Ct^{s/2}\int_0^t (t-\tau)^{(\sigma_1-s)/2-1}\||y(\tau)|^{\lambda-1}y(\tau)\|_{\sigma_1-2}\,d\tau \\ &\quad + Ct^{s/2}\int_0^t (t-\tau)^{(\sigma_2-s)/2-1}\|u(\tau)\|_{\sigma_2-2}\,d\tau \\ &\le C(C_q) + Ct^{s/2}\int_0^t (t-\tau)^{(\sigma_1-s)/2-1}\|y(\tau)\|_s^\lambda\,d\tau \\ &\quad + Ct^{s/2}\int_0^t (t-\tau)^{(\sigma_2-s)/2-1}\|u(\tau)\|_{L_2(\Omega)}\,d\tau \\ &\le C(C_q) + CR^\lambda t^{s/2}\int_0^t (t-\tau)^{(\sigma_1-s)/2-1}\tau^{-s\lambda/2}\,d\tau \\ &\quad + CC_r t^{s/2}\Big(\int_0^t (t-\tau)^{r'[(\sigma_2-s)/2-1]}\,d\tau\Big)^{1/r'}, \end{aligned}$$

which shows $t^{s/2}\|Ky(t)\|_s < R/2$ if $R = R(C_q)$ is large enough and $t = t(R, C_r)$ is small enough. Similar arguments show the same bound for $\|Ky(t)\|_{L_q(\Omega)}$ and the fact that K is a contraction. Obviously, the fixed point of K is a solution of our problem. Uniqueness of this solution in the class (3.4) can be proved in the same way as in [7, pp. 295–296].

We have $W_{q,\mathcal{B}}^s \hookrightarrow L_{q_1}(\Omega)$ whenever $n/q_1 > n/q - s$. Due to the upper bound for s in (3.3), q_1 is restricted by the conditions

$$\frac{n}{q_1} > -2 + \frac{2}{r} + \frac{n}{2} \quad \text{and} \quad \frac{n}{q_1} > \frac{n}{q} - \varepsilon(q), \tag{3.11}$$

where

$$\varepsilon(q) := \frac{2}{\lambda} + \frac{n}{\lambda q} + 2 - \frac{2}{r} - \frac{n}{2} > 0.$$

Let $\hat{q} \ge q$ satisfy (3.6). If $n/\hat{q} > n/q - \varepsilon(q)$ then $W_{q,\mathcal{B}}^s \hookrightarrow L_{\hat{q}}(\Omega)$ since the second inequality in (3.6) guarantees that the first condition in (3.11) is satisfied with $q_1 = \hat{q}$. Consequently, (3.7) and (3.8) follow from (3.4) and (3.5). If $n/\hat{q} \le n/q - \varepsilon(q)$ then we fix $q_1 > q$ satisfying (3.11) (this is possible due to (2.9)). Now the first part of the Lemma with q replaced by q_1 (and $t = 0$ replaced by $t = \delta_1$, where $\delta_1 > 0$ is small) implies $y \in C((\delta_1, \tau^*], W_{q_1,\mathcal{B}}^{s_1})$. Similarly as above, $W_{q_1,\mathcal{B}}^{s_1} \hookrightarrow L_{q_2}(\Omega)$, where

$$\frac{n}{q_2} > -2 + \frac{2}{r} + \frac{n}{2} \quad \text{and} \quad \frac{n}{q_2} > \frac{n}{q_1} - \varepsilon(q_1).$$

Repeating this bootstrapping argument finitely many times, we obtain (3.7) and (3.8).

It remains to prove (3.9) and the corresponding bound. Fix $\delta \in (0, \tau^*)$ and set $t_0 := \delta/2$, $J_0 := [t_0, \tau^*]$ and $J^* := [\delta, \tau^*]$. Taking $R > 1$ large and $\hat{q}$ close to its upper bound, we have $\hat{q} > \lambda$ and $|y|^\lambda \in L_R(J_0, L_{\hat{q}/\lambda}(\Omega))$. Set $f_1 := |y|^{\lambda-1}y$ and $f_2 := u$. Writing $y = y_1 + y_2 + y_3$, where $\mathcal{B}y_i = 0$, $i = 1, 2, 3$, and

$$\left.\begin{array}{lll}\partial_t y_1 - \Delta y_1 = f_1 & \text{in } \Omega\times J_0, & y_1(t_0)=0,\\ \partial_t y_2 - \Delta y_2 = f_2 & \text{in } \Omega\times J_0, & y_1(t_0)=0,\\ \partial_t y_3 - \Delta y_3 = 0 & \text{in } \Omega\times J_0, & y_1(t_0)=y(t_0),\end{array}\right\} \tag{3.12}$$

the maximal Sobolev regularity implies

$$y_1 \in W^1_R(J_0, L_{\hat q/\lambda}(\Omega)) \cap L_R(J_0, W^2_{\hat q/\lambda,\mathcal{B}}) \hookrightarrow C(J_0, W^1_{2,\mathcal{B}})$$

since we can take $\hat q > 2\lambda n/(n+2)$ and R arbitrarily large. Similar arguments guarantee $y_2 \in C(J_0, W^1_{2,\mathcal{B}})$ and $y_3 \in C(J^*, W^1_{2,\mathcal{B}})$, hence $y \in C(J^*, W^1_{2,\mathcal{B}})$ (and the corresponding estimate in this space is valid).

Choose $k > 1$ such that $\hat q > (\lambda - 1/k)n/2$ and fix $m \in \mathbb{N}$ such that $k^m \hat q > 2\lambda$. Choose also $R > r\lambda^{m+1}$. Set $t_i := \delta/2 + i\delta/(2m+2)$, $J_i := [t_i, \tau^*]$ and $\hat q_i := k^i \hat q$, $i = 1, 2, \dots, m$. Notice that $y_2, y_3 \in L_{r\lambda}(J_1, L_{2\lambda}(\Omega))$ and $W^2_{\hat q/\lambda,\mathcal{B}} \hookrightarrow L_{\hat q_1}(\Omega)$, hence $y_1 \in L_R(J_1, L_{\hat q_1}(\Omega))$. Consequently, $|y|^\lambda$ can be written in the form

$$|y|^\lambda = \tilde f_1 + \tilde f_2, \qquad \tilde f_1 \in L_{R/\lambda}(J_1, L_{\hat q_1/\lambda}(\Omega)), \quad \tilde f_2 \in L_r(J_1, L_2(\Omega)).$$

Writing $y = \tilde y_1 + \tilde y_2 + \tilde y_3$, where $\mathcal{B}\tilde y_i = 0$, $i = 1, 2, 3$, and $\tilde y_1, \tilde y_2, \tilde y_3$ satisfy (3.12) with f_1, f_2, J_0, t_0 replaced by $\tilde f_1, \tilde f_2, J_1, t_1$, respectively, we obtain as above $\tilde y_2, \tilde y_3 \in L_{r\lambda}(J_2, L_{2\lambda}(\Omega))$ and $\tilde y_1 \in L_{R/\lambda}(J_2, L_{\hat q_2}(\Omega))$. Repeating this argument m times we get

$$y \in L_{r\lambda}(J^*, L_{2\lambda}(\Omega)) + L_{R/\lambda^m}(J^*, L_{\hat q_m}(\Omega)) = L_{r\lambda}(J^*, L_{2\lambda}(\Omega))$$

(and the corresponding estimates), which concludes the proof. □

Lemma 3.2. *Let λ, q, r be as in Theorem 2.3. Let $t_1 > 0$, $u \in L_r([0, T + t_1], L_2(\Omega))$ and let its norm in this space be bounded by a positive constant C_r. Assume that y is a global solution of* (2.5) *(with J replaced by $[0, T + t_1]$) and $y^0 \in W^2_q(\Omega)$, $\|y^0\|_{W^2_q(\Omega)} \le C_q$. Then there exists a constant $C = C(C_r, C_q, t_1)$ such that $\|y(t)\|_{L_q(\Omega)} \le C$ for any $t \in [0, T]$.*

Proof. The proof is a modification of the proof of the main result in [25] (cf. also [26] and [27, the proof of Theorem 5.1]).

All our constants (and bounds) in this proof may change from line to line and may depend on C_r, C_q, t_1. First we deduce from Lemma 3.1 and the beginning of the proof of Theorem 2.3 that $y \in C([0, T + t_1], W^1_{2,\mathcal{B}})$ and there exists $\tau > 0$ such that

$$y \text{ is bounded in } C([0,\tau], L_q(\Omega)) \text{ by a constant } C = C(C_q, C_r). \tag{3.13}$$

Denote

$$V(t) = \frac{1}{2}\int_\Omega |\nabla y(x,t)|^2\,dx - \frac{1}{\lambda+1}\int_\Omega |y(x,t)|^{\lambda+1}\,dx.$$

If u is smooth then

$$V'(t) = -\int_\Omega (\partial_t y)^2\,dx + \int_\Omega u\partial_t y\,dx \le \frac{1}{2}\int_\Omega u^2\,dx - \frac{1}{2}\int_\Omega (\partial_t y)^2\,dx,$$

hence

$$V(\tau_2) - V(\tau_1) \le C - \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_\Omega (\partial_t y)^2\,dx\,dt. \tag{3.14}$$

Now let u be general. Approximating u by smooth functions u_k we see that (3.14) remains true for any $u \in L_r([0, T + t_1], L_2(\Omega))$.

We will show that $V(t)$ is bounded for $t \in [0,T]$. The upper estimate for $V(t)$ follows immediately from (3.14). To prove the lower estimate we assume on the contrary that $V(t_0) \le -(C+K)$ for some $t_0 \in [0,T]$, where C is from (3.14) and $K \gg 1$. Then (3.14) guarantees $V(t) \le -K$ for all $t \ge t_0$. Multiplying the equation in (2.5) by y and integrating over Ω we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega y^2\,dx &= -2V(t) + c_1\int_\Omega |y|^{\lambda+1}\,dx + \int_\Omega uy\,dx \\ &\ge K + c_2\Big(\int_\Omega y^2\,dx\Big)^{(\lambda+1)/2} - C_2\int_\Omega u^2\,dx,\end{aligned} \tag{3.15}$$

where the inequality is true for all $t \ge t_0$. Denote $Y(t) = \int_{t_0}^t\int_\Omega y^2\,dx\,dt$. Then integrating estimate (3.15) we get

$$Y' \ge c_3 Y^{(\lambda+1)/2} + 2K(t-t_0) - C_3.$$

Let $K \ge 10C_3/t_1$. Integrating the inequality $Y' \ge 2K(t-t_0) - C_3$ on $[t_0, t_0+t_1/2]$ we obtain

$$Y(t_0 + t_1/2) \ge K\frac{t_1^2}{4} - C_3\frac{t_1}{2} \ge K\frac{t_1^2}{5}.$$

We also have

$$Y' \ge c_3 Y^{(\lambda+1)/2} \quad \text{for} \quad t \ge t_0 + \frac{t_1}{2}.$$

Since the solution of the equation $Z'(t) = c_3 Z^{(\lambda+1)/2}(t)$ for $t \ge 0$, $Z(0) = Kt_1^2/5$, blows up at $t < t_1/2$ if K is large enough, the function $Y(t) \ge Z(t-t_0-t_1/2)$ blows up at some $t < T + t_1$ which yields a contradiction. Hence K has to be bounded by a constant depending only on c_3, C_3, t_1 and λ. Consequently, V is bounded on $[0,T]$ and (3.14) provides a bound for y in the space $W_2^1([0,T], L_2(\Omega))$.

If $\lambda < 1 + 4/n$ then Lemma 3.1 with q replaced by $\tilde q := 2$ and $\hat q$ replaced by q guarantees a bound for y in $L_\infty([\tau,T], L_q(\Omega))$ which (together with (3.13)) implies the assertion.

Let $\lambda \ge 1 + 4/n$. Since y is bounded in $W_2^1([0,T], L_2(\Omega)) \hookrightarrow L_\infty([0,T], L_2(\Omega))$, we have

$$\int_0^T \|uy\|_{L_1(\Omega)}^z\,dt \le C\int_0^T \|u\|_{L_2(\Omega)}^z\,dt \le C, \qquad z \le r.$$

Using this bound and the boundedness of V on $[0,T]$, we obtain from the equality in (3.15)

$$\int_0^T \|y(t)\|_{L_{\lambda+1}(\Omega)}^{z(\lambda+1)}\,dt \le C\Big(1 + \int_0^T \|\partial_t y(t) y(t)\|_{L_1(\Omega)}^z\,dt\Big).$$

In particular, if $z = 2$ then this estimate, the bound for y in $W_2^1([0,T], L_2(\Omega))$ and

$$\|\partial_t y(t) y(t)\|_{L_1(\Omega)} \le \|\partial_t y(t)\|_{L_2(\Omega)}\|y(t)\|_{L_2(\Omega)} \le C\|\partial_t y\|_{L_2(\Omega)}$$

guarantee a uniform bound for y in

$$X_z := L_{z(\lambda+1)}([0,T], L_{\lambda+1}(\Omega)).$$

Interpolating between the bound of y in X_z and in $W_2^1([0,T], L_2(\Omega))$ yields a bound in $L_\infty([0,T], L_m(\Omega))$ provided

$$m < \lambda + 1 - \frac{\lambda-1}{z+1} \tag{3.16}$$

(cf. [25, (12)]). If $r > 2$ then we will use the bootstrapping procedure in [25] in order to get these estimates for some $z > 2$. Replacing u by y, p by λ, q by z, $\tilde{q}$ by $\tilde{z}$, and λ by m in [25], denoting

$$\lambda_1 := (\lambda+1)/\lambda, \qquad \theta := \frac{\lambda+1}{\lambda-1}\,\frac{m-2}{m}, \qquad \beta := \frac{2}{(1-\theta)\tilde{z}},$$

and assuming the estimate in X_z for some $z \geq 2$, we get for $\tilde{z} > z$

$$\begin{aligned}\int_0^T \|y(t)\|_{L_{\lambda+1}(\Omega)}^{\tilde{z}(\lambda+1)}\,dt &\leq C\Big(1+\int_0^T \|\partial_t y(t)y(t)\|_{L_1(\Omega)}^{\tilde{z}}\,dt\Big)\\ &\leq C\Big(1+\int_0^T \|\partial_t y(t)\|_{L_{m'}(\Omega)}^{\tilde{z}}\,dt\Big)\\ &\leq C\Big(1+\int_0^T \|\partial_t y(t)\|_{L_{\lambda_1}(\Omega)}^{\theta\tilde{z}}\|\partial_t y(t)\|_{L_2(\Omega)}^{(1-\theta)\tilde{z}}\,dt\Big)\\ &\leq C\Big(1+\Big(\int_0^T \|\partial_t y(t)\|_{L_{\lambda_1}(\Omega)}^{\theta\beta'\tilde{z}}\,dt\Big)^{1/\beta'}\Big)\\ &\leq C\Big(1+\Big(\int_0^T \|y(t)\|_{L_{\lambda+1}(\Omega)}^{\theta\beta'\tilde{z}\lambda}\,dt\Big)^{1/\beta'}\Big),\end{aligned}$$

provided $\tilde{z} \leq r$ and

$$u \in L_{\theta\beta'\tilde{z}}(J, L_{\lambda_1}(\Omega)). \tag{3.17}$$

Recall from [25] that the bootstrap condition $\theta\beta' \leq \lambda_1$ is satisfied if m is chosen close to its upper bound and $\tilde{z}$ is close to z. For such m and $\tilde{z}$, one can even check that $\theta\beta' < (\lambda+1)r/(\lambda r+2)$ provided $\tilde{z} < r$. Consequently, $\theta\beta'\tilde{z} \vee \tilde{z} < r$ (and (3.17) is true) whenever $\tilde{z} < (\lambda r+2)/(\lambda+1)$. Hence, we obtain a bound for y in X_z for any

$$z < (\lambda r+2)/(\lambda+1). \tag{3.18}$$

Recall that this guarantees a bound in $L_\infty([0,T], L_m(\Omega))$ for any m satisfying (3.16). Using (2.10) we can find z satisfying (3.18) and $m \in ((\lambda-1)n/2, q]$ such that (3.16) is true. Now we can use Lemma 3.1 with q replaced by m and $\hat{q}$ replaced by q to get a bound for y in $L_\infty([\tau,T], L_q(\Omega))$ which (together with (3.13)) concludes the proof. □

Remark 3.3. We announced in Remark 2.4(i) that Theorem 2.3 remains true if we replace the nonlinearity $|y|^{\lambda-1}y$ with $|y|^\lambda$. Let us sketch the proof of this statement.

Since y satisfies

$$\partial_t y - \Delta y = |y|^\lambda + u \geq u \qquad \text{in } Q,$$

the parabolic maximum principle implies $y \geq y_L$, where y_L is the solution of the linear problem

$$\left.\begin{aligned}\partial_t y_L - \Delta y_L &= u && \text{in } Q,\\ \mathcal{B}y_L &= 0 && \text{on } \Sigma,\\ y_L(\cdot,0) &= y^0 && \text{in } \Omega.\end{aligned}\right\}$$

Using the same arguments as in (3.1) we see that $y_L \in L_{r\lambda}(J, L_{2\lambda}(\Omega))$ and that the norm of y_L in this space can be bounded by the norm of u in $L_r(J, L_2(\Omega))$ and a suitable norm of y^0. Notice that

$$|y|^\lambda = |y|^{\lambda-1}y + 2|y^-|^{\lambda-1}y^-,$$

where $y^- := -\min(0,y)$ is bounded above by $|y_L|$, hence $2|y^-|^{\lambda-1}y^-$ is bounded in $L_r(J, L_2(\Omega))$. Consequently, replacing u with $\tilde{u} := u + 2|y^-|^{\lambda-1}y^-$, we can repeat word by word the proof of Theorem 2.3. □

Optimality of the Growth Bounds

Remarks 3.4. (i) Consider problem (2.5) with Ω being the unit ball in R^n, $n \geq 3$, $\mathcal{B} = \gamma$ and $\lambda > (n+2)/(n-2)_+$. If $n > 10$ then assume also

$$\lambda < 1 + 4\frac{n-4+2\sqrt{n-1}}{(n-2)(n-10)}. \tag{3.19}$$

Choose a smooth radial, radially decreasing function $\psi : \bar{\Omega} \to \mathbb{R}^+$ satisfying $\psi(0) > 0$ and $\psi(x) = 0$ for $x \in \Gamma$ and denote by w_α the (classical) solution of (2.5) with $u = 0$ and $y^0 = \alpha\psi$, $\alpha \geq 0$. We deduce from [22] and an obvious modification of [20] that there exists $\alpha^* > 0$ with the following property: if $\alpha < \alpha^*$ then $w_\alpha(t)$ exists for all $t \in \mathbb{R}^+$ and $w_\alpha(t) \to 0$ as $t \to \infty$; if $\alpha > \alpha^*$ then this solution blows up in finite time completely.

From now on fix $y^0 = \alpha^*\psi$. Let y_k be the solution of (2.5) with $u = 0$ and the nonlinearity y^λ replaced by $\min(y^\lambda, k)$, $k = 1, 2, \ldots$ Then y_k are globally defined classical solutions, $y_{k+1} \geq y_k$. Set $y^*(t) = \lim_{k\to\infty} y_k(t)$. The results in [22] and [16] guarantee that $y^* \in L_{p,\mathrm{loc}}([0,\infty), L_p(\Omega))$ is a weak solution solution of (2.5) with $u = 0$ and there exists $T^* \in (0,\infty)$ such that y^* is a classical solution on $(0, T^*)$ but it blows up at $t = T^*$ in the $L_\infty(\Omega)$-norm. In particular, $w_{\alpha^*} = y^*|_{[0,T^*)}$. Next [12] shows that y^* is a classical solution for all t except for finitely many points $T_0 = T^* < T_1 < \cdots < T_k$. Choose $T > T^*$ such that $T \neq T_j$ for any j and let $y_d(x) := y^*(x, T)$. Choose also $0 < t_1 < t_2 < T^*$ and a smooth function $U : \bar{\Omega} \times [0, T] \to [0, \infty)$ with support $K_U \subset \Omega \times [t_1, t_2]$, $K_U \neq \emptyset$, and denote by y^*_β the solution of (2.5) with $u = \beta U$ and $y^0 = \alpha^*\psi$.

Since $y^* > 0$ in K_U and $y^*_{-\beta} \to y^*$ uniformly in K_U as $\beta \to 0+$, fixing $b > 0$ small we have $|y^*_{-b}|^{\lambda-1}y^*_{-b} - bU \geq 0$ in K_U. Consequently, the maximum principle implies $y^*_{-b} \geq 0$. Choose $\beta \in (0, b]$. Since $y^*(t_2) - y^*_{-\beta}(t_2)$ belongs to the interior of the positive cone in $C^1(\bar{\Omega})$ and $w_\alpha(t_2) \to w_{\alpha^*}(t_2) = y^*(t_2)$ in $C^1(\bar{\Omega})$ as $\alpha \to \alpha^*-$, there exists $\alpha < \alpha^*$ such that $y^*_{-\beta}(t_2) \leq w_\alpha(t_2)$. Now the maximum principle implies $y^*_{-\beta}(t) \leq w_\alpha(t)$ for any $t \geq t_2$ and $y^*_{-\beta} \geq y^*_{-b} \geq 0$ for any $t \geq 0$, hence $y^*_{-\beta}$ is a global nonnegative classical solution. On the other hand, if $\beta \geq 0$ then $y^*_\beta \geq y^*$, hence y^*_β blows up at finite time $T_\beta \leq T^*$ in the $L_\infty(\Omega)$-norm and, consequently, in $L_q(\Omega)$-norm for any $q > n(\lambda-1)/2$ (cf. [14], [30]).

Let $\mathbb{U}_{\mathrm{ad}} = \{\beta U\,;\, \beta \in [-b, b]\}$. Fix $q > n(\lambda-1)/2$ and set $\mathbb{J}(y, u) = \int_\Omega |y(T) - y_d|^q\, dx$. The above arguments show that y^*_β is a global $L_\infty(L_q)$-solution of (2.5) if and only if $\beta < 0$. Moreover, $\beta \mapsto \mathbb{J}(y^*_\beta, \beta U)$ is decreasing on $[-b, 0)$. Hence the optimal control problem (1.3) does not have a solution with $y \in L_\infty(J, L_q(\Omega))$.

(ii) Consider problem (2.5) with Ω being the unit ball in R^n, $\mathcal{B} = \gamma$ and let $1 \leq q < (\lambda-1)n/2$. Then there exists a smooth radial positive function y^0 such that the solution y of (2.5) with $u = 0$ blows up at $t = T$ in the L_∞-norm and satisfies $\partial_t y \geq 0$, $y_d := y(\cdot, T) \in L_q(\Omega)$ (see [14]). Let U be a smooth nonnegative function with support $K \subset \{(x,t)\,;\, |x| < 1/2\}$, $K \neq \emptyset$, and $u_c := cU$. Then there exists $\varepsilon > 0$ such that the solution y of (2.5) with u replaced by $u_{-\varepsilon}$ remains positive.

Let $\mathbb{U}_{\rm ad} = \{u_c\,;\, c \in [-\varepsilon, 0]\}$ and

$$\mathbb{J}(y,u) = \Big| \int_\Omega |y(x,T)|^q\,dx - \int_\Omega y_d^q\,dx \Big|.$$

Then $(y(u_{-1/k}), u_{-1/k})$, $k \geq k_0$, is obviously a minimizing sequence for the control problem (1.3) but $y(u_0)$ is not a (classical) global solution of (2.5). □

4. Proof of the Optimality Conditions

We start with the following technical lemma concerning linear problems.

Lemma 4.1. *Suppose that $\beta > 2 \vee (n+2)/2$ and $2 \leq q < 2n/(n-2)_+$. Given $a \in L_\beta(Q)$, $u \in L_2(Q)$ and $y^0 \in L_{q'}(\Omega)$, problem*

$$\left.\begin{aligned} \partial_t y - \Delta y &= ay + u && \text{in } Q,\\ \mathcal{B}y &= 0 && \text{on } \Sigma,\\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{4.1}$$

has a unique solution

$$y \in C([0,T], L_{q'}(\Omega)) \cap C((0,T], L_q(\Omega)) \cap L_2(Q).$$

The map

$$L_\beta(Q) \times L_2(Q) \times L_{q'}(\Omega) \to L_2(Q) \times L_q(\Omega), \qquad (a,u,y^0) \mapsto (y, y(T)),$$

is analytic and bounded on bounded sets.

Proof. (i) Writing (4.1) in the abstract form

$$\dot y + Ay = ay + u \quad \text{in } (0,T], \qquad y(0) = y^0,$$

and denoting $U(t) := e^{-tA}$, we see that we have to prove the unique solvability of

$$y = U * (ay) + U * u + Uy^0 \tag{4.2}$$

in appropriate spaces.

(ii) Fix $s \in [0,1) \setminus \{1/q'\}$ such that $q \leq 2n/(n-2s)_+$. Then $W^s_{q',\mathcal{B}} \hookrightarrow L_2(\Omega)$. Hence we infer from (2.3) (with q replaced by q' and $r := 0$) that

$$\|U(t)y^0\|_{L_2(\Omega)} \leq c\|U(t)y^0\|_{W^s_{q',\mathcal{B}}} \leq ct^{-s/2}\|y^0\|_{L_{q'}(\Omega)}, \qquad 0 < t \leq T.$$

Since $s < 1$ it follows that

$$(y^0 \mapsto Uy^0) \in \mathcal{L}\big(L_{q'}(\Omega), L_2((0,T), L_2(\Omega))\big) = \mathcal{L}\big(L_{q'}(\Omega), L_2(Q)\big),$$

where $\mathcal{L}(X,Y)$ denotes the space of continuous linear operators from X to Y.

(iii) It is easy to see that

$$(u \mapsto U * u) \in \mathcal{L}(L_2(Q)).$$

(iv) Put $1/r := 1/\beta + 1/2 < 1$ and note that

$$L_r(\Omega) \hookrightarrow W^{-2+\gamma}_{2,\mathcal{B}} \quad \text{if} \quad 1/2 \geq 1/r + (\gamma-2)/n,$$

that is, if $0 \leq \gamma \leq 2 - n/\beta$.

(v) For $m \in \mathbb{R}$ we write $L_{2,m}(Q)$ for $L_2(Q)$ endowed with the equivalent norm

$$y \mapsto \Big(\int_0^T e^{-2mt}\|y(t)\|^2_{L_2(\Omega)}\,dt\Big)^{1/2}.$$

From (iv), Hölder's inequality, and (2.3) (with $q = 2$ and $r := \gamma - 2$) we infer that

$$\begin{aligned}\|U * (ay)(t)\|_{L_2(\Omega)} &\le c\int_0^t (t-\tau)^{\gamma/2-1}\|a(\tau)\|_{L_\beta(\Omega)}\|y(\tau)\|_{L_2(\Omega)}\,d\tau \\ &= ce^{mt}\int_0^t (t-\tau)^{\gamma/2-1}e^{-m(t-\tau)}\|a(\tau)\|_{L_\beta(\Omega)}e^{-m\tau}\|y(\tau)\|_{L_2(\Omega)}\,d\tau.\end{aligned}$$

Thus, by Young's inequality for convolutions (cf. the proof of [5, Lemma 3]), followed by Hölder's inequality,

$$\begin{aligned}\|U * (ay)\|_{L_{2,m}(Q)} &\le cI(m)\Big(\int_0^T \big(\|a(\tau)\|_{L_\beta(\Omega)}e^{-m\tau}\|y(\tau)\|_{L_2(\Omega)}\big)^r\,d\tau\Big)^{1/r} \\ &\le cI(m)\|a\|_{L_\beta(Q)}\|y\|_{L_{2,m}(Q)},\end{aligned}$$

where

$$I(m) := \Big(\int_0^T t^{(\gamma/2-1)\beta'}e^{-\beta' mt}\,dt\Big)^{1/\beta'},$$

provided $\gamma > 2/\beta$. Such a choice is possible by (iv), thanks to $2/\beta < 2 - n/\beta$.

(vi) For $a \in L_\beta(Q)$ set $T_a(y) := U * (ay)$. Then (v) implies

$$(a \mapsto T_a) \in \mathcal{L}\big(L_\beta(Q), \mathcal{L}(L_{2,m}(Q))\big)$$

and

$$\|T_a\|_{\mathcal{L}(L_{2,m}(Q))} \le cI(m)\|a\|_{L_\beta(Q)}.$$

Note that, by Lebesgue's theorem, $I(m) \to 0$ as $m \to \infty$. Thus, given $R > 0$, there exists $m := m_R > 0$ such that $\|T_a\|_{\mathcal{L}(L_{2,m}(Q))} \le 1/2$ for all $a \in L_\beta(Q)$ satisfying $\|a\|_{L_\beta(Q)} \le R$. Consequently, $1 - T_a$ has a bounded inverse on $L_{2,m}(Q)$, and the map $a \mapsto (1 - T_a)^{-1}$ is analytic for $\|a\|_{L_\beta(Q)} \le R$. Hence, by (4.2),

$$y = (1 - T_a)^{-1}(U * u + Uy^0) \in L_2(Q)$$

for $\|a\|_{L_\beta(Q)} \le R$, thanks to (ii) and (iii), and the map

$$L_\beta(Q) \times L_2(Q) \times L_{q'}(\Omega) \to L_2(Q), \quad (a, u, y^0) \mapsto y$$

is analytic and bounded on bounded sets.

(vii) Let

$$q' \le q_1 \le 2 \le q_2 \le q \quad \text{with} \quad \frac{1}{n} > \frac{1}{q_1} - \frac{1}{q_2}.$$

Choose s such that

$$1 + n\Big(\frac{1}{2} - \frac{1}{q_1}\Big) > s > n\Big(\frac{1}{2} - \frac{1}{q_2}\Big). \tag{4.3}$$

Then there exists $\xi \in (1/2, 1)$ such that $2 - 2\xi + n(1/2 - 1/q_1) > s$. This choice of s, ξ guarantees

$$W^s_{2,\mathcal{B}} \hookrightarrow L_{q_2}(\Omega) \tag{4.4}$$

and

$$L_{q_1}(\Omega) \hookrightarrow W^{s-2+2\xi}_{2,\mathcal{B}}. \tag{4.5}$$

(viii) Let q_1, q_2, s, ξ be as in (vii). For $m \in \mathbb{R}$ we denote by $C_{1-\xi,m}((0,T], L_{q_2}(\Omega))$ the Banach space of all $v \in C((0,T], L_{q_2}(\Omega))$ such that $\sup_{0<t\le T} t^{1-\xi}\|v(t)\|_{L_{q_2}(\Omega)} < \infty$, endowed with the norm

$$\|v\|_{C_{1-\xi,m}} := \sup_{0<t\le T} t^{1-\xi}e^{-mt}\|v(t)\|_{L_{q_2}(\Omega)}.$$

It is an easy consequence of (2.3), (4.4) and (4.5) that

$$(y^0 \mapsto Uy^0) \in \mathcal{L}(L_{q_1}(\Omega), C_{1-\xi,m}((0,T], L_{q_2}(\Omega)).$$

(ix) Let q_1, q_2, s, ξ be as in (vii). Using (2.3) we get

$$\begin{aligned}\|U * u(t)\|_{L_{q_2}(\Omega)} &\le c\|U * u(t)\|_{W^s_{2,\mathcal{B}}} \le c\int_0^t (t-\tau)^{-s/2}\|u(\tau)\|_{L_2(\Omega)}\,d\tau \\ &\le ct^{(1-s)/2}\|u\|_{L_2(Q)} \le c\|u\|_{L_2(Q)}\end{aligned}$$

for $0 < t \le T$. In particular,

$$(u \mapsto U * u) \in \mathcal{L}\big(L_2(Q), C_{1-\xi,m}((0,T], L_{q_2}(\Omega))\big).$$

(x) Let q_1, q_2, s, ξ be as in (vii) such that s satisfies also

$$2 - \frac{n+2}{\beta} > s - n\Big(\frac{1}{2} - \frac{1}{q_2}\Big).$$

Then there exists $\eta > 1/\beta$ such that

$$2 - \frac{n}{\beta} - 2\eta > s - n\Big(\frac{1}{2} - \frac{1}{q_2}\Big).$$

Hence

$$L_r(\Omega) \hookrightarrow W^{s-2+2\eta}_{2,\mathcal{B}}, \tag{4.6}$$

where $1/r := 1/\beta + 1/q_2$. With this choice it follows that

$$\begin{aligned}e^{-mt}\|U * (ay)(t)\|_{L_{q_2}(\Omega)} &\le ce^{-mt}\|U * (ay)(t)\|_{W^s_{2,\mathcal{B}}} \\ &\le ce^{-mt}\int_0^t (t-\tau)^{\eta-1}\|a(\tau)\|_{L_\beta(\Omega)}\|y(\tau)\|_{L_{q_2}(\Omega)}\,d\tau \\ &\le c\int_0^t (t-\tau)^{\eta-1}\tau^{\xi-1}e^{-m(t-\tau)}\|a(\tau)\|_{L_\beta(\Omega)}\,d\tau\ \|y\|_{C_{1-\xi,m}}\end{aligned}$$

for $0 < t \le T$. Thus, by Hölder's inequality,

$$t^{1-\xi}e^{-mt}\|U * (ay)(t)\|_{L_{q_2}(\Omega)} \le cK(t,m)\|a\|_{L_\beta(Q)}\|y\|_{C_{1-\xi,m}},$$

where

$$\begin{aligned}K(m,t) &:= t^{1-\xi}\Big(\int_0^t (t-\tau)^{(\eta-1)\beta'}\tau^{(\xi-1)\beta'}e^{-\beta' m(t-\tau)}\,d\tau\Big)^{1/\beta'} \\ &= t^{\eta-1/\beta}\Big(\int_0^1 (1-\sigma)^{(\eta-1)\beta'}\sigma^{(\xi-1)\beta'}e^{-\beta' mt(1-\sigma)}\,d\sigma\Big)^{1/\beta'}.\end{aligned}$$

Fix any $\delta \in (0,T)$. Then $K(t,m) \to 0$ as $m \to \infty$ by Lebesgue's theorem, uniformly with respect to $t \in [\delta, T]$. If $0 < t \le \delta$ then

$$K(t,m) \le c\delta^{\eta-1/\beta}.$$

Thus, given $R > 0$, it follows that we can fix $m > 0$ such that

$$\|T_a\|_{\mathcal{L}(C_{1-\xi,m}((0,T],L_{q_2}(\Omega)))} \le 1/2$$

for all $a \in L_\beta(Q)$ satisfying $\|a\|_{L_\beta(Q)} < R$. Now we infer from (viii) and (ix) that

$$y = (1 - T_a)^{-1}(U * u + Uy^0) \in C_{1-\xi,m}((0,T], L_{q_2}(\Omega))$$

for $y^0 \in L_{q_1}(\Omega)$, $\|a\|_{L_\beta(Q)} < R$ and that the map

$$L_\beta(Q) \times L_2(Q) \times L_{q_1}(\Omega) \to C_{1-\xi,m}((0,T], L_{q_2}(\Omega)), \qquad (a, u, y^0) \mapsto y$$

is analytic and bounded on bounded sets. Using this property for the couple $(q_1, q_2) := (q', 2)$ and, subsequently, for $(q_1, q_2) := (2, q)$, we see that the map

$$L_\beta(Q) \times L_2(Q) \times L_{q'}(\Omega) \to L_q(\Omega), \qquad (a, u, y^0) \mapsto y(T)$$

is analytic and bounded on bounded sets. This concludes the proof. □

Remark 4.2. Lemma 4.1 guarantees the solvability of (2.13): notice that $r = 2$ and (2.9) imply $q < 2n/(n-2)_+$, that $a := \lambda |y|^{\lambda-1} \in L_\beta(Q)$ for some $\beta > 2 \vee (n+2)/2$ due to $y \in L_{2\lambda}(Q)$ and $\lambda < (n+2)/(n-2)_+$, and that $p(\cdot, T) \in L_{q'(\Omega)}$ due to $y(\cdot, T) \in L_q(\Omega)$. □

Proof of Theorem 2.6. Choose $v \in \mathbb{U}_{\rm ad}$, $\mu \in [0,1]$ and let y_μ be the solution of (2.5) with u replaced by $u + \mu(v-u)$. If μ is small enough, say $\mu \le \mu_0$, then due to the stability estimates in Theorem A.1 and the regularity results in Theorem 2.3, the solution y_μ is global and satisfies

$$\|y_\mu - y\|_{L_{2\lambda}(Q)} + \|y_\mu(\cdot, T) - y(\cdot, T)\|_{L_q(\Omega)} \le C\mu \|v - u\|_{L_2(Q)}. \tag{4.7}$$

Assume $\mu \le \mu_0$ and set $z_\mu := (y_\mu - y)/\mu$. Then z_μ solves the problem

$$\left.\begin{aligned} \partial_t z_\mu - \Delta z_\mu &= a_\mu z_\mu + (v - u), && x \in \Omega,\ t \in J, \\ \mathcal{B} z_\mu &= 0, && x \in \Gamma,\ t \in J, \\ z_\mu(\cdot, 0) &= 0, \end{aligned}\right\} \tag{4.8}$$

where $a_\mu := \lambda \int_0^1 |y + \theta(y_\mu - y)|^{\lambda-1}\, d\theta$. Let z be the solution of

$$\left.\begin{aligned} \partial_t z - \Delta z &= a z + (v - u), && x \in \Omega,\ t \in J, \\ \mathcal{B} z &= 0, && x \in \Gamma,\ t \in J, \\ z(\cdot, 0) &= 0, \end{aligned}\right\}$$

where $a := \lambda |y|^{\lambda-1}$. Set $\beta := 2\lambda/(\lambda - 1)$. Since $a_\mu \to a$ in $L_\beta(Q)$ as $\mu \to 0$, Lemma 4.1 implies

$$z_\mu(\cdot, T) \to z(\cdot, T) \quad \text{in } L_q(\Omega). \tag{4.9}$$

Set

$$\mathcal{I}_1(\mu) := \int_\Omega |y_\mu(\cdot, T) - y^*|^q\, dx, \qquad \mathcal{I}_2(\mu) := N \int_Q (u + \mu(v-u))^2\, dx\, dt.$$

The mapping $L_q(\Omega) \to \mathbb{R} : \varphi \mapsto \int_\Omega |\varphi - y^*|^q\, dx$ is convex. Hence

$$\begin{aligned} q \int_\Omega |y(\cdot, T) - y^*|^{q-2}(y(\cdot, T) - y^*) z_\mu(\cdot, T)\, dx &\le \frac{\mathcal{I}_1(\mu) - \mathcal{I}_1(0)}{\mu} \\ &\le q \int_\Omega |y_\mu(\cdot, T) - y^*|^{q-2}(y_\mu(\cdot, T) - y^*) z_\mu(\cdot, T)\, dx. \end{aligned}$$

Since (4.7) implies

$$|y_\mu(\cdot, T) - y^*|^{q-2}(y_\mu(\cdot, T) - y^*) \to |y(\cdot, T) - y^*|^{q-2}(y(\cdot, T) - y^*)$$

in $L_{q'}(\Omega)$ and (4.9) is true, we see that $\mathcal{I}_1$ is right differentiable at 0 and $\mathcal{I}_1'(0+) = \int_\Omega p(\cdot, T) z(\cdot, T)\, dx$. We have also $\mathcal{I}_2'(0) = 2N \int_Q u(v-u)\, dx\, dt$ and

$$(\mathcal{I}_1 + \mathcal{I}_2)(\mu) = \mathbb{J}(y_\mu, u + \mu(v-u)) \ge J(y, u) = (\mathcal{I}_1 + \mathcal{I}_2)(0),$$

hence

$$\int_\Omega p(\cdot, T) z(\cdot, T)\, dx + 2N \int_Q u(v-u)\, dx\, dt \ge 0.$$

Consequently, it is sufficient to show that

$$\int_\Omega p(\cdot,T)z(\cdot,T)\,dx = \int_Q p(v-u)\,dx\,dt.$$

Let $\varphi_k \in \mathcal{D}(\Omega)$ be such that $\varphi_k \to p(\cdot,T)$ in $L_{q'}(\Omega)$ and $a_k \in \mathcal{D}(Q)$ be such that $a_k \to a$ in $L_\beta(Q)$. Let p_k be the solution of (2.13) with $a = \lambda|y|^{\lambda-1}$ replaced by a_k and the final condition replaced by $p_k(\cdot,T) = \varphi_k$. Then p_k is smooth and $p_k \to p$ in $L_2(Q)$ due to Lemma 4.1. Notice that $z \in L_{2\lambda}(Q)$ due to Theorem A.1 (cf. the beginning of the proof of Theorem 2.3), hence $az \in L_2(Q)$ and the maximal Sobolev regularity implies $\partial_t z, \Delta z \in L_2(Q)$. We have

$$\begin{aligned}
\int_Q p_k(v-u)\,dx\,dt &= \int_Q p_k(\partial_t z - \Delta z - az)\,dx\,dt \\
&= \int_Q (-\partial_t p_k - \Delta p_k - ap_k)z\,dx\,dt + \int_\Omega \varphi_k z(\cdot,T)\,dx \\
&= \int_Q (a_k - a)p_k z\,dx\,dt + \int_\Omega \varphi_k z(\cdot,T)\,dx \to \int_\Omega p(\cdot,T)z(\cdot,T)\,dx,
\end{aligned}$$

since p_k stay bounded in $L_2(Q)$ due to Lemma 4.1. Now

$$\int_Q p_k(v-u)\,dx\,dt \to \int_Q p(v-u)\,dx\,dt$$

concludes the proof. □

5. The Case of a Multiplicative Control

Proof of Theorem 2.8. The proof is almost the same as in Theorem 2.3 (but the solutions y are more regular now). The only nontrivial modification is required in the estimate of the function V and the $L_2(Q)$-norm of $\partial_t y$ in the proof of Lemma 3.2.

Hence, assume $y \in C([0,T+t_1], W^1_{2,\mathcal{B}})$ is a solution of (2.14), where $t_1 > 0$ is fixed. Since $\mathbb{U}_{\mathrm{ad}}$ is bounded in $L_\infty(Q)$, there exists a constant M such that $\|u\|_{L_\infty(Q)} \le M$ for all $u \in \mathbb{U}_{\mathrm{ad}}$. Let V be defined as in the proof of Lemma 3.2. Then

$$\begin{aligned}
V'(t) &= -\int_\Omega (\partial_t y)^2(t)\,dx + \int_\Omega uy\partial_t y(t)\,dx \\
&\le \frac{M^2}{2}\int_\Omega y^2(t)\,dx - \frac{1}{2}\int_\Omega (\partial_t y)^2(t)\,dx.
\end{aligned} \tag{5.1}$$

Let $\tau < 1$, $\tau \le T + t_1$ and $t \in [0,\tau]$. Denoting $C_0 := \int_\Omega y^2(x,0)\,dx$, we have

$$\int_\Omega y^2(t)\,dx = C_0 + 2\int_0^t\int_\Omega y\partial_t y\,dx\,dt \le C_0 + \int_0^\tau\int_\Omega y^2\,dx\,dt + \int_0^\tau\int_\Omega (\partial_t y)^2\,dx\,dt.$$

Integrating this estimate over $t \in [0,\tau]$, we get

$$\int_0^\tau\int_\Omega y^2\,dx\,dt \le C_0\tau + \tau\int_0^\tau\int_\Omega y^2\,dx\,dt + \tau\int_0^\tau\int_\Omega (\partial_t y)^2\,dx\,dt,$$

hence

$$\int_0^\tau\int_\Omega y^2\,dx\,dt \le \frac{C_0\tau}{1-\tau} + \frac{\tau}{1-\tau}\int_0^\tau\int_\Omega (\partial_t y)^2\,dx\,dt. \tag{5.2}$$

Let $\tau_1 \in (0,1)$ be defined by $\frac{\tau_1}{1-\tau_1}M^2 = \frac{1}{2}$ and $\tau \in [0,\tau_1]$ (enlarging M we may assume $\tau_1 \le T + t_1$). Then integrating (5.1) and using (5.2) we arrive at

$$V(\tau) - V(0) \le \frac{C_0}{4} - \frac{1}{4}\int_0^\tau \int_\Omega (\partial_t y)^2\,dx\,dt, \qquad \tau \in [0,\tau_1]. \tag{5.3}$$

This estimate guarantees $V(t) \le V(0) + C_0/4$ on $[0,\tau_1]$.

Fix $\delta \in (0, t_1 \wedge \tau_1)$ and assume $V(t_0) \ll -1$ for some $t_0 \in [0,\tau_1 - \delta]$. Then (5.3) implies $V(t) \le -K \ll -1$ for all $t \in [\tau_1 - \delta, \tau_1]$. As in (3.15) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega y^2\,dx &= -2V(t) + c_1\int_\Omega |y|^{\lambda+1}\,dx + \int_\Omega uy^2\,dx \\ &\ge K + c_2\Bigl(\int_\Omega y^2\,dx\Bigr)^{(\lambda+1)/2}\end{aligned}$$

for any $t \in [\tau_1 - \delta, \tau_1]$. In the same way as in the proof of Lemma 3.2, this inequality yields a contradiction if $K = K(\lambda, c_2, \delta)$ is large enough. Consequently,

$$V(t) \ge -C \qquad \text{for all } t \in [0, \tau_1 - \delta]. \tag{5.4}$$

Now (5.3) implies $\int_0^{\tau_1-\delta}\int_\Omega (\partial_t y)^2\,dx\,dt \le C$, hence $\int_\Omega y^2(t)\,dx \le C$ for t belonging to $[0,\tau_1-\delta]$. In particular, $\int_\Omega y^2(\tau_1-\delta)\,dx \le C_1$, where C_1 does not depend on u.

Repeating the estimates above on the interval $[\tau_1-\delta, 2\tau_1-\delta]$ instead of $[0,\tau_1]$ and then on $[2\tau_1-2\delta, 3\tau_1-2\delta]$ etc, we obtain the desired bounds for $V(t)$, $\|y(t)\|_{L_2(\Omega)}$, $t \in J$, and $\|\partial_t y\|_{L_2(Q)}$. □

6. Parabolic Systems

Proof of Theorem 2.10. Let $\varphi_1 > 0$ be an eigenfunction corresponding to the first eigenvalue μ_1 of the problem $-\Delta\varphi = \mu\varphi$ in Ω, $\mathcal{B}_1\varphi = 0$ on Γ. Notice that φ_1 is a positive constant if $\mathcal{B}_1 = \partial_\nu$, hence the weighted Lebesgue space $L_1(\Omega, \varphi_1(x)\,dx)$ equals $L_1(\Omega)$ in this case.

We shall prove that

(i) any bound of $y_1(t)$ in $L_p(\Omega,\varphi_1(x)\,dx)$ or $L_p(\Omega)$, $p \ge 1$, implies a bound of $y_2(t)$ in the same space;

(ii) the space $X := L_1(\Omega,\varphi_1(x)\,dx) \times L_1(\Omega,\varphi_1(x)\,dx)$ is a **continuation space** for problem (2.15), that is, if the solution y is defined on $[0,T^*]$, $T^* > 0$, and $\|y(T^*)\|_X \le M$ then this solution can be continued for $t \in [T^*, T^*+\tau]$, where $\tau = \tau(M) > 0$. In addition, $\|u(t)\|_{L_\infty(\Omega)\times L_\infty(\Omega)} \le C(\delta, M)$ for any $t \in [T^*+\delta, T^*+\tau]$ and $\delta > 0$;

(iii) all global solutions of problem (2.15) with u bounded in $L_r(J, L_z^+(\Omega))$ and $y_1(T)$ bounded in $L_q(\Omega)$ are uniformly bounded in $L_\infty(Q)$.

Then the conclusion follows similarly as in the proof of Theorem 2.3.

(i) Let $u \in \mathbb{U}_{\rm ad}$. Set $w := y_2^2/2 - by_2 - ay_1$. One can easily verify

$$\partial_t w - \Delta w \le -au \le 0,$$

hence the comparison principle guarantees $w \le C$ in Q, where C does not depend on u. This estimate implies

$$y_2^2 \le C(1 + y_1), \tag{6.1}$$

and the conclusion follows.

(ii) Set $z := ay_1 + by_2$. Then

$$\partial_t z - \Delta z = ay_1y_2 + au \le C(1 + z^{3/2}) + au. \tag{6.2}$$

Since $n < 4$ if $\mathcal{B} = \partial_\nu$ and $n < 3$ if $\mathcal{B} = \gamma$, the problem $\partial_t \tilde{z} - \Delta \tilde{z} = C(1 + |\tilde{z}|^{3/2}) + au$, $\mathcal{B}\tilde{z} = 0$, is well posed in $X_1 := L_1(\Omega, \varphi_1(x)\,dx)$ due to [30] and [13], respectively. More precisely, if $\|\tilde{z}(0)\|_{X_1} \le M$ then there exists $\tau = \tau(M) > 0$ such that the solution $\tilde{z}$ exists on $[0,\tau]$ and satisfies $\|\tilde{z}(t)\|_{L_\infty(\Omega)} \le C(\delta, M)$ for any $t \in [\delta, \tau]$ and $\delta > 0$. A comparison argument shows that the same estimate is true for the function z. In particular, the space X is a continuation space for (2.15) in the sense described above.

(iii) Now assume that u belongs to a bounded set in $\mathbb{U}^G_{\rm ad} \subset L_r(J, L^+_z(\Omega))$ and $y_1(T)$ is bounded in $L_1(\Omega)$. The above arguments show that the solution y can be continued for $t \in [0, T+\tau]$, where $\tau > 0$ does not depend on u and $u(x,t) := 0$ if $t > T$. Multiplying the second equation in (2.15) with φ_1 and using (6.1) we obtain

$$\begin{aligned}\partial_t \int_\Omega y_2\varphi_1\,dx + \mu_1 \int_\Omega y_2\varphi_1\,dx = a\int_\Omega y_1\varphi_1\,dx &\ge c\int_\Omega y_2^2\varphi_1\,dx - C \\ &\ge c\Big(\int_\Omega y_2\varphi_1\,dx\Big)^2 - C\end{aligned}$$

for any $t \in [0, T+\tau]$. Using standard blow-up arguments (cf. the arguments following (3.15) in the proof of Lemma 3.2), this estimate guarantees a uniform bound for $y_2(t),\ t \in [0, T+\tau/2]$, in the weighted space $L_1(\Omega, \varphi_1(x)\,dx)$. Integrating the second equation in (2.15) we obtain now

$$\int_0^{T+\tau/2}\int_\Omega y_1\varphi_1\,dx\,dt \le C. \tag{6.3}$$

The first equation in (2.15) implies

$$\int_\Omega y_1\varphi_1\,dx\Big|_{t_1}^{t_2} + (\mu_1 + b)\int_{t_1}^{t_2}\int_\Omega y_1\varphi_1\,dx\,dt \ge 0,$$

hence using (6.3) we deduce

$$\int_\Omega y_1(t_2)\varphi_1\,dx \ge \int_\Omega y_1(t_1)\varphi_1\,dx - C \tag{6.4}$$

for any $t_1, t_2 \in [0, T+\tau/2],\ t_2 > t_1$.

Obviously, (6.3) and (6.4) imply a uniform estimate for $y_1(t),\ t \in J$, in the space $L_1(\Omega, \varphi_1(x)\,dx)$. Now (i) and (ii) imply uniform bounds for y_1, y_2 in $L_\infty([\delta, T]\times\Omega)$ for any $\delta > 0$. Since the bounds for y_1, y_2 in $L_\infty([0,\delta]\times\Omega)$ for $\delta > 0$ small enough are guaranteed by the well posedness of (2.15) in $L_\infty(\Omega)\times L_\infty(\Omega)$ and the boundedness of u in $L_r(J, L_z(\Omega))$, the conclusion follows. □

Appendix: The Basic Existence, Uniqueness, and Stability Theorem for Semilinear Problems

For the reader's convenience we collect here the main existence, uniqueness and stability results for strong solutions of the semilinear problem

$$\dot{y} + Ay = F(y) \text{ in } [0,T], \qquad y(0) = y^0, \tag{A.1}$$

where $A = A_s$ be the isomorphism between $W^s_{q,\mathcal{B}}$ and $W^{s-2}_{q,\mathcal{B}}$ mentioned in Section 2. They follow from [6, Theorems 3.3, 3.4] and [5, Theorems 5, 7(ii)]. Analogous results are true in the case of systems.

We write $C_b^{1-}(Y,X)$ for the space of all maps from Y into X which are uniformly Lipschitz continuous on bounded sets. If X and Y are paces of functions defind on $[0,T]$, then $F : X \to Y$ is said to possess the Volterra property if, given any $u \in X$ and $t \in (0,T)$, the restriction of $F(u)$ to $[0,t]$ depends on the values of $u \,|\, [0,t]$ only.

Theorem A.1. *Assume*

$$s, \sigma \notin S_q, \quad 0 \le s < \sigma < 2. \tag{A.2}$$

and suppose that $r > 1$, $r \neq 2/(\sigma - s)$, $\sigma - 2/r \notin S_q$, $y^0 \in Y^0 := W_{q,\mathcal{B}}^{\sigma-2/r}$. *Denote* $X_t := L_r([0,t], W_{q,\mathcal{B}}^{\sigma-2})$.

If $r < 2/(\sigma-s)$ *fix* $p \in [1, 2/(s-\sigma+2/r))$ *and set* $Y_t := L_p([0,t], W_{q,\mathcal{B}}^s)$,
if $r > 2/(\sigma-s)$ *fix* $\rho \in [0, (\sigma-s-2/r)/2)$ *and set* $Y_t := C^\rho([0,t], W_{q,\mathcal{B}}^s)$.

Let $F \in C_b^{1-}(Y_T, X_T)$ *have the Volterra property. If* $r < 2/(\sigma-s)$ *or* $r > 2/(\sigma-s)$ *then* (A.1) *has a unique strong* $L_p(W_q^s)$*- or* $C^\rho(W_q^s)$*-solution* $y(y^0,F)$*, respectively, defined on the maximal existence interval* $[0, t(y^0,F))$*. If* $y(y^0,F) \in Y_{t(y^0,F)}$ *or* $F(y(y^0,F)) \in X_{t(y^0,F)}$ *then* $y(y^0,F)$ *is global.*

The map $(y^0,F) \mapsto y(y^0,F)$ *is Lipschitz continuous in the following sense: Fix* $t < t(y^0,F)$ *(we can take* $t = t(y^0,F) = T$ *if* $y(y^0,F)$ *is global). Let* $\omega_1 > 0$*, and let* $\omega_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function,*

$$\left.\begin{aligned} &\|y^0\|_{Y^0} + \|F(0)\|_{X_T} \le \omega_1, \\ &\|F(y_1) - F(y_2)\|_{X_T} \le \omega_2(R)\|y_1 - y_2\|_{Y_T}, \end{aligned}\right\} \tag{A.3}$$

for any $R > 0$ *and* $y_1, y_2 \in Y_T$ *whose norms are bounded by* R*. Fix* $R > \|y(y^0,F)\|_{Y_t}$*. Then there exist positive constants* ε, c *(depending only on* R, t, ω_1, ω_2*) with the following property: If* $\tilde{y}^0 \in Y^0$*,* $\tilde{F} \in C_b^{1-}(Y_T, X_T)$ *has the Volterra property,* $\tilde{y}^0$ *and* $\tilde{F}$ *satisfy* (A.3) *and*

$$\|y^0 - \tilde{y}^0\|_{Y^0} + \sup_{\|y\|_{Y_T} \le R} \|(F - \tilde{F})(y)\|_{X_T} \le \varepsilon,$$

then $t \le t(\tilde{y}^0, \tilde{F})$*,* $y(\tilde{y}^0, \tilde{F}) \in Y_t$ *and*

$$\|y(y^0,F) - y(\tilde{y}^0,\tilde{F})\|_{Y_t} \le c\Big(\|y^0 - \tilde{y}^0\|_{Y^0} + \sup_{\|y\|_{Y_T} \le R} \|(F - \tilde{F})(y)\|_{X_T}\Big).$$

If $y = y(y^0,F)$ *is global then*

$$y \in L_r(J, W_{q,\mathcal{B}}^{\tilde{\sigma}}) \cap W_r^1(J, W_{q,\mathcal{B}}^{\tilde{\sigma}-2}) \tag{A.4}$$

for any $\tilde{\sigma} < \sigma$*, and the norm of* y *in this space can be estimated by a constant* $C = C(\|F(y)\|_{X_T}, \|y^0\|_{Y^0})$.

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