Compact Embeddings of Vector-Valued Sobolev and Besov Spaces

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In memoriam Branko Najman

1. Introduction and Main Results

Let E, E_0, and E_1 be Banach spaces such that

$$E_1 \overset{d}{\hookrightarrow}\!\!\hookrightarrow E \hookrightarrow E_0 \ , \tag{1.1}$$

with $\hookrightarrow$ and $\hookrightarrow\!\!\hookrightarrow$ denoting continuous and compact embedding, respectively. Suppose that $p_0, p_1 \in [1,\infty]$ and $T > 0$, that

$$\mathcal{V} \text{ is a bounded subset of } L_{p_1}\big((0,T),E_1\big) \ , \tag{1.2}$$

and that

$$\partial\mathcal{V} := \{\, \partial v \ ; \ v \in \mathcal{V} \,\} \text{ is bounded in } L_{p_0}\big((0,T),E_0\big) \ , \tag{1.3}$$

where ∂ denotes the distributional derivative. Then the well-known 'Aubin lemma', more precisely, the 'Aubin-Dubinskii lemma' guarantees that

$$\mathcal{V} \text{ is relatively compact in } L_{p_1}\big((0,T),E\big) \ . \tag{1.4}$$

This result is proven in [Aub63, Théorème 1] and also in [Lio69, Théorème I.5.1], provided E_0 and E_1 are reflexive and $p_0, p_1 \in (1,\infty)$. It has also been derived by Dubinskii [Dub65] (see [Lio69, Théorème I.12.1]) with the same restrictions for p_0 and p_1, but without the reflexivity hypothesis. (In fact, Dubinskii proves a slightly more sophisticated theorem in which the L_{p_1}-norm in (1.2) is replaced by a more general functional.)

A proof of (1.4), given assumptions (1.2) and (1.3) only, is due to Simon (see [Sim87, Corollary 4]). In fact, this author oberves that (1.3) can be replaced by

$$\lim_{h\to 0+} \|v(\cdot + h) - v\|_{L_{p_1}((0,T-h),E_0)} = 0 \ , \qquad \text{uniformly for } v \in \mathcal{V} \ , \tag{1.5}$$

(see [Sim87, Theorem 5]). Note that the integrability exponents in (1.2) and (1.5) are equal.

Compactness theorems of 'Aubin-Dubinskii type' are very useful in the theory of nonlinear evolution equations and are employed in numerous research papers. Typical situations are as follows: (u_k) is a sequence of approximate solutions to a given nonlinear evolution equation. If it is possible to bound this sequence in $L_{p_1}(X,E_1)$ and if one can bound the sequence (∂u_k) in $L_{p_0}(X, E_0)$, then the Aubin-Dubinskii lemma guarantees that one can extract a subsequence which converges in $L_{p_1}(X, E)$. If it is then possible to pass to the limit in the approximating problems, whose solutions are the u_k, and if the limiting equation coincides with the original evolution equation, then the existence of a solution to the original problem has been established (cf. [Lio69] for an exposition of this technique). In many concrete cases it is rather difficult, if not impossible, to pass to the limit in nonlinear equations if (∂u_k) is only known to converge in $L_{p_1}(X, E)$. Convergence in 'better spaces', whose elements are more regular (in space or in time), is needed. Even if convergence in $L_{p_1}(X, E)$ is sufficient, it is often important to know that the limiting element belongs to a space with more regularity.

It is the purpose of this paper to prove compact embedding theorems of 'Aubin-Dubinskii type' involving spaces of higher regularity. For this we observe that in most practical cases it is possible to squeeze an interpolation space between E and E_1 (see Remark 7.4). Thus we replace assumption (1.1) by the slightly more restrictive condition:

$$E_1 \overset{d}{\hookrightarrow} E_0 \quad \text{and} \quad (E_0, E_1)_{\theta,1} \hookrightarrow E \hookrightarrow E_0 \text{ for some } \theta \in (0,1) \ , \tag{1.6}$$

where $(\cdot,\cdot)_{\theta,q}$ denote the real interpolation functors (cf. [BL76] or [Tri78] for the basic facts of interpolation theory; also see [Ama95, Section I.2] for a summary). Note that the compactness assumption in (1.6) is weaker than the one in (1.1). Moreover, it is well-known that $(E_0, E_1)_{\theta,1} \hookrightarrow E \hookrightarrow E_0$ iff $E_1 \hookrightarrow E \hookrightarrow E_0$ and

$$\|x\|_E \le c\, \|x\|_{E_0}^{1-\theta}\, \|x\|_{E_1}^{\theta} \ , \qquad x \in E_1 \ ,$$

(e.g., [BL76, Theorem 3.5.2] or [Tri78, Lemma 1.10.1]). Here and below c denotes positive constants which may differ from formula to formula. Intuitively, the parameter $1-\theta$ measures the 'distance' between E_1 and E.

In order to formulate our main result involving assumptions (1.2) and (1.6) we need some notation. Throughout this paper it is always assumed that $p, p_0, p_1 \in [1,\infty]$, unless explicit restrictions are given, and that $0 < \theta < 1$. Then

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \ .$$

Given $s \in \mathbb{R}^+ := [0,\infty)$, we denote by $W_p^s\big((0,T), E\big)$ the Sobolev-Slobodeckii space of order s of E-valued distributions on $(0,T)$, which is defined in analogy to the scalar case (see Section 2). We also put $c^0\big([0,T], E\big) := C\big([0,T], E\big)$; and $c^s\big([0,T], E\big)$ is, for $0 < s < 1$, the Banach space of all s-Hölder-continuous E-valued functions on $[0,T]$ satisfying

$$\lim_{r\to 0} \sup_{\substack{0<x,y<T \\ 0<|x-y|<r}} \frac{\|u(x)-u(y)\|}{|x-y|^s} = 0 \ ,$$

the 'little Hölder space' of order s.

Theorem 1.1. *Let* (1.2) *and* (1.6) *be satisfied. Suppose that either*

$$s_0 := 1 \quad \textit{and} \quad (1.3) \textit{ is true,} \tag{1.7}$$

or

$$\left.\begin{aligned} &0 < s_0 < 1, \quad p_0 \le p_1, \ \textit{and} \\ &\|v(\cdot + h) - v\|_{L_{p_0}((0,T-h),E_0)} \le c h^{s_0}, \quad 0 < h < T, \quad v \in \mathcal{V} \ . \end{aligned}\right\} \tag{1.8}$$

Then $\mathcal{V}$ is relatively compact in

$$W_p^s\big((0,T),E\big) \qquad \text{if } 0 \le s < (1-\theta)s_0 \text{ and } s - 1/p < (1-\theta)s_0 - 1/p_\theta \ , \tag{1.9}$$

and in

$$c^s\big([0,T],E\big) \qquad \text{if } 0 \le s < (1-\theta)s_0 - 1/p_\theta \ . \tag{1.10}$$

Let (1.2), (1.3), and (1.6) be satisfied. In [Sim87, Corollary 8] it is shown that $\mathcal{V}$ is relatively compact in

$$L_p\big((0,T),E\big) \qquad \text{if } 1-\theta \le 1/p_\theta < 1/p \ , \tag{1.11}$$

and in

$$C\big([0,T],E\big) \qquad \text{if } 1-\theta > 1/p_\theta \ . \tag{1.12}$$

Note that (1.9) implies in this case that $\mathcal{V}$ is relatively compact in $L_p\big((0,T),E\big)$ if

$$1/p_\theta - (1-\theta) < 1/p \ .$$

Hence we can admit values $p > p_\theta$ if $1-\theta < 1/p_\theta$, in contrast to (1.11) where $p < p_\theta$ is required. Furthermore, (1.9) implies in the present situation that $\mathcal{V}$ is relatively compact in

$$W_{p_\theta}^s\big((0,T),E\big) \qquad \text{if } 0 \le s < 1-\theta \ .$$

Since (1.10) shows that $\mathcal{V}$ is relatively compact in $c^s\big([0,T],E\big)$ if $0 \le s < 1-\theta-1/p_\theta$, we see that Theorem 1.1 is a substantial improvement over Simon's extension of the Aubin-Dubinskii lemma, provided condition (1.6) is satisfied.

In [Sim87, Theorem 7] it is also shown that $\mathcal{V}$ is relatively compact in $L_{p_\theta}\big((0,T),E\big)$ if (1.2), (1.5), and (1.6) are true. Theorem 1.1 gives a considerable sharpening of this result, provided (1.5) is replaced by its quantitative version (1.8).

Suppose that V and H are Hilbert spaces such that $V \overset{d}{\hookrightarrow} H$. Then, identifying H with its (anti-)dual H', it follows that $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$. It is known (e.g., [LM72]) that $H = (V', V)_{1/2,2}$. Hence, letting $(E_0, E_1) := (V', V)$ and $E := H$, condition (1.6) is satisfied with $\theta := 1/2$. Setting $p_0 := p_1 := 2$, we infer from (1.9) that $\mathcal{V}$ is relatively compact in $L_p\big((0,T),H\big)$ for $1 \le p < \infty$. It is also known that $\mathcal{V}$ is continuously — but not compactly — injected in $C\big([0,T],H\big)$ (see [Mig95]). This shows that Theorem 1.1 is sharp. It should be noted that Simon's result (1.11) guarantees only that $\mathcal{V}$ is relatively compact in $L_p\big((0,T),H\big)$ for $1 \le p < 2$.

Theorem 1.1 is a special case of much more general results which are also valid if $(0,T)$ is replaced by a sufficiently regular bounded open subset of $\mathbb{R}^n$. Its proof is given in Section 5.

In the next section we introduce vector-valued Besov spaces on $\mathbb{R}^n$ and recall some of their basic properties. In particular, we prove an interpolation theorem extending an earlier result due to Grisvard. In Section 4 we discuss vector-valued Besov spaces on X and prove compact embedding theorems for them. In Section 5 we derive an analogue of the Rellich-Kondrachov theorem for vector-valued Sobolev spaces on X as well as a compact embedding theorem for intersections of Sobolev-Slobodeckii spaces. The last section contains a renorming result for Sobolev-Slobodeckii spaces. We close this paper by commenting on the regularity assumptions for X.

We are indebted to E. Maître for bringing [Mig95] to our attention.

2. Some Function Spaces

Let X be an open subset of $\mathbb{R}^n$. Suppose that E is a Banach space, that $1 \le p \le \infty$, and $m \in \mathbb{N}$. Then the Sobolev space $W_p^m(X,E)$ is the Banach space of all $u \in L_p(X,E)$ such that the distributional derivatives $\partial^\alpha u$ belong to $L_p(X,E)$ for $|\alpha| \le m$, endowed with the usual norm $\|\cdot\|_{m,p}$. Furthermore, $BUC^m(X,E)$ is the closed linear subspace of $W_\infty^m(X,E)$ consisting of all u such that $\partial^\alpha u$ is bounded and uniformly continuous on X, that is, $\partial^\alpha u \in BUC(X,E)$, for $|\alpha| \le m$.

If $0 < \theta < 1$, put

$$[u]_{\theta,p} := \begin{cases} \Big[\displaystyle\int_{X\times X} \Big(\frac{\|u(x)-u(y)\|_E}{|x-y|^\theta}\Big)^p \frac{d(x,y)}{|x-y|^n}\Big]^{1/p} , & p < \infty , \\ \displaystyle\sup_{\substack{x,y\in X \\ x\ne y}} \frac{\|u(x)-u(y)\|_E}{|x-y|^\theta} , & p = \infty . \end{cases}$$

Then we set

$$W_p^{m+\theta}(X,E) := \Big(\big\{ u \in W_p^m(X,E) \ ; \ \|u\|_{m+\theta,p} < \infty \big\},\ \|\cdot\|_{m+\theta,p}\Big) ,$$

where

$$\|u\|_{m+\theta,p} := \|u\|_{m,p} + \max_{|\alpha|=m} [\partial^\alpha u]_{\theta,p} .$$

If $p < \infty$ then $W_p^{m+\theta}(X,E)$ is a vector-valued Slobodeckii space, and

$$W_\infty^{m+\theta}(X,E) = BUC^{m+\theta}(X,E) ,$$

the subspace of $BUC^m(X,E)$ consisting of all u such that $\partial^\alpha u$ is uniformly θ-Hölder continuous for $|\alpha| = m$.

If $m > 0$ and $0 \le \theta < 1$ then $W_p^{-m+\theta}(X,E)$ [resp. $BUC^{-m}(X,E)$] is the Banach space of all E-valued distributions u on X having a representation

$$u = \sum_{|\alpha|\le m} \partial^\alpha u_\alpha$$

with $u_\alpha \in W_p^\theta(X,E)$ [resp. $u_\alpha \in BUC^\theta(X,E)$], equipped with the norm

$$u \mapsto \|u\|_{-m+\theta,p} := \inf\Big(\sum_{|\alpha|\le m} \|u_\alpha\|_{\theta,p}\Big) ,$$

the infimum being taken over all such representations, and p being equal to ∞ if $u_\alpha \in BUC^\theta(X,E)$. Thus the 'Sobolev-Slobodeckii scale' $W_p^s(X,E)$, $s \in \mathbb{R}$, is well-defined for each $p \in [1,\infty]$, as is the 'Hölder scale' $BUC^s(X,E)$, $s \in \mathbb{R}$. Moreover,

$$\mathcal{D}(X,E) \hookrightarrow W_p^s(X,E) \cap BUC^s(X,E) \hookrightarrow W_p^s(X,E) + BUC^s(X,E) \hookrightarrow \mathcal{D}'(X,E)$$

for $s \in \mathbb{R}$. Here $\mathcal{D}(X,E)$ is the space of all E-valued test functions on X endowed with the usual inductive limit topology, and $\mathcal{D}'(X,E) = \mathcal{L}\big(\mathcal{D}(X),E\big)$ is the space of E-valued distributions on X, with $\mathcal{L}$ denoting the space of continuous linear maps, equipped with the topology of uniform convergence on bounded sets.

We also define the scale of 'little Hölder spaces' $buc^s(X,E)$, $s \in \mathbb{R}$, by setting

$$buc^m(X,E) := BUC^m(X,E)$$

and by denoting by

$$buc^{m+\theta}(X,E) \text{ the closure of } BUC^{m+1}(X,E) \text{ in } BUC^{m+\theta}(X,E)$$

for $m \in \mathbb{Z}$ and $\theta \in (0,1)$. Then $u \in BUC^{m+\theta}(X,E)$ belongs to $buc^{m+\theta}(X,E)$ iff

$$\lim_{r \to 0} \sup_{\substack{x,y \in X \\ 0<|x-y|<r}} \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|_E}{|x-y|^\theta} = 0 \ , \qquad |\alpha| = m \ ,$$

(cf. [Lun95, Proposition 0.2.1], for example).

Throughout the remainder of this paper we suppose that

$$X \text{ is a smoothly bounded open subset of } \mathbb{R}^n,$$

which means that $\overline{X}$ is a compact n-dimensional C^∞-submanifold of $\mathbb{R}^n$ with boundary. This assumption is imposed for convenience and can be considerably relaxed (see the last paragraph of Section 7).

It follows that $BUC^s(X,E) = C^s(\overline{X},E)$ for $s \in \mathbb{R}^+$ by identifying $u \in BUC^s(X,E)$ with its unique continuous extension $\overline{u} \in C^s(\overline{X},E)$. For this reason we put

$$C^s(\overline{X},E) := BUC^s(X,E) \ , \quad c^s(\overline{X},E) := buc^s(X,E)$$

for all $s \in \mathbb{R}$.

Henceforth, we always suppose that E, E_0, and E_1 are complex Banach spaces. The real case can be covered by complexification. We also suppose that $s, s_0, s_1 \in \mathbb{R}$ and put $s_\theta := (1-\theta)s_0 + \theta s_1$.

3. Besov Spaces on $\mathbb{R}^n$

Fix a radial $\psi := \psi_0 \in \mathcal{D}(\mathbb{R}^n) := \mathcal{D}(\mathbb{R}^n, \mathbb{C})$ with $\psi(\xi) = 1$ for $|\xi| < 1$ and $\psi(\xi) = 0$ for $|\xi| \geq 2$. Put

$$\psi_k(\xi) := \psi(2^{-k}\xi) - \psi(2^{-k+1}\xi) \ , \qquad \xi \in \mathbb{R}^n \ , \quad k \in \mathbb{N}\backslash\{0\} \ ,$$

and $\psi_k(D) := \mathcal{F}^{-1}\psi_k\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'(\mathbb{R}^n,E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n),E)$ and $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$. Then the Besov space $B^s_{p,q}(\mathbb{R}^n,E)$ of E-valued distributions on $\mathbb{R}^n$ is defined to be the vector subspace of $\mathcal{S}'(\mathbb{R}^n,E)$ consisting of all u satisfying

$$\|u\|_{s,p,q} := \left\| \left(2^{sk} \|\psi_k(D)\|_{L_p(\mathbb{R}^n,E)}\right)_{k \in \mathbb{N}} \right\|_{\ell_q} < \infty \ .$$

It is a Banach space with this norm, and different choices of ψ lead to equivalent norms.

In this section we simply write $\mathfrak{F}$ for $\mathfrak{F}(\mathbb{R}^n,E)$ if the latter is a locally convex space of E-valued distributions on $\mathbb{R}^n$, that is, $\mathfrak{F}(\mathbb{R}^n,E) \hookrightarrow \mathcal{D}'(\mathbb{R}^n,E)$, and no confusion seems likely.

It follows that

$$\mathcal{S} \hookrightarrow B^{s_1}_{p,q_1} \hookrightarrow B^{s_0}_{p,q_0} \hookrightarrow \mathcal{S}' \ , \qquad s_1 > s_0 \ , \tag{3.1}$$

and

$$B^{s}_{p,q_0} \hookrightarrow B^{s}_{p,q_1} \ , \qquad q_0 < q_1 \ . \tag{3.2}$$

Moreover,

$$B^{s_1}_{p_1,q} \hookrightarrow B^{s_0}_{p_0,q} \ , \qquad s_1 > s_0 \ , \quad s_1 - n/p_1 = s_0 - n/p_0 \ . \tag{3.3}$$

Besov spaces are stable under real interpolation, that is, if $0 < \theta < 1$ then

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} \doteq B^{s_\theta}_{p,q} \ , \qquad s_0 \neq s_1 \ . \tag{3.4}$$

They are related to Slobodeckii and Hölder spaces by

$$B^{s}_{p,p} \doteq W^{s}_{p} \ , \qquad s \in \mathbb{R}\backslash\mathbb{Z} \ , \tag{3.5}$$

and

$$B^{m}_{p,1} \hookrightarrow W^{m}_{p} \hookrightarrow B^{m}_{p,\infty} \ , \qquad m \in \mathbb{Z} \ , \quad p < \infty \ . \tag{3.6}$$

Moreover, $B^{m}_{p,p} \neq W^{m}_{p}$ for $m \in \mathbb{Z}$ unless $p = 2$ and E is a Hilbert space. Note that (3.4)–(3.6) imply

$$(W^{s_0}_{p}, W^{s_1}_{p})_{\theta,q} \doteq B^{s_\theta}_{p,q} \ , \qquad s_0 \neq s_1 \ , \quad p < \infty \ . \tag{3.7}$$

It is also true that

$$B^{m}_{\infty,1} \hookrightarrow BUC^{m} \hookrightarrow B^{m}_{\infty,\infty} \ , \qquad m \in \mathbb{Z} \ , \tag{3.8}$$

and $B^{m}_{\infty,\infty}$ is the Zygmund space $\mathcal{C}^m$ for $m \in \mathbb{N}\backslash\{0\}$ (e.g., [Tri83] for the scalar case). Hence we infer from (3.4) and (3.5) that

$$(BUC^{s_0}, BUC^{s_1})_{\theta,q} \doteq B^{s_\theta}_{\infty,q} \ . \tag{3.9}$$

The definition and the above properties of vector-valued Besov spaces are literally the same as in the scalar case (for which we refer to [Tri78], [Tri83], [Tri92], and [BL76]). The proofs carry over from the scalar to the vector-valued setting by employing the Fourier multiplier theorem of Propostion 4.5 of [Ama97]. A detailed and coherent treatment containing many additional results will be given in [Ama99]. For earlier (partial) results and different approaches we refer to [Gri66], [Sch86], and [Tri97, Section 15], as well as to the other references cited in [Ama97]. Embedding theorems for vector-valued Besov and Slobodeckii spaces on an interval are also derived in [Sim90], but with s, s_0, and s_1 restricted to the interval $[0,1]$.

We define the little Besov space $b^{s}_{p,q}$ to be the closure of $B^{s+1}_{p,q}$ in $B^{s}_{p,q}$. Then

$$b^{s}_{p,q} := \begin{cases} B^{s}_{p,q} \ , & p \vee q < \infty \ , \quad s \in \mathbb{R} \ , \\ buc^{s} \ , & p = q = \infty \ , \quad s \in \mathbb{R}\backslash\mathbb{Z} \ , \end{cases} \tag{3.10}$$

and

$$b^{s}_{p,q} \text{ is the closure of } B^{t}_{p,q} \text{ in } B^{s}_{p,q} \text{ for } t > s \tag{3.11}$$

(see [Ama97, Propositions 5.3 and 5.4 and Remark 5.5(b)] and [Ama99]). Denoting by $\overset{d}{\hookrightarrow}$ dense embedding, it follows that

$$\mathcal{S} \overset{d}{\hookrightarrow} B^{s_1}_{p,q_1} \overset{d}{\hookrightarrow} B^{s_0}_{p,q_0} \overset{d}{\hookrightarrow} b^{s_0}_{p,\infty} \overset{d}{\hookrightarrow} \mathcal{S}' \ , \qquad p < \infty \ , \tag{3.12}$$

if either $s_1 = s_0$ and $1 \leq q_1 \leq q_0 < \infty$, or $s_1 > s_0$ and $q_0 \vee q_1 < \infty$ (see [Ama97, Remark 5.5(a)]).

The following interpolation theorem for vector-valued Besov spaces will be of particular importance for us.

Theorem 3.1. *Let (E_0, E_1) be an interpolation couple and suppose that $s_0 \neq s_1$ and $p_0, p_1, q_0, q_1 \in [1, \infty)$. Then*

$$\left(B^{s_0}_{p_0,q_0}(\mathbb{R}^n, E_0), B^{s_1}_{p_1,q_1}(\mathbb{R}^n, E_1)\right)_{\theta,q_\theta} \doteq B^{s_\theta}_{p_\theta,q_\theta}\left(\mathbb{R}^n, (E_0, E_1)_{\theta,q_\theta}\right) ,$$

provided $p_\theta = q_\theta$.

Proof We denote by $\ell^s_q(E)$ the subspace of $E^{\mathbb{N}}$ consisting of all $u = (u_k)$ satisfying

$$\|u\|_{\ell^s_q(E)} := \|(2^{sk} u_k)_{k\in\mathbb{N}}\|_{\ell_q} < \infty .$$

It is a Banach space with this norm. If (F_0, F_1) is an interpolation couple then

$$\left(\ell^{s_0}_{q_0}(F_0), \ell^{s_1}_{q_1}(F_1)\right)_{\theta,q_\theta} \doteq \ell^{s_\theta}_{q_\theta}\left((F_0, F_1)_{\theta,q_\theta}\right) \tag{3.13}$$

(e.g., [BL76, Theorem 5.6.2] or [Tri78, Theorem 1.18.1]). Furthermore ([Tri78, Theorem 1.18.4]),

$$\left(L_{p_0}(\mathbb{R}^n, E_0), L_{p_1}(\mathbb{R}^n, E_1)\right)_{\theta,p_\theta} \doteq L_{p_\theta}\left(\mathbb{R}^n, (E_0, E_1)_{\theta,p_\theta}\right) . \tag{3.14}$$

From [Ama97, Lemma 5.1] we know that $B^s_{p,q}$ is a retract of $\ell^s_q(L_p)$. Hence the assertion follows from (3.13), (3.14), and [Tri78, Theorem 1.2.4] or [Ama95, Proposition I.2.3.2]. ∎

Theorem 3.1 generalizes a result of Grisvard [Gri66, formula (6.9) on p. 179] who considers the case $p_j = q_j$ and $n = 1$. It should be noted that Grisvard's proof does not extend to $n > 1$ since, in general, $W^m_p(\mathbb{R}^n, E)$ is not isomorphic to $L_p(\mathbb{R}^n, E)$.

4. Besov Spaces on X

We denote by $r_{\overline{X}} \in \mathcal{L}\big(C(\mathbb{R}^n, E), C(\overline{X}, E)\big)$ the operator of point-wise restriction, $u \mapsto u|\overline{X}$, and recall that $r_X \in \mathcal{L}\big(\mathcal{D}'(\mathbb{R}^n, E), \mathcal{D}'(X, E)\big)$ is the restriction operator in the sense of distribution, that is,

$$r_X u(\varphi) := u(\varphi) , \qquad u \in \mathcal{D}'(\mathbb{R}^n, E) , \quad \varphi \in \mathcal{D}(X) .$$

Observe that coretractions for $r_{\overline{X}}$ and r_X are extension operators.

The following extension theorem is of basic importance for the study of spaces of distributions on X. Here and below we set

$$\mathcal{W}^s_p(Y, E) := \begin{cases} W^s_p(Y, E) , & p < \infty , \\ BUC^s(Y, E) , & p = \infty , \end{cases}$$

for $s \in \mathbb{R}$ and $Y \in \{\mathbb{R}^n, X\}$.

Theorem 4.1. *r_X is a retraction from $\mathcal{S}'(\mathbb{R}^n, E)$ onto $\mathcal{D}'(X, E)$ and there exists a coretraction e_X for r_X which is independent of E. Moreover, $r_X \supset r_{\overline{X}}$, and r_X belongs to*

$$\mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E), C^\infty(\overline{X}, E)\big) \cap \mathcal{L}\big(\mathcal{W}^s_p(\mathbb{R}^n, E), \mathcal{W}^s_p(X, E)\big) \cap \mathcal{L}\big(buc^s(\mathbb{R}^n, E), c^s(\overline{X}, E)\big) .$$

Furthermore, e_X is an element of

$$\mathcal{L}\big(C^\infty(\overline{X},E),\mathcal{S}(\mathbb{R}^n,E)\big) \cap \mathcal{L}\big(\mathcal{W}_p^s(X,E),\mathcal{W}_p^s(\mathbb{R}^n,E)\big) \cap \mathcal{L}\big(c^s(\overline{X},E),buc^s(\mathbb{R}^n,E)\big) \ ,$$

and it is a coretraction for r_X in each case.

Proof By a standard partition of unity argument the proof is reduced to establishing a corresponding statement if X is replaced by a half-space of $\mathbb{R}^n$. In this case the theorem is deduced by constructing an extension operator along the lines of [Ham75, Part II]. For details and generalizations we refer to [Ama99]. ■

Now we define the Besov spaces of E-valued distributions on X by

$$B_{p,q}^s(X,E) := r_X B_{p,q}^s(\mathbb{R}^n,E) \ ,$$

equipped with the obvious quotient space topology.

Proposition 4.2. *r_X is a retraction from $B_{p,q}^s(\mathbb{R}^n,E)$ onto $B_{p,q}^s(X,E)$ and e_X is a corresponding coretraction.*

Proof Fix $s_0 < s < s_1$ and put $\theta := (s-s_0)/(s_1-s_0)$. Then

$$\big(\mathcal{W}_p^{s_0}(\mathbb{R}^n,E),\mathcal{W}_p^{s_1}(\mathbb{R}^n,E),\big)_{\theta,q} \doteq B_{p,q}^s(\mathbb{R}^n,E)$$

thanks to (3.7) and (3.9). By Theorem 4.1 the diagrams of continuous linear maps

$$\begin{array}{ccc} \mathcal{W}_p^{s_j}(\mathbb{R}^n,E) & \xrightarrow{\ r_X\ } & \mathcal{W}_p^{s_j}(X,E) \\ {\scriptstyle e_X}\nwarrow & & \swarrow{\scriptstyle \mathrm{id}} \\ & \mathcal{W}_p^{s_j}(X,E) & \end{array}$$

are commutative. Hence the assertion follows by interpolation. ■

Corollary 4.3. *Assertions* (3.1)–(3.12) *as well as Theorem* 3.1 *remain valid if $\mathbb{R}^n$ is replaced by X, provided we substitute $C^\infty(\overline{X},E)$ and $\mathcal{D}'(X,E)$ for $\mathcal{S}$ and $\mathcal{S}'$, respectively.*

Proof This is deduced from Proposition 4.2 by standard arguments. ■

In the following (4.x), where $\mathrm{x} \in \{1,\ldots,12\}$, denotes the analogue of formula (3.x) with $\mathbb{R}^n$ replaced by X, as well as $\mathcal{S}$ and $\mathcal{S}'$ replaced by $C^\infty(\overline{X},E)$ and $\mathcal{D}'(X,E)$, respectively.

Now it is easy to prove the following compact embedding theorem.

Theorem 4.4. *Suppose that $E_1 \overset{d}{\hookrightarrow} E_0$. Then*

$$B_{p,q}^{s_1}(X,E_1) \overset{d}{\hookrightarrow} B_{p,q}^{s_0}(X,E_0) \ , \qquad s_1 > s_0 \ .$$

Proof Fix $\sigma_0 < s_0 < s_1 < \sigma_1$ and $\sigma \in (0,1)$ such that $\sigma_0 < 0$ and $\sigma < \sigma_1 - n/p$. Then we infer from (4.1)–(4.3) and (4.5), (4.6) that

$$B_{p,q}^{\sigma_1}(X,E_1) \hookrightarrow B_{\infty,\infty}^{\sigma_1-n/p}(X,E_1) \hookrightarrow C^\sigma(\overline{X},E_1)$$

and

$$C(\overline{X}, E_0) \hookrightarrow L_p(X, E_0) \hookrightarrow B^{\sigma_0}_{p,q}(X, E_0) \ .$$

Since, by the Arzéla-Ascoli theorem, $C^\sigma(\overline{X}, E_1)$ is compactly embedded in $C(\overline{X}, E_0)$, it follows that $B^{\sigma_1}_{p,q}(X, E_1) \lhook\joinrel\longrightarrow B^{\sigma_0}_{p,q}(X, E_0)$. Now the assertion is a consequence of (4.4) and the Lions-Peetre compactness theorem for the real interpolation method. ■

Corollary 4.5. (i) *Suppose that* $E_1 \lhook\joinrel\longrightarrow E_0$*. If* $s_1 > s_0$ *and* $s_1 - n/p_1 > s_0 - n/p_0$ *then*

$$B^{s_1}_{p_1,q_1}(X, E_1) \lhook\joinrel\longrightarrow b^{s_0}_{p_0,q_0}(X, E_0) \ .$$

(ii) *Suppose that*

$$E_1 \hookrightarrow E_0 \quad \textit{and} \quad (E_0, E_1)_{\theta,p_\theta} \lhook\joinrel\longrightarrow E \ .$$

If $s_\theta > s$ *and* $s_\theta - n/p_\theta > s - n/p$ *then*

$$B^{s_0}_{p_0,q_0}(X, E_0) \cap B^{s_1}_{p_1,q_1}(X, E_1) \lhook\joinrel\longrightarrow b^{s}_{p,q}(X, E) \ .$$

Proof (i) Since $X$is bounded, it is obvious that

$$C^m(\overline{X}, E) \hookrightarrow W^m_p(X, E) \hookrightarrow W^m_{\overline{p}}(X, E) \ , \qquad 1 \le \overline{p} < p \ , \quad m \in \mathbb{Z} \ .$$

Thus it is an easy consequence of (4.1), (4.5), (4.7), and (4.9) that

$$B^s_{p,q}(X, E) \hookrightarrow B^s_{\overline{p},q}(X, E) \ , \qquad 1 \le \overline{p} < p \ .$$

Fix $p \in [1, p_1]$ and $s \in (s_0, s_1)$ such that $t := s - n(1/p - 1/p_0) < s$ and suppose that $s_0 < \sigma < \tau < t$. Then we infer from (4.1)–(4.3), Theorem 4.4, and the above embedding that

$$\begin{aligned} B^{s_1}_{p_1,q_1}(X, E_1) \hookrightarrow B^{s}_{p,q_1}(X, E_1) \hookrightarrow B^{t}_{p_0,q_1}(X, E_1) \hookrightarrow B^{\tau}_{p_0,q_0}(X, E_1) \\ \lhook\joinrel\longrightarrow B^{\sigma}_{p_0,q_0}(X, E_0) \hookrightarrow b^{s_0}_{p_0,q_0}(X, E_0) \ , \end{aligned}$$

where the last embedding follows from (4.11).

(ii) Fix $\sigma_j < s_j$ such that $s - n/p < \sigma_\theta - n/p_\theta$. Then

$$B^{s_0}_{p_0,q_0}(X, E_0) \cap B^{s_1}_{p_1,q_1}(X, E_1) \hookrightarrow B^{\sigma_0}_{p_0,p_0}(X, E_0) \cap B^{\sigma_1}_{p_1,p_1}(X, E_1) \ .$$

Since

$$B^{\sigma_0}_{p_0,p_0}(X, E_0) \cap B^{\sigma_1}_{p_1,p_1}(X, E_1) \hookrightarrow B^{\sigma_j}_{p_j,p_j}(X, E_j) \ , \qquad j = 0, 1 \ ,$$

interpolation gives

$$\begin{aligned} B^{\sigma_0}_{p_0,p_0}(X, E_0) \cap B^{\sigma_1}_{p_1,p_1}(X, E_1) &\hookrightarrow \big(B^{\sigma_0}_{p_0,p_0}(X, E_0), B^{\sigma_1}_{p_1,p_1}(X, E_1)\big)_{\theta,p_\theta} \\ &= B^{\sigma_\theta}_{p_\theta,p_\theta}\big(X, (E_0, E_1)_{\theta,p_\theta}\big) \ , \end{aligned}$$

where the last equality follows from Theorem 3.1 and Corollary 4.3. Now it suffices to apply (i). ■

5. Sobolev-Slobodeckii Spaces on X

As an easy consequence of the preceding results we obtain the following vector-valued version of the Rellich-Kondrachov theorem.

Theorem 5.1. *Suppose that* $E_1 \hookrightarrow\hookrightarrow E_0$. *If* $s_1 > s_0$ *and* $s_1 - n/p_1 > s_0 - n/p_0$ *then*

$$W_{p_1}^{s_1}(X, E_1) \hookrightarrow\hookrightarrow W_{p_0}^{s_0}(X, E_0) \ .$$

If $0 \leq s < s_1 - n/p_1$ *then*

$$W_{p_1}^{s_1}(X, E_1) \hookrightarrow\hookrightarrow c^s(\overline{X}, E_0) \ .$$

Proof Fix $\sigma_0, \sigma_1 \in (s_0, s_1)$ with $\sigma_1 > \sigma_0$ such that $\sigma_1 - n/p_1 > \sigma_0 - n/p_0$. Then (4.5), (4.6), and Corollary 4.5(i) imply

$$W_{p_1}^{s_1}(X, E_1) \hookrightarrow B_{p_1,p_1}^{\sigma_1}(X, E_1) \hookrightarrow\hookrightarrow b_{p_0,p_0}^{\sigma_0}(X, E_0) \ .$$

Now the assertion follows from (4.10) and (4.5). ∎

It is also easy to prove a compact embedding theorem involving intersections of Sobolev-Slobodeckii spaces as well as interpolation spaces E_θ.

Theorem 5.2. *Suppose that*

$$E_1 \hookrightarrow\hookrightarrow E_0 \quad \textit{and} \quad (E_0, E_1)_{\theta, p_\theta} \hookrightarrow E \hookrightarrow E_0 \ . \tag{5.1}$$

Then

$$W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1) \hookrightarrow\hookrightarrow W_p^s(X, E) \ , \tag{5.2}$$

provided

$$s < s_\theta \quad \textit{and} \quad s - n/p < s_\theta - n/p_\theta \ . \tag{5.3}$$

If $0 \leq s < s_\theta - n/p_\theta$ *then*

$$W_{p_0}^{s_0}(X, E_0) \cap W_{p_1}^{s_1}(X, E_1) \hookrightarrow\hookrightarrow c^s(\overline{X}, E) \ . \tag{5.4}$$

Proof Since $E_1 \hookrightarrow\hookrightarrow E_0$, interpolation theory guarantees that

$$E_1 \hookrightarrow\hookrightarrow (E_0, E_1)_{\vartheta, p_\vartheta} \hookrightarrow\hookrightarrow (E_0, E_1)_{\theta,1} \ , \qquad \theta < \vartheta < 1 \ .$$

Hence (4.2) and the second part of (5.1) show that $(E_0, E_1)_{\vartheta, p_\vartheta} \hookrightarrow\hookrightarrow E$. Fix $\vartheta \in (\theta, 1)$ sufficiently close to θ such that $s - n/p < s_\vartheta - n/p_\vartheta$ if (5.3) holds, and such that $s < p_\vartheta - n/p_\vartheta$ if $s_\theta - n/p_\theta > 0$. Now the assertion is an easy consequence of Corollary 4.5(ii) and (4.1), (4.5), and (4.6). ∎

Remarks 5.3.

(a) Suppose that H is a Hilbert space. Then u belongs to $W_2^s(\mathbb{R}^n, H)$, where $s \in \mathbb{R}^+$, iff $u \in L_2(\mathbb{R}^n, H)$ and

$$\left(\xi \mapsto |\xi|^{2s}\, \widehat{u}(\xi)\right) \in L_2(\mathbb{R}^n, H) \ ,$$

with $\widehat{u}$ denoting the Fourier transform of u. Thus assumption (5.1), modulo Theorem 5.2, generalizes a result of J.-L. Lions (cf. [Lio61, Théorème IV.2.2] and [Lio69, Théorème I.5.2]), who considers the case $n=1$, $p=2$, and $s_1=0$ with E, E_0, and E_1 being Hilbert spaces satisfying $E_1 \overset{d}{\hookrightarrow} E \hookrightarrow E_0$.

(b) Theorem 1.1 also improves Corollary 9 of [Sim87] which, for $n=1$, guarantees the validity of (5.2)–(5.4) for $s=0$.

(c) Observe that there are no sign restrictions for s, s_0, and s_1 in (5.3). Hence the first part of Theorem 5.2 is also valid if $s_0<0$, for example. In this connection it is important to know that, similarly as in the scalar case, Sobolev-Slobodeckii spaces of negative order can be characterized by duality. More precisely: Denote by $\mathring{W}_p^s(X,E)$ the closure of $\mathcal{D}(X,E)$ in $W_p^s(X,E)$. Then, given a reflexive Banach space F,

$$W_p^{-s}(X,F) \doteq \bigl[\mathring{W}_{p'}^s(X,F')\bigr]' , \qquad 1<p<\infty ,$$

and

$$W_1^{-s}(X,F) \doteq \bigl[c^s(\overline{X},F')\bigr]' , \qquad s\in \mathbb{R}^+\setminus\mathbb{N} ,$$

with respect to the duality pairing induced by

$$\langle u',u\rangle := \int_X \bigl\langle u'(x),u(x)\bigr\rangle_{F'}\,dx , \qquad u,u'\in\mathcal{D}(X,E) , \tag{5.5}$$

where $\langle\cdot,\cdot\rangle_{F'} : F\times F'\to\mathbb{K}$ is the duality pairing between F and F'.

Consequently, if $1<p<\infty$ then a subset $\mathcal{V}$ of $W_p^{-s}(X,F)$ is bounded iff there exists a constant c such that

$$|\langle v,\varphi\rangle| \le c\,\|\varphi\|_{s,p'} , \qquad \varphi\in\mathcal{D}(X,F') , \quad v\in\mathcal{V} . \tag{5.6}$$

Similarly, a subset $\mathcal{V}$ of $W_1^{-s}(X,F)$ is bounded iff (5.6) holds for all $\varphi\in C^\infty(\overline{X},F')$. In concrete situations, estimates of this type are often rather easy to establish.

Proof Note that (5.5) extends by continuity from $\mathcal{D}(X,F)\times\mathcal{D}(X,F')$ to a bilinear form on $W_p^{-s}(X,F)\times W_{p'}^s(X,F')$ and from $\mathcal{D}(X,F)\times C^\infty(\overline{X},F')$ to such a form on $W_1^{-s}(X,F)\times c^s(\overline{X},F')$. For a proof of the duality assertions we refer to [Ama99, Chapter VII]. ∎

(d) Suppose that (5.1) is satisfied and $\alpha\in\mathbb{N}^n$. Then

$$\partial^\alpha : W_{p_0}^{s_0}(X,E_0)\cap W_{p_1}^{s_1}(X,E_1) \to W_p^s(X,E) \text{ compactly} ,$$

provided

$$s<s_\theta \quad\text{and}\quad s-n/p < s_\theta - |\alpha| - n/p_\theta .$$

If $0\le s< s_\theta-|\alpha|-n/p_\theta$ then

$$\partial^\alpha : W_{p_0}^{s_0}(X,E_0)\cap W_{p_1}^{s_1}(X,E_1) \to c^s(\overline{X},E) \text{ compactly} .$$

This generalizes Théorème 2 of [Aub63] as well as Simon's extension of it [Sim87, Corollary 10].

Proof Since

$$\partial^\alpha \in \mathcal{L}\bigl(W_{p_0}^{s_0}(X,E_0)\cap W_{p_1}^{s_1}(X,E_1), W_{p_0}^{s_0-|\alpha|}(X,E_0)\cap W_{p_1}^{s_1-|\alpha|}(X,E_1)\bigr) ,$$

the assertion follows from Theorem 5.2. ∎

6. Proof of Theorem 1.1

In order to derive Theorem 1.1 from the preceding results we need some preparation.

Lemma 6.1. *Set*

$$V := V_{p_0,p_1}(E_0,E_1) := \big\{ v \in L_{p_1}\big((0,T),E_1\big) \ ; \ \partial v \in L_{p_0}\big((0,T),E_0\big) \big\} \ .$$

Then $V \doteq W^1_{p_0}\big((0,T),E_0\big) \cap L_{p_1}\big((0,T),E_1\big)$.

Proof It is clear that V is a Banach space and that

$$W^1_{p_0}\big((0,T),E_0\big) \cap L_{p_1}\big((0,T),E_1\big) \hookrightarrow V \ .$$

Moreover,

$$V \hookrightarrow C\big([0,T],E_0\big) \hookrightarrow L_{p_0}\big((0,T),E_0\big) \ ,$$

where we refer to [Tri78, Lemma 1.8.1], for example, for a proof of the first embedding. Now the assertion is obvious. ■

Put $X_h := X \cap (X-h)$ for $h \in \mathbb{R}^n$ and suppose that $p < \infty$. Also set

$$[u]_{\theta,p,\infty} := \sup_{\substack{h\in\mathbb{R}^n \\ h\neq 0}} \frac{\|u(\cdot+h)-u\|_{L_p(X_h,E)}}{|h|^\theta}$$

and, given $m \in \mathbb{N}$,

$$N^{m+\theta}_p(X,E) := \Big(\big\{ u \in L_p(X,E) \ ; \ [\partial^\alpha u]_{\theta,p,\infty} < \infty, \ |\alpha| = m \big\}, \ \|\cdot\|_{m+\theta,p,\infty} \Big) \ ,$$

where

$$\|u\|_{m+\theta,p,\infty} := \|u\|_p + \max_{|\alpha|=m} [\partial^\alpha u]_{\theta,p,\infty} \ .$$

Then $N^s_p(X,E)$, $s \in \mathbb{R}^+ \setminus \mathbb{N}$, are the Nikol'skii spaces of E-valued distributions on X. The proof for the scalar case (e.g., [Tri78, Section 2.5.1]) carries over to the vector-valued case to show that

$$N^s_p(X,E) \doteq B^s_{p,\infty}(X,E) \ , \qquad s \in \mathbb{R}^+ \setminus \mathbb{N} \ , \tag{6.1}$$

(cf. [Ama99, Section VII.3].

Proof of Theorem 1.1. Clearly, we can assume that $p_0 \vee p_1 < \infty$.

Let (1.7) be satisfied. Then (1.2), (1.3), and Lemma 6.1 imply that $\mathcal{V}$ is bounded in $W^1_{p_0}\big((0,T),E_0\big) \cap L_{p_1}\big((0,T),E_1\big)$. Hence the assertion is entailed by Theorem 5.2.

Suppose that assumption (1.8) is fulfilled. Then (6.1) shows that $\mathcal{V}$ is bounded in $B^{s_0}_{p_0,\infty}\big((0,T),E_0\big)$. Hence it is bounded in $B^{s_0}_{p_0,\infty}\big((0,T),E_0\big) \cap L_{p_1}\big((0,T),E_1\big)$ by (1.6). Thus (4.1) and (4.6) imply that $\mathcal{V}$ is bounded in $B^{s_0}_{p_0,\infty}\big((0,T),E_0\big) \cap B^{s_1}_{p_1,p_1}\big((0,T),E_1\big)$ for each $s_1 < 0$. Now the assertion follows from Corollary 4.5(ii) by means of the arguments used in the proof of Theorem 5.2. ■

7. Final Remarks

So far we have not put any restriction, like reflexivity for example, on the Banach spaces under consideration. However, in order to prove an n-dimensional analogue to Lemma 6.1 we need such an additional assumption. For this we recall that a Banach space F is a UMD space if the Hilbert transform is a continuous self-map of $L_2(\mathbb{R}^n, F)$. Every UMD space is reflexive (but not conversely), and every Hilbert space is a UMD space. The class of UMD spaces enjoys many useful permanence properties. For example, each closed subspace of a UMD space is again a UMD space. For details and more information we refer to [Ama95, Subsection III.4.5].

Example 7.1. Suppose that Ω is an open subset of some euclidean space. Then $W_p^s(\Omega)$ and every closed linear subspace thereof are UMD spaces, provided $1 < p < \infty$.

Proof If $m \in \mathbb{N}$ then $W_p^m(\Omega)$ is well-known to be isomorphic to a closed linear subspace of the M-fold product of $L_p(\Omega)$, where $M := \sum_{|\alpha| \le m} 1$. Hence $W_p^m(\Omega)$ is a UMD space by Theorem III.4.5.2 in [Ama95]. Consequently, $\mathring{W}_p^m(\Omega)$ is a UMD space as well. Thus $W_p^{-m}(\Omega) = \big[\mathring{W}_{p'}^m(\Omega)\big]'$ is also a UMD space, as follows from part (v) of Theorem III.4.5.2 in [Ama95]. Finally, part (vii) of that theorem, together with (3.5) and (3.7), implies the assertion. ∎

If F is a UMD space then the Sobolev-Slobodeckii spaces $W_p^s(X, F)$ possess essentially the same properties as their scalar ancestors, provided $1 < p < \infty$. This is seen, for example, by the following proposition.

Proposition 7.2. *Suppose that F is a UMD space and $1 < p < \infty$. Then, given $s \in \mathbb{R}$ and $m \in \dot{\mathbb{N}}$,*

$$u \mapsto \|u\|_{s,p} + \sum_{|\alpha|=m} \|\partial^\alpha u\|_{s,p}$$

is an equivalent norm for $W_p^{s+m}(X, F)$.

Proof If F is a UMD space then Mikhlin's multiplier theorem is valid in $L_p(\mathbb{R}^n, F)$ for $1 < p < \infty$ (and scalar symbols) (e.g., [Ama95, Theorem III.4.4.3]). Thus the well-known proof for scalar Sobolev spaces extends to the vector-valued setting in this case. ∎

Corollary 7.3. *Suppose that E_0 is a UMD space and $1 < p_0 < \infty$. Then*

$$W_{p_0}^m(X, E_0) \cap L_{p_1}(X, E_1) = \big\{\, u \in L_{p_1}(X, E_1) \ ; \ \partial^\alpha u \in L_{p_0}(X, E_0),\ |\alpha| = m \,\big\}$$

for $m \in \mathbb{N}$ and $1 \le p_1 \le \infty$.

Lastly, we show that, in practice, the assumption that we can squeeze an interpolation space between E and E_1 is no serious restriction. In other words: in most applications assumption (1.6) is satisfied.

Remark 7.4. In concrete applications it is most often the case that $E_j := W_{r_j}^{\sigma_j}(\Omega)$ for $j = 0, 1$ and $E := W_r^\sigma(\Omega)$, where Ω is a bounded smooth open subset of $\mathbb{R}^d$, σ_0 and σ_1

are real numbers with $\sigma_0 < \sigma < \sigma_1$, and $r, r_0, r_1 \in [1, \infty)$. Thanks to the classical Rellich-Kondrachov theorem $E_1 \hookrightarrow\hookrightarrow E_0$. Suppose that $\sigma_0 - d/r_0 < \sigma - d/r < \sigma_1 - d/r_1$. Fix $\vartheta \in (0,1)$ such that

$$\sigma - d/r < \sigma_\vartheta - d/r_\vartheta < \sigma_1 - d/r_1 \ , \qquad \sigma < \sigma_\vartheta < \sigma_1 \ ,$$

and $\sigma_\vartheta \notin \mathbb{Z}$. Then we infer from (4.1) and (4.7) that

$$E_1 \hookrightarrow (E_0, E_1)_{\vartheta,1} \hookrightarrow (E_0, E_1)_{\vartheta, r_\vartheta} \doteq W^{\sigma_\vartheta}_{r_\vartheta}(\Omega) \hookrightarrow E \ ,$$

since, by making σ_1 slightly smaller and σ_0 slightly bigger, if necessary, we can suppose that $W^{\sigma_j}_{r_j}(\Omega) = B^{\sigma_j}_{r_j,r_j}(\Omega)$ for $j = 0, 1$. ■

For simplicity, we presupposed throughout that X be smooth. However, everything remains valid if we drop this hypothesis and assume instead that r_X possesses a coretraction with the properties stated in Theorem 4.1. This is known to be the case for a much wider class of subdomains of $\mathbb{R}^n$. We do not go into detail but refer to [Ama99]. The same observation applies to Ω, of course.

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