On a quasilinear coagulation-fragmentation model with diffusion

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Abstract We consider a system of a very large number of particles of very different sizes, suspended in a carrier fluid. These particles move due to diffusion and superimposed transport processes, merge to form larger clusters, or fragment into smaller ones.

In the present paper a mathematical model for such processes is derived, consisting of an infinite quasilinear reaction-diffusion system, coupled to the Navier Stokes equations for the motion of the suspension. We prove the well-posedness of this problem, derive a positivity result, and show that the total mass of the suspended particles is conserved.

1 Introduction

The aim of this paper is to discuss the well-posedness of a mathematical model describing cluster growth. More precisely, we consider a very large number of particles of very different sizes, suspended in a carrier fluid. These particles, which are also called clusters, can coagulate to form larger particles, or fragment into smaller ones. Moreover, the clusters move due to diffusion and superimposed transport processes.

We describe this system by means of the particle size distribution function, which is a density measuring the number of clusters of a given size y at place x and time t. Two situations are considered simultaneously: the discrete case, where each cluster consists of an integer number of elementary particles, and the continuous case. Accordingly, in the discrete case the variable y runs through $\mathbb{N} \setminus \{0\}$. Here the particle size distribution function has to satisfy a system of countably many coupled reaction-diffusion equations. In the continuous case the variable y takes on values in $(0, \infty)$. Then the particle size distribution function is described by an integro-differential equation, which can be viewed as an uncountable reaction-diffusion system. In both cases we also take the motion of the suspension into account. It is described by the Navier-Stokes equations, coupled to the reaction-diffusion system.

In recent years the mathematical theory of coagulation-fragmentation processes has made considerable progress. Many contributions to this field, however, are confined to the discrete case and to kinetic models, in which neither diffusion nor the motion of the suspension is taken into account. Such problems were considered, for instance, by M. Aizenman and T. A. Bak (cf. [AB79]), I. W. Stewart (cf. [Ste89]-[Ste91]), or D. J. McLaughlin, W. Lamb, and A. C. McBride (cf. [MLM95]-[MLM98]). The first mathematically rigorous treatment of countable reaction-diffusion systems, describing coagulation-fragmentation processes, is due to Ph. Bénilan and D. Wrzosek (cf. [BW97]). Further results for such models were obtained by D. Wrzosek and Ph. Laurençot (cf. [Wrz97] and [LW98a]-[LW99]). Uncountable reaction-diffusion systems are investigated in the recent paper [Ama00a] by the first author. There the continuous case is treated simultaneously with the discrete one by considering the coagulation-fragmentation models as semilinear evolution equations in infinite-dimensional state spaces. However, as in the other papers mentioned above, the motion of the suspension is not taken into account.

Although coagulation-fragmentation processes with diffusion have also been considered in the physical literature, we are not aware of a rigorous derivation of those models. In the present paper suspensions of particles are viewed as a mixture with countably or uncountably many components. Then we fall back on basic principles from the classical theory of mixtures with finitely many components (cf. e.g. [LL66], [dGM84] or [RT95]). This leads to a (countable or uncountable) quasilinear reaction-diffusion system governing the behaviour of the particle size distribution function. In order to describe the motion of the suspension, we assume it to be a Newtonian fluid. Thus, the baricentric velocity of the mixture has to satisfy the Navier-Stokes equations, coupled to the reaction-diffusion system. This approach is justified by our subsequent analysis which shows that the model possesses a unique solution, where the particle size distribution function remains non-negative, and the flow is mass preserving.

Our paper is organized as follows. In Section 2 we derive the model and formulate the main results (in a simplified form). Section 3 is devoted to the proof that the principal part of our system, that is, the diffusion operator in an infinite-dimensional state space, generates an analytic semigroup. This derivation essentially relies on Fourier multiplier theorems in vector-valued Besov spaces. Note that an application of such tools in (less difficult) vector-valued L_p-spaces requires an additional geometric assumption on the underlying Banach space, which is too restrictive for our problem. Having established the generation result, we show in Section 4 that our model can be formulated as a quasilinear Cauchy problem of parabolic type. Then existence of a unique maximal solution follows from the general abstract theory for those equations. Moreover, we derive a regularity result which enables us to prove conservation of the mass of all suspended particles. Finally, it is shown that the particle size distribution function remains positive for positive initial values.

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Notations

In this section, we collect several basic notations used throughout the paper.

For x, $y \in \mathbb{R}$ we put $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. The set $\mathbb{R}^{n\times n}_{\text{sym}}$ consists of all symmetric matrices in $\mathbb{R}^{n\times n}$.

Let E, $E_0, ..., E_m$ be Banach spaces. Then $\mathcal{L}(E_1, ..., E_m; E_0)$ denotes the Banach space of all m-linear continuous maps from $E_1 \times \cdots \times E_m$ into E_0. If $m \geq 2$, elements of $\mathcal{L}(E_1, ..., E_m; E_0)$ are called *multiplications*. In the case $m = 1$ we put $\mathcal{L}(E_1, E_0) := \mathcal{L}(E_1; E_0)$, and $\mathcal{L}(E) := \mathcal{L}(E, E)$. The set $\{\mathcal{A} \in \mathcal{L}(E_1, E_0)\,;\ \mathcal{A} \text{ bijective}, \mathcal{A}^{-1} \in \mathcal{L}(E_0, E_1)\}$ is denoted by $\mathcal{L}\text{is}(E_1, E_0)$, and $\mathcal{L}\text{aut}(E)$ stands for $\mathcal{L}\text{is}(E, E)$.

If E_1 is continuously injected in E_0, we write $E_1 \hookrightarrow E_0$. Moreover, the additional "d" in $E_1 \overset{d}{\hookrightarrow} E_0$ indicates the density of this imbedding. The notation $E_0 \doteq E_1$ means that $E_1 \hookrightarrow E_0$ and $E_0 \hookrightarrow E_1$ hold.

An interpolation functor $(\cdot,\cdot)_\theta$ is said to be *admissible*, if $E_1 \overset{d}{\hookrightarrow} E_0$ implies $E_1 \overset{d}{\hookrightarrow} (E_0, E_1)_\theta$ for $\theta \in (0, 1)$. Note that the standard real interpolation functor $(\cdot,\cdot)_{\theta,p}$ of exponent $\theta \in (0, 1)$ and parameter $p \in [1, \infty)$, and the standard complex interpolation functor $[\cdot,\cdot]_\theta$, $\theta \in (0, 1)$, satisfy this condition.

In our paper we employ various spaces $\mathfrak{F}(J, E)$ of E-valued functions on a perfect interval $J \subset \mathbb{R}$, where $\mathfrak{F} \in \{C^k, C^\gamma\,;\ k \in \mathbb{N}, \gamma \in (0, 1) \cup \{1-\}\}$. Here C^k stands for k-times continuously differentiable maps, and $C := C^0$. Furthermore, C^γ means γ-Hölder continuity if $\gamma \in (0, 1)$, and Lipschitz continuity if $\gamma = 1-$.

Let (M, μ) be a measure space and $p \in [1, \infty]$. Then $L_p(M, E; \mu)$ is the usual Lebesgue space of (equivalence classes of) E-valued integrable functions on M. In the case of $M = \mathbb{R}^n$, if we use Lebesgue's measure, then we write $L_p(\mathbb{R}^n, E)$. By $W^s_p(\mathbb{R}^n, E)$, we denote the usual Sobolev-Slobodeckii space of order $s \geq 0$ and integrability index $p \in [1, \infty]$. Let $k \in \mathbb{N}$. Then $\text{BUC}^k(\mathbb{R}^n, E)$ consists of all functions in $C^k(\mathbb{R}^n, E)$ whose derivatives are bounded and uniformly continuous. It is a Banach space with the norm

$$u \longmapsto \|u\|_{\mathrm{BUC}^k(\mathbb{R}^n,E)} := \max_{|\nu|\le k} \sup_{x\in\mathbb{R}^n} \|\partial^\nu u(x)\|_E \,.$$

If $s \in (k, k+1)$, then $\mathrm{BUC}^s(\mathbb{R}^n, E)$ consists of all $u \in \mathrm{BUC}^k(\mathbb{R}^n, E)$ satisfying

$$\|u\|_{\mathrm{BUC}^s(\mathbb{R}^n,E)} := \|u\|_{\mathrm{BUC}^k(\mathbb{R}^n,E)} + \max_{|\nu|=k} \sup_{x\neq y} \frac{\|\partial^\nu u(x) - \partial^\nu u(y)\|_E}{|x-y|^{s-k}} < \infty \,.$$

By $B^s_{p,q}(\mathbb{R}^n, E)$, we denote the Besov space of order $s \in \mathbb{R}$ and integrability indices p, $q \in [1,\infty]$ consisting of E-valued distributions on $\mathbb{R}^n$. It is known that

$$B^s_{p,p}(\mathbb{R}^n, E) \doteq \begin{cases} W^s_p(\mathbb{R}^n, E) & , p \in [1,\infty) \\ \mathrm{BUC}^s(\mathbb{R}^n, E) & , p = \infty \end{cases} \tag{1.1}$$

holds for $s \in (0,\infty) \setminus \mathbb{N}$.

The letter c is often used to denote an arbitrary constant. If it depends upon additional parameters, say t, we sometimes indicate this by $c(t)$.

2 The model and the main result

We consider a system of a very large number of particles, which are also called clusters. They are suspended in a carrier fluid filling all $\mathbb{R}^n$, where $n \in \mathbb{N} \setminus \{0\}$. The "size" of a particle is measured by the variable y, which, according to the concrete physical situation, takes its values either in $Y = \dot{\mathbb{N}} = \mathbb{N} \setminus \{0\}$ or in $Y = (0,\infty)$. In the *discrete case*, $Y = \dot{\mathbb{N}}$, each cluster consists of an integer number of uniform elementary particles, called monomers. Accordingly, y is proportional to the number of monomers, forming the clusters. In order to describe situations where the particles are not supposed to be integer multiples of elementary units, we consider the *continuous case*, $Y = (0,\infty)$, as well. Here the size y of a cluster represents its volume. Note that the use of infinite domains Y is a common and convenient practice to avoid (a priori given) upper bounds for the particle size.

To obtain a uniform model for both the discrete and the continuous case, which allows a simultaneous treatment, we introduce the measure space (Y, μ), where μ denotes the counting measure if $Y = \dot{\mathbb{N}}$, and the Lebesgue measure if $Y = (0,\infty)$.

The system is described by the *particle size distribution function*, $u(y) = u(t, x, y)$, depending on time t, spatial variable $x \in \mathbb{R}^n$, and $y \in Y$. Thus, for any domain $X \subset \mathbb{R}^n$ and $y_0, y_1 \in Y$ with $y_0 \le y_1$, the integral

$$\int_X \int_{y_0}^{y_1} u(t,x,y)\, \mathrm{d}\mu(y)\, \mathrm{d}x$$

gives the total number of clusters of size $y \in [y_0, y_1]$, contained in X at time t. Note that in the discrete case this expression takes the form

$$\int_X \sum_{y=y_0}^{y_1} u(t,x,y)\,\mathrm{d}x\,.$$

The basic artifice, which we use to treat the suspension of clusters, consists of considering the whole configuration as a mixture of infinitely many fluids. The components of this mixture are the "fluids of type y", containing all particles of a given size $y \in Y$, as well as the carrier fluid, labelled by $y = 0$. For a uniform description we set $Y_0 := \{0\} \cup Y$ and define a measure dy on Y_0 by setting

$$\int_{Y_0} F(y)\,\mathrm{d}y := F(0) + \int_Y F(y)\,\mathrm{d}\mu(y)$$

for each continuous function F with compact support in Y_0.

Let $\varrho(t,x,y)$ be the *density* of the fluid of type $y \in Y_0$ at time t and place x. We assume that there exists a positive constant η such that

$$\varrho(t,x,y) = \eta\, y\, u(t,x,y)\ ,\ t \geq 0\,,\ x \in \mathbb{R}^n\ ,\ y \in Y. \tag{2.1}$$

This means homogeneity of the suspension in the sense that the mass of each cluster is proportional to its size. Moreover, we assume the *total density* of the mixture to be constant: there exits a $\varrho > 0$ such that

$$\int_{Y_0} \varrho(t,x,y)\,\mathrm{d}y = \varrho(t,x,0) + \int_Y \varrho(t,x,y)\,\mathrm{d}\mu(y) = \varrho \tag{2.2}$$

holds for $t \geq 0$ and $x \in \mathbb{R}^n$. This means that the mixture, considered as a whole, is incompressible.

Let $\vec{\omega}(t,x,y)$ denote the *velocity* of the fluid of type $y \in Y_0$. Then,

$$\vec{v}(t,x) := \frac{1}{\varrho}\int_{Y_0} \varrho(t,x,y)\,\vec{\omega}(t,x,y)\,\mathrm{d}y$$

is the *baricentric velocity* of the mixture at time $t \geq 0$ and place $x \in \mathbb{R}^n$.

Remark Hereafter we refrain from indicating dependencies on $(t,x) \in [0,\infty) \times \mathbb{R}^n$, except where this may cause confusion.

Conservation of mass

To obtain the equations governing the behaviour of the particle size distribution function, we fall back on arguments originally used for mixtures with a finite number of components (cf. e.g. [LL66], [dGM84], or [RT95]). Accordingly, $m(t,x,y)$ denotes the *mass production density* of the fluid of type $y \in Y_0$ at time t and place x. We assume the condition

$$\int_{Y_0} m(y)\,\mathrm{d}y = m(0) + \int_Y m(y)\,\mathrm{d}\mu(y) = 0 \tag{2.3}$$

to be satisfied, which means that the *total mass* of the mixture is conserved at each moment $t \geq 0$ and at each position $x \in \mathbb{R}^n$.

Due to standard arguments (cf. [dGM84] or [Ser59]), the continuity equation

$$\partial_t \varrho(y) + \operatorname{div}\left[\varrho(y)\,\vec{\omega}(y)\right] = m(y) \tag{2.4}$$

holds for each $y \in Y_0$. We now integrate (2.4) over Y_0. In view of the hypotheses (2.2) and (2.3), this yields

$$\operatorname{div} \vec{v} = 0\,. \tag{2.5}$$

For each fluid of type $y \in Y_0$ the *diffusive flux* $\vec{\jmath}(t,x,y)$ with respect to the baricentric velocity is defined as $\vec{\jmath}(y) := \varrho(y)\,[\vec{\omega}(y) - \vec{v}]$. Inserting this into (2.4) and using (2.5) we obtain

$$\partial_t \varrho(y) + \operatorname{div} \vec{\jmath}(y) + \vec{v} \cdot \operatorname{grad} \varrho(y) = m(y)\ ,\ y \in Y_0\,. \tag{2.6}$$

Since (2.2) implies

$$\varrho(0) = \varrho - \int_Y \varrho(y)\,\mathrm{d}\mu(y),$$

it suffices to study system (2.6) for $y \in Y$. Thus, taking (2.1) into consideration, we arrive at

$$\partial_t u(y) + \frac{1}{\eta\, y} \operatorname{div} \vec{\jmath}(y) + \vec{v} \cdot \operatorname{grad} u(y) = \frac{1}{\eta\, y} m(y)\ ,\ y \in Y. \tag{2.7}$$

Finally, the diffusive flux $\vec{\jmath}$ and the mass production density m remain to be specified. For that purpose, we now make some constitutive assumptions.

The diffusive flux $\vec{\jmath}$ is supposed to be given by

$$\begin{aligned} \vec{\jmath}(y) &:= -\mathbf{a}(u,y)\operatorname{grad}\varrho(y) - \int_Y \mathbf{b}(u,y,y')\operatorname{grad}\varrho(y')\,\mathrm{d}\mu(y') \\ &= -\eta\, y\,\mathbf{a}(u,y)\operatorname{grad} u(y) - \eta \int_Y \mathbf{b}(u,y,y')\operatorname{grad} u(y')\, y'\,\mathrm{d}\mu(y') \end{aligned} \tag{2.8}$$

for $y \in Y$, where $\mathbf{a}$ and $\mathbf{b}$ also depend, of course, on the spatial variable $x \in \mathbb{R}^n$. We assume that the "principal diffusion coefficient" $\mathbf{a}$ is positive, and dominates the "cross diffusion coefficient" $\mathbf{b}$ in a way that will be specified later (cf. Definition 3.10).

The mass production density m is supposed to be of the form

$$m(t,x,y) := \eta\, y\,\left[c_\psi(u)(t,x,y) + f_\phi(u)(t,x,y)\right]\ ,\ y \in Y, \tag{2.9}$$

where c_ψ and f_ϕ are reaction terms, describing the kinetic behaviour of the clusters due to coagulation and fragmentation, respectively.

First we consider the coagulation term. It is defined as

$$\begin{aligned} c_\psi(u)(y) := & \frac{1}{2}\int_0^y \psi(y-y',y')\, u(y-y')\, u(y')\, \mathrm{d}\mu(y') \\ & - u(y)\int_Y \psi(y,y')\, u(y')\, \mathrm{d}\mu(y') \quad , \; y \in Y, \end{aligned} \tag{2.10}$$

where we set $\mu(\{0\}) = 0$ if μ denotes the counting measure on Y. The coagulation kernel $\psi(x, y, y')$ describes the rate of coalescence of clusters of sizes y and y' at place x. This motivates the hypothesis

$$0 \le \psi(x,y,y') = \psi(x,y',y) \quad , \; x \in \mathbb{R}^n \; , \; y, y' \in Y. \tag{2.11}$$

The first integral on the right hand side of (2.10) expresses the fact that a particle of size y comes into being if two clusters of sizes $y - y'$ and y' merge, where the factor $1/2$ ensures that each combination is counted only once. Here we neglect simultaneous coalescence of more than two particles. The last term in (2.10) says that a cluster of size y disappears from the fluid of type y by merging into another one. Since coagulation does not produce or destroy mass, we ought to have

$$\int_Y c_\psi(u)(x,y)\, y\, \mathrm{d}\mu(y) = 0 \quad , \; x \in \mathbb{R}^n.$$

This can actually be verified, provided the map $(y, y') \longmapsto y\, \psi(y, y')\, u(y)\, u(y')$ is integrable with respect to $\mu \otimes \mu$.

The fragmentation term f_ϕ is given by

$$f_\phi(u)(y) := \int_y^\infty \phi(y',y)\, u(y')\, \mathrm{d}\mu(y') - \frac{u(y)}{y}\int_0^y \phi(y,y')\, y'\, \mathrm{d}\mu(y') \tag{2.12}$$

for $y \in Y$, where the first integral describes the generation of clusters of size y by fragmentation of particles of larger size. We assume that the fragmentation rate $\phi(x, y', y)$ satisfies

$$0 \le \phi(x,y',y) \quad , \; x \in \mathbb{R}^n \; , \; (y',y) \in Y_\Delta^2 := \{(z',z) \in Y^2;\, 0 < z \le z'\}. \tag{2.13}$$

The second term on the right hand side of (2.12) guarantees the disappearance of clusters of size y by splitting into smaller ones. Note that (2.12) describes multiple fragmentation. As in the case of coagulation, fragmentation does not create or annihilate mass. In fact, the identity

$$\int_Y f_\phi(u)(x,y)\, y\, \mathrm{d}\mu(y) = 0 \quad , \; x \in \mathbb{R}^n,$$

can easily be verified, if the map $(y, y') \longmapsto y\, \phi(y', y)\, u(y')$ is integrable with respect to $\mu \otimes \mu$.

The motion of the mixture

In order to describe the baricentric velocity of the suspension, we assume the mixture to be a Newtonian fluid. Consequently, $\vec{v}$ has to satisfy the Navier-Stokes equations

$$\partial_t \vec{v} + (\vec{v} \cdot \text{grad})\, \vec{v} \; - \nu\, \Delta \vec{v} = -\varrho^{-1}\, \text{grad}\, \mathsf{p} + \vec{f} \tag{2.14}$$

with *pressure* p, constant *viscosity* $\nu > 0$, and *specific external force field* $\vec{f}$ acting on the mixture. Here $\vec{f}$ is given by

$$\vec{f}(t,x) := \frac{1}{\varrho} \int_{Y_0} \varrho(t,x,y)\, \vec{\varphi}(t,x,y)\, \mathrm{d}y,$$

where $\vec{\varphi}(y) = \vec{\varphi}(t,x,y)$ denotes the *density of the specific external force field acting on the fluid of type* $y \in Y_0$. From (2.1) and (2.2) we infer that

$$\vec{f} = \vec{f}(u) = \left[1 - \frac{\eta}{\varrho} \int_Y u(y)\, y \, \mathrm{d}\mu(y)\right] \vec{\varphi}(0) + \frac{\eta}{\varrho} \int_Y u(y)\, \vec{\varphi}(y)\, y\, \mathrm{d}\mu(y). \tag{2.15}$$

Formulation of the problem and the main result

Summarizing (2.5), (2.7)-(2.9), and (2.14), we can now formulate the following initial value problem.

Problem 2.1 Let the reaction terms c_ψ, f_ϕ, and the external force field $\vec{f}$ be given by (2.10), (2.12), and (2.15), respectively.

Then we search for functions $u : J \times \mathbb{R}^n \times Y \to \mathbb{R}$, $\vec{v} : J \times \mathbb{R}^n \to \mathbb{R}^n$, and $\mathsf{p} : J \times \mathbb{R}^n \to \mathbb{R}$ on a time interval $J = [0,T)$ with $T \in (0, \infty]$, satisfying the equations

$$\begin{aligned} \partial_t u(y) - \text{div}\, [\alpha(u)\, \text{grad}\, u]\, (y) + \vec{v} \cdot \text{grad}\, u(y) \;&=\; c_\psi(u)(y) + f_\phi(u)(y) \;\;,\; y \in Y, \\ \partial_t \vec{v} - \nu\, \Delta \vec{v} + (\vec{v} \cdot \text{grad})\, \vec{v} \;&=\; \vec{f}(u) - \varrho^{-1}\, \text{grad}\, \mathsf{p} \;, \\ \text{div}\, \vec{v} \;&=\; 0 \;, \end{aligned}$$

in $J \times \mathbb{R}^n$, as well as the initial conditions

$$u(0,\cdot,y) = u_0(\cdot,y) \;\;,\; y \in Y \quad \text{and} \quad \vec{v}(0,\cdot) = \vec{v}_0 \quad \text{in } \mathbb{R}^n,$$

where

$$[\alpha(w)\, v](y) := \mathbf{a}(w)(y)\, v(y) + \frac{1}{y} \int_Y \mathbf{b}(w)(y,y')\, v(y')\, y'\, \mathrm{d}\mu(y'). \tag{2.16}$$

In Section 4 this model is investigated analytically. Our results, obtained there, are summarized in the following theorem. Note that, for simplicity's sake, this formulation is confined to smooth data; to the most important case of diffusion operators with vanishing cross diffusion coefficient $\mathbf{b}$; and to the (practically relevant) spatial dimensions $n \in \{2,3\}$.

Theorem 2.2 *Let $n \in \{2,3\}$, and assume that the mappings*

$$\begin{aligned}
&\mathbf{a} : \mathbb{R} \times \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^{n\times n}_{\mathrm{sym}}\,, \quad \psi : \mathbb{R}^n \times Y^2 \longrightarrow \mathbb{R}\,, \quad \phi : \mathbb{R}^n \times Y^2_\Delta \longrightarrow \mathbb{R}\,,\\
&\vec{\varphi} : \mathbb{R}^n \times Y_0 \longrightarrow \mathbb{R}^n\,, \qquad\qquad u_0 : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}\,, \quad \vec{v}_0 : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n\,,
\end{aligned}$$

are bounded smooth functions satisfying the following conditions.

(C_1) $\xi \cdot \mathbf{a}(\zeta, x, y)\, \xi \geq a_0\, |\xi|^2\ \ , \zeta \in \mathbb{R},\ x \in \mathbb{R}^n\,,\ y \in Y\,,\ \xi \in \mathbb{R}^n$.

(C_2) *The coagulation kernel and the fragmentation rate fulfil* (2.11) *and* (2.13), *respectively. Moreover, there is a constant $c > 0$ such that*

$$\frac{1}{y} \int_Y \phi(x, y, y')\, y'\ d\mu(y') \leq c < \infty\ ,\ x \in \mathbb{R}^n\ ,\ y \in Y.$$

(C_3) *The density $\vec{\varphi}(\cdot, 0)$ of the specific external force field acting on the carrier fluid is compactly supported in $\mathbb{R}^n$.*

(C_4) *The initial particle size distribution $x \longmapsto u_0(x, \cdot)$ has a compact support in the space $E := L_1(Y; (1+y)\, d\mu(y))$. Moreover, $u_0(x, y) \geq 0$ holds for $x \in \mathbb{R}^n$ and almost all $y \in Y$. The initial baricentric velocity $\vec{v}_0$ of the suspension is divergence free and compactly supported.*

Then problem 2.1, *considered for the diffusion coefficient*

$$[\alpha(w)\, v](y)\ :\ x \longmapsto \mathbf{a}(w, x, y)\, v(x, y),$$

possesses a unique solution $(u, \vec{v}, \mathsf{p})$ with

$$\begin{aligned}
&u \in C^1(J, W^s_p(\mathbb{R}^n, E) \cap W^s_1(\mathbb{R}^n, E)) \cap C(J, W^{s+2}_p(\mathbb{R}^n, E) \cap W^{s+2}_1(\mathbb{R}^n, E))\,,\\
&\vec{v} \in C^1(J, L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap C(J, W^2_p(\mathbb{R}^n, \mathbb{R}^n))\,,\\
&\mathsf{p} \in C(J, W^1_p(\mathbb{R}^n, \mathbb{R})) \quad \text{and} \quad \int_{\mathbb{R}^n} \mathsf{p}(t, x)\, \mathrm{d}x = 0\,,\ t \in J\,,
\end{aligned}$$

where $0 < s < 1$ and $p > n/s$, on a maximal time interval $J = [0, T)$. The particle size distribution function has the property

$$u(t, x, y) \geq 0\ ,\ t \in J\ ,\ x \in \mathbb{R}^n\ ,\ a.a.\, y \in Y.$$

Moreover, the total mass of the suspended particles,

$$\mathfrak{M}(t) = \eta \int_{\mathbb{R}^n} \int_Y u(t, x, y)\, y\, \mathrm{d}\mu(y) \mathrm{d}x, \tag{2.17}$$

is conserved, i.e. $\mathfrak{M}(t) = \mathfrak{M}(0)$, $t \in J$.

For the proof we refer to Section 4. Existence of a unique maximal solution and conservation of mass follow from Theorems 4.7 and 4.13, respectively. Note that both of these statements include the case of a non-vanishing cross diffusion coefficient $\mathbf{b}$, dominated by $\mathbf{a}$. Positivity of the particle size distribution function is shown in Theorem 4.16.

Remark 2.3 Theorem 2.2 remains valid for time dependent functions $\mathbf{a}$, ψ, ϕ, $\vec{\varphi}$, provided these dependencies are sufficiently smooth, and assumptions (C_1)-(C_3) of Theorem 2.2 hold uniformly with respect to t (cf. Remark 4.8).

3 Differential operators with operator-valued coeffcients

In this section we consider the differential operator

$$\mathcal{A}_\alpha : u \longmapsto \{x \longmapsto -\mathrm{div}\,[\alpha(x)\,\mathrm{grad}\,u(x)]\} \tag{3.1}$$

for functions $u : \mathbb{R}^n \longrightarrow E$, where E denotes some Banach space. Our aim is to specify conditions on $\alpha = (\alpha_{ij}) : \mathbb{R}^n \longrightarrow \mathcal{L}(E^n)$, which ensure that $-\mathcal{A}_\alpha$ generates an analytic semigroup of bounded linear operators on $B^s_{p,q}(\mathbb{R}^n, E)$.

This is done in two steps. Using a Fourier multiplier theorem, we first obtain a generation result for the special case of spatially constant $\alpha \in \mathcal{L}(E^n)$. By means of a localization procedure, combined with perturbation arguments, this statement can be extended to the general case of variable coefficients.

Remark 3.1 Throughout the remainder of this paper, we work in various spaces, $\mathfrak{F}(\mathbb{R}^n, E)$, of E-valued distributions on $\mathbb{R}^n$. In order to simplify the notation of these spaces, the specification $(\mathbb{R}^n, E)$ shall be omitted, if no misunderstandings seem likely. So, for example, we often write $B^s_{p,q}$ instead of $B^s_{p,q}(\mathbb{R}^n, E)$.

3.1 Preliminaries

In the following discussion we provide the basic concepts and technical tools, needed in the sequel. Throughout this section $E, E_0, ..., E_m$ are Banach spaces.

Let $E_1 \overset{d}{\hookrightarrow} E_0$. Then $\mathcal{H}(E_1, E_0)$ denotes the set of $\mathcal{A} \in \mathcal{L}(E_1, E_0)$ such that $-\mathcal{A}$ generates an analytic semigroup $\{\mathrm{e}^{-t\mathcal{A}}; t \geq 0\}$ in $\mathcal{L}(E_0)$.

Moreover, the class $\mathcal{A} \in \mathcal{H}(E_1, E_0; \kappa, \omega)$ with $\kappa \geq 1$ and $\omega > 0$ consists of all operators $\mathcal{A} \in \mathcal{L}(E_1, E_0)$ which satisfy $\omega + \mathcal{A} \in \mathcal{L}\mathrm{is}(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(\lambda + \mathcal{A})u\|_{E_0}}{|\lambda|\,\|u\|_{E_0} + \|u\|_{E_1}} \leq \kappa\ ,\ u \in E_1 \setminus \{0\}\,,\ \mathrm{Re}\lambda \geq \omega\,. \tag{3.2}$$

Observe that $\mathcal{H}(E_1, E_0) = \bigcup \{\mathcal{H}(E_1, E_0\,;\ \kappa, \omega); \kappa \geq 1, \omega > 0\}$ (cf. [Ama95, I.1.2.2 Theorem]).

Remark 3.2 Condition (3.2) can be weakened as follows.

If there are constants $\kappa \geq 1$, $\omega > 0$ *such that* $\mathcal{A} \in \mathcal{L}(E_1, E_0)$ *satisfies* $\omega + \mathcal{A} \in \mathcal{L}\mathrm{is}(E_1, E_0)$ *and one of the estimates*

$$|\lambda| \, \|u\|_{E_0} \leq \kappa \, \|(\lambda + \mathcal{A})u\|_{E_0} \quad \text{or} \quad \|u\|_{E_1} \leq \kappa \, \|(\lambda + \mathcal{A})u\|_{E_0} \ , \ u \in E_1 \ , \ \mathrm{Re}\lambda \geq \omega \ ,$$

then $\mathcal{A} \in \mathcal{H}(E_1, E_0)$ (cf. [Ama95, I.1.2.1 Remark (a)]).

The following useful perturbation result for the class $\mathcal{H}(E_1, E_0)$ is proven in [Ama95, I.1.3.1 Theorem].

Lemma 3.3 *Let* $\mathcal{A} \in \mathcal{H}(E_1, E_0; \kappa, \omega)$, *and assume that* $\mathcal{B} \in \mathcal{L}(E_1, E_0)$ *satisfies*

$$\|\mathcal{B}u\|_{E_0} \leq \varepsilon \, \|u\|_{E_1} + c \, \|u\|_{E_0} \ \ , \ u \in E_1 \ ,$$

for some $\varepsilon \in (0, 1/\kappa)$ *and* $c \geq 0$. *Then we have* $\mathcal{A} + \mathcal{B} \in \mathcal{H}(E_1, E_0; \kappa_*, \omega_*)$ *with* $\omega_* := \omega \vee (c/\varepsilon)$ *and* $\kappa_* := \kappa/(1 - \kappa\, \varepsilon)$.

Let $(\cdot,\cdot)_\theta$ be an admissible interpolation functor, and define $E_\theta := (E_0, E_1)_\theta$ for $\theta \in (0,1)$. Then the following statement can be shown.

Corollary 3.4 *Assume that* $\mathcal{A} \in \mathcal{H}(E_1, E_0)$ *and* $\mathcal{B} \in \mathcal{L}(E_\theta, E_0)$ *for a given* $\theta \in [0,1)$. *Then* $\mathcal{A} + \mathcal{B} \in \mathcal{H}(E_1, E_0)$.

Proof. Using Young's inequality, we see that the estimate

$$\|\mathcal{B}u\|_{E_0} \leq c \, \|u\|_{E_\theta} \leq c \, \|u\|_{E_0}^{1-\theta} \, \|u\|_{E_1}^{\theta} \leq \varepsilon \, \|u\|_{E_1} + c(\varepsilon) \, \|u\|_{E_0} \ , \ u \in E_1 \ ,$$

holds for each $\varepsilon > 0$. Thus, our assertion follows from Lemma 3.3. □

Next we consider pointwise multiplications in Besov spaces.

Proposition 3.5 *For* $j \in \{1, ..., m\}$, *let* p, p_j, q, $q_j \in [1, \infty]$ *and* s, $s_j \in (0, \infty)$. *Moreover, assume that* $[(\varphi_1, ..., \varphi_m) \longmapsto \varphi_1 \bullet \cdots \bullet \varphi_m] \in \mathcal{L}(E_1, ..., E_m; E)$.

Then the following statements are valid.

(S$_1$) *The conditions* $s \leq \min\{s_j\}$, $1/p \leq \sum_{j=1}^m 1/p_j$, $q \geq \max\{q_j \, ; \, s_j = s\}$, *and*

$$s - n/p < \begin{cases} \sum_{s_j < n/p_j} (s_j - n/p_j) & \text{if } \min\{s_j - n/p_j\} < 0 \\ \min\{s_j - n/p_j\} & \text{otherwise} \end{cases}$$

are supposed to be satisfied. Then the pointwise product

$$(v_1, ..., v_m) \longmapsto [x \longmapsto v_1(x) \bullet \cdots \bullet v_m(x)] \tag{3.3}$$

is m-linear and continuous from $\prod_{j=1}^m B_{p_j,q_j}^{s_j}(\mathbb{R}^n, E_j)$ *into* $B_{p,q}^s(\mathbb{R}^n, E)$.

(S₂) *If* $s_2 = \cdots = s_m > n/p$ *and* $0 < s < s_1 \wedge s_2$ *hold, then* (3.3) *is an* m*-linear continuous map* $\mathrm{BUC}^{s_1}(\mathbb{R}^n, E_1) \times \prod_{j=2}^m B_{p,q}^{s_2}(\mathbb{R}^n, E_j) \longrightarrow B_{p,q}^s(\mathbb{R}^n, E)$.

(S_3) *In the case of* $m = 2$, *statement* (S_2) *holds for* $0 < s_2 = s < s_1$.

PROOF. Our first assertion is a special case of [Ama91, Theorem 4.1]. Using (1.1), statement (S_2) follows from (S_1). For the proof of (S_3) we refer to [Ama91, Remark 4.2 (b)]. □

Remark 3.6 For $p \in [1, \infty)$ and $s \in (0, 1]$, let $W_p^{-s}(\mathbb{R}^n, E)$ contain all E-valued distributions u on $\mathbb{R}^n$ which have a representation

$$u = \sum_{|\nu| \leq 1} (-1)^{|\nu|} \partial^\nu u_\nu \tag{3.4}$$

with $\{u_\nu\}_{|\nu| \leq 1} \in W_p^{1-s}(\mathbb{R}^n, E)^{n+1}$. Equipped with the norm

$$u \longmapsto \|u\|_{W_p^{-s}} := \inf \left\{ \left(\sum_{|\nu| \leq 1} \|u_\nu\|^p_{W_p^{1-s}(\mathbb{R}^n, E)} \right)^{1/p} \right\},$$

where the infimum is taken over all those representations, $W_p^{-s}(\mathbb{R}^n, E)$ is a Banach space. Assume that $[(\varphi_1, \varphi_2) \longmapsto \varphi_1 \bullet \varphi_2] \in \mathcal{L}(E_1, E_2; E)$. Then a pointwise product of a distribution $u \in W_p^{-1}(\mathbb{R}^n, E_1)$ and a function $v \in W_\infty^1(\mathbb{R}^n, E_2)$ can be introduced as follows. Using the representation (3.4) of u we set

$$u \bullet v := \sum_{|\nu| \leq 1} (-1)^{|\nu|} \partial^\nu w_\nu \, ,$$

where $w_\nu \in L_p(\mathbb{R}^n, E)$ is given by

$$w_\nu(x) := \begin{cases} u_\nu(x) \bullet v(x) + \sum_{|\eta|=1} u_\eta(x) \bullet \partial^\eta v(x) & , \ |\nu| = 0 \\ u_\nu(x) \bullet v(x) & , \ |\nu| = 1 \end{cases} .$$

It is easily verified that this definition does not depend on the representation (3.4) of u. Moreover,

$$[(u, v) \longmapsto u \bullet v] \in \mathcal{L}(W_p^{-1}(\mathbb{R}^n, E_1), W_\infty^1(\mathbb{R}^n, E_2); W_p^{-1}(\mathbb{R}^n, E)),$$

and $u \bullet v$ coincides with the usual pointwise product if u and v are smooth.

For $p \in [1, \infty)$, we now introduce the Banach space

$$\ell_p(E) := \left\{ (u_j) \in E^{\mathbb{N}} \, ; \ \|(u_j)\|_{\ell_p(E)} := \left[\sum_{j \in \mathbb{N}} \|u_j\|_E^p \right]^{1/p} < \infty \right\} .$$

The following interpolation result is a special case of [Tri78, 1.18.1 Theorem].

Lemma 3.7 *Let* $E_1 \hookrightarrow E_0$. *Then* $(\ell_p(E_0), \ell_p(E_1))_{\theta,p} \doteq \ell_p((E_0, E_1)_{\theta,p})$ *holds for* $p \in [1, \infty)$ *and* $\theta \in (0, 1)$.

3.2 Spatially constant coefficients

In the following, we consider the differential operator $\mathcal{A}_\alpha$, given by (3.1), for spatially constant $\alpha \in \mathcal{L}(E^n)$. Our aim is to formulate conditions on α, which guarantee that

$$\mathcal{A}_\alpha := -\mathrm{div} \circ \alpha \circ \mathrm{grad} = -\textstyle\sum_{i,j} \alpha_{ij}\, \partial_{ij}^2$$

lies in $\mathcal{H}(B_{p,q}^{s+2}, B_{p,q}^s)$ for $p, q \in [1,\infty)$ and $s \in \mathbb{R}$. We first introduce the *symbol*

$$\mathfrak{a}_\alpha \;:\; \mathbb{R}^n \longrightarrow \mathcal{L}(E)\ ,\ \xi \longmapsto \xi \cdot \alpha\xi = \textstyle\sum_{i,j=1}^n \xi_i\xi_j\, \alpha_{ij}$$

of $\mathcal{A}_\alpha$ and define the following subset of $\mathcal{L}(E^n)$.

Definition 3.8 Let $M \geq 1$ be given. Then $\mathcal{E}\ell\ell(E;M) = \mathcal{E}\ell\ell(M)$ denotes the class of all $\alpha \in \mathcal{L}(E^n)$ whose symbols $\mathfrak{a}_\alpha$ satisfy the condition

$$\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq 0\} \subset \rho(-\mathfrak{a}_\alpha(\xi))\ \ ,\ \xi \in \mathrm{S}^{n-1} := \{x \in \mathbb{R}^n\ ;\ |x| = 1\},$$

(where $\rho(-\mathfrak{a}_\alpha(\xi))$ is the resolvent set of $-\mathfrak{a}_\alpha(\xi)$), and

$$(1+|\lambda|)\left\|(\lambda + \mathfrak{a}_\alpha(\xi))^{-1}\right\|_{\mathcal{L}(E)} \leq M\ ,\ \mathrm{Re}\,\lambda \geq 0\,,\ \xi \in \mathrm{S}^{n-1}.$$

We now are able to prove a generation result for $\mathcal{A}_\alpha$ with constant $\alpha \in \mathcal{E}\ell\ell(M)$.

Proposition 3.9 *Assume that p, $q \in [1,\infty)$ and $s \in \mathbb{R}$. Then, for given $M \geq 1$ and $\omega > 0$, there exists a constant $\kappa = \kappa(M,\omega)$ such that*

$$\mathcal{A}_\alpha = -\mathrm{div} \circ \alpha \circ \mathrm{grad} \in \mathcal{H}(B_{p,q}^{s+2}, B_{p,q}^s; \kappa, \omega)$$

holds for all $\alpha \in \mathcal{E}\ell\ell(E;M)$ with $\|\alpha\|_{\mathcal{L}(E^n)} \leq M$.

Proof. Since $\xi \longmapsto \mathfrak{a}_\alpha(\xi)$ is a homogeneous polynomial of degree two, it follows that

$$\|\partial^\nu \mathfrak{a}_\alpha(\xi)\|_{\mathcal{L}(E)} \leq c\,(1+|\xi|)^{2-|\nu|}\ ,\ \xi \in \mathbb{R}^n\,. \tag{3.5}$$

Setting $\xi' := \xi/|\xi|$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and recalling the assumption $\alpha \in \mathcal{E}\ell\ell(E;M)$, we moreover obtain

$$\begin{aligned}\left\|(\lambda + \mathfrak{a}_\alpha(\xi))^{-1}\right\|_{\mathcal{L}(E)} &= |\xi|^{-2}\left\|(\lambda|\xi|^{-2} + \mathfrak{a}_\alpha(\xi'))^{-1}\right\|_{\mathcal{L}(E)} \\ &\leq M\,(|\xi|^2 + |\lambda|)^{-1}\ ,\ \xi \in \mathbb{R}^n \setminus \{0\}\,,\ \mathrm{Re}\lambda \geq 0\,.\end{aligned}$$

Thus, for each $\omega > 0$ there exists a $c(\omega) > 0$ such that

$$\left(1+|\lambda|+|\xi|^2\right)\left\|(\lambda + \mathfrak{a}_\alpha(\xi))^{-1}\right\|_{\mathcal{L}(E)} \leq c(\omega)\,M\ ,\ \xi \in \mathbb{R}^n \setminus \{0\}\,,\ \mathrm{Re}\lambda \geq \omega\,. \tag{3.6}$$

Estimates (3.5) and (3.6) enable us now to apply [Ama97, Theorems 7.1 and 7.3] to the symbol $\mathfrak{a}_\alpha$. This shows that $\mathfrak{a}_\alpha$ is a Fourier multiplier for $B^s_{p,q}$, and $\mathcal{F}^{-1}\mathfrak{a}_\alpha\mathcal{F} = \mathfrak{a}_\alpha(D) = \mathcal{A}_\alpha$ has the asserted property. □

The remainder of Section 3.2 is devoted to an example which arises from the diffusion operator in Problem 2.1. Namely, we consider the mapping

$$\alpha = \alpha[\mathbf{a},\mathbf{b}] \,:\, u \longmapsto \left[y \longmapsto \mathbf{a}(y)u(y) + \frac{1}{y}\int_Y \mathbf{b}(y,y')u(y')\,y'\,\mathrm{d}\mu(y')\right]$$

on E^n with $E := L_1(Y;(1+y)\,\mathrm{d}\mu(y))$, where

$$\mathbf{a} \in E_{\mathbf{a}} := L_\infty(Y,\mathbb{R}^{n\times n}_{\mathrm{sym}}) \;\text{ and }\; \mathbf{b} \in E_{\mathbf{b}} := L_1(Y, L_\infty(Y,\mathbb{R}^{n\times n}_{\mathrm{sym}});(1+1/y)\,\mathrm{d}\mu(y)).$$

These conditions ensure that $\alpha = \alpha[\mathbf{a},\mathbf{b}]$ belongs to $\mathcal{L}(E^n)$, and satisfies

$$\|\alpha\|_{\mathcal{L}(E^n)} \leq \|\mathbf{a}\|_{E_{\mathbf{a}}} + \|\mathbf{b}\|_{E_{\mathbf{b}}}.$$

Definition 3.10 Let $a_0 > 0$ and $q \in [0,1)$ be given. Then $\alpha = \alpha[\mathbf{a},\mathbf{b}]$ with $(\mathbf{a},\mathbf{b}) \in E_{\mathbf{a}} \times E_{\mathbf{b}}$ is said to be of class $\mathcal{ell}(a_0,q)$, if

$$\underline{\mathbf{a}}(y) := \min\left\{\xi\cdot\mathbf{a}(y)\xi\,;\xi\in \mathrm{S}^{n-1}\right\} \geq a_0 \tag{3.7}$$

holds for almost all $y \in Y$, and $\overline{\mathbf{b}}(y,y') := \max\left\{\xi\cdot\mathbf{b}(y,y')\xi\,;\xi\in\mathrm{S}^{n-1}\right\}$ satisfies

$$\int_Y \frac{1}{\underline{\mathbf{a}}(y)}\left\|\overline{\mathbf{b}}(y,\cdot)\right\|_{L_\infty(Y)}(1+1/y)\,\mathrm{d}\mu(y) \leq q\,. \tag{3.8}$$

We now prove that $\mathcal{ell}(a_0,q)$ is contained in a class $\mathcal{Ell}(L_1(Y;(1+y)\,\mathrm{d}\mu(y));M)$.

Proposition 3.11 *For given $a_0 > 1$ and $q \in [0,1)$, there exists a constant $M = M(a_0,q)$ such that $\mathcal{ell}(a_0,q) \hookrightarrow \mathcal{Ell}(L_1(Y;(1+y)\,d\mu(y));M)$.*

PROOF. We represent the symbol of α by $\mathfrak{a}_\alpha = \mathfrak{a}_{\mathbf{a}} + \mathfrak{a}_{\mathbf{b}}$ with

$$\mathfrak{a}_{\mathbf{a}}(\xi)u := \{y \longmapsto [\xi\cdot\mathbf{a}(y)\xi]u(y)\}\,,\xi\in\mathbb{R}^n,$$

and

$$\mathfrak{a}_{\mathbf{b}}(\xi)u := \left\{y \longmapsto \frac{1}{y}\int_Y [\xi\cdot\mathbf{b}(y,y')\xi]u(y')\,y'\,\mathrm{d}\mu(y')\right\},\xi\in\mathbb{R}^n,$$

where the conditions $\mathbf{a}\in E_{\mathbf{a}}$, $\mathbf{b}\in E_{\mathbf{b}}$ ensure that $\mathfrak{a}_{\mathbf{a}}(\xi)$, $\mathfrak{a}_{\mathbf{b}}(\xi)\in\mathcal{L}(E)$ hold for $\xi\in\mathbb{R}^n$. Let us first consider $\mathfrak{a}_{\mathbf{a}}$. Using (3.7) we obtain the estimate

$$\begin{aligned}|\lambda + \xi\cdot\mathbf{a}(y)\xi| &\geq c\,(|\lambda| + \xi\cdot\mathbf{a}(y)\xi)\\ &\geq c\,(a_0 + |\lambda|) \geq M(a_0)\,(1+|\lambda|)\,,\ \mathrm{Re}\lambda\geq 0\,,\ \xi\in\mathrm{S}^{n-1},\end{aligned}$$

which implies

$$\{\mathbf{a} : u \longmapsto [y \longmapsto \mathbf{a}(y)u(y)]\} \in \mathcal{E}\ell\ell(M(a_0)). \tag{3.9}$$

Next the operators $Q_\lambda(\xi) := (\lambda + \mathfrak{a}_\mathbf{a}(\xi))^{-1}\mathfrak{a}_\mathbf{b}(\xi) \in \mathcal{L}(E)$, $\mathrm{Re}\lambda \geq 0$, $\xi \in \mathrm{S}^{n-1}$, represented by

$$Q_\lambda(\xi)u : y \longmapsto \frac{1}{y\,[\lambda + \xi \cdot \mathbf{a}(y)\xi]} \int_Y [\xi \cdot \mathbf{b}(y, y')\xi] u(y')\, y' \,\mathrm{d}\mu(y'),$$

are considered. From condition (3.8) it follows that

$$\|Q_\lambda(\xi)u\|_E \leq \int_Y \int_Y \left| \frac{\overline{\mathbf{b}}(y, y')}{\underline{\mathbf{a}}(y)}\, u(y') \right| (1 + 1/y)\, y' \,\mathrm{d}\mu(y') \,\mathrm{d}\mu(y) \leq q\, \|u\|_E$$

holds for $\mathrm{Re}\lambda \geq 0$, $\xi \in \mathrm{S}^{n-1}$, and $u \in E$. Hence we obtain $I + Q_\lambda(\xi) \in \mathcal{L}\mathrm{aut}(E)$, and therefore,

$$\lambda + \mathfrak{a}_\alpha(\xi) = (\lambda + \mathfrak{a}_\mathbf{a}(\xi))\,(I + Q_\lambda(\xi)) \in \mathcal{L}\mathrm{aut}(E)\,,\ \mathrm{Re}\lambda \geq 0\,,\ \xi \in \mathrm{S}^{n-1}.$$

Thus, in view of (3.9), the resolvents satisfy the estimate

$$\begin{aligned} \left\|(\lambda + \mathfrak{a}_\alpha(\xi))^{-1}\right\|_{\mathcal{L}(E)} &\leq \left\|(\lambda + \mathfrak{a}_\mathbf{a}(\xi))^{-1}\right\|_{\mathcal{L}(E)} \left\|(I + Q_\lambda(\xi))^{-1}\right\|_{\mathcal{L}(E)} \\ &\leq \frac{M(a_0)}{1-q}\,(1 + |\lambda|)^{-1}\,,\ \mathrm{Re}\lambda \geq 0\,,\ \xi \in \mathrm{S}^{n-1}\,, \end{aligned}$$

which proves our assertion. □

3.3 Spatially variable coefficients

We now consider the general case of operator-valued coefficients α, which depend on the spatial variable $x \in \mathbb{R}^n$.

According to Proposition 3.5 (S_3), the pointwise product

$$(\alpha, v) \longmapsto \alpha \bullet v := [x \longmapsto \alpha(x)v(x)]$$

is a bilinear continuous mapping from $\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n)) \times B^s_{p,q}(\mathbb{R}^n, E^n)$ into $B^s_{p,q}(\mathbb{R}^n, E^n)$, provided $0 < s < t$. In particular, this implies

$$\{\alpha \longmapsto \Lambda_\alpha := [v \longmapsto \alpha \bullet v]\} \in \mathcal{L}(\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(B^s_{p,q}(\mathbb{R}^n, E^n))) \tag{3.10}$$

for $0 < s < t$. As a consequence, we obtain the following result.

Lemma 3.12 *Let p, $q \in [1, \infty]$, $t \in (-1, \infty)$ and $s \in (-1, t)$. Then,*

$$[\alpha \longmapsto \mathcal{A}_\alpha = -\mathrm{div} \circ \Lambda_\alpha \circ \mathrm{grad}] \in \mathcal{L}(\mathrm{BUC}^{t+1}(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(B^{s+2}_{p,q}, B^s_{p,q}))\,.$$

PROOF. Since $\operatorname{grad} \in \mathcal{L}(B^{s+2}_{p,q}, B^{s+1}_{p,q}(\mathbb{R}^n, E^n))$ and $\operatorname{div} \in \mathcal{L}(B^{s+1}_{p,q}(\mathbb{R}^n, E^n), B^s_{p,q})$ (cf. [Ama97, Theorem 6.2]), our assertion follows from (3.10). □

Remark 3.13 In addition to Λ_α we introduce the mappings

$$\tilde{\Lambda}_\alpha v := \Big[x \longmapsto \textstyle\sum_{i,j=1}^n \alpha_{ij}(x) \bullet v_{ij}(x)\Big] \ , \ \hat{\Lambda}_\beta w := \Big[x \longmapsto \textstyle\sum_{j=1}^n \beta_j(x) \bullet w_j(x)\Big]$$

for functions α_{ij}, $\beta_j : \mathbb{R}^n \longrightarrow \mathcal{L}(E)$ and v_{ij}, $w_j : \mathbb{R}^n \longrightarrow E$. Similarly, as above, it follows that

$$\Big[\alpha \longmapsto \tilde{\Lambda}_\alpha\Big] \in \mathcal{L}(\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(B^s_{p,q}(\mathbb{R}^n, E^{n\times n}), B^s_{p,q}))$$

and

$$\Big[\beta \longmapsto \hat{\Lambda}_\beta\Big] \in \mathcal{L}(\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n, E)), \mathcal{L}(B^s_{p,q}(\mathbb{R}^n, E^n), B^s_{p,q}))$$

hold for $p, q \in [1, \infty]$ and $0 < s < t$. Consequently, we are able to rewrite $\mathcal{A}_\alpha$ (in non-divergence form) as

$$\mathcal{A}_\alpha = -\tilde{\Lambda}_\alpha \circ D^2 - \hat{\Lambda}_{\operatorname{div}\alpha} \circ \operatorname{grad}$$

with $D^2 u := (\partial^2_{ij} u)$, $\operatorname{div} \alpha := (\sum_{i=1}^n \partial_i \alpha_{ij})$, where

$$\left.\begin{aligned} &\Big[\alpha \longmapsto \tilde{\Lambda}_\alpha \circ D^2\Big] \in \mathcal{L}(\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(B^{s+2}_{p,q}, B^s_{p,q})), \\ &\Big[\alpha \longmapsto \hat{\Lambda}_{\operatorname{div}\alpha} \circ \operatorname{grad}\Big] \in \mathcal{L}(\mathrm{BUC}^{t+1}(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(B^{s+1}_{p,q}, B^s_{p,q})), \end{aligned}\right\} \tag{3.11}$$

provided $0 < s < t$.

The aim of our following considerations is to extend the generation result of Proposition 3.9 to the case $\alpha \in \mathrm{BUC}^{1+t}(\mathbb{R}^n, \mathcal{E}\ell\ell(M))$. For that purpose, we fix an arbitrary $\varepsilon \in (0, 1]$ and define the open covering

$$\mathcal{U} = \{U_j \ ; \ j \in \mathbb{N}\} := \{x_j + \varepsilon\, Q \ ; \ x_j \in \varepsilon\, \mathbb{Z}^n\}$$

of $\mathbb{R}^n$, where Q denotes the open ball $\{x \in \mathbb{R}^n \, ; |x| < \sqrt{n}\,\}$, and the enumeration of the U_j is chosen in such a manner that $j \geq k$ implies $|x_j| \geq |x_k|$. Note that $\mathcal{U}$ has finite multiplicity, i.e.

$$\exists\, \kappa = \kappa(n) \ : \ \operatorname{card}\{j \in \mathbb{N};\ x \in U_j\} \leq \kappa \ , \ x \in \mathbb{R}^n. \tag{3.12}$$

For each $j \in \mathbb{N}$, we introduce the smooth diffeomorphism

$$\varphi_j \ : \ U_j \longleftrightarrow Q \, , \, x \longmapsto (x - x_j)/\varepsilon \, .$$

Moreover, let $\pi \in C^\infty(\mathbb{R}^n, [0, 1])$ be a function with $\operatorname{supp}(\pi) \subset\subset Q$ and $\pi = 1$ on $Q/2$. Then we define

$$\pi_j := \pi \circ \varphi_j \left[\textstyle\sum_{i\in\mathbb{N}} (\pi \circ \varphi_i)^2\right]^{-1/2} .$$

It is easily verified that

$$\pi_j \in C^\infty(\mathbb{R}^n, [0,1]) \, , \, \mathrm{supp}\,(\pi_j) \subset\subset U_j \, , \, j \in \mathbb{N}, \ \text{ and } \sum_{j\in\mathbb{N}} \pi_j^2(x) = 1 \, , x \in \mathbb{R}^n .$$

Moreover, for each $m \in \mathbb{N}$, there exists a constant $c = c(m)$ such that

$$\sup_{x\in\mathbb{R}^n} |\partial^\nu \pi_j(x)| \le c\,\varepsilon^{-|\nu|} \, , \, j \in \mathbb{N}, \ |\nu| \le m. \tag{3.13}$$

Finally, we choose $\chi \in C^\infty(\mathbb{R}^n, [0,1])$ with $\mathrm{supp}\,(\chi) \subset\subset Q$ and $\chi = 1$ on $\mathrm{supp}\,(\pi)$. Then it follows that the functions $\chi_j := \chi \circ \varphi_j \in C^\infty(\mathbb{R}^n, [0,1])$ satisfy

$$\mathrm{supp}\,(\chi_j) \subset\subset U_j \ , \ \chi_j = 1 \ \text{ on } \ \mathrm{supp}\,(\pi_j) \ , \ \pi_j \le \chi_j \le 1 \ , \ j \in \mathbb{N}.$$

Lemma 3.14 *Let $p \in [1,\infty)$ and $s \in [-1,\infty)$. Then,*

$$r \, : \, \left(W_p^s\right)^{\mathbb{N}} \longrightarrow W_p^s \, , \, (u_j) \longmapsto \sum_{j\in\mathbb{N}} \pi_j \, u_j$$

is a retraction in $\mathcal{L}(\ell_p(W_p^s), W_p^s)$. A co-retraction is given by

$$r^c \, : \, W_p^s \longrightarrow \left(W_p^s\right)^{\mathbb{N}} \, , \, u \longmapsto (\pi_j \, u),$$

i.e., $r^c \in \mathcal{L}(W_p^s, \ell_p(W_p^s))$ and $r \circ r^c = \mathrm{id}_{W_p^s}$.

Proof. From [AHS94, Proofs of Lemma 9.1 and Corollary 9.2] we already know that

$$r \in \mathcal{L}(\ell_p(W_p^s), W_p^s) \ \text{ and } \ r^c \in \mathcal{L}(W_p^s, \ell_p(W_p^s)) \tag{3.14}$$

hold for $p \in [1,\infty)$ and $s \in [0,\infty)$. The aim of the following steps is to extend this result to $s = -1$, and then by interpolation to $s \in [-1, 0)$.

First we consider the mapping r. Let (u_j) be a sequence in $\ell_p(W_p^{-1})$. According to the definition of W_p^{-1}, there are functions $u_{j,\nu} \in L_p$ such that

$$u_j = \sum_{|\nu|\le 1} (-1)^{|\nu|} \, \partial^\nu u_{j,\nu} \, , \, j \in \mathbb{N}. \tag{3.15}$$

Hence the products $\pi_j \, u_j$ may be represented by

$$\pi_j \, u_j = \sum_{|\nu|\le 1} (-1)^{|\nu|} \, \partial^\nu v_{j,\nu} \tag{3.16}$$

with

$$v_{j,\nu} = \begin{cases} \pi_j \, u_{j,\nu} + \sum_{|\eta|=1} \partial^\eta \pi_j \, u_{j,\eta} & , \, |\nu| = 0 \\ \pi_j \, u_{j,\nu} & , \, |\nu| = 1 \end{cases} \in L_p$$

(cf. Remark 3.6). Thus, we obtain the identity

$$\sum_{j=k}^{k+\ell} \pi_j \, u_j = \sum_{|\nu|\le 1} (-1)^{|\nu|} \, \partial^\nu \sum_{j=k}^{k+\ell} v_{j,\nu} \, , \, k, \ell \in \mathbb{N},$$

which shows that $\sum_{j=k}^{k+\ell} \pi_j\, u_j \in W_p^{-1}$ holds for $k,\ell \in \mathbb{N}$. Moreover, the finite multiplicity (3.12) of $\mathcal{U}$, and inequality (3.13) enable us to derive the uniform estimate

$$\Big\| \sum_{j=k}^{k+\ell} \pi_j\, u_j \Big\|_{W_p^{-1}}^p \le \sum_{|\nu|\le 1} \Big\| \sum_{j=k}^{k+\ell} v_{j,\nu} \Big\|_{L_p}^p \le \kappa^{p-1} \sum_{|\nu|\le 1} \sum_{j=k}^{k+\ell} \|v_{j,\nu}\|_{L_p}^p$$
$$\le c(\varepsilon) \sum_{j=k}^{k+\ell} \sum_{|\nu|\le 1} \|u_{j,\nu}\|_{L_p}^p\ ,\ k,\ell \in \mathbb{N}.$$

Since the function $u_{j,\nu} \in L_p$, representing u_j in (3.15), were chosen arbitrarily, we arrive at

$$\Big\| \sum_{j=k}^{k+\ell} \pi_j\, u_j \Big\|_{W_p^{-1}} \le c(\varepsilon) \Big\{ \sum_{j=k}^{k+\ell} \|u_j\|_{W_p^{-1}}^p \Big\}^{1/p}\ ,\ k,\ell \in \mathbb{N}.$$

Hence, $r(u_j) = \sum_{j\in\mathbb{N}} \pi_j\, u_j$ exists in W_p^{-1}, and satisfies the estimate

$$\|r(u_j)\|_{W_p^{-1}} \le c(\varepsilon)\ \|(u_j)\|_{\ell_p(W_p^{-1})}\ .$$

This proves $r \in \mathcal{L}(\ell_p(W_p^{-1}), W_p^{-1})$.

Let us consider now the mapping r^c. We choose $u \in W_p^{-1}$ and $u_\nu \in L_p$ such that

$$u = \sum_{|\nu|\le 1} (-1)^{|\nu|}\, \partial^\nu u_\nu\ . \tag{3.17}$$

Then the products $\pi_j\, u$ are represented by formula (3.16) (with u and u_ν instead of u_j and $u_{j,\nu}$, respectively), so that estimate (3.13) leads to

$$\|\pi_j\, u\|_{W_p^{-1}}^p \le c \sum_{|\nu|\le 1} \|v_{j,\nu}\|_{L_p}^p \le c(\varepsilon) \sum_{|\nu|\le 1} \|\chi_j\, u_\nu\|_{L_p}^p\ ,\ j \in \mathbb{N}.$$

Recalling the finite multiplicity (3.12) of $\mathcal{U}$, we consequently obtain

$$\|r^c u\|_{\ell_p(W_p^{-1})}^p \le c \sum_{|\nu|\le 1} \sum_{j\in\mathbb{N}} \|\chi_j\, u_\nu\|_{L_p}^p \le c\,\kappa \sum_{|\nu|\le 1} \|u_\nu\|_{L_p}^p\ .$$

Since the representation (3.17) of u was chosen arbitrarily, this leads to

$$\|r^c u\|_{\ell_p(W_p^{-1})} \le c\, \|u\|_{W_p^{-1}}\ ,$$

and therefore to $r^c \in \mathcal{L}(W_p^{-1}, \ell_p(W_p^{-1}))$.

Using $(W_p^{-1}, L_p)_{s+1,p} \doteq W_p^s$ for $s \in (-1,0)$ (cf. [Ama, VII.3.2.2 Theorem]), as well as Lemma 3.7, we see that (3.14) holds for $p \in [1,\infty)$ and $s \in [-1,\infty)$. Since $r \circ r^c = \mathrm{id}_{W_p^s}$ is obvious, this completes our proof. □

Remark 3.15 Analogous to the proof of Lemma 3.14, the following statement can be derived. *Let $p \in [1,\infty)$ and $s \in [-1,\infty)$. Then,*

$$[u \longmapsto (\chi_j\, u)] \in \mathcal{L}(W_p^s, \ell_p(W_p^s))$$

and

$$\Big[(u_j) \longmapsto \sum_{j\in\mathbb{N}} \chi_j\, u_j \Big] \in \mathcal{L}(\ell_p(W_p^s), W_p^s).$$

We now are able to derive the main result of Section 3.

Theorem 3.16 *Assume that $p,\ q \in [1,\infty)$, $t \in (-1,1]$ and $s \in (-1,t)$.*

Then, for a given $M \geq 1$, there exist $\kappa \geq 1$ and $\omega > 0$ such that

$$[\alpha \longmapsto \mathcal{A}_\alpha] \in C^{1-}(\mathrm{BUC}^{t+1}(\mathbb{R}^n, \mathcal{E}\ell\ell(M)), \mathcal{H}(B^{s+2}_{p,q}, B^s_{p,q}; \kappa, \omega)) .$$

Proof. **(a)** Let ρ be a function in $\mathrm{BUC}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\rho(x) = \begin{cases} x, & \text{if } x \in \operatorname{supp}(\pi) \subset\subset Q, \\ x/|x|, & \text{if } x \in \mathbb{R}^n \setminus Q. \end{cases}$$

We set $\rho_j : x \longmapsto x_j + \varepsilon\, \rho(\varphi_j(x))$ and consider $\mathcal{A}_{\alpha_j}$ with $\alpha_j := \alpha \circ \rho_j$ for $j \in \mathbb{N}$. The aim of this part of our proof is to show that $\mathcal{A}_{\alpha_j} \in \mathcal{H}(W^{s+2}_p, W^s_p; \kappa_*, \omega_*)$, where κ_* and ω_* are independent of j.

We first consider the operators $\mathcal{A}_{\alpha^0_j}$, whose (spatially constant) coefficients are defined by $\alpha^0_j := \alpha \circ \rho^0_j$ with $\rho^0_j : x \longmapsto \rho_j(x_j) = x_j$. Proposition 3.9 ensures that, for a given $\omega_0 > 0$, there exists $\kappa_0 = \kappa_0(M, \omega_0)$ such that

$$\mathcal{A}_{\alpha^0_j} \in \mathcal{H}(B^{s+2}_{p,q}, B^s_{p,q}; \kappa_0, \omega_0)\ ,\ j \in \mathbb{N}. \tag{3.18}$$

To derive the desired result for $\mathcal{A}_{\alpha_j}$ we employ the perturbation Lemma 3.3. This requires an estimate of $\mathcal{A}_{\alpha_j} - \mathcal{A}_{\alpha^0_j}$ which is uniform with respect to $j \in \mathbb{N}$. Setting $r := 1 \wedge (1+t)$ and fixing some $r_0 \in (0,r)$, we obtain

$$\left\| \left(\alpha_j - \alpha^0_j\right)(x) \right\|_{\mathcal{L}(E^n)} \leq c\, |\rho_j(x) - x_j|^r = c\, \varepsilon^r\, |\rho(\varphi_j(x))|^r \leq c\, \varepsilon^r$$

as well as

$$\begin{aligned} \left\| \left(\alpha_j - \alpha^0_j\right)(x) - \left(\alpha_j - \alpha^0_j\right)(y) \right\|_{\mathcal{L}(E^n)} &\leq c\, \varepsilon^r\, |\rho_j(x) - \rho_j(y)|^r \\ &\leq c\, \varepsilon^r\, |\rho(\varphi_j(x)) - \rho(\varphi_j(y))|^{r_0}\, \left(|\rho(\varphi_j(x))| + |\rho(\varphi_j(y))|\right)^{r-r_0} \\ &\leq c\, \varepsilon^r\, |\varphi_j(x) - \varphi_j(y)|^{r_0} = c\, \varepsilon^{r-r_0}\, |x-y|^{r_0} \end{aligned}$$

for $x, y \in \mathbb{R}^n$ and $j \in \mathbb{N}$. Both inequalities lead to the estimate

$$\left\| \alpha_j - \alpha^0_j \right\|_{\mathrm{BUC}^{r_0}(\mathbb{R}^n, \mathcal{L}(E^n))} \leq c\, \varepsilon^{r-r_0}\ ,\ j \in \mathbb{N}, \tag{3.19}$$

which enables us to investigate the differences $\mathcal{A}_{\alpha_j} - \mathcal{A}_{\alpha^0_j}$. We first consider the case $t \in (0,1]$. According to Remark 3.13,

$$\mathcal{A}_{\alpha_j} - \mathcal{A}_{\alpha^0_j} = \mathcal{A}_{\alpha_j - \alpha^0_j} = -\tilde{\Lambda}_{\alpha_j - \alpha^0_j} \circ D^2 - \hat{\Lambda}_{\operatorname{div}\alpha_j} \circ \operatorname{grad}\ .$$

We fix $t_0 \in (s,t)$ with $t_0 > 0$ and apply (3.11). In view of the inequality (3.19) and the interpolation result $(B^s_{p,q}, B^{s+2}_{p,q})_{1/2,q} \doteq B^{s+1}_{p,q}$, this yields

$$\begin{aligned} \left\| \mathcal{A}_{\alpha_j - \alpha_j^0} u \right\|_{B^s_{p,q}} &\leq c \left\| \alpha_j - \alpha_j^0 \right\|_{\mathrm{BUC}^{t_0}(\mathbb{R}^n, \mathcal{L}(E^n))} \|u\|_{B^{s+2}_{p,q}} \\ &\quad + c \left\| \alpha_j \right\|_{\mathrm{BUC}^{t+1}(\mathbb{R}^n, \mathcal{L}(E^n))} \|u\|_{B^{s+1}_{p,q}} \\ &\leq c\, \varepsilon^{t-t_0} \|u\|_{B^{s+2}_{p,q}} + c(\varepsilon) \|u\|_{B^{s+1}_{p,q}} \\ &\leq c\, \varepsilon^{t-t_0} \|u\|_{B^{s+2}_{p,q}} + c(\varepsilon) \|u\|_{B^{s}_{p,q}} \end{aligned} \tag{3.20}$$

for $u \in B^{s+2}_{p,q}$, $j \in \mathbb{N}$, and $\varepsilon > 0$. In the case $t \in (-1, 0]$, we represent the difference $\mathcal{A}_{\alpha_j} - \mathcal{A}_{\alpha_j^0}$ as $\mathcal{A}_{\alpha_j - \alpha_j^0} = -\mathrm{div} \circ \Lambda_{\alpha_j - \alpha_j^0} \circ \mathrm{grad}$. For $t_0 \in (s,t)$, Lemma 3.12 and inequality (3.19) imply

$$\left\| \mathcal{A}_{\alpha_j - \alpha_j^0} \right\|_{\mathcal{L}(B^{s+2}_{p,q}, B^s_{p,q})} \leq c \left\| \alpha_j - \alpha_j^0 \right\|_{\mathrm{BUC}^{t_0+1}(\mathbb{R}^n, \mathcal{L}(E^n))} \leq c\, \varepsilon^{t-t_0} \ , \ j \in \mathbb{N}, \ \varepsilon > 0 \, ,$$

so that (3.20) remains valid.

Recalling (3.18) and fixing $\varepsilon < 1/\kappa_0$ in (3.20), we now see that Lemma 3.3 leads to the desired statement: there are $\kappa_* \geq 1$ and $\omega_* > 0$ such that

$$\mathcal{A}_{\alpha_j} = \mathcal{A}_{\alpha_j^0} + \mathcal{A}_{\alpha_j - \alpha_j^0} \in \mathcal{H}(B^{s+2}_{p,q}, B^s_{p,q}; \kappa_*, \omega_*) \ , \ j \in \mathbb{N}. \tag{3.21}$$

(b) Next we consider the mapping

$$\mathbb{A} : \left(W^{s+2}_p\right)^{\mathbb{N}} \longrightarrow \left(W^s_p\right)^{\mathbb{N}} \ , \ (u_j) \longmapsto (\mathcal{A}_{\alpha_j} u_j) \ .$$

Since (3.21) holds uniformly with respect to $j \in \mathbb{N}$, it can easily be verified that

$$\mathbb{A} \in \mathcal{H}(\ell_p(W^{s+2}_p), \ell_p(W^s_p); \kappa_*, \omega_*).$$

(c) In the following, we construct a left inverse to $\lambda + \mathcal{A}_\alpha$ for $\mathrm{Re}\lambda \geq \omega_L$ with a suitable $\omega_L > 0$. For that purpose,

$$B : W^{s+2}_p \longrightarrow \left(W^s_p\right)^{\mathbb{N}} \ , \ u \longmapsto B_j u := \big(\pi_j \, \mathcal{A}_\alpha u - \mathcal{A}_{\alpha_j}(\pi_j \, u)\big)$$

is considered. In view of $\alpha - \alpha_j|_{\mathrm{supp}\,(\pi_j)} = 0$, it follows that

$$B_j u := (\mathrm{div} \circ \Lambda_{\alpha_j})(u \, \mathrm{grad}\pi_j) + \mathrm{grad}\pi_j \cdot [(\Lambda_\alpha \circ \mathrm{grad})u] \ , \ j \in \mathbb{N}.$$

Since (3.10) implies

$$[\alpha \longmapsto \mathrm{div} \circ \Lambda_\alpha] \in \mathcal{L}(\mathrm{BUC}^{t+1}(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(W^{s+1}_p(\mathbb{R}^n, E^n), W^s_p))$$

as well as

$$[\alpha \longmapsto \Lambda_\alpha \circ \mathrm{grad}] \in \mathcal{L}(\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n)), \mathcal{L}(W^{s+1}_p, W^s_p(\mathbb{R}^n, E^n))),$$

and ε is fixed (cf. part (a)), we obtain

$$\begin{aligned}\|B_j u\|_{W_p^s} &\le \big\|(\mathrm{div}\circ\Lambda_{\alpha_j})(u\,\mathrm{grad}\pi_j)\big\|_{W_p^s} + c\,\|(\Lambda_\alpha\circ\mathrm{grad})\,(\chi_j\,u)\|_{W_p^s(\mathbb{R}^n,E^n)}\\ &\le c\left(\|\alpha_j\|_{\mathrm{BUC}^{t+1}(\mathbb{R}^n,\mathcal{L}(E^n))}+\|\alpha\|_{\mathrm{BUC}^{t}(\mathbb{R}^n,\mathcal{L}(E^n))}\right)\|\chi_j\,u\|_{W_p^{s+1}}\\ &\le c\,\|\chi_j\,u\|_{W_p^{s+1}}\ ,\ u\in W_p^{s+1}\,,\ j\in\mathbb{N}.\end{aligned}\tag{3.22}$$

According to Remark 3.15 this leads to

$$\|Bu\|_{\ell_p(W_p^s)}\le c\,\|(\chi_j\,u)\|_{\ell_p(W_p^{s+1})}\le c\,\|u\|_{W_p^{s+1}}\ ,\ u\in W_p^{s+1}\,,$$

so that $B\in\mathcal{L}(W_p^{s+1},\ell_p(W_p^s))$. Thus, the perturbation result of Corollary 3.4 ensures that $\mathbb{A}+B\circ r\in\mathcal{H}(\ell_p(W_p^{s+2}),\ell_p(W_p^s);\kappa_L,\omega_L)$ holds for a $\kappa_L\ge 1$ and an $\omega_L>0$. In particular, this implies

$$\lambda+\mathbb{A}+B\circ r\in\mathcal{L}\mathrm{is}(\ell_p(W_p^{s+2}),\ell_p(W_p^s))\ ,\ \mathrm{Re}\lambda\ge\omega_L\,,$$

so that the operator

$$L_\lambda := r\circ(\lambda+\mathbb{A}+B\circ r)^{-1}\circ r^c\in\mathcal{L}(W_p^s,W_p^{s+2})\ ,\ \mathrm{Re}\lambda\ge\omega_L\,,$$

can be defined. As a consequence of

$$\begin{aligned}\pi_j\,(\lambda+\mathcal{A}_\alpha)u &= (\lambda+\mathcal{A}_{\alpha_j})(\pi_j u)+\pi_j\,\mathcal{A}_\alpha u-\mathcal{A}_{\alpha_j}(\pi_j u)\\ &= (\lambda+\mathcal{A}_{\alpha_j})(\pi_j u)+(Bu)_j\ ,\ u\in W_p^{s+2}\,,\ j\in\mathbb{N},\end{aligned}$$

we moreover obtain $r^c\circ(\lambda+\mathcal{A}_\alpha)=(\lambda+\mathbb{A}+B\circ r)\circ r^c$. Thus, L_λ has the desired property $L_\lambda\circ(\lambda+\mathcal{A}_\alpha)=r\circ r^c=\mathrm{id}_{W_p^{s+2}}$, $\mathrm{Re}\lambda\ge\omega_L$.

(d) Analogous to the preceding part of our proof, a right inverse to $\lambda+\mathcal{A}_\alpha$ can be constructed. We consider the operator

$$C\ :\ \left(W_p^{s+2}\right)^{\mathbb{N}}\longrightarrow W_p^s\,,\ (u_j)\longmapsto\textstyle\sum_{j\in\mathbb{N}}\left[\mathcal{A}_\alpha(\pi_j\,u_j)-\pi_j\,\mathcal{A}_{\alpha_j}u_j\right]\ ,$$

which can be rewritten as $C:(u_j)\longmapsto\sum_{j\in\mathbb{N}}C_j u_j$ with

$$C_j u := -(\mathrm{div}\circ\Lambda_\alpha)(u\,\mathrm{grad}\pi_j)-\mathrm{grad}\pi_j\cdot\left[(\Lambda_{\alpha_j}\circ\mathrm{grad})u\right]\ ,\ j\in\mathbb{N}.$$

Analogous to (3.22) we derive

$$\|C_j u\|_{W_p^s}\le c\,\|\chi_j\,u\|_{W_p^{s+1}}\le c\,\|u\|_{W_p^{s+1}}\ ,\ u\in W_p^{s+1}\,,\ j\in\mathbb{N},$$

so that, by finite multiplicity of $\mathcal{U}$,

$$\|C(u_j)\|_{W_p^s}=\Big\|\textstyle\sum_{j\in\mathbb{N}}\chi_j\,(C_j u_j)\Big\|_{W_p^s}\le c\ \|(C_j u_j)\|_{\ell_p(W_p^s)}\le c\ \|(u_j)\|_{\ell_p(W_p^{s+1})}$$

follows for $(u_j) \in \ell_p(W_p^{s+1})$. Thus, C belongs to $\mathcal{L}(\ell_p(W_p^{s+1}), W_p^s)$. Since Lemma 3.7 implies $(\ell_p(W_p^s), \ell_p(W_p^{s+2}))_{1/2,p} \doteq \ell_p(W_p^{s+1})$, Corollary 3.4 ensures that

$$\mathbb{A} + r^c \circ C \in \mathcal{H}(\ell_p(W_p^{s+2}), \ell_p(W_p^s); \kappa_R, \omega_R)$$

holds for a $\kappa_R \geq 1$ and an $\omega_R > 0$. This enables us to define

$$R_\lambda := r \circ (\lambda + \mathbb{A} + r^c \circ C)^{-1} \circ r^c \in \mathcal{L}(W_p^s, W_p^{s+2}) \ , \ \mathrm{Re}\lambda \geq \omega_R \, .$$

In view of

$$\begin{aligned} \sum_{j\in\mathbb{N}} (\lambda + \mathcal{A}_\alpha)(\pi_j \, u_j) &= \sum_{j\in\mathbb{N}} \left[\pi_j \, (\lambda + \mathcal{A}_{\alpha_j}) u_j + \mathcal{A}_\alpha(\pi_j \, u_j) - \pi_j \, \mathcal{A}_{\alpha_j} u_j\right] \\ &= [r \circ (\lambda + \mathbb{A}) + C](u_j) \ , \ (u_j) \in \ell_p(W_p^{s+2}) \, , \end{aligned}$$

we moreover obtain the identity $(\lambda + \mathcal{A}_\alpha) \circ r = r \circ (\lambda + \mathbb{A} + r^c \circ C)$ which leads to the desired property $(\lambda + \mathcal{A}_\alpha) \circ R_\lambda = r \circ r^c = \mathrm{id}_{W_p^{s+2}}$, $\mathrm{Re}\lambda \geq \omega_R$.

(e) From the preceding parts of our proof it follows that

$$\lambda + \mathcal{A}_\alpha \in \mathcal{L}\mathrm{is}(W_p^{s+2}, W_p^s) \ , \ \mathrm{Re}\lambda \geq \omega := \omega_L \vee \omega_R \, ,$$

where $(\lambda + \mathcal{A}_\alpha)^{-1} = L_\lambda = R_\lambda$. Using $\mathbb{A} + B \circ r \in \mathcal{H}(\ell_p(W_p^{s+2}), \ell_p(W_p^s); \kappa_L, \omega_L)$, and Lemma 3.14, we obtain

$$\begin{aligned} |\lambda| \left\|(\lambda + \mathcal{A}_\alpha)^{-1} u\right\|_{W_p^s} &= |\lambda| \left\| \left[r \circ (\lambda + \mathbb{A} + B \circ r)^{-1} \circ r^c\right] u \right\|_{W_p^s} \\ &\leq c \, \kappa_L \, \|r^c u\|_{\ell_p(W_p^s)} \leq c \, \|u\|_{W_p^s} \ , \ u \in W_p^s \ , \ \mathrm{Re}\lambda \geq \omega \, . \end{aligned}$$

In view of Remark 3.2, this ensures that $\mathcal{A}_\alpha \in \mathcal{H}(W_p^{s+2}, W_p^s)$. Since this holds for an arbitrary $s \in (-1, t)$, we also have

$$\mathcal{A}_\alpha \in \mathcal{H}(W_p^{s+\delta+2}, W_p^{s+\delta}) \cap \mathcal{H}(W_p^{s-\delta+2}, W_p^{s-\delta}) \, ,$$

provided $0 < \delta < (t - s) \wedge (s + 1)$. Consequently, the interpolation result

$$\left(W_p^{s-\delta}, W_p^{s+\delta}\right)_{1/2,q} \doteq B_{p,q}^s \ , \ p, q \in [1, \infty) \ , \ s \in \mathbb{R} \, ,$$

leads to $\mathcal{A}_\alpha \in \mathcal{H}(B_{p,q}^{s+2}, B_{p,q}^s)$. This completes our proof. □

Remark 3.17 The basic idea of the construction of resolvents, carried out in parts (c)-(e) of the preceding proof, goes back to [AHS94, Proposition 3.2].

We now are able to extend the result of Theorem 3.16 to operators

$$\mathcal{A}_{\alpha,\beta} : u \longmapsto \{x \longmapsto -\mathrm{div}\,[\alpha(x)\,\mathrm{grad}\,u(x)] + \beta(x)\,\mathrm{grad}\,u(x)\}$$

with $\beta = (\beta_i) : \mathbb{R}^n \longrightarrow \mathcal{L}(E^n, E)$. Namely, the following can be proven.

Corollary 3.18 *Let* $p,\ q \in [1,\infty)$, $t \in (0,1]$ *and* $s \in (0,t)$.

Then, given $M \geq 1$, *there are* $\kappa \geq 1$ *and* $\omega > 0$ *such that* $[(\alpha,\beta) \longmapsto \mathcal{A}_{\alpha,\beta}]$ *is Lipschitz continuous from* $\mathrm{BUC}^{t+1}(\mathbb{R}^n, \mathcal{E}\ell\ell(M)) \times \mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n, E))$ *into* $\mathcal{H}(B^{s+2}_{p,q}, B^s_{p,q}; \kappa, \omega)$.

PROOF. Since Proposition 3.5 (S_3) ensures that $(\beta, v) \longmapsto [x \longmapsto \beta(x)\, v(x)]$ is a bilinear continuous mapping from $\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n, E)) \times B^s_{p,q}(\mathbb{R}^n, E^n)$ into $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n, E)$, we obtain

$$\{\beta \longmapsto [u \longmapsto \beta \operatorname{grad} u]\} \in \mathcal{L}(\mathrm{BUC}^t(\mathbb{R}^n, \mathcal{L}(E^n, E)), \mathcal{L}(B^{s+1}_{p,q}, B^s_{p,q})).$$

Hence our assertion follows by the perturbation result of Lemma 3.4. □

4 Existence and Uniqueness

In the following, we return to the coagulation-fragmentation model of Section 2. The aim is to show that Problem 2.1 possesses a unique maximal solution, where the sought particle size distribution function remains positive for positive initial values.

Our proof can be outlined as follows. Choosing suitable Banach spaces $\mathbb{E}_i = E_i \times F_i$, $i \in \{0,1\}$, and an admissible interpolation functor $(\cdot,\cdot)_\theta$, we will show (in Section 4.1) that there is some $\beta \in (0,1)$ such that the operator family $\mathfrak{A}(v)$, formally given by

$$v = \begin{pmatrix} u \\ \vec{v} \end{pmatrix} \longmapsto \mathfrak{A}(v) := \begin{pmatrix} -\operatorname{div} \alpha(u) \operatorname{grad} & 0 \\ 0 & -\nu\,\Delta \end{pmatrix},$$

has the property $[v \longmapsto \mathfrak{A}(v)] \in C^{1-}((\mathbb{E}_0, \mathbb{E}_1)_\beta, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0))$. Then (in Section 4.2) the mapping

$$F(v) = \begin{pmatrix} F_1(u, \vec{v}) \\ F_2(u, \vec{v}) \end{pmatrix} := \begin{pmatrix} c_\psi(u) + f_\phi(u) - \vec{v} \cdot \operatorname{grad} u \\ -P(\vec{v} \cdot \operatorname{grad})\, \vec{v} + P\vec{f}(u) \end{pmatrix},$$

where P denotes the Helmholtz projection, is investigated. For a suitable $\gamma \in (0,\beta)$, we derive $[v \longmapsto F(v)] \in C^{1-}((\mathbb{E}_0, \mathbb{E}_1)_\beta, (\mathbb{E}_0, \mathbb{E}_1)_\gamma)$. Now our coagulation-fragmentation model can be formulated as an abstract quasilinear Cauchy problem

$$\dot{v}(t) + \mathfrak{A}(v(t))v(t) = F(v(t))\ ,\ t > 0\,, \quad v(0) = v_0,$$

in $\mathbb{E}_0$, and an existence result, stated in [Ama93], may be applied. This ensures that the problem possesses a unique maximal solution $v \in C^1(J, \mathbb{E}_0) \cap C(J, \mathbb{E}_1)$ (on the maximal time interval $J = [0, T(v_0))$), provided $v_0 \in \mathbb{E}_1$.

4.1 Diffusion and Stokes operator

This section is devoted to the second order differential operators on the left-hand side of Problem 2.1.

First we consider $\mathcal{A}_{\alpha(w)}u : \mathbb{R}^n \longrightarrow E\,,\ x \longmapsto -\mathrm{div}\,[\alpha(x,w)\,\mathrm{grad}\,u(x)]$, where E denotes a Banach space of real-valued functions on Y (specified in 4.2), and $w \longmapsto \alpha(w) = \alpha(\cdot, w)$ is supposed to fulfil the following condition.

Assumption 4.1 There are constants σ and τ with $0 < \sigma \leq \tau < \sigma + 1 \leq 3$, such that $\alpha \in C^{1-}(\mathrm{BUC}^\tau(\mathbb{R}^n, E), \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{E}\ell\ell(E; M)))$.

We fix $s, p \in \mathbb{R}$ satisfying

$$-1 \vee (\tau - 2) < s < \sigma - 1 \ \ , \ \ p > n/(2 + s - \tau)\,, \tag{4.1}$$

and introduce the spaces

$$E_0 := B^s_{p,p} = B^s_{p,p}(\mathbb{R}^n, E) \ \ , \ \ E_1 := B^{s+2}_{p,p} = B^{s+2}_{p,p}(\mathbb{R}^n, E).$$

Moreover, E_θ is the real interpolation space $(E_0, E_1)_{\theta,p}$ of exponent $\theta \in (0,1)$, which can be characterised as

$$E_\theta \doteq B^{s+2\theta}_{p,p} \ \ , \ \theta \in (0,1) \tag{4.2}$$

(cf. e.g. [Ama, VII.1.3.6 Theorem]). Since (4.1) implies

$$1/2 < (n/p - s + \tau)/2 < 1 \quad \text{and} \quad (1-s)/2 < 1\,, \tag{4.3}$$

we are able to fix $\beta > 1/2$ with

$$[(1-s)/2] \vee [(n/p - s + \tau)/2] < \beta < 1\,. \tag{4.4}$$

This choice guarantees, in particular, that $\tau < s + 2\beta - n/p$, and therefore,

$$E_\beta \doteq B^{s+2\beta}_{p,p} \hookrightarrow \mathrm{BUC}^\tau \quad \text{(cf. e.g. [Ama91, (48)])}. \tag{4.5}$$

Hence Assumption 4.1 implies $[w \longmapsto \alpha(w)] \in C^{1-}(E_\beta, \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{E}\ell\ell(E; M)))$. Recalling the generation result of Theorem 3.16 we arrive at

$$\big[w \longmapsto \mathcal{A}_{\alpha(w)}\big] \in C^{1-}(E_\beta, \mathcal{H}(B^{s+2}_{q,q}, B^s_{q,q}))\ ,\ q \in [1, \infty). \tag{4.6}$$

Now the Helmholtz projection and the Stokes operator are introduced. To that end, let

$$F_0 := L_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)$$

be the closure of $\{\vec{v} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)\,;\, \mathrm{div}\,\vec{v} = 0\}$ in $L_p(\mathbb{R}^n, \mathbb{R}^n)$, and

$$F_1 := W^2_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n) := W^2_p(\mathbb{R}^n, \mathbb{R}^n) \cap L_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n).$$

Then the *Helmholtz projection* P_p of $L_p(\mathbb{R}^n,\mathbb{R}^n)$ onto $L_{p,\sigma}(\mathbb{R}^n,\mathbb{R}^n)$ is given by

$$P_p = I - (R_j R_k)_{j,k\in\{1,\dots,n\}}\,,$$

where R_j, $j\in\{1,...,n\}$, are the Riesz transforms $R_j := \mathcal{F}^{-1} r_j \mathcal{F}$ with symbol $r_j(\xi)=\xi_j/|\xi|$. Since $R_j \in \mathcal{L}(W_p^m(\mathbb{R}^n))$, $m\in\mathbb{N}$, holds by Mikhlin's multiplier theorem, this implies

$$P_p \in \mathcal{L}(W_p^{2i}(\mathbb{R}^n,\mathbb{R}^n),F_i)\;,\; i\in\{0,1\}.$$

Using the property $B^{2\theta}_{p,p}(\mathbb{R}^n,\mathbb{R}^n) \doteq (L_p(\mathbb{R}^n,\mathbb{R}^n),W_p^2(\mathbb{R}^n,\mathbb{R}^n))_{\theta,p}$, $\theta\in(0,1)$, we arrive at

$$P_p \in \mathcal{L}(B^{2\theta}_{p,p}(\mathbb{R}^n,\mathbb{R}^n),F_\theta), \tag{4.7}$$

where the spaces $F_\theta := (F_0,F_1)_{\theta,p}$, $\theta\in(0,1)$, are characterised by

$$F_\theta \doteq B^{2\theta}_{p,p,\sigma}(\mathbb{R}^n,\mathbb{R}^n) := B^{2\theta}_{p,p}(\mathbb{R}^n,\mathbb{R}^n)\cap L_{p,\sigma}(\mathbb{R}^n,\mathbb{R}^n) \tag{4.8}$$

(cf. [Ama00b, Theorem 3.4]). It is known that the *Stokes operator*,

$$\mathcal{S}_p : F_1 \longrightarrow F_0\,,\; u \longmapsto -\nu\,\Delta\, u\,,$$

has the property

$$\mathcal{S}_p \in \mathcal{H}(F_1,F_0). \tag{4.9}$$

For further references and a more comprehensive treatment of the Stokes scale we refer to [Ama00b, Section 3].

Our considerations are summarized in the following statement.

Lemma 4.2 *Assume that* $w \longmapsto \alpha(w)$ *satisfies Assumption* 4.1, *and let*

$$\begin{aligned}\mathbb{E}_0 &:= E_0\times F_0 = B^s_{p,p}(\mathbb{R}^n,E)\times L_{p,\sigma}(\mathbb{R}^n,\mathbb{R}^n),\\ \mathbb{E}_1 &:= E_1\times F_1 = B^{s+2}_{p,p}(\mathbb{R}^n,E)\times W^2_{p,\sigma}(\mathbb{R}^n,\mathbb{R}^n),\end{aligned}$$

where $s,p\in\mathbb{R}$ *fulfil* (4.1). *Then the real interpolation spaces* $\mathbb{E}_\theta := (\mathbb{E}_0,\mathbb{E}_1)_{\theta,p}$ *of exponent* $\theta\in(0,1)$ *are characterised by*

$$\mathbb{E}_\theta \doteq B^{s+2\theta}_{p,p}(\mathbb{R}^n,E)\times B^{2\theta}_{p,p,\sigma}(\mathbb{R}^n,\mathbb{R}^n),$$

and

$$\left[v = \begin{pmatrix} w \\ \vec{v}\end{pmatrix} \longmapsto \mathfrak{A}(v) := \begin{pmatrix} \mathcal{A}_{\alpha(w)} & 0 \\ 0 & \mathcal{S}_p\end{pmatrix}\right] \in C^{1-}(\mathbb{E}_\beta,\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0))$$

holds for $\beta\in\mathbb{R}$ *with* $(n/p-s+\tau)/2<\beta<1$.

Proof. According to [Ama95, I.2.3.3 Proposition], the characterization of the interpolation spaces $\mathbb{E}_\theta$, $\theta\in(0,1)$, follows from (4.2) and (4.8). Our second assertion is a consequence of (4.6), (4.9), and [Ama95, I.1.6.1 Theorem]. □

4.2 Reaction terms, convection terms, and external forces

We now consider the right-hand side of the coagulation-fragmentation model. For that purpose, let the space E be given by

$$E := L_1(Y; (1+y)\,\mathrm{d}\mu(y)) = L_1(Y; \mathrm{d}\mu(y)) \cap L_1(Y; y\,\mathrm{d}\mu(y)).$$

For a motivation of this choice we refer to Section 4.4.

Let us first consider the coagulation term c_ψ, given by (2.10), where ψ is supposed to satisfy the following condition.

Assumption 4.3 The coagulation kernel $\psi : \mathbb{R}^n \times Y \times Y \longrightarrow \mathbb{R}$ belongs to $\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n, K_c)$, where $K_c = K_c(Y \times Y)$ denotes the closed subspace

$$\{k \in L_\infty(Y \times Y)\,;\, k(y, y') = k(y', y) \text{ for almost all } y, y' \in Y\}$$

of $L_\infty(Y \times Y)$, and σ is the real number specified in Assumption 4.1.

For $k \in K_c$ and $v, w \in E$, we introduce the mapping

$$\begin{aligned} \mathsf{c}_k(v,w)(y) := &\ \frac{1}{2} \int_0^y k(y-y', y')\, v(y-y')\, w(y')\ \mathrm{d}\mu(y') \\ & - v(y) \int_0^\infty k(y, y')\, w(y')\ \mathrm{d}\mu(y')\ ,\ y \in Y, \end{aligned}$$

which has the property $[(k, v, w) \longmapsto \mathsf{c}_k(v, w)] \in \mathcal{L}(K_c, E, E; E)$. Since (4.1) and (4.4) imply

$$n/p < s + 2\beta < \sigma + 1\,, \tag{4.10}$$

Proposition 3.5 (S_2) ensures that the pointwise product

$$\mathfrak{c}_k(v, w)\ :\ x \longmapsto \mathsf{c}_{k(x)}(v(x), w(x)) \tag{4.11}$$

of functions $(k, v, w) : \mathbb{R}^n \longrightarrow K_c \times E \times E$ is a multiplication

$$[(k, v, w) \longmapsto \mathfrak{c}_k(v, w)] \in \mathcal{L}(\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n, K_c), B^{s+2\beta}_{p,p}, B^{s+2\beta}_{p,p}; B^{s+2\gamma}_{p,p}), \tag{4.12}$$

provided $\gamma < \beta$. Recalling (4.2) and Assumption 4.3, we consequently obtain

$$[u \longmapsto c_\psi(u) = \mathfrak{c}_\psi(u, u)] \in C^\infty(E_\beta, E_\gamma)\,,\ \gamma \in (0, \beta). \tag{4.13}$$

Now the fragmentation term f_ϕ, defined by (2.12), can be treated analogously. To that end, we impose the following condition on ϕ.

Assumption 4.4 Let Y_Δ^2 be the set $\{(y,y')\in Y\times Y\,;\,0<y'\le y\}$. Moreover, $K_f=K_f(Y_\Delta^2)$ denotes the Banach space

$$\Big\{k\in L_\infty(Y_\Delta^2)\,;\,\Big[y\longmapsto \Phi_k(y):=\frac{1}{y}\int_0^y k(y,y')\,y'\,\mathrm{d}\mu(y')\Big]\in L_\infty(Y)\Big\}$$

with norm $k\longmapsto \|k\|_{L_\infty(Y_\Delta^2)}+\|\Phi_k\|_{L_\infty(Y)}$. Then the fragmentation kernel $\phi:\mathbb{R}^n\times Y_\Delta^2\longrightarrow\mathbb{R}$ is supposed to lie in $\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n,K_f)$.

It is easily seen now that

$$\mathsf{f}_k(v)\;:\;y\longmapsto\int_y^\infty k(y',y)\,v(y')\,\mathrm{d}\mu(y')-\Phi_k(y)\,v(y)\;,\;y\in Y\,,$$

defines a multiplication $[(k,v)\longmapsto \mathsf{f}_k(v)]\in\mathcal{L}(K_f,E;E)$. Then, according to (4.10), Proposition 3.5 (S_3) shows that

$$\big\{(k,v)\longmapsto\big[f_k(v)\;:\;x\longmapsto \mathsf{f}_{k(x)}(v(x))\big]\big\}\in\mathcal{L}(\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n,K_f),B_{q,q}^{s+2\beta};B_{q,q}^{s+2\beta})$$

holds for $q\in[1,\infty)$. By virtue of Assumption 4.4, this implies

$$[u\longmapsto f_\phi(u)]\in\mathcal{L}(B_{q,q}^{s+2\beta})\;,\;q\in[1,\infty). \tag{4.14}$$

We now consider $\vec{v}\cdot\operatorname{grad}u\,:\,y\longmapsto\vec{v}\cdot\operatorname{grad}u(y)$. To that end, $\gamma\in(0,\beta)$ is assumed to satisfy the additional condition

$$0<\gamma<\beta-1/2\,, \tag{4.15}$$

which ensures that $s+2\gamma<s+2\beta-1<2\beta$. Moreover, $s+2\beta-1>0$ and

$$2\beta>2\beta-1>n/p+\tau-s-1>n/p \tag{4.16}$$

hold by (4.1), (4.4), and Assumption 4.1. Hence Proposition 3.5 (S_1) implies

$$[(w,\vec{v})\longmapsto\vec{v}\cdot w]\in\mathcal{L}(B_{p,p}^{s+2\beta-1}(\mathbb{R}^n,E^n),B_{p,p}^{2\beta}(\mathbb{R}^n,\mathbb{R}^n);B_{p,p}^{s+2\gamma}),$$

and therefore,

$$[(u,\vec{v})\longmapsto\vec{v}\cdot\operatorname{grad}u]\in\mathcal{L}(B_{p,p}^{s+2\beta},B_{p,p}^{2\beta}(\mathbb{R}^n,\mathbb{R}^n);B_{p,p}^{s+2\gamma}). \tag{4.17}$$

In view of $F_\beta\doteq B_{p,p,\sigma}^{2\beta}(\mathbb{R}^n,\mathbb{R}^n)\hookrightarrow B_{p,p}^{2\beta}(\mathbb{R}^n,\mathbb{R}^n)$, it follows that

$$[(u,\vec{v})\longmapsto\vec{v}\cdot\operatorname{grad}u]\in C^\infty(\mathbb{E}_\beta,E_\gamma). \tag{4.18}$$

We now consider the convection term $\vec{v}\longmapsto(\vec{v}\cdot\operatorname{grad})\,\vec{v}$. In view of (4.16) and the inequality $2\gamma<2\beta-1$, which holds by (4.15), Proposition 3.5 (S_1) ensures that

$$[(v,w)\longmapsto v\,w]\in\mathcal{L}(B_{p,p}^{2\beta}(\mathbb{R}^n,\mathbb{R}),B_{p,p}^{2\beta-1}(\mathbb{R}^n,\mathbb{R});B_{p,p}^{2\gamma}(\mathbb{R}^n,\mathbb{R})).$$

This implies

$$[(v, w) \longmapsto v\, \partial_i w] \in \mathcal{L}(B^{2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}), B^{2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}); B^{2\gamma}_{p,p}(\mathbb{R}^n, \mathbb{R}))\ ,\ i \in \{1, ..., n\},$$

and therefore,

$$[(\vec{v}, \vec{w}\,) \longmapsto (\vec{v} \cdot \mathrm{grad})\vec{w}\,] \in \mathcal{L}(B^{2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n), B^{2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n); B^{2\gamma}_{p,p}(\mathbb{R}^n, \mathbb{R}^n)).$$

In view of (4.7), we consequently obtain

$$[\vec{v} \longmapsto P_p(\vec{v} \cdot \mathrm{grad})\, \vec{v}\,] \in C^\infty(F_\beta, F_\gamma). \tag{4.19}$$

Finally, the external force field $\vec{f}$, given by (2.15), is considered under the following condition.

Assumption 4.5 Let $\vec{\varphi} : \mathbb{R}^n \times Y \longrightarrow \mathbb{R}^n$ be in $\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n, L_\infty(Y, \mathbb{R}^n))$. Furthermore, there is some $\delta > 0$ such that $\vec{\varphi}(\cdot, 0) \in \mathrm{BUC}^{\sigma+1}(\mathbb{R}^n, \mathbb{R}^n) \cap B^{\delta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n)$.

First we introduce

$$\left[(\vec{w}, u) \longmapsto \mathfrak{b}(\vec{w}, u) := \frac{\eta}{\varrho} \int_Y \vec{w}(y)\, u(y)\, y\, \mathrm{d}\mu(y)\right] \in \mathcal{L}(L_\infty(Y, \mathbb{R}^n), E; \mathbb{R}^n).$$

Then, in view of (4.10), Proposition 3.5 (S_3) ensures that $\mathfrak{b}(\vec{w}, u) : x \longmapsto \mathfrak{b}(\vec{w}(x), u(x))$ is a multiplication

$$[(\vec{w}, u) \longmapsto \mathfrak{b}(\vec{w}, u)] \in \mathcal{L}(\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n, L_\infty(Y, \mathbb{R}^n)), B^{s+2\beta}_{p,p}; B^{s+2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n)).$$

Since $2\gamma < s + 1 + 2\gamma < s + 2\beta$ holds by (4.15), we may apply the continuous imbedding $B^{s+2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n) \hookrightarrow B^{2\gamma}_{p,p}(\mathbb{R}^n, \mathbb{R}^n)$, which, together with Assumption 4.5, leads to

$$[u \longmapsto \mathfrak{b}(\vec{\varphi} - \vec{\varphi}(\cdot, 0), u)] \in C^\infty(E_\beta, B^{2\gamma}_{p,p}(\mathbb{R}^n, \mathbb{R}^n)).$$

In the following, let γ satisfy the additional assumption $\gamma < \delta/2$, ensuring that

$$\vec{\varphi}(\cdot, 0) \in B^{\delta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n) \hookrightarrow B^{2\gamma}_{p,p}(\mathbb{R}^n, \mathbb{R}^n).$$

Consequently,

$$\left[u \longmapsto \vec{f}(u) = \vec{\varphi}(\cdot, 0) + \mathfrak{b}(\vec{\varphi} - \vec{\varphi}(\cdot, 0), u)\right] \in C^\infty(E_\beta, B^{2\gamma}_{p,p}(\mathbb{R}^n, \mathbb{R}^n))$$

is valid. Recalling property (4.7), we finally arrive at

$$\left[u \longmapsto P_p \vec{f}(u)\right] \in C^\infty(E_\beta, F_\gamma). \tag{4.20}$$

Our considerations are summarized in the following statement.

Lemma 4.6 *Let Assumptions* 4.3*,* 4.4*, and* 4.5 *be satisfied. Then there are* $\beta, \gamma \in \mathbb{R}$ *with* $0 < \gamma < \beta < 1$ *such that*

$$\left[v = \begin{pmatrix} u \\ \vec{v} \end{pmatrix} \longmapsto F(v) := \begin{pmatrix} c_\psi(u) + f_\phi(u) - \vec{v} \cdot \mathrm{grad}\, u \\ -P_p(\vec{v} \cdot \mathrm{grad})\, \vec{v} + P_p \vec{f}(u) \end{pmatrix}\right] \in C^\infty(\mathbb{E}_\beta, \mathbb{E}_\gamma).$$

PROOF. Our assertion follows from (4.13), (4.14), (4.18), (4.19), and (4.20). □

4.3 Existence of a unique maximal solution

The results obtained in Sections 4.1 and 4.2 enable us to formulate the following existence theorem for Problem 2.1. Here (and in the sequel) we do not explicitly mention the pressure p, since it is well-known that it can be recovered from the velocity $\vec{v}$ of the suspension by

$$\mathsf{p} = -\sum_{j,k=1}^{n} R_j R_k (v_j v_k).$$

Theorem 4.7 *Assume that the diffusion coefficient α, the coagulation kernel ψ, the fragmentation rate ϕ, and the external force field $\vec{f}$ satisfy Assumptions* 4.1, 4.3, 4.4, *and* 4.5, *respectively. Moreover, $s, p \in \mathbb{R}$ are supposed to fulfil* (4.1), *and E denotes the space $L_1(Y, (1+y)\,\mathrm{d}\mu(y))$.*

Then, for each initial value $\upsilon_0 = (u_0, \vec{v}_0) \in B^{s+2}_{p,p}(\mathbb{R}^n, E) \times W^2_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)$, Problem 2.1 *possesses a unique maximal solution $\upsilon = (u, \vec{v})$ with*

$$\upsilon \in C^1(J, B^s_{p,p}(\mathbb{R}^n, E) \times L_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)) \cap C(J, B^{s+2}_{p,p}(\mathbb{R}^n, E) \times W^2_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)),$$

where $J = J(\upsilon_0)$ denotes the maximal existence interval $[0, T(\upsilon_0))$.

Proof. Applying the Helmholtz projection P_p to the Navier-Stokes equations, and using the notation introduced in Sections 4.1 and 4.2, we formulate the coagulation-fragmentation model 2.1 as the abstract Cauchy problem

$$\dot{\upsilon}(t) + \mathfrak{A}(\upsilon(t))\upsilon(t) = F(\upsilon(t)) \,,\ t > 0 \,, \quad \upsilon(0) = \upsilon_0, \tag{4.21}$$

in $\mathbb{E}_0$. Now Lemmas 4.2 and 4.6 guarantee the existence of $\beta, \gamma \in \mathbb{R}$ with $0 < \gamma < \beta < 1$ such that

$$[\upsilon \longmapsto (\mathfrak{A}(\upsilon), F(\upsilon))] \in C^{1-}(\mathbb{E}_\beta, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0) \times \mathbb{E}_\gamma).$$

As a consequence, the existence result [Ama93, 12.1 Theorem] applies to the Cauchy problem (4.21), and proves our assertion. □

Remark 4.8 Existence of a unique maximal solution to Problem 2.1 can also be proven for time-dependent data. In that case, our coagulation-fragmentation model is formulated as the abstract Cauchy problem

$$\dot{\upsilon}(t) + \mathfrak{A}(t, \upsilon(t))\upsilon(t) = F(t, \upsilon(t)) \,,\ t > 0 \,, \quad \upsilon(0) = \upsilon_0,$$

and the desired statement follows from [Ama93, 12.2 Remark (a)], provided

$$[(t, \upsilon) \longmapsto (\mathfrak{A}(t, \upsilon), F(t, \upsilon))] \in C^{1-}(J \times \mathbb{E}_\beta, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0) \times \mathbb{E}_\gamma).$$

Remark 4.9 The maximal solution $\upsilon = (u, \vec{v})$ to Problem 2.1 has the Hölder regularity $\upsilon \in C^{1-\theta}(J, B^{s+2\theta}_{p,p}(\mathbb{R}^n, E) \times B^{2\theta}_{p,p,\sigma}(\mathbb{R}^n, \mathbb{R}^n))$, $\theta \in (0,1)$.

This property follows from [Ama95, II.1.1.2 Proposition] and Theorem 4.7.

Remark 4.10 The maximal existence interval $J = [0, T(v_0))$ of the solution $v = (u, \vec{v})$, obtained by Theorem 4.7, does not depend on the choice of s. This can be proven by means of the following bootstrapping argument.

For $i \in \{0,1\}$ we define $\mathbb{F}_i := B^{s'+2i}_{p,p} \times W^{2i}_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)$, and set

$$\mathbb{F}_\theta := (\mathbb{F}_0, \mathbb{F}_1)_{\theta,p} \doteq B^{s'+2\theta}_{p,p} \times B^{2\theta}_{p,p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)\ ,\ \theta \in (0,1)\,,$$

where $s' \in \mathbb{R}$ satisfies $s < s' < (s+1) \wedge (\sigma - 1)$. Together with (4.1) and (4.3), this condition enables us to choose a $\beta' > 1/2$ with

$$[(1-s')/2] \vee [(n/p - s' + \tau)/2] < \beta' < 1 - (s'-s)/2 < 1\,.$$

By Lemmas 4.2 and 4.6 we then obtain

$$[w \longmapsto (\mathfrak{A}(w), F(w))] \in C^{1-}(\mathbb{F}_{\beta'}, \mathcal{H}(\mathbb{F}_1, \mathbb{F}_0) \times \mathbb{F}_0).$$

Since $\mathbb{E}_{1-\varepsilon} \hookrightarrow \mathbb{F}_{\beta'}$ holds for $\varepsilon := 1 - (s'-s)/2 - \beta' \in (0,1)$, Remark 4.9 implies

$$[t \longmapsto (\mathfrak{A}(v(t)), F(v(t)))] \in C^{\varepsilon}(J, \mathcal{H}(\mathbb{F}_1, \mathbb{F}_0) \times \mathbb{F}_0).$$

It follows by [Ama95, II.1.2.1 Theorem] that the linear Cauchy problem

$$\dot{w}(t) + \mathfrak{A}(v(t))w(t) = F(v(t)))\ ,\ t \in J \setminus \{0\}, \quad w(0) = v_0\,,$$

has a unique solution $w \in C^1(J, \mathbb{F}_0) \cap C(J, \mathbb{F}_1)$. Thus, $v - w \in C^1(J, \mathbb{E}_0) \cap C(J, \mathbb{E}_1)$ fulfils

$$\dot{z}(t) + \mathfrak{A}(v(t))z(t) = 0\ ,\ t \in J \setminus \{0\}, \quad z(0) = 0\,. \tag{4.22}$$

Since Lemma 4.2 and Remark 4.9 imply $[t \longmapsto \mathfrak{A}(v(t)] \in C^{1-\beta}(J, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0))$, the existence and uniqueness result of [Ama95, II.1.2.1 Theorem] ensures that (4.22) possesses only the trivial solution in $C^1(J, \mathbb{E}_0) \cap C(J, \mathbb{E}_1)$. Consequently, the function v equals w, and therefore, it belongs to $C^1(J, \mathbb{F}_0) \cap C(J, \mathbb{F}_1)$. This proves our assertion.

4.4 Conservation of mass

It is the aim of this section to show that the total mass of all suspended particles, given by (2.17), is finite at each moment $t \in J$, and conserved. For that purpose, we first derive the following regularity result.

Proposition 4.11 *Let the assumptions of Theorem* 4.7 *be satisfied, ensuring that Problem* 2.1 *possesses the maximal solution* $v = (u, \vec{v})$ *on the time interval* J. *Then the particle size distribution function* u *has the property*

$$u \in C^1(J, B^s_{p,p}(\mathbb{R}^n, E) \cap B^s_{1,1}(\mathbb{R}^n, E)) \cap C(J, B^{s+2}_{p,p}(\mathbb{R}^n, E) \cap B^{s+2}_{1,1}(\mathbb{R}^n, E)).$$

PROOF. Our proof is based on a bootstrapping argument, where the unique maximal solution $v = (u, \vec{v})$, obtained by Theorem 4.7, is inserted into the nonlinearities of the considered reaction-diffusion equation. This leads to a linear problem, whose solution has the desired regularity, and can be identified with u.

First we consider the operator family $\{\mathcal{A}(t) := \mathcal{A}_{\alpha(u(t))}\,;\, t \in J\}$. From (4.6) it follows that

$$\big[u \longmapsto \mathcal{A}_{\alpha(u)}\big] \in C^{1-}(E_\beta, \mathcal{H}(B^{s+2}_{p,p}, B^s_{p,p}) \cap \mathcal{H}(B^{s+2}_{1,1}, B^s_{1,1})).$$

Setting $X_i := B^{s+2i}_{p,p} \cap B^{s+2i}_{1,1}$, $i \in \{0,1\}$, and using the easily verified imbedding

$$\mathcal{H}(B^{s+2}_{p,p}, B^s_{p,p}) \cap \mathcal{H}(B^{s+2}_{1,1}, B^s_{1,1}) \hookrightarrow \mathcal{H}(X_1, X_0),$$

we obtain $\big[u \longmapsto \mathcal{A}_{\alpha(u)}\big] \in C^{1-}(E_\beta, \mathcal{H}(X_1, X_0))$. In view of Remark 4.9, this implies

$$\big[t \longmapsto \mathcal{A}(t) = \mathcal{A}_{\alpha(u(t))}\big] \in C^{1-\beta}(J, \mathcal{H}(X_1, X_0)). \tag{4.23}$$

For a treatment of the coagulation term, we consider $\mathfrak{c}_k(v, w)$ defined by (4.11). Analogous to (4.12), it follows by means of Proposition 3.5 (S_1) and (1.1) that

$$[(k, u, w) \longmapsto \mathfrak{c}_k(u, w)] \in \mathcal{L}(\mathrm{BUC}^{\sigma+1}(\mathbb{R}^n, K_c), B^{s+2\beta}_{p,p}, B^{s+2\beta}_{1,1}; B^s_{1,1}).$$

Thus,

$$[(u, w) \longmapsto \mathfrak{c}_\psi(u, w)] \in \mathcal{L}(E_\beta, B^{s+2\beta}_{p,p} \cap B^{s+2\beta}_{1,1}; B^s_{p,p} \cap B^s_{1,1}) \tag{4.24}$$

holds. We now observe that, $X_i \hookrightarrow B^{s+2i}_{q,q}$ for $i \in \{0,1\}$ and $q \in \{1,p\}$ implies

$$X_\theta := [X_0, X_1]_\theta \hookrightarrow [B^s_{q,q}, B^{s+2}_{q,q}]_\theta \doteq B^{s+2\theta}_{q,q}\ ,\ q \in \{1,p\}\ ,\ \theta \in (0,1)$$

(cf. e.g. [Ama, Chapter VII]). This leads to the relation

$$X_\theta \hookrightarrow [B^s_{p,p}, B^{s+2}_{p,p}]_\theta \cap [B^s_{1,1}, B^{s+2}_{1,1}]_\theta \doteq B^{s+2\theta}_{p,p} \cap B^{s+2\theta}_{1,1} \hookrightarrow X_0\ ,\ \theta \in (0,1), \tag{4.25}$$

whose application to (4.24) yields

$$\{u \longmapsto [w \longmapsto \mathfrak{c}_\psi(u, w)]\} \in \mathcal{L}(E_\beta, \mathcal{L}(X_\beta, X_0)). \tag{4.26}$$

From (4.14), we analogously infer

$$\{w \longmapsto f_\phi(w)\} \in \mathcal{L}(X_\beta, B^{s+2\beta}_{p,p} \cap B^{s+2\beta}_{1,1}) \hookrightarrow \mathcal{L}(X_\beta, X_0). \tag{4.27}$$

Finally, the term $w \longmapsto \vec{v} \cdot \operatorname{grad} w$ is considered. According to Proposition 3.5 (S_1) we have

$$[(w, \vec{v}) \longmapsto w\,\vec{v}] \in \mathcal{L}(B^{s+2\beta-1}_{1,1}(\mathbb{R}^n, E^n), B^{2\beta}_{p,p}(\mathbb{R}^n, \mathbb{R}^n); B^s_{1,1}),$$

so that $[(w,\vec{v}) \longmapsto \vec{v}\cdot \operatorname{grad} w] \in \mathcal{L}(B^{s+2\beta}_{1,1}, B^{2\beta}_{p,p}(\mathbb{R}^n,\mathbb{R}^n); B^s_{1,1})$ holds. In view of (4.17) this leads to

$$\{\vec{v} \longmapsto [w \longmapsto \vec{v}\cdot \operatorname{grad} w]\} \in \mathcal{L}(F_\beta, \mathcal{L}(B^{s+2\beta}_{p,p} \cap B^{s+2\beta}_{1,1}, B^s_{p,p} \cap B^s_{1,1})).$$

Consequently, (4.25) implies

$$\{\vec{v} \longmapsto [w \longmapsto \vec{v}\cdot \operatorname{grad} w]\} \in \mathcal{L}(F_\beta, \mathcal{L}(X_\beta, X_0)). \tag{4.28}$$

Summarizing (4.26), (4.27), and (4.28), we obtain the property

$$\{v \longmapsto [w \longmapsto g(v)w := \mathfrak{c}_\psi(u,w) + f_\phi(w) - \vec{v}\cdot \operatorname{grad} w]\} \in \mathcal{L}(\mathbb{E}_\beta, \mathcal{L}(X_\beta, X_0)),$$

which, by virtue of Remark 4.9, leads to

$$[t \longmapsto \mathcal{B}(t) := -g(v(t))] \in C^{1-\beta}(J, \mathcal{L}(X_\beta, X_0)).$$

Due to this property and (4.23), Corollary 3.4 ensures that

$$[t \longmapsto L(t) := \mathcal{A}(t) + \mathcal{B}(t)] \in C^{1-\beta}(J, \mathcal{H}(X_1, X_0)).$$

As a consequence, the existence result of [Ama95, II.1.2.1 Theorem] applies to the linear Cauchy problem

$$\dot{w}(t) + L(t)w(t) = 0\,,\ t \in J\setminus\{0\}, \quad w(0) = u_0, \tag{4.29}$$

in X_0, and guarantees a unique solution $w \in C^1(J, X_0) \cap C(J, X_1)$.

Hence it remains to be proven that w equals u. Since u obviously fulfils (4.29) as well, the difference $u - w \in C^1(J, E_0) \cap C(J, E_1)$ is a solution to the homogeneous problem

$$\dot{z}(t) + L(t)z(t) = 0\,,\ t \in J\setminus\{0\}, \quad z(0) = 0\,. \tag{4.30}$$

On the other hand, we also have $[t \mapsto L(t)] \in C^{1-\beta}(J, \mathcal{H}(E_1, E_0))$. Thus, [Ama95, II.1.2.1 Theorem] ensures that (4.30) possesses only the trivial solution in $C^1(J, E_0) \cap C(J, E_1)$. Consequently, u equals w, and therefore, it has the asserted regularity. □

Corollary 4.12 *For $\theta \in [0, \tau/2)$, where $\tau > 0$ is given in Assumption* 4.1*, the particle size distribution function u belongs to $C^\theta(J, L_1(\mathbb{R}^n, E))$. In particular, the total mass $\mathfrak{M}(t)$ of all suspended particles is finite at each moment $t \in J$, and has the property $\mathfrak{M} \in C^\theta(J)$.*

Proof. According to [Ama95, II.1.1.2 Proposition], it follows that

$$u \in C^1(J, B^s_{1,1}) \cap C(J, B^{s+2}_{1,1}) \hookrightarrow C^\theta(J, B^{s+2(1-\theta)}_{1,1}).$$

From (4.1), we moreover infer the inequality $2\theta \le \tau < s+2$, which implies $s + 2(1-\theta) > 0$, and therefore, $B^{s+2(1-\theta)}_{1,1} \hookrightarrow L_1$. This proves our assertion. □

We now are able to prove the following conservation law.

Theorem 4.13 *Let the hypotheses of Theorem 4.7 be satisfied, where the constant σ, given in Assumption 4.1, is supposed to satisfy the additional condition $\sigma > 1$. Then, the total mass $\mathfrak{M}(t)$ of all suspended particles is finite and conserved on the maximal time interval J of existence, i.e.*

$$\mathfrak{M}(t) = \mathfrak{M}(0) = \eta \int_{\mathbb{R}^n} \int_Y u_0(x,y)\, y \,\mathrm{d}\mu(y)\mathrm{d}x \ , \ t \in J.$$

Proof. In view of the additional assumption $\sigma > 1$, the constant s can be chosen in such a manner that it satisfies both (4.1) and $s > 0$. Thus, the particle size distribution function u has the regularity

$$u \in C^1(J, B^s_{p,p} \cap B^s_{1,1}) \cap C(J, B^{s+2}_{p,p} \cap B^{s+2}_{1,1}) \hookrightarrow C^1(J, L_1(\mathbb{R}^n, E)),$$

which implies $\mathfrak{M} \in C^1(J)$. From (4.23), (4.26), (4.27), and (4.28), it follows that

$$\mathcal{A}_{\alpha(u)}u, c_\psi(u), f_\phi(u), \vec{v} \cdot \operatorname{grad} u \in C(J, B^s_{p,p} \cap B^s_{1,1}) \hookrightarrow C(J, L_1(\mathbb{R}^n, E)).$$

Hence, the reaction-diffusion equations in Problem 2.1 may be integrated over $Y \times \mathbb{R}^n$ with respect to $y\,\mathrm{d}\mu(y) \otimes \mathrm{d}x$. Since the arguments of [Ama00a, Proof of Lemma 7.1] lead to

$$\int_{\mathbb{R}^n} \int_Y \left[(\mathcal{A}_{\alpha(u)}u)(t,x,y) + (\vec{v} \cdot \operatorname{grad} u)(t,x,y) \right] y \,\mathrm{d}\mu(y)\,\mathrm{d}x = 0 \ , \ t \in J,$$

and it can easily be verified that

$$\int_{\mathbb{R}^n} \int_Y \left[c_\psi(u)(t,x,y) + f_\phi(u)(t,x,y) \right] y \,\mathrm{d}\mu(y)\,\mathrm{d}x = 0 \ , \ t \in J,$$

this yields $\dot{\mathfrak{M}}(t) = 0$, $t \in J$. □

Remark 4.14 In view of hypothesis (2.3), Theorem 4.13 implies

$$\int_{\mathbb{R}^n} m(t,x,0)\,\mathrm{d}x = 0 \ , \ t \in J \,,$$

i.e. the mass of the carrier fluid is conserved. This reflects the fact that the particles do not interact with the carrier fluid during the process.

4.5 Positivity of the particle size distribution function

In this section our considerations on the coagulation-fragmentation model, Problem 2.1, are confined to the spatial dimensions $n \in \{2,3\}$, and to diffusion operators with vanishing cross diffusion coefficient $\mathbf{b}$. The aim is to prove that, in this special case, the particle size distribution function $u(t) = u(t,\cdot,\cdot)$ remains non-negative for $t \in J$, provided the initial value u_0 is positive.

First we specify a suitable condition on the diffusion coefficient.

Assumption 4.15 There are constants σ, τ with $1 < \sigma \le \tau < 2 < \sigma + 1 \le 3$, and $a_0 > 0$, such that $\alpha \in C^{1-}(\mathrm{BUC}^\tau(\mathbb{R}^n, E), \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{e}\ell\ell(a_0, 0))$.

This means that the diffusion coefficient (2.16) is of the form $[\alpha(w)\, v](y) := \mathbf{a}(w)(y)\, v(y)$, where $\mathbf{a} \in C^{1-}(\mathrm{BUC}^\tau(\mathbb{R}^n, E), \mathrm{BUC}^\sigma(\mathbb{R}^n, L_\infty(Y, \mathbb{R}^{n\times n}_{\mathrm{sym}})))$ is positive definite.

Theorem 4.16 *Let $n \in \{2,3\}$, and suppose that the hypotheses of Theorem 4.7 are satisfied, where Assumption 4.1 is substituted by the (stronger) Assumption 4.15. Moreover, the additional conditions $\psi(x,y,y') \ge 0$ and $\phi(x,y,y') \ge 0$ are assumed to be satisfied for $x \in \mathbb{R}^n$, a.a. $(y,y') \in Y \times Y$, and a.a. $(y,y') \in Y^2_\Delta$, respectively.*

Then, $u(t,x,y) \ge 0$ holds for $t \in J$, $x \in \mathbb{R}^n$, and a.a. $y \in Y$, provided the initial value $u(0,x,y) = u_0(x,y)$ is non-negative.

Proof. Since Proposition 3.11 implies $\mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{e}\ell\ell(a_0,0)) \hookrightarrow \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{E}\ell\ell(M))$ for an $M = M(a_0)$, Theorem 4.7 and Proposition 4.11 guarantee existence of a maximal solution

$$v = (u, \vec{v}\,) \in C(J, (B^{s+2}_{p,p} \cap B^{s+2}_{1,1}) \times W^2_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n)) \;\cap\; C^1(J, (B^s_{p,p} \cap B^s_{1,1}) \times L_{p,\sigma}(\mathbb{R}^n, \mathbb{R}^n))$$

to Problem 2.1, where $s \in (0, \sigma - 1)$ and $p > n/(2-\tau)$ can be chosen. In view of Remark 4.9 and (4.5), the function u belongs to $C^{1-\beta}(J, \mathrm{BUC}^\tau(\mathbb{R}^n, E))$. Hence, Assumption 4.15 implies

$$\alpha(u) \in C^{1-\beta}(J, \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{e}\ell\ell(a_0, 0))).$$

Since $p > n/(2-\tau) \ge n/(2-\sigma)$ guarantees $F_1 \hookrightarrow W^2_p(\mathbb{R}^n, \mathbb{R}^n) \hookrightarrow \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, we obtain

$$\vec{v} \in C(J, F_1) \hookrightarrow C(J, \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathbb{R}^n)).$$

Consequently,

$$(\alpha(u(t)), \vec{v}(t)) \in \mathrm{BUC}^\sigma(\mathbb{R}^n, \mathcal{e}\ell\ell(a_0, 0) \times \mathbb{R}^n)\ ,\ t \in J. \tag{4.31}$$

We now observe that $u \in C(J, B^{s+2}_{1,1}) \cap C^1(J, B^s_{1,1})$ solves the semilinear Cauchy problem

$$\dot{z}(t) + \mathcal{A}_v(t) z(t) = c_\psi(z(t)) + f_\phi(z(t))\ ,\ t \in J \setminus \{0\}, \quad z(0) = u_0\,,$$

with

$$\mathcal{A}_v(t)\ :\ w \longmapsto \mathcal{A}_{\alpha(u(t))} w + \vec{v} \cdot \operatorname{grad} w\ ,\ t \in J\,,$$

which coincides with the model considered in [Ama00a]. Thus, in view of (4.31) and our assumptions on the kernels ψ, ϕ, the result of [Ama00a, Theorem 6.3] can be applied. This yields our assertion. □

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