Coagulation-Fragmentation Processes

Herbert Amann

Abstract. We study the well-posedness of coagulation-fragmentation models with diffusion for large systems of particles. The continuous and the discrete case are considered simultaneously. In the discrete situation we are concerned with a countable system of coupled reaction-diffusion equations, whereas the continuous case amounts to an uncountable system of such equations. These problems can be handled by interpreting them as abstract vector-valued parabolic evolution equations, where the dependent variables take values in infinite-dimensional Banach spaces. Given suitable assumptions, we prove existence and uniqueness in the class of volume preserving solutions. We also derive sufficient conditions for global existence.

1. Introduction

In recent years, much effort has been put into the mathematical foundation of cluster growth. In this theory it is assumed that the system under consideration consists of a very large number of particles that can coagulate to form clusters, which in turn, can merge to form larger clusters or can break apart into smaller ones. Models of cluster growth arise in a variety of situations, for example in aerosol science, atmospheric physics, astrophysics, colloidal chemistry, polymer science, hematology, and biology. The aim of the theory is the description of the particle size distribution as a function of time and space as the system undergoes changes due to various physical influences (see [28] for a description of the forces dominating processes of this type).

In a multitude of situations, namely in the case of thermal coagulation, the movement of the clusters is governed by diffusion. It can be promoted or hindered by free fields between the particles as well as by external fields due to electric, magnetic, gravitational, or centrifugal forces. If the particles are suspended in a gas or a fluid, the movement of that medium can influence the coagulation process too (e.g., [10]).

The theory of coagulation processes originated in the work of M.V. Smoluchowski [25], [26], who derived an infinite system of ordinary differential equations for describing the coagulation of colloids moving according to Brownian motion. That model has since been widely extended and generalized. In particular, continuous models formulated in terms of integro-differential equations have been derived which are used to approximate physical situations involving a very large number of particles of very different sizes. We refer to [8] for an extensive survey of the various models and their derivations, and for a description of the mathematical results during the first three quarters of this century, as well as to [10].

This paper deals with the mathematical foundation of coagulation-fragmentation processes taking into account the movement of the clusters due to diffusion and superimposed transport processes. In many concrete situations, like aerosol physics, atmospheric physics, or astrophysics, say, it is very difficult, if not impossible, to describe a boundary and boundary conditions for the domain under consideration. In these cases it is a very reasonable approximation to consider distribution functions defined throughout the space, but being 'small at infinity'. For this — as well as for mathematical — reason(s) we consider full-space problems in this paper.

Formally, the equations under consideration take the form of an initial value problem of reaction-diffusion type:

$$\begin{aligned} \partial_t u + \mathcal{A}(x,t,y)u &= r(x,t,y,u) \ , \qquad x \in \mathbb{R}^n \ , \quad t > 0 \ , \\ u(x,0,y) &= u^0(x,y) \ , \qquad x \in \mathbb{R}^n \ , \end{aligned} \tag{1.1}$$

where $n = 1, 2$, or 3, depending on an additional real parameter y, the volume (which is used as the characteristic size. Thus by the size of a cluster we mean its volume.). Here $\mathcal{A}$ are diffusion-convection operators, the 'reaction term' r describes kinetic behavior of the process, and u is the particle size distribution function. Thus

$$u(x,t,y) \geq 0 \tag{1.2}$$

and

$$\int_X \int_{y_0}^{y_1} u(x,t,y)\, dy\, dx \tag{1.3}$$

is the total number of particles with volumes belonging to the interval $[y_0, y_1] \subset \mathbb{R}^+$ and being at time t contained in the space region $X \subset \mathbb{R}^n$. The measure dy is either Lebesgue's measure on $\mathbb{R}^+$ or the counting measure on $\dot{\mathbb{N}} := \{1, 2, 3, \ldots\}$. In the latter case only clusters can occur whose sizes are integer multiples of an 'elementary unit'. In this case all integrals with respect to dy reduce to sums, of course, so that (1.3) takes the form

$$\int_X \sum_{y=y_0}^{y_1} u(x,t,y)\, dx \ .$$

This situation corresponds to *discrete* coagulation-fragmentation processes, the former one to *continuous* models. Note that we treat both cases simultaneously.

To be more precise, we begin by describing the kinetic part r. It is a sum of three terms:

$$r(x,t,y,u) = c(x,t,y,u) + f(x,t,y,u) + h(x,t,y) \ , \tag{1.4}$$

accounting for coagulation, fragmentation, and particle input, respectively. The coagulation term is of the form

$$\begin{aligned} c(x,t,y,u) := \frac{1}{2}\int_0^y &\gamma(x,t,y-y',y')u(y-y')u(y')\,dy' \\ - u(y)\int_0^\infty &\gamma(x,t,y,y')u(y')\,dy' \ . \end{aligned} \tag{1.5}$$

The coagulation kernel γ, where $\gamma(x,t,y,y')$ describes the rate of coalescences of clusters of sizes y and y' at time t and position x, is supposed to satisfy

$$0 \le \gamma(x,t,y,y') = \gamma(x,t,y',y) \ , \qquad y,y' \in Y \ , \tag{1.6}$$

where Y is the support of dy and, here and below, (x,t) runs through $\mathbb{R}^n \times \mathbb{R}^+$. Thus we neglect triple and higher collisions assuming them to be rare, and take account of binary coagulation only. The first integral in (1.5) expresses the fact that a cluster of size y can only come into existence if two clusters of volumes $y-y'$ and y' collide. The factor $1/2$ guarantees that each combination is counted only once. The last term in (1.5) says that a cluster of size y disappears from 'level y' if it coagulates with a cluster of any volume.

The fragmentation term f is given by

$$f(x,t,y,u) := \int_y^\infty \varphi(x,t,y',y)u(y')\,dy' - \Phi(x,t,y)u(y) \ , \tag{1.7}$$

where the fragmentation kernel φ satisfies

$$0 \le \varphi(x,t,y,y') \ , \qquad 0 < y' \le y < \infty \ , \quad y,y' \in Y \ , \tag{1.8}$$

and

$$\Phi(x,t,y) := \frac{1}{y}\int_0^y \varphi(x,t,y,y')y'\,dy' \ , \qquad y \in Y \ . \tag{1.9}$$

The integral in (1.7) accounts for the production of clusters of size y by the break-up of clusters of larger volumes. The last term takes care of the disappearance of y-clusters by their fragmentation into smaller ones. Note that we allow multiple fragmentation processes. The binary case, in which each splitting produces two clusters only, can be subsumed in this formulation by a suitable change of dependent variables (see [17]).

Lastly, the source term satisfies

$$h(x,t,y) \geq 0 \ , \qquad y \in Y \ , \tag{1.10}$$

and accounts for creation of clusters of size y at time t and position x due to particle input, for example.

There are several models for the dependence of the coagulation and fragmentation rates on the particle volume. Most of them amount to bounds of the form

$$\gamma(x,t,y,y') \leq c\big[(1+y)^{\alpha_0} + (1+y')^{\alpha_0}\big] \ , \qquad \varphi(x,t,y,y') \leq c(1+y+y')^{\alpha_1} \ ,$$

where α_0 and α_1 are positive constants. This means that these rates can become arbitrarily large if arbitrarily large clusters are involved. However, it has to be kept in mind that *laws of this type are mathematical idealizations which apply to a finite range of the particle volume only, since there are no infinitely large clusters in nature.* Thus, since coagulation-fragmentation processes are of real interest for clusters up to a fixed — albeit rather large — size only, we can assume, without losing physical significance, that the coagulation and fragmentation rates are bounded.

Hence we impose the hypothesis that there exists a positive constant β such that

$$\gamma(x,t,y,y') \leq \beta \ , \quad \varphi(x,t,y,y') \leq \beta \tag{1.11}$$

for all possible arguments (x,t,y,y') of γ and φ, respectively. We also suppose that

$$\varPhi(x,t,y) \leq \beta \ , \quad (x,t,y) \in \mathbb{R}^n \times \mathbb{R}^+ \times Y \ , \tag{1.12}$$

which means that the volume rate of change in the fragmentation process is bounded as well. Note that assumptions (1.6), (1.8), and (1.10)–(1.12) hold in virtue of their physical significance. For mathematical reasons we suppose that $\gamma(x,t,\cdot,\cdot)$, $\varphi(x,t,\cdot,\cdot)$, and $f(x,t,\cdot)$ are measurable for $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and sufficiently smooth with respect to (x,t).

As for the operator $\mathcal{A}$, we assume that

$$\begin{aligned} \mathcal{A}(x,t,y)u := &- \operatorname{div}\big(\boldsymbol{a}(x,t,y)\operatorname{grad} u + \vec{a}(x,t,y)u\big) \\ &+ \vec{b}(x,t,y)\cdot\operatorname{grad} u + a_0(x,t,y)u \ , \end{aligned} \tag{1.13}$$

where div and grad are taken with respect to $x \in \mathbb{R}^n$. The diffusion matrix $\boldsymbol{a}$, the drift vectors $\vec{a}$ and $\vec{b}$, and the absorption rate a_0 are sufficiently smooth functions of (x,t) and measurable with respect to y. We also assume that

$$\begin{aligned} &\boldsymbol{a}(x,t,y) \text{ is symmetric and positive definite,} \\ &\text{uniformly with respect to } (x,t,y) \in \mathbb{R}^n \times \mathbb{R}^+ \times Y \ . \end{aligned} \tag{1.14}$$

In summary, using standard notation and suppressing the independent variables, system (1.1) can be written in the conventional form

$$\partial_t u - \nabla\cdot(\boldsymbol{a}\nabla u + \vec{a}u) + \vec{b}\cdot\nabla u + a_0 u = [\partial_t u]_{\mathrm{coag}} + [\partial_t u]_{\mathrm{frag}} + h$$

with

$$[\partial_t u]_{\mathrm{coag}} = \frac{1}{2}\int_0^y \gamma(y-y',y')u(y-y')u(y')\,dy' - u(y)\int_0^\infty \gamma(y,y')u(y')\,dy'$$

and

$$[\partial_t u]_{\mathrm{frag}} = \int_y^\infty \varphi(y',y)u(y')\,dy' - u(y)\int_0^y \varphi(y,y')\frac{y'}{y}\,dy' \ .$$

In most concrete situations $\boldsymbol{a} = a \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the identity matrix, and a is a decreasing function of y. It is often approximated by an expression of the form

$$\left[\alpha y^{-1/3} + y^{-2/3}\left(\beta + \delta e^{-\varepsilon y^{1/3}}\right)\right]T(x,t) \ , \tag{1.15}$$

where α, β, δ, and ε are positive constants and T is the temperature, which in this model is supposed to be given (cf. [10, Sect. 2.3]). This is consistent with intuition which suggests that large clusters diffuse more slowly than small ones. However, again it has to be kept in mind that such laws apply only to clusters whose sizes belong to a finite range. Since in nature there do exist neither infinitely large nor infinitely small clusters we do not lose any physical significance if we assume that $\boldsymbol{a}(x,t,y)$ is uniformly bounded and uniformly positive definite (where the constant of uniform definiteness can be very small, of course).

The drift vector field $\vec{a}$ describes the particle transport due to outer forces such as gravitational, electrical or thermal fields, where the last one is produced by temperature gradients in the gas or fluid in which the particles are being suspended. These fields — as well as the temperature distribution — are supposed to be explicitly given in order to simplify the presentation. Of course, in general $\vec{a}$ is determined by a set of partial differential equations which is coupled to the reaction-diffusion system (1.1).

Lastly, if the particles are being suspended in a flowing fluid, that is, if we consider convective diffusion, then $\vec{b}$ is the velocity of the fluid. Thus in this case (1.1) has to be complemented by the Navier-Stokes equations for the vector field $\vec{b}$. In the incompressible case $\operatorname{div}\vec{b} = 0$ so that $\vec{b}$ can be subsumed in $\vec{a}$ if it is regular enough. If we assume that the suspended particles have no effect on the velocity distribution — which is true for low aerosol concentration, for example — then we can solve the Navier-Stokes equations for $\vec{b}$ and substitute the result in (1.1). In other words, in many cases of physical interest, $\vec{b}$ can also be considered to be a given vector field. This is the position adopted in this paper.

Most of the mathematical research on the coagulation-fragmentation equations is concerned with the discrete case. Within this class the *kinetic* coagulation-fragmentation equations

$$\dot{u} = r(t,y,u) \ , \quad t > 0 \ , \qquad u(0,y) = u^0(y) \ , \tag{1.16}$$

in which diffusion is neglected, attracted most attention. Note that in this case (1.16) reduces to an infinite system of ordinary integro-differential equations. We refer in particular to [5], [6], and the references therein for some of the more recent results.

Relatively little is known about the discrete coagulation-fragmentation equations with diffusion. This case amounts to an infinite system of coupled reaction-diffusion equations. An investigation of basic existence and uniqueness questions has been initiated only very recently in [7] and has been continued in a series of papers and preprints by Ph. Laurençot and D. Wrzosek ([11]–[14], [27]). In all these papers, $\mathcal{A}(x,t,y)u = -a(y)\triangle u$ with $a(y)$ being non-negative constants for $y \in \mathbb{N}$. Furthermore, the equations are supposed to hold in a bounded domain under no-flux boundary conditions.

In the case of discrete coagulation-fragmentation systems the technique used in practically all papers is the natural one: first one studies finite systems obtained by truncating to the first N equations and, after having established suitable a priori bounds, passes to the limit as N tends to infinity. The situation is complicated by the fact that the authors allow unbounded coagulation and fragmentation rates.

Much less seems to be known for the case of continuous coagulation-fragmentation models, that is, if dy is Lebesgue's measure on $\mathbb{R}^+$. The kinetic equations have first been studied by Melzak [20], [21] under assumptions (1.6), (1.8), (1.11), and (1.12) with $h = 0$. He proved the existence of a unique positive global solution by means of series expansions. Melzak's ideas have been extended by Marcus to include a transport term in one space dimension, which depends on y only, that is, $\mathcal{A}(x,t,y)u := b(y)\partial_x u$ (see [8, Sect. 3.4]).

A different approach has been initiated by Aizenman and Bak [1]. These authors consider the autonomous kinetic continuous coagulation-fragmentation equations with bounded coagulation and fragmentation rates, but without assumption (1.12). By means of semigroup techniques they establish the existence of a unique non-negative volume-preserving solution. This semigroup approach has lately been extended in a series of papers by McLaughlin, Lamb, and McBride ([15]–[19]) to include certain classes of unbounded kernels as well. For further results we refer to the papers by Dubovskiĭ and Stewart (see [9] and the references therein).

Although diffusion and convection are fundamental in coagulation-fragmentation processes, there are no rigorous results for the continuous case in presence of diffusion. The mathematical analysis is restricted to some formal manipulations involving additional ad hoc hypotheses (see [8, pp. 336–373], [22]–[24]).

The reason for this lack of a mathematical theory becomes apparent by looking at the simplest continuous coagulation model with diffusion. It is given by the autonomous partial integro-differential equation

$$\begin{aligned}\partial_t u - a(y)\triangle u = \frac{1}{2}\int_0^y \gamma(y-y',y')u(y-y')u(y')\,dy' \\ - u(y)\int_0^\infty \gamma(y,y')u(y')\,dy'\end{aligned}$$

on $\mathbb{R}^n$ with y running through $\mathbb{R}^+$ and $a(y)$ being a positive constant for fixed y. The above equation can be viewed as a coupled system of uncountably many reaction-diffusion equations, which sheds some light on the inherent difficulties. It seems to be natural to approximate such a system by finitely — or even countably — many equations and try to do a limit argument. Even if such an approach could be carried through, it is unlikely that it would lead to optimal results. In fact, even in the discrete models involving diffusion not much is known so far about continuous dependence on the data, say. For example, uniqueness already poses serious problems in certain cases.

Our approach is a completely different one. Namely, we consider problem (1.1) as a single semilinear evolution equation

$$\dot{u} + \mathcal{A}(t)u = R(t,u)\ , \quad t > 0\ , \qquad u(0) = u^0\ ,$$

where u is a Banach-space-valued function of $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. In other words, we interprete (1.1) as a vector-valued evolution equation which we are able to handle thanks to recent Fourier multiplier theorems for operator-valued symbols and Banach-space-valued distributions, which we obtained earlier [4].

Finally, we comment once more on our assumptions. As already mentioned, we impose — besides of mild regularity hypotheses — the physical conditions (1.6) and (1.8)–(1.12) for the kinetic part. Of course, the fact that y runs from 0 to ∞ is a mathematical abstraction since there exist no arbitrarily large masses or infinitely small particle volumes in the physical world. *This abstraction is made for convenience and does not influence physical models since we can always assume that $\mathcal{A}$, γ, φ, and h vanish identically for sufficiently large or small values of y.* In this case condition (1.12) is automatically satisfied, given the boundedness and non-negativity of γ. These remarks also show that *unbounded coagulation and fragmentation kernels are artifacts* which, by the way, enhance the mathematical difficulties considerably.

Of course, setting up infinite systems of differential equations in the discrete case, or approximating reality by continuous coagulation-fragmentation models are mathematical idealizations as well. They simply serve as approximations of very large systems of reaction-diffusion equations, having the advantage that 'one and the same law holds everywhere' and one does not have to worry about 'non-uniform behavior near the boundaries of the system'. In the discrete case, or in the continuous kinetic case when no diffusion is taken into consideration, it is mathematically intriguing to admit unbounded coagulation and fragmentation kernels, and — from the mathematical point of view — this would also be of interest in the continuous model involving diffusion. However, in the latter case the mathematical difficulties are insurmountable, at present. We emphasize again that this paper contains the first rigorous results on continuous coagulation-fragmentation equations *with* diffusion.

We close this introduction with a somewhat informal statement of the main result of this paper.

Theorem 1.1. *Suppose that* $\vec{b} = 0$, *and* $u^0 \geq 0$ *satisfies*

$$\int_{\mathbb{R}^n} \int_Y \left|\partial_x^\alpha u^0(x,y)\right| (1+y)\, dy\, dx < \infty \ , \qquad |\alpha| \leq 2 \ .$$

Then there exists a maximal $T > 0$ *such that the coagulation-fragmentation system* (1.1) *possesses a unique solution* u *on* $[0,T)$ *satisfying* (1.2) *and*

$$\int_{\mathbb{R}^n} \int_Y u(x,t,y)(1+y)\, dy\, dx < \infty \ , \qquad 0 \leq t < T \ . \tag{1.17}$$

It is a smooth function of x *for* $t > 0$ *and depends continuously on all data.*

If there is neither absorption nor particle input then the total volume is being conserved, that is,

$$\int_{\mathbb{R}^n} \int_Y u(x,t,y) y\, dy\, dx = \int_{\mathbb{R}^n} \int_Y u^0(x,y) y\, dy\, dx \ , \qquad 0 < t < T \ .$$

Lastly, $T = \infty$, *that is,* u *is a global solution if either* $n = 1$ *or* $\boldsymbol{a}$ *is independent of* y *or coagulation does not take place.*

Note that condition (1.17) means that the total number of particles as well as the total volume stay finite during time evolution.

It should be remarked that the assumption that $\vec{b}$ vanishes is only needed to prove volume preservance and the global existence result.

Proofs and precise statements are given in the following sections.

2. Preliminaries

In this paper all vector spaces are over the reals. If there occurs a complex number in a given formula then it is understood that the latter is interpreted as the corresponding complexification.

Let $E, E_0, \ldots, E_m$ be Banach spaces. Then $\mathcal{L}(E_1, \ldots, E_m; E_0)$ ist the Banach space of all continuous m-linear maps from $E_1 \times \cdots \times E_m$ into E_0, and

$$\mathcal{L}^m(E; E_0) := \mathcal{L}(E_1, \ldots, E_m; E_0) \qquad \text{if} \quad E_1 = \cdots = E_m = E \ .$$

Moreover, $\mathcal{L}(E, E_0) := \mathcal{L}^1(E; E_0)$ and $\mathcal{L}(E) := \mathcal{L}(E, E)$. Elements of the space $\mathcal{L}(E_1, \ldots, E_m; E_0)$ are sometimes simply denoted by

$$(e_1, \ldots, e_m) \mapsto e_1 \bullet \cdots \bullet e_m$$

and are said to be multiplications.

Let (M, μ) be a measure space. Then $L_p(M, \mu; E)$ is the usual Lebesgue space of (equivalence classes of) E-valued integrable functions on M for $1 \leq p \leq \infty$, and $L_p(M, \mu) := L_p(M, \mu; \mathbb{R})$. We write $L_p[E]$ for $L_p(\mathbb{R}^n; E)$, where it is understood that Lebesgue's measure is used. If no confusion seems likely, the norm in $L_p(M, \mu; E)$ is simply denoted by $\|\cdot\|_p$.

If $m \in \mathbb{N}$ then $W_1^m[E] := W_1^m(\mathbb{R}^n; E)$ is the Sobolev space of order m of E-valued functions on $\mathbb{R}^n$ whose distributional derivatives of order $\leq m$ belong to $L_1[E]$, endowed with the norm

$$u \mapsto \|u\|_{m,1} := \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_1 \ .$$

If $m < s < m+1$ then $u \in W_1^s[E]$ iff $u \in W_1^m[E]$ and $[\partial^\alpha u]_{s-m,1} < \infty$ for $|\alpha| = m$, where

$$[u]_{\sigma,1} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+\sigma}} \, d(x,y) \ , \qquad 0 < \sigma < 1 \ .$$

This Slobodeckii space is a Banach space with the norm

$$u \mapsto \|u\|_{s,1} := \|u\|_{m,1} + \sum_{|\alpha| = m} [\partial^\alpha u]_{s-m,1} \ .$$

Finally, if $m - 1 < s \leq m$ for some $m \in \dot{\mathbb{N}}$ then $W_1^{-s}[E]$ consists of all E-valued distributions u on $\mathbb{R}^n$ such that there exist $u_\alpha \in W_1^{m-s}[E]$ for $|\alpha| \leq m$ satisfying

$$u = \sum_{|\alpha| \leq m} \partial^\alpha u_\alpha \ . \tag{2.1}$$

It is a Banach space with the norm

$$u \mapsto \|u\|_{-s,1} := \inf\Bigl(\sum_{|\alpha| \leq m} \|u_\alpha\|_{m-s,1} \Bigr) \ ,$$

where the infimum is taken over all representations (2.1). It follows that

$$W_1^s[E] \overset{d}{\hookrightarrow} W_1^t[E] \ , \qquad -\infty < t < s < \infty \ , \tag{2.2}$$

where $\hookrightarrow$ denotes 'continuous injection' and 'd' stands for 'dense'. Moreover,

$$\partial^\alpha \in \mathcal{L}\bigl(W_1^{s+|\alpha|}[E], W_1^s[E]\bigr) \ , \qquad \alpha \in \mathbb{N}^n \ , \quad s \in \mathbb{R} \ . \tag{2.3}$$

We denote for $m \in \mathbb{N}$ by $BUC^m[E]$ the Banach space consisting of all $u \in C^m(\mathbb{R}^n, E)$ for which all derivatives of order at most m are bounded and uniformly continuous, endowed with the norm

$$u \mapsto \|u\|_{m,\infty} := \max_{|\alpha|\le m} \|\partial^\alpha u\|_\infty \ .$$

If $m < s < m+1$ then $u \in BUC^s[E]$ iff $u \in BUC^m[E]$ and

$$\|u\|_{s,\infty} := \|u\|_{m,\infty} + \max_{|\alpha|=m} [\partial^\alpha u]_{s-m,\infty} < \infty \ ,$$

where $[\cdot]_{\sigma,\infty}$ is the Hölder seminorm defined by

$$[u]_{\sigma,\infty} := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\sigma} \ , \qquad 0<\sigma<1 \ .$$

It is known that

$$BUC^s[E] \hookrightarrow BUC^t[E] \ , \qquad 0 \le t \le s < \infty \ . \tag{2.4}$$

(We refer to [3, vol. II] for a thorough treatment of vector-valued distributions and related vector spaces.)

Let

$$E_1 \times \cdots \times E_m \to E_0 \ , \quad (e_1,\ldots,e_m) \mapsto e_1 \bullet \cdots \bullet e_m \tag{2.5}$$

be a multiplication. For $u_j \in E_j^{\mathbb{R}^n}$ we define $u_1 \bullet \cdots \bullet u_m \in E_0^{\mathbb{R}^n}$, the point-wise product induced by (2.5), by

$$u_1 \bullet \cdots \bullet u_m(x) := u_1(x) \bullet \cdots \bullet u_m(x) \ , \qquad x \in \mathbb{R}^n \ . \tag{2.6}$$

Let $\mathfrak{F}_j[E_j]$ be Banach spaces of E_j-valued functions on $\mathbb{R}^n$ for $0 \le j \le m$. Then we write

$$\mathfrak{F}_1[E_1] \bullet \cdots \bullet \mathfrak{F}_m[E_m] \hookrightarrow \mathfrak{F}_0[E_0]$$

if the point-wise product (2.6) defines a continuous m-linear map

$$\mathfrak{F}_1[E_1] \times \cdots \times \mathfrak{F}_m[E_m] \to \mathfrak{F}_0[E_0] \ , \quad (u_1,\ldots,u_m) \mapsto u_1 \bullet \cdots \bullet u_m \ ,$$

the point-wise multiplication induced by (2.5).

In the next lemma we collect those properties of point-wise multiplication which we shall need below.

Lemma 2.1. (i) *Suppose* $E_1 \times E_2 \to E_0$, $(e_1,e_2) \mapsto e_1 \bullet e_2$ *is a multiplication. Then*

$$BUC^s[E_1] \bullet W_p^t[E_2] \hookrightarrow W_p^t[E_0] \ , \qquad 0 \le t < s < \infty \ , \quad 1 \le p < \infty \ .$$

(ii) *Let* $E_1 \times E_2 \times E_3 \to E_0$, $(e_1, e_2, e_3) \mapsto e_1 \bullet e_2 \bullet e_3$ *be a multiplication. Then*

$$BUC^r[E_1] \bullet W_1^s[E_2] \bullet W_1^s[E_3] \hookrightarrow W_1^t[E_0]$$

provided $2n > 2s > t + n \geq n$ *and* $r > t$, *and*

$$BUC^r[E_1] \bullet W_1^s[E_2] \bullet W_1^s[E_3] \hookrightarrow W_1^s[E_0]$$

if $r > s > n$.

Proof. For $\sigma \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ we denote by $B^\sigma_{p,q}[E] := B^\sigma_{p,q}(\mathbb{R}^n, E)$ the Besov space of order σ and integrability indices p and q consisting of E-valued distributions on $\mathbb{R}^n$. It is known that

$$B^\sigma_{p,p}[E] \doteq \begin{cases} W_1^\sigma[E] \ , & p = 1 \ , \quad \sigma \in \mathbb{R} \backslash \mathbb{Z} \ , \\ BUC^\sigma[E] \ , & p = \infty \ , \quad \sigma \in \mathbb{R}^+ \backslash \mathbb{N} \ , \end{cases} \tag{2.7}$$

where $\doteq$ means 'equivalent norms' (cf. [3, vol. II], [4]). Using these facts, (2.2), and

$$B^s_{p,1}[E] \hookrightarrow B^s_{p,q}[E] \hookrightarrow B^s_{p,\infty}[E] \hookrightarrow B^t_{p,1}[E]$$

for $-\infty < t < s < \infty$ and $1 \leq p, q \leq \infty$, the assertions are easy consequences of [2, Theorems 2.1 and 4.1 and Remark 4.2(b)]. (Note that the results in [4] are also valid without the assumption of finite dimensionality of the Banach spaces $E_0, \ldots, E_m$ (see [3, vol. II]).) □

3. The Coagulation and Fragmentation Terms

We set

$$\mathbb{F} := L_1\big(Y, (1+y)\, dy\big) = L_1(Y, dy) \cap L_1(Y, y\, dy) \ ,$$

and denote by K_{coag} the closed linear subspace of $L_\infty(Y^2, d^2y)$ consisting of all γ satisfying

$$\gamma(y, y') = \gamma(y', y) \ , \qquad \text{a.a. } y, y' \in Y \ .$$

Given $\gamma \in K_{\text{coag}}$, we put

$$c_\gamma(v, w)(y) := \frac{1}{2} \int_0^y \gamma(y-y', y') v(y-y') w(y')\, dy' - v(y) \int_0^\infty \gamma(y, y') w(y')\, dy'$$

for $v, w \in \mathbb{F}$ and a.a. $y \in Y$. It is easily verified that

$$\big((\gamma, v, w) \mapsto c_\gamma(v, w)\big) \in \mathcal{L}(K_{\text{coag}}, \mathbb{F}, \mathbb{F}; \mathbb{F}) \ . \tag{3.1}$$

Moreover,

$$\int_Y c_\gamma(v, v)\, dy = -\frac{1}{2} \int_{Y^2} \gamma\, v \otimes v\, d^2y \tag{3.2}$$

and

$$\int_Y c_\gamma(v,v)y\,dy = 0 \tag{3.3}$$

for $v \in \mathbb{F}$.

Set $Y_\triangle^2 := \{\,(y,y') \in Y^2 \;;\; 0 \le y' \le y\,\}$ and define

$$(\varphi \mapsto \Phi_\varphi) \in \mathcal{L}\big(L_\infty(Y_\triangle^2, d^2y), L_{1,\mathrm{loc}}(Y,dy)\big)$$

by

$$\Phi_\varphi(y) := \frac{1}{y}\int_0^y \varphi(y,y')y'\,dy' \ , \qquad \text{a.a. } y \in Y \ ,$$

with the understanding that $\Phi_\varphi(0) := 0$. Then

$$K_{\mathrm{frag}} := \{\,\varphi \in L_\infty(Y_\triangle^2, d^2y) \;;\; \Phi_\varphi \in L_\infty(Y,dy)\,\}$$

is a Banach space with the norm $\varphi \mapsto \|\varphi\|_\infty + \|\Phi_\varphi\|_\infty$. For each $\varphi \in K_{\mathrm{frag}}$ we set

$$f_\varphi(v)(y) := \int_y^\infty \varphi(y',y)v(y')\,dy' - \Phi_\varphi(y)v(y) \ , \qquad v \in \mathbb{F} \ , \quad \text{a.a. } y \in Y \ .$$

It is obvious that

$$\big((\varphi,v) \mapsto f_\varphi(v)\big) \in \mathcal{L}(K_{\mathrm{frag}}, \mathbb{F}; \mathbb{F}) \ . \tag{3.4}$$

It is also easy to see that

$$\int_Y f_\varphi(v)\,dy = \int_Y \int_0^y (1 - y'/y)\varphi(y,y')\,dy'\,v(y)\,dy \tag{3.5}$$

and

$$\int_Y f_\varphi(v)y\,dy = 0 \tag{3.6}$$

for $\varphi \in K_{\mathrm{frag}}$ and $v \in \mathbb{F}$.

Suppose that (γ,u,v) is a map from $\mathbb{R}^n$ into $K_{\mathrm{coag}} \times \mathbb{F} \times \mathbb{F}$. Then we put

$$c_\gamma(u,v)(x) := c_{\gamma(x)}\big(u(x),v(x)\big) \ , \qquad x \in \mathbb{R}^n \ . \tag{3.7}$$

Similarly, if (φ,u) maps $\mathbb{R}^n$ into $K_{\mathrm{frag}} \times \mathbb{F}$ then

$$f_\varphi(u)(x) := f_{\varphi(x)}\big(u(x)\big) \ , \qquad x \in \mathbb{R}^n \ . \tag{3.8}$$

The following lemma establishes continuity properties of these maps.

Lemma 3.1. (i) *If $0 \le \tau < r < n$ and $\tau + n < 2\sigma < 2n$ then*

$$\big((\gamma,u,v) \mapsto c_\gamma(u,v)\big) \in \mathcal{L}\big(BUC^r[K_{\mathrm{coag}}], W_1^\sigma[\mathbb{F}], W_1^\sigma[\mathbb{F}]; W_1^\tau[\mathbb{F}]\big) \ .$$

If $n < \tau < r < \infty$ then

$$\big((\gamma,u,v) \mapsto c_\gamma(u,v)\big) \in \mathcal{L}\big(BUC^r[K_{\mathrm{coag}}], W_1^\tau[\mathbb{F}], W_1^\tau[\mathbb{F}]; W_1^\tau[\mathbb{F}]\big) \ .$$

(ii) *Suppose that* $0 \le \tau < r < \infty$. *Then*

$$\big((\varphi, u) \mapsto f_\varphi(u)\big) \in \mathcal{L}\big(BUC^r[K_{\text{frag}}], W_1^\tau[\mathbb{F}]; W_1^\tau[\mathbb{F}]\big) \ .$$

Proof. Thanks to (3.7) and (3.8) the maps

$$(\gamma, u, v) \mapsto c_\gamma(u, v) \quad \text{and} \quad (\varphi, u) \mapsto f_\varphi(u)$$

are point-wise multiplications induced by (3.1) and (3.4), respectively. Hence the assertion is a consequence of Lemma 2.1. □

Throughout the remainder J denotes a closed subinterval of $\mathbb{R}^+$ containing 0 and more than one point. For each subinterval J' of J we put $\dot{J}' := J' \backslash \{0\}$. Moreover, $\xi^+ := \xi \vee 0$ for $\xi \in \mathbb{R}$.

Corollary 3.2. *Suppose that* $\tau \in (-1, r)\backslash\mathbb{N}$ *with* $r > 0$ *and that*

$$\tau^+ + n < 2\sigma < 2n \qquad \text{if} \quad \tau < n \ ,$$

whereas $\sigma := \tau$ *if* $\tau > n$. *Also suppose that*

$$\big(t \mapsto \big(\gamma(t), \varphi(t)\big)\big) \in C^\rho\big(J, BUC^r[K_{\text{coag}} \times K_{\text{frag}}]\big)$$

for some $\rho \in \mathbb{R}^+$. *Then*

$$\big(t \mapsto (c_{\gamma(t)}, f_{\varphi(t)})\big) \in C^\rho\Big(J, \mathcal{L}^2\big(W_1^\sigma[\mathbb{F}], W_1^\tau[\mathbb{F}]\big) \times \mathcal{L}\big(W_1^\sigma[\mathbb{F}], W_1^\tau[\mathbb{F}]\big)\Big) \ .$$

In the following, we set

$$\chi(x,t) := \chi(t)(x) \ , \quad \chi(x,t,y,y') := \chi(x,t)(y,y')$$

for $\chi \in \{\gamma, \varphi\}$, $(x,t) \in \mathbb{R}^n \times J$, and $(y,y') \in Y \times Y$. We also put

$$c(x,t,y,u) := c_{\gamma(x,t)}(u,u)(y) \ , \quad f(x,t,y,u) := f_{\varphi(x,t)}(u)(y)$$

for $(x,t) \in \mathbb{R}^n \times J$, $y \in Y$, and $u \in \mathbb{F}$. Finally, $C(t,\cdot)$ and $F(t,\cdot)$ denote the Nemytskii operators induced by $c(\cdot,t,\cdot,\cdot)$ and $f(\cdot,t,\cdot,\cdot)$, respectively, that is,

$$C(t,u)(x) := c\big(x,t,\cdot,u(x)\big) \ , \quad F(t,u)(x) := f\big(x,t,\cdot,u(x)\big)$$

for $u \colon \mathbb{R}^n \to \mathbb{F}$ and $(x,t) \in \mathbb{R}^n \times J$. Then it follows that, given the hypotheses of Corollary 3.2,

$$\big(t \mapsto C(t,\cdot) + F(t,\cdot)\big) \in C^\rho\Big(J, C_b^\infty\big(W_1^\sigma[\mathbb{F}], W_1^\tau[\mathbb{F}]\big)\Big) \ , \tag{3.9}$$

where $C_b^\infty(E_1, E_0)$ is the vector space $C^\infty(E_1, E_0)$ endowed with the topology of uniform convergence of all derivatives on bounded subsets of E_1.

4. The Diffusion-Convection Semigroup

For $k \in \dot{\mathbb{N}}$ we set

$$L_\infty^k := L_\infty(Y, dy; \mathbb{R}^k)$$

and

$$L_{\infty,\mathrm{sym}}^{n\times n} := L_\infty(Y, dy; \mathbb{R}_{\mathrm{sym}}^{n\times n}) ,$$

where $\mathbb{R}_{\mathrm{sym}}^{n\times n}$ is the space of all real symmetric $(n \times n)$-matrices. Then, given $\sigma \in \mathbb{R}^+$,

$$\mathbb{E}^\sigma := BUC^\sigma[L_{\infty,\mathrm{sym}}^{n\times n}] \times BUC^\sigma[L_\infty^n] \times BUC^\sigma[L_\infty^n] \times BUC^{(\sigma-1)^+}[L_\infty^1] .$$

We put

$$\mathcal{A}[\boldsymbol{e}]u := -\nabla \cdot (\boldsymbol{a}\nabla u + \vec{a}u) + \vec{b} \cdot \nabla u + a_0 u , \qquad \boldsymbol{e} := (\boldsymbol{a}, \vec{a}, \vec{b}, a_0) \in \mathbb{E}^\sigma .$$

In the following, we write $\boldsymbol{e}(x, y)$ for $\boldsymbol{e}(x)(y)$ for $x \in \mathbb{R}^n$ and $y \in Y$.

Lemma 4.1. *If* $-1 \le s < \sigma - 1 < \infty$ *then*

$$(\boldsymbol{e} \mapsto \mathcal{A}[\boldsymbol{e}]) \in \mathcal{L}\Big(\mathbb{E}^\sigma, \mathcal{L}\big(W_1^{s+2}[\mathbb{F}], W_1^s[\mathbb{F}]\big)\Big) .$$

Proof. First observe that

$$L_\infty(Y, dy) \times \mathbb{F} \to \mathbb{F} , \quad (a, u) \mapsto au ,$$

with $au(y) := a(y)u(y)$ for a.a. $y \in Y$, is a muliplication. Now the assertion is an easy consequence of (2.2), (2.3), and Lemma 2.1. □

Suppose that $E_1 \overset{d}{\hookrightarrow} E_0$. Given $\kappa \ge 1$ and $\omega > 0$, we write

$$A \in \mathcal{H}(E_1, E_0; \kappa, \omega)$$

iff $A \in \mathcal{L}(E_1, E_0)$ with $\omega + A$ being an isomorphism from E_1 onto E_0 and

$$\kappa^{-1} \le \frac{\|(\lambda + A)u\|_0}{|\lambda|\,\|u\|_0 + \|u\|_1} \le \kappa , \qquad \mathrm{Re}\,\lambda \ge \omega , \quad u \in E_1\backslash\{0\} ,$$

where $\|\cdot\|_j$ is the norm in E_j. We also set

$$\mathcal{H}(E_1, E_0) := \bigcup_{\substack{\kappa\ge 1\\ \omega>0}} \mathcal{H}(E_1, E_0; \kappa, \omega) .$$

Then $\mathcal{H}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$, and $A \in \mathcal{H}(E_1, E_0)$ iff $-A$, considered as a linear operator in E_0 with domain E_1, generates a strongly continuous analytic semigroup on E_0, that is, in $\mathcal{L}(E_0)$ (see [3, Sect. I.1]).

After these preparations we can formulate the following basic generation result.

Theorem 4.2. *Suppose that $s \in (-1,\infty)\backslash\mathbb{N}$ with $\sigma > s+1$, and $\underline{\alpha}, M > 0$. Then there exist $\kappa \geq 1$ and $\omega > 0$ such that*

$$\mathcal{A}[\boldsymbol{e}] \in \mathcal{H}\big(W_1^{s+2}[\mathbb{F}], W_1^s[\mathbb{F}]; \kappa, \omega\big) \ ,$$

whenever $\boldsymbol{e} = (\boldsymbol{a}, \vec{a}, \vec{b}, a_0) \in \mathbb{E}^\sigma$ satisfies $\|\boldsymbol{e}\|_{\mathbb{E}^\sigma} \leq M$ and

$$\boldsymbol{a}(x,y)\xi\cdot\xi \geq \underline{\alpha}\,|\xi|^2 \ , \qquad x \in \mathbb{R}^n \ , \quad a.a.\ y \in Y \ , \quad \xi \in \mathbb{R}^n \ . \tag{4.1}$$

Proof. If $\boldsymbol{a}$ is a constant with respect to $x \in \mathbb{R}^n$ and $(\vec{a}, \vec{b}, a_0) = 0$ then the assertion follows from [4, Theorem 7.3], thanks to (2.7). The case of x-dependent $\boldsymbol{a}$ is then handled — as in the finite-dimensional case — by freezing the coefficients and a partition of unity argument. Finally, the lower order terms are included by employing a standard perturbation technique. Details are left to the reader (and will be given in [3, vol. II]). □

Suppose that $t \mapsto \boldsymbol{e}(t) : J \to \mathbb{E}^\sigma$. Then we put

$$(\boldsymbol{a}, \vec{a}, \vec{b}, a_0)(x,t) := \boldsymbol{e}(t)(x) \ , \qquad x \in \mathbb{R}^n \ , \quad t \in J \ ,$$

and

$$\mathcal{A}(t) := \mathcal{A}\big[\boldsymbol{e}(t)\big] \ , \qquad t \in J \ .$$

Using these notations we can prove the following theorem which is the basis for our further investigations.

Theorem 4.3. *Suppose that $s \in (-1,r)\backslash\mathbb{N}$ with $r > 0$ and*

$$\big(t \mapsto \boldsymbol{e}(t)\big) \in C^\rho(J, \mathbb{E}^{1+r})$$

for some $\rho \in \mathbb{R}^+$. Also suppose that there exists $\underline{\alpha} > 0$ such that

$$\boldsymbol{a}(x,t,y)\xi\cdot\xi \geq \underline{\alpha}\,|\xi|^2 \ , \qquad (x,t) \in \mathbb{R}^n \times J \ , \quad a.a.\ y \in Y \ , \quad \xi \in \mathbb{R}^n \ . \tag{4.2}$$

Then

$$\big(t \mapsto \mathcal{A}(t)\big) \in C^\rho\Big(J, \mathcal{H}\big(W_1^{s+2}[\mathbb{F}], W_1^s[\mathbb{F}]\big)\Big) \ . \tag{4.3}$$

Proof. This is an easy consequence of Lemma 4.1 and Theorem 4.2. □

Finally, suppose that

$$\big(t \mapsto h(t)\big) \in C^\rho\big(J, W_1^\tau[\mathbb{F}]\big)$$

for some $\tau \in (s,r)$, and put

$$R(t,u) := C(t,u) + F(t,u) + h(t) \ .$$

Then, given the hypotheses of Corollary 3.2 and Theorem 4.3, the initial value problem (1.1) can be rewritten as the semilinear parabolic evolution equation

$$\dot{u} + \mathcal{A}(t)u = R(t,u) \ , \quad t \in \dot{J} \ , \qquad u(0) = u^0 \tag{4.4}$$

in the Banach space $W_1^s[\mathbb{F}]$, where $\mathcal{A}$ satisfies (4.3) and

$$\big(t \mapsto R(t,\cdot)\big) \in C^\rho\Big(J, C_b^\infty\big(W_1^\sigma[\mathbb{F}], W_1^\tau[\mathbb{F}]\big)\Big) \ . \tag{4.5}$$

5. Existence, Uniqueness, And Regularity

By an admissible interpolation functor $(\cdot,\cdot)_\theta$ we mean an interpolation functor of exponent θ such that $E_1 \overset{d}{\hookrightarrow} (E_0, E_1)_\theta$ whenever $E_1 \overset{d}{\hookrightarrow} E_0$. Note that, in particular, the real interpolation functor $(\cdot,\cdot)_{\theta,1}$ has this property (We refer to [3, Sect. I.2] for a summary of the basic facts of interpolation theory).

We denote by $C_b^{1\text{-}}(E_1, E_0)$ the set of all maps from E_1 into E_0, which are uniformly Lipschitz continuous on bounded subsets of E_1. It is a locally convex space endowed with the family of seminorms

$$u \mapsto \sup_{x\in B} \|u(x)\|_0 + \sup_{\substack{x,y\in B \\ x\neq y}} \frac{\|u(x)-u(y)\|_0}{\|x-y\|_1} \ ,$$

where B runs through the family of all bounded subsets of E_1. As an easy consequence of the mean-value theorem we obtain

$$C_b^1(E_1, E_0) \hookrightarrow C_b^{1\text{-}}(E_1, E_0) \ , \tag{5.1}$$

where $C_b^1(E_1, E_0)$ is $C^1(E_1, E_0)$ endowed with the topology of uniform convergence of the functions and their first derivatives on bounded subsets of E_1.

Suppose that X is a locally convex space. Then $C^\rho(J, X)$ is for $0<\rho<1$ a locally convex space as well, where the topology is induced by the family of seminorms

$$u \mapsto \max_{0\le t\le T} p\big(u(t)\big) + \sup_{0\le s<t\le T} \frac{p\big(u(s)-u(t)\big)}{|s-t|^\rho} \ , \qquad T\in J \ ,$$

with p running through a family of seminorms defining the topology of X.

After these preparations we can prove the following fundamental existence, uniqueness, and continuity theorem for semilinear parabolic evolution equations.

Theorem 5.1. *Suppose that* $E_1 \overset{d}{\hookrightarrow} E_0$, *that* $0<\gamma\le\beta<\alpha<1$, *and that* $(\cdot,\cdot)_\theta$ *are admissible interpolation functors for* $\theta\in\{\alpha,\beta,\gamma\}$. *Put*

$$E_\theta := (E_0, E_1)_\theta$$

and suppose that

$$\Big(t\mapsto \big(A(t), g(t,\cdot)\big)\Big) \in C^\rho\big(J, \mathcal{H}(E_1,E_0)\times C_b^{1\text{-}}(E_\beta, E_\gamma)\big)$$

for some $\rho\in(0,1)$.

Then, given $u^0\in E_\alpha$, *the initial value problem*

$$\dot u + A(t)u = g(t,u) \ , \quad t\in \dot J \ , \qquad u(0)=u^0 \tag{5.2}$$

has a unique maximal solution

$$u(\cdot,u^0):=u(\cdot,u^0,A,g)\in C\big(J(u^0),E_\alpha\big)\cap C\big(\dot J(u^0),E_1\big)\cap C^1\big(\dot J(u^0),E_0\big)\ .$$

The maximal interval of existence, $J(u^0):=J(u^0,A,g)$*, is open in* J*. If*

$$\sup_{t\in J(u^0)\cap[0,T]}\|u(t,u^0)\|_\alpha<\infty \tag{5.3}$$

for each $T\in J$ *then* $u(\cdot,u^0)$ *is a global solution, that is,* $J(u^0)=J$.

For each $T\in\dot J(u^0)$ *there exists a neighborhood* $\mathcal{U}$ *of* (u^0,A,g) *in*

$$E_\alpha\times C^\rho\big(J,\mathcal{H}(E_1,E_0)\big)\times C^\rho\big(J,C_b^{1\text{-}}(E_\beta,E_\gamma)\big)$$

such that $J(\widetilde u^0,\widetilde A,\widetilde g)\supset[0,T]$ *for* $(\widetilde u^0,\widetilde A,\widetilde g)\in\mathcal{U}$ *and such that*

$$u(\cdot,\widetilde u^0,\widetilde A,\widetilde g)\to u(\cdot,u^0,A,g)\ \text{in}\ C\big([0,T],E_\alpha\big)$$

as $(\widetilde u^0,\widetilde A,\widetilde g)\to(u^0,A,g)$ *in* $\mathcal{U}$.

Proof. Put $\delta:=\rho\wedge(\alpha-\beta)$. Fix $T\in\dot J$ and set

$$g_v(t):=g\big(t,v(t)\big)\ ,\qquad 0\le t\le T\ ,$$

for $v\in C^\delta\big([0,T],E_\beta\big)$. Then $g_v\in C^\delta\big([0,T],E_\gamma\big)$ and [3, Theorems II.1.2.1 and II.5.3.1] guarantee the existence of a unique solution

$$u(\cdot;v)\in C\big([0,T],E_\alpha\big)\cap C\big((0,T],E_1\big)\cap C^1\big((0,T],E_0\big) \tag{5.4}$$

of the linear Cauchy problem

$$\dot u+A(t)u=g_v(t)\ ,\quad 0<t\le T\ ,\qquad u(0)=u^0\ . \tag{5.5}$$

If $w\in C^\delta\big([0,T],E_\beta\big)$ then [3, Theorem II.5.2.1] implies

$$\|u(t;v)-u(t;w)\|_\beta\le cT^{1-\beta}\,\|u-v\|_{C([0,T],E_\beta)}\ ,\qquad 0\le t\le T\ ,$$

where c is independent of v and w if $v\big([0,T]\big)$ and $w\big([0,T]\big)$ remain in a given bounded subset of E_β. Thus, by making T smaller, if necessary, the contraction mapping principle implies the existence of a fixed point $\overline u\in C\big([0,T],E_\beta\big)$ of $v\mapsto u(\cdot;v)$. Next we infer from [3, Theorem II.5.3.1] that $\overline u\in C\big([0,T],E_\alpha\big)\cap C^\delta\big([0,T],E_\beta\big)$. Hence $\overline u=u(\cdot;\overline u)$ and (5.4) imply that $\overline u$ is a solution of (5.2) on $[0,T]$. Now a standard continuation argument shows that $\overline u$ has an extension $u(\cdot,u^0)$ to a maximal solution of (5.2), and that the corresponding maximal interval of existence is open in J. The uniqueness assertion is obvious.

Suppose that (5.3) is satisfied for each $T\in J$ and $J(u^0)\ne J$. Then the extension argument can be applied to the initial value $u(t^*,u^0)$, where t^* is sufficiently close to the right end point of $J(u^0)$, to obtain an extension of $u(\cdot,u^0)$ over an interval which is strictly larger than $J(u^0)$. Since this contradicts the maximality of $J(u^0)$ it follows that (5.3) implies $J(u^0)=J$.

Lastly, it is not difficult to deduce the stated continuity assertion from [3, Theorem II.5.2.1]. (Recall that $\mathcal{H}(E_1,E_0)$ is open in $\mathcal{L}(E_1,E_0)$.) For details we refer to [3, vol. II]. □

It is now easy to establish the well-posedness of system (1.1).

Theorem 5.2. *Suppose that* $r,\rho>0$ *and*

$$(-2+n/2)\vee(-1)<s<r\ , \qquad s\notin\mathbb{N}\ . \tag{5.6}$$

Also suppose that

$$\Bigl(t\mapsto\bigl(\boldsymbol{e},(\gamma,\varphi),h\bigr)(t)\Bigr)\in C^\rho\bigl(J,\mathbb{E}^{1+r}\times BUC^r[K_{\rm coag}\times K_{\rm frag}]\times W_1^\tau[\mathbb{F}]\bigr)$$

for some $\tau>s$*, such that* (4.2) *is satisfied for some* $\underline{\alpha}>0$*. Finally, assume that*

$$(s^++n)/2<\sigma<n\wedge(s+2)\ , \qquad s<n\ , \tag{5.7}$$

and

$$s<\sigma<s+2\ , \qquad s>n\ , \tag{5.8}$$

with $\sigma\notin\mathbb{N}$*. Then, given any* $u^0\in W_1^\sigma[\mathbb{F}]$*, the coagulation-fragmentation system* (1.1)*, that is, problem* (4.4)*, has a unique maximal solution*

$$u(\cdot,u^0)\in C\bigl(J(u^0),W_1^\sigma[\mathbb{F}]\bigr)\cap C\bigl(\dot J(u^0),W_1^{s+2}[\mathbb{F}]\bigr)\cap C^1\bigl(\dot J(u^0),W_1^s[\mathbb{F}]\bigr)\ , \tag{5.9}$$

where the maximal interval of existence, $J(u^0)$*, is open in* J*.*

This solution, $u(\cdot,u^0,\boldsymbol{e},\gamma,\varphi,h):=u(\cdot,u^0)$*, depends continuously on the data in the following sense: given* $T\in J(u^0)$*, there exists a neighborhood* $\mathcal{U}$ *of* $\bigl(u^0,\bigl(\boldsymbol{e},(\gamma,\varphi),h\bigr)\bigr)$ *in*

$$W_1^\sigma[\mathbb{F}]\times BUC^\rho\bigl(J,\mathbb{E}^{1+r}\times BUC^r[K_{\rm coag}\times K_{\rm frag}]\times W_1^\tau[\mathbb{F}]\bigr)$$

such that $u(\cdot,\widetilde u^0,\widetilde{\boldsymbol{e}},\widetilde\gamma,\widetilde\varphi,\widetilde h)$ *exists on* $[0,T]$ *and*

$$u(\cdot,\widetilde u^0,\widetilde{\boldsymbol{e}},\widetilde\gamma,\widetilde\varphi,\widetilde h)\to u(\cdot,u^0,\boldsymbol{e},\gamma,\varphi,h)\quad\text{in } C\bigl([0,T],W_1^\sigma[\mathbb{F}]\bigr)$$

as $(\widetilde u^0,\widetilde{\boldsymbol{e}},\widetilde\gamma,\widetilde\varphi,\widetilde h)\to(u^0,\boldsymbol{e},\gamma,\varphi,h)$ *in* $\mathcal{U}$*.*

Proof. First note that (5.6) implies $s>-1$ if $n=1,2$, and $s>-1/2$ if $n=3$. Moreover, (5.6) guarantees that condition (5.7) is not void.

By making τ smaller, if necessary, we can assume that

$$(\tau^++n)/2<\sigma<n\wedge(s+2) \qquad \text{if} \quad s<n\ ,$$

and that $\tau<\sigma$ if $s>n$. Then we can fix σ_1 such that

$$\tau\vee(\tau^++n)/2<\sigma_1<\sigma<n\wedge(s+2)\ , \qquad s<n\ ,$$

and

$$s < \tau < \sigma_1 < \sigma \ , \qquad s > n \ .$$

We can also assume that $\tau, \sigma_1 \notin \mathbb{Z}$.

We set $E_0 := W_1^s[\mathbb{F}]$ and $E_1 := W_1^{s+2}[\mathbb{F}]$. We also set $E_\theta := (E_0, E_1)_{\theta,1}$ for $0 < \theta < 1$. Then it follows from (2.7) and [4, formula (5.7)] (also see [3, vol. II]) that

$$E_\theta \doteq W_1^{s+2\theta}[\mathbb{F}] \ , \qquad s + 2\theta \notin \mathbb{Z} \ . \tag{5.10}$$

Put $\alpha := (\sigma - s)/2$, $\beta := (\sigma_1 - s)/2$, $\gamma := (\tau - s)/2$. Then Theorem 4.3 and assertions (4.5) and (5.1) imply that problem (4.4) satisfies the hypotheses of Theorem 5.1 (with g replaced by R, of course). This proves everything. $\square$

The following proposition shows that $u(t, u^0)$ is independent of the choice of s and σ, provided $t > 0$, and that problem (4.4) enjoys a smoothing property.

Proposition 5.3. *Presuppose the hypotheses of Theorem* 5.2 *and fix* $\overline{\sigma}$ *in* $(n/2, n \wedge 2)$. *Then, given* $u^0 \in W_1^{\overline{\sigma}}[\mathbb{F}]$, *problem* (4.4) *has a unique maximal solution*

$$u(\cdot, u^0) \in C\big(J(u^0), W_1^{\overline{\sigma}}[\mathbb{F}]\big) \cap C\big(\dot{J}(u^0), W_1^{s+2}[\mathbb{F}]\big) \cap C^1\big(\dot{J}(u^0), W_1^{s}[\mathbb{F}]\big) \ ,$$

and $J(u^0)$ *is independent of* s *satisfying* (5.6).

Proof. Fix $\overline{s} \in (-1/2, 0)$ such that $\overline{\sigma} < n \wedge (\overline{s} + 2)$. Then Theorem 5.2 guarantees the existence of a unique maximal solution $u(\cdot, u^0)$ in

$$C\big(J(u^0), W_1^{\overline{\sigma}}[\mathbb{F}]\big) \cap C\big(\dot{J}(u^0), W_1^{\overline{s}+2}[\mathbb{F}]\big) \cap C^1\big(\dot{J}(u^0), W_1^{\overline{s}}[\mathbb{F}]\big) \ . \tag{5.11}$$

If $\overline{s} < s$ then we fix any $t^* \in \dot{J}(u^0)$ and put $u^* := u(t^*, u^0)$. For some sufficiently small $\delta \in (0, \rho)$ we set $\sigma := \overline{s} + 2 - 2\delta$ and choose $s_1 \in \mathbb{R}^+ \setminus \mathbb{N}$ such that it satisfies (5.7) or (5.8), respectively. Then, setting

$$J^* := \big\{\, t \in \mathbb{R}^+ \ ; \ t + t^* \in J(u^0) \,\big\} \ , \quad \mathcal{A}^*(t) := \mathcal{A}(t + t^*)$$

and

$$R^*(t) := R\big(t + t^*, u(t + t^*, u^0)\big) \ , \qquad t \in J^* \ ,$$

we consider the linear initial value problem

$$\dot{v} + \mathcal{A}^*(t) v = R^*(t) \ , \quad t \in \dot{J}^* \ , \qquad v(0) = u^* \ . \tag{5.12}$$

From (5.10) and [3, Proposition II.1.1.2] we infer that

$$\big(t \mapsto u(t + t^*, u^0)\big) \in C^\delta\big(J^*, W_1^{\sigma}[\mathbb{F}]\big) \ .$$

Thus (3.9) and (5.1) imply $R^* \in C^\delta\big(J^*, W_1^{s_1}[\mathbb{F}]\big)$. Consequently, a unique solution

$$v \in C\big(J^*, W_1^{s_1}[\mathbb{F}]\big) \cap C\big(\dot{J}^*, W_1^{s_1+2}[\mathbb{F}]\big) \cap C^1\big(\dot{J}^*, W_1^{s_1}[\mathbb{F}]\big) \tag{5.13}$$

of (5.12) is guaranteed by [3, Theorem II.1.2.1]. Clearly, v is also the unique solution of (5.12) in

$$C\big(J^*, W_1^{\overline{s}}[\mathbb{F}]\big) \cap C\big(\dot{J}^*, W_1^{\overline{s}+2}[\mathbb{F}]\big) \cap C^1\big(\dot{J}^*, W_1^{\overline{s}}[\mathbb{F}]\big) \ .$$

In this class $t \mapsto u(t+t^*, u^0)$ is a solution of (5.12) as well. Hence

$$v(t) = u(t+t^*, u^0) \ , \qquad t \in J^* \ ,$$

by uniqueness. This proves that $t \mapsto u(t+t^*, u^0)$ possesses the regularity properties (5.13), where $s_1 > \overline{s}$. Now it is easy to see that we can repeat this bootstrapping argument a finite number of times to reach s, which proves the assertion. □

We denote by $C_0^m[E] := C_0^m(\mathbb{R}^n, E)$ the closed subspace of $BUC^m[E]$ consisting of all u such that $\partial^\alpha u$ vanishes at infinity for $|\alpha| \le m$. Moreover,

$$C_0^\infty[E] := \bigcap_{m \ge 0} C_0^m[E] \ ,$$

equipped with the natural projective limit topology. Similar definitions apply to $\mathfrak{F}^\infty$ for $\mathfrak{F} \in \{BUC, W_1\}$ and to $\mathbb{E}^\infty$.

Corollary 5.4. *Suppose that* $\rho > 0$ *and*

$$\Big(t \mapsto \big(\boldsymbol{e}, (\gamma, \varphi), h\big)(t)\Big) \in C^\rho\big(J, \mathbb{E}^\infty \times BUC^\infty[K_{\mathrm{coag}} \times K_{\mathrm{frag}}] \times W_1^\infty[\mathbb{F}]\big)$$

such that (4.2) *is satisfied. Then, if* $u^0 \in W_1^\sigma[\mathbb{F}]$ *for some* $\sigma \in (n/2, n \wedge 2)$*, the unique maximal solution of* (4.4) *belongs to* $C^1\big(\dot{J}(u^0), C_0^\infty[\mathbb{F}]\big)$.

Proof. This follows from the preceding proposition and the Sobolev embedding

$$W_1^s[E] \overset{d}{\hookrightarrow} C_0^m[E] \ , \qquad s > m+n \ , \quad m \in \mathbb{N} \ , \tag{5.14}$$

which is also valid in the case of an arbitrary Banach space E (cf. [3, vol. II]). □

It should be observed that this corollary applies, in particular, if all data are independent of $x \in \mathbb{R}^n$. Moreover, it can also be shown that the solution is more regular in the time variable than stated here. Roughly speaking, $\dot{u}$ is ρ-Hölder continuous with respect to $t > 0$ (see [3, Theorem II.1.2.1]). We do not go into details.

6. Positivity

Let X be a vector space ordered by a proper cone X^+, the positive cone of X, by means of $x \le y$ iff $y - x \in X^+$. If S is any set, then X^S is also an ordered vector space with respect to the 'natural' (that is, point-wise) order induced by the positive cone $(X^+)^S$. Thus $u \le v$ for $u, v \in X^S$ iff $u(s) \le v(s)$ for $s \in S$.

If X and Y are ordered vector spaces, their product $X \times Y$ is an ordered vector space as well with respect to the 'natural product order', whose positive cone is $X^+ \times Y^+$.

If X is a locally convex space then X is an ordered locally convex space if X is an ordered vector space whose positive cone is closed. If X is an ordered locally convex space and Y is a locally convex space with $Y \hookrightarrow X$, then Y is given its 'natural' order induced by X, whose positive cone equals $Y \cap X^+$. Note that Y is then an ordered locally convex space as well. Finally, the real line is always given its natural order whose positive cone is $\mathbb{R}^+$.

From these definitions it follows that each one of the spaces $L_p(M, \mu; E)$, $1 \le p \le \infty$, and $BUC^s[E]$, $W_1^s[E]$, $s \in \mathbb{R}^+$, is an ordered Banach space, provided E is an ordered Banach space. (Of course, in the case of $L_p(M, \mu; E)$ the point-wise order refers to the point-wise order μ-a.a., that is, the point-wise order of the equivalence classes.) In particular, $\mathbb{F}$ is an ordered Banach space with respect to the natural order induced by the positive cone $\mathbb{F}^+ = L_1^+\big(Y, (1+y)dy\big)$, and all function spaces considered below are given their natural orders.

After these preparations we prove an approximation result for positive cones. For this we denote by $C_c(Y)$ the space of all continuous functions on Y with compact supports.

Lemma 6.1. *$\mathcal{D}^+(\mathbb{R}^n) \otimes C_c^+(Y)$ is dense in $W_1^s[\mathbb{F}]$ for $s \in \mathbb{R}^+$.*

Proof. Thanks to (2.2) it suffices to consider $s \in \mathbb{N}$. Standard cutting and mollification arguments show that $\mathcal{D}^+[\mathbb{F}]$ is dense in $W_1^s[\mathbb{F}]^+$. From the proof of [3, Proposition V.2.4.1] we infer that $\mathcal{D}^+(\mathbb{R}^n) \otimes \mathbb{F}^+$ is dense in $\mathcal{D}^+[\mathbb{F}]$. Now the assertion follows from the well-known fact that $C_c^+(Y)$ is dense in $\mathbb{F}^+$. $\square$

A bounded linear operator B on an ordered Banach space E is positive (in symbols: $B \ge 0$) if $B(E^+) \subset E^+$. A closed linear operator A in E is resolvent positive if there exists $\lambda_0 \ge 0$ such that $[\lambda_0, \infty)$ belongs to the resolvent set $\rho(-A)$ of $-A$ and $(\lambda + A)^{-1} \ge 0$ for $\lambda \ge \lambda_0$.

Proposition 6.2. *Suppose that $s \in (0, r)\backslash\mathbb{N}$, and let $\boldsymbol{e} \in \mathbb{E}^{1+r}$ satisfy* (4.1)*. Then $\mathcal{A}[\boldsymbol{e}]$ is resolvent positive on $W_1^s[\mathbb{F}]$.*

Proof. Theorem 4.2 implies that $\mathcal{A} := \mathcal{A}[\boldsymbol{e}]$ is a closed linear operator in $W_1^s[\mathbb{F}]$ with $[\omega, \infty) \subset \rho(-\mathcal{A})$ for some $\omega > 0$.

(i) Suppose that $s > n$ and put $\mathbb{F}_\infty := L_\infty(Y, dy)$. Then the proof of Theorem 4.2 applies to give

$$\mathcal{A} \in \mathcal{H}\big(W_1^{s+2}[\mathbb{F}_\infty], W_1^s[\mathbb{F}_\infty]\big) \ .$$

Hence there exists $\omega_\infty > 0$ such that $[\omega_\infty, \infty) \subset \rho(-\mathcal{A}_\infty)$, where $\mathcal{A}_\infty$ denotes $\mathcal{A}$, but considered as a linear operator in $W_1^s[\mathbb{F}_\infty]$. Put

$$\lambda_0 := \omega \vee \omega_\infty \vee \big(\|a_0\|_{BUC[L_\infty^1]} + \|\nabla \cdot \vec{a}\|_{BUC[L_\infty^1]}\big) \ .$$

Fix $\lambda \geq \lambda_0$ and $v \in \mathcal{D}^+(\mathbb{R}^n) \otimes C_c(Y)$, and put $u := (\lambda + \mathcal{A}_\infty)^{-1} v$. Then $u \in W_1^{s+2}[\mathbb{F}_\infty]$ and

$$\big(\lambda + \mathcal{A}(x,y)\big)u(x,y) = v(x,y) \ , \qquad x \in \mathbb{R}^n \ , \quad \text{a.a. } y \in Y \ .$$

Note that (5.14) implies $u \in C_0^2[\mathbb{F}_\infty]$. Thus it follows that, for a.a. $y \in Y$, the function $u(\cdot, y)$ belongs to $C_0^2(\mathbb{R}^n)$ and satisfies the elliptic differential inequality

$$-\boldsymbol{a}(\cdot,y) : \nabla^2 u(\cdot,y) + \vec{c}(\cdot,y) \cdot \nabla u(\cdot,y) + d(\cdot,y)u(\cdot,y) \geq 0 \tag{6.1}$$

on $\mathbb{R}^n$ where $d := \lambda + a_0 - \nabla \cdot \vec{a} \geq 0$. Here $\nabla^2 w$ denotes the Hessian of w and $A : B$ is the trace of the matrix product $AB^\top$. The coefficients of (6.1) are uniformly bounded on $\mathbb{R}^n$. Since $u(\cdot, y)$ vanishes at infinity, the classical maximum principle implies that $u(\cdot, y)$ is nonnegative. This being true for a.a. $y \in Y$, we see that $u \in W_1^{2+s}[\mathbb{F}_\infty]^+$.

Since $\mathcal{D}^+(\mathbb{R}^n) \otimes C_c^+(Y) \subset W_1^s[\mathbb{F}]^+$ and $\lambda > \omega$, Theorem 4.2 guarantees that u belongs to $W_1^{s+2}[\mathbb{F}]$ as well. Consequently,

$$(\lambda + \mathcal{A})^{-1}\big(\mathcal{D}^+(\mathbb{R}^n) \otimes C_c^+(Y)\big) \subset W_1^s[\mathbb{F}]^+ \ , \qquad \lambda \geq \lambda_0 \ .$$

Now we infer from Lemma 6.1, the continuity of $(\lambda + \mathcal{A})^{-1}$ on $W_1^s[\mathbb{F}]^+$, and the closedness of the positive cone that $\mathcal{A}$ is resolvent positive on $W_1^s[\mathbb{F}]$.

(ii) Suppose that $s < n$. Fix $n < t < r_1 < \infty$ with $t \notin \mathbb{N}$ and suppose that $\boldsymbol{e} \in \mathbb{E}^{1+r_1}$. It follows from (i) that $\mathcal{A}$ is resolvent positive on $W_1^t[\mathbb{F}]$. Lemma 6.1 also implies that $W_1^t[\mathbb{F}]^+$ is dense in $W_1^s[\mathbb{F}]^+$. Thus, since $(\lambda + \mathcal{A})^{-1}$ exists and is continuous on $W_1^s[\mathbb{F}]$ for sufficiently large λ, we see, once more by approximation, that $\mathcal{A}$ is resolvent positive on $W_1^s[\mathbb{F}]$.

Finally, fix $r_0 \in (s, r)$ if $r < r_1$ and suppose that $\boldsymbol{e} \in \mathbb{E}^{1+r}$. Then it is well-known that there exists a sequence $(\boldsymbol{e}_j)$ in $\mathbb{E}^{1+r_1}$ converging in $\mathbb{E}^{1+r_0}$ towards $\boldsymbol{e}$. Hence we deduce from Lemma 4.1 and the continuity of the inversion map $B \to B^{-1}$ that

$$\big(\lambda + \mathcal{A}[\boldsymbol{e}_j]\big)^{-1} \to (\lambda + \mathcal{A})^{-1} \quad (j \to \infty)$$

in $\mathcal{L}\big(W_1^s[\mathbb{F}]\big)$ for sufficiently large λ, since we can assume that $\boldsymbol{e}_j$ satisfies (4.1) for all $j \in \mathbb{N}$ with $\underline{\alpha}$ replaced by some smaller positive number. Thus the resolvent positivity follows in this case also. $\square$

After these preparations we can prove the main result of this section, namely that the solution $u(\cdot, u^0)$ of (4.1) is positive whenever $u^0 \geq 0$ and (γ, φ) and h are positive. (Recall that an element x of an ordered vector space is positive iff $x \geq 0$.)

Theorem 6.3. *Let the assumptions of Theorem 5.2 be satisfied and suppose that $(\gamma, \varphi) \geq 0$ and $h \geq 0$. Then $u^0 \geq 0$ implies $u(\cdot, u^0) \geq 0$.*

Proof. (i) Suppose that $s > n$. Then Theorem 5.2 and (5.14) imply

$$u := u(\cdot, u^0) \in C\big(J(u^0), C_0[\mathbb{F}]\big) \ .$$

Fix $T \in \dot{J}(u^0)$ and put $\omega_0 := \|\gamma\|_\infty \max_{0 \leq t \leq T} \|u(t)\|_{C_0[\mathbb{F}]}$. Then

$$\Big| \int_0^\infty \gamma(x, t, y, y') u(x, t, y')\, dy' \Big| \leq \omega_0 \tag{6.2}$$

for $(x, t) \in \mathbb{R}^n \times [0, T]$ and a.a. $y \in Y$. Set

$$p_\gamma(v, w)(y) := \frac{1}{2} \int_0^y \gamma(y - y', y') v(y - y') w(y')\, dy'$$

and

$$q_\gamma(v, w)(y) := v(y) \int_0^\infty \gamma(y, y') w(y')\, dy'$$

for $y \in Y$ and $v, w \in \mathbb{F}$. Also put $\omega := \omega_0 + \|\Phi_\varphi\|_\infty$ and

$$G(t, v) := p_{\gamma(t)}(v, v) - q_{\gamma(t)}(v, u) + \omega v + F(t, v) + h(t)$$

for $0 \leq t \leq T$ and $v \in \mathbb{F}$. Then

$$G\big(t, u(t)\big) = R\big(t, u(t)\big) + \omega u(t) \ , \qquad 0 \leq t \leq T \ ,$$

and (6.2) and the structure of F imply

$$G\big(t, v(t)\big) \geq 0 \ , \qquad v \in C\big([0, T], C^+[\mathbb{F}]\big) \ , \quad 0 \leq t \leq T \ . \tag{6.3}$$

Lastly, set $\mathcal{A}_\omega := \omega + \mathcal{A}$. Then u is the unique solution of the initial value problem

$$\dot{v} + \mathcal{A}_\omega(t) v = G(t, v) \ , \quad 0 \leq t \leq T \ , \qquad v(0) = u^0 \tag{6.4}$$

in $W_1^s[\mathbb{F}]$.

Denote by U the parabolic evolution operator for $\mathcal{A}_\omega$, whose existence is guaranteed by [3, Corollary II.4.4.2]. Put

$$V(v)(t) := \int_0^t U(t, \tau) G\big(\tau, v(\tau)\big)\, d\tau \ , \qquad v \in W_1^\sigma[\mathbb{F}] \ , \quad 0 \leq t \leq T \ .$$

Then (6.4) implies that u solves the nonlinear Volterra integral equation

$$u = U(\cdot, 0) u^0 + V(u) \tag{6.5}$$

in $C([0,T], W_1^\sigma[\mathbb{F}])$. If T is sufficiently small then equation (6.5) can be solved by the method of successive approximations, that is, the sequence (u_n), determined by $u_0 := u^0$ and

$$u_{n+1} := U(\cdot,0)u^0 + V(u_n) \ , \qquad n \in \mathbb{N} \ ,$$

converges in $W_1^\sigma[\mathbb{F}]$ towards u. Since $\mathcal{A}_\omega$ is resolvent positive by Proposition 6.2, it follows from [3, Theorems II.6.4.1 and II.6.4.2] that U is positive. Thus (5.14) and (6.3) entail that $u_n \geq 0$ for $n \in \mathbb{N}$. Consequently, $u \geq 0$.

These considerations show that there exists $T \in \dot{J}(u^0)$ such that $u^0 \geq 0$ implies $u(t,u^0) \geq 0$ for $0 \leq t \leq T$. Set

$$T^* := \max\{\, T \in \dot{J}(u^0) \ ; u(t,u^0) \geq 0 \,\} \ .$$

If $T^* < \sup J(u^0)$ then we apply the above reasoning to the initial value problem

$$\dot{v} + \mathcal{A}(t+T^*)v = R(t+T^*, v) \ , \quad t \in J(u^0) - T^* \ , \qquad v(0) = u(T^*, u^0)$$

to find that $u(t,u^0) \geq 0$ on $[0, T^* + T^{**}]$ for some $T^{**} > 0$. Since this contradicts the choice of T^*, we see that $T^* = \sup J(u^0)$, that is, $u(\cdot,u^0) \geq 0$.

(ii) Suppose that $s < n$ and $r = \infty$. Fix $s_1, \sigma_1 \notin \mathbb{N}$ with

$$n < s_1 < \sigma_1 < s_1 + 2$$

and suppose that $u^0 \in W_1^{\sigma_1}[\mathbb{F}]^+$. Then it follows from (i) that $u(\cdot,u^0) \geq 0$ in $W_1^{\sigma_1}[\mathbb{F}]$, hence in $W_1^\sigma[\mathbb{F}]$ by (2.2). Since $W_1^{\sigma_1}[\mathbb{F}]^+$ is dense in $W_1^\sigma \mathbb{F}]^+$, the continuous dependence of $u(\cdot,u^0)$ on u^0 in $W_1^\sigma[\mathbb{F}]$, as guaranteed by Theorem 5.2, implies $u(\cdot,u^0) \geq 0$ in $W_1^\sigma[\mathbb{F}]$ for $u^0 \in W_1^\sigma[\mathbb{F}]^+$.

(iii) Lastly, suppose that $s < n$ and $r > s$. Then, as in step (ii) of the proof of Proposition 6.2, we approximate $(u^0, \boldsymbol{e}, (\gamma,\varphi), h)$ by smooth functions and derive the positivity of $u(\cdot,u^0)$ from its continuous dependence on the data and from (ii). □

7. Conservation of Volume

Throughout this section we suppose that

$$\left.\begin{array}{l} r,\rho,\tau > 0, \text{ and } t \mapsto \big(\boldsymbol{e},(\gamma,\varphi),h\big)(t) \text{ belongs to} \\ C^\rho\big(J, \mathbb{E}^{1+r} \times BUC^r[K_{\mathrm{coag}}^+ \times K_{\mathrm{frag}}^+] \times W_1^\tau[\mathbb{F}]^+\big) \\ \text{with (4.2) being satisfied. Moreover,} \\ \vec{b} = 0 \ , \quad n/2 < \sigma < n \ , \quad u^0 \in W_1^\sigma[\mathbb{F}]^+ \ . \end{array}\right\} \tag{7.1}$$

We fix $s \in \big(0, \tau \wedge (2\sigma - n) \wedge r\big)$ and denote by $u := (\cdot, u^0)$ the unique maximal solution of the coagulation-fragmentation system (1.1). Theorem 5.2 implies that u is well-defined and satisfies (5.9). Thus

$$u \in C\big(\dot{J}(u^0), W_1^2[\mathbb{F}]\big) \cap C^1\big(\dot{J}(u^0), L_1[\mathbb{F}]\big) \ , \tag{7.2}$$

and Theorem (6.3) guarantees $u \geq 0$.

Lemma 7.1. *If* $v \in W_1^2[\mathbb{F}]$ *then*

$$\int_{\mathbb{R}^n} \int_Y \mathcal{A}(t) v\, y^j \,dy\,dx = \int_{\mathbb{R}^n} \int_Y a_0 v\, y^j \,dy\,dx \ , \qquad j = 0, 1 \ ,$$

for $t \in J$.

Proof. Choose $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi(x) = 1$ for $|x| \leq 1$ and set $\chi_\varepsilon(x) := \chi(\varepsilon x)$ for $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Then

$$-\int_{\mathbb{R}^n} \int_Y \nabla \cdot (\boldsymbol{a} \nabla v + \vec{a} v)\, y^j \,dy\, \chi_\varepsilon \,dx = \int_{|x| \geq 1/\varepsilon} \int_Y (\boldsymbol{a} \nabla v + \vec{a} v)\, y^j \,dy \cdot \nabla \chi_\varepsilon \,dx \ .$$

Since $\|\nabla \chi_\varepsilon\|_\infty \leq \varepsilon \|\nabla \chi\|_\infty$, we see that the last integral tends to zero as $\varepsilon \to 0$. Now the assertion follows. $\square$

We denote by

$$V(t) := \int_{\mathbb{R}^n} \int_Y u(t)\, y dy\,dx \ , \qquad t \in J(u^0) \ ,$$

the total particle volume at time t. Similarly,

$$A_0(t) := \int_{\mathbb{R}^n} \int_Y a_0(t) u(t)\, y dy\,dx$$

and

$$H(t) := \int_{\mathbb{R}^n} \int_Y h(t)\, y dy\,dx$$

are the total absorbed particle volume and the total particle input, respectively, at time t.

The following theorem shows, in particular, that the total particle volume is conserved if neither absorption nor particle input takes place.

Theorem 7.2.

$$V(t) = V(0) + \int_0^t \big(H(\tau) - A_0(\tau)\big)\,d\tau \ , \qquad t \in J(u^0) \ .$$

Proof. By integrating

$$\dot{u}(t) + \mathcal{A}(t) u(t) = R\big(t, u(t)\big) \tag{7.3}$$

over $\mathbb{R}^n \times Y$ with respect to the measure $dx \otimes y dy$ and taking (3.3) and (3.6) into account we obtain

$$\dot{V}(t) = H(t) - A_0(t) \ , \tag{7.4}$$

thanks to (7.2) and Lemma 7.1. Now the assertion follows by integrating (7.4) from t_0 to t, where $0 < t_0 < t$, and letting t_0 tend to zero. $\square$

Corollary 7.3. *Put* $\alpha^\pm := \|a_0^\pm\|_\infty$. *Then*

$$e^{-\alpha^+ t}V(0)+\int_0^t e^{-\alpha^+(t-\tau)}H(\tau)\,d\tau \le V(t) \le e^{\alpha^- t}V(0)+\int_0^t e^{\alpha^-(t-\tau)}H(\tau)\,d\tau$$

for $t \in J(u^0)$.

Proof. Note that

$$-\alpha^- V(t) \le A_0(t) \le \alpha^+ V(t)\ , \qquad t \in J(u^0)\ .$$

Thus (7.3) entails the differential inequalities

$$H(t) - \alpha^+ V(t) \le \dot V(t) \le H(t) + \alpha^- V(t)\ , \qquad t \in J(u^0)\ ,$$

which imply the assertion. □

8. Global Existence

Finally, we discuss the problem of global existence, that is, the question whether $J(u^0) = J$.

Theorem 8.1. *Let assumption* (7.1) *be satisfied. Then* $u := u(\cdot, u^0)$ *exists globally, provided one of the following assumptions is satisfied:*

(i) *There is no coagulation, that is,* $\gamma = 0$.
(ii) $n = 1$.
(iii) $\mathcal{A}$ *is independent of* $y \in Y$.

Proof. (i) is obvious since in this case (4.4) is a linear evolution equation.

(ii) Set $\mathbb{F}_j := L_1(Y, y^j dy)$ for $j = 0, 1$. Then, by integrating (7.3), we infer from Lemma 7.1, the positivity of u, and (3.2) and (3.5) that

$$\begin{aligned}\|u(t)\|^{\cdot}_{L_1[\mathbb{F}_0]} &= \int_{\mathbb{R}^n}\int_Y \dot u(t)\,dy\,dx \\ &\le \|a_0\|_\infty\,\|u(t)\|_{L_1[\mathbb{F}_0]} + \frac{\|\varphi\|_\infty}{2}V(t) + \|h(t)\|_{L_1[\mathbb{F}_0]}\end{aligned}$$

for $t \in \dot J(u^0)$. Thus we deduce from Corollary 7.3 that there exist $\alpha > 0$ and $\beta \in C^+(J)$ such that $\xi := \|u(\cdot)\|_{L_1[\mathbb{F}_0]}$ satisfies the differential inequality

$$\dot\xi \le \alpha\xi + \beta(t)\ , \qquad t \in \dot J(u^0)\ .$$

Since $\xi \in C\big(J(u^0)\big) \cap C^1\big(\dot J(u^0)\big)$ it follows that

$$\|u(t)\|_{L_1[\mathbb{F}_0]} \le c(T)\ , \qquad t \in J(u^0) \cap [0, T]\ , \quad T \in J\ .$$

Thus, by taking $V(t) = \|u(t)\|_{L_1[\mathbb{F}_0]}$ into account and applying Corollary 7.3 once more,

$$\|u(t)\|_{L_1[\mathbb{F}]} \le c(T) , \qquad t \in J(u^0) \cap [0,T] , \quad T \in J . \tag{8.1}$$

From (3.1) and (3.4) we deduce that

$$\big\|C\big(t,u(t)\big)\big\|_1 \le c\,\|u(t)\|_\infty\,\|u(t)\|_1$$

and

$$\big\|F\big(t,u(t)\big)\big\|_1 \le c\,\|u(t)\|_\infty$$

for $t \in J(u^0)$ (where $\|\cdot\|_{\lambda,q}$ is the norm in $W_q^\lambda[\mathbb{F}]$ and $\|\cdot\|_q := \|\cdot\|_{0,q}$). Hence we infer from (8.1) that

$$\big\|R\big(t,u(t)\big)\big\|_1 \le c(T)\big(\|u(t)\|_\infty + 1\big) , \qquad t \in J(u^0) \cap [0,T] , \quad T > 0 . \tag{8.2}$$

Fix $\overline{s} \in (-1,0)$ and $\overline{\sigma} \in \mathbb{R}^+ \setminus \mathbb{N}$ with $1 < \overline{\sigma} < \overline{s} + 2$. Then (5.14), the injection $L_1[\mathbb{F}] \hookrightarrow W_1^{\overline{s}}[\mathbb{F}]$, and (8.2) imply

$$\big\|R\big(t,u(t)\big)\big\|_{\overline{s},1} \le c(T)\big(\|u(t)\|_{\overline{\sigma},1} + 1\big) \tag{8.3}$$

for $t \in J(u^0) \cap [0,T]$ and $T \in J$. Theorem 5.2 guarantees that u is a solution on $J(u^0)$ of the linear initial value problem

$$\dot{v} + \mathcal{A}(t)v = R\big(t,u(t)\big) , \quad t \in \dot{J}(u^0) , \qquad v(0) = u^0 ,$$

where $R\big(\cdot,u(\cdot)\big) \in C\big(J(u^0), W_1^{\overline{\tau}}[\mathbb{F}]\big)$ with $\overline{s} < \overline{\tau} < 0$. Consequently, u satisfies in $W_1^{\overline{s}}[\mathbb{F}]$ the integral equation

$$u(t) = U(t,0)u^0 + \int_0^t U(t,\tau)R\big(\tau,u(\tau)\big)\,d\tau , \qquad t \in J(u^0) . \tag{8.4}$$

Hence it follows from [3, Lemma II.5.1.3] and

$$\|u(t)\|_{\overline{\sigma},1} \le c(T)\Big(t^{\sigma-\overline{\sigma}}\,\|u^0\|_{\sigma,1} + \int_0^t (t-\tau)^{(\overline{s}-\overline{\sigma})/2}\big(\|u(\tau)\|_{\overline{\sigma},1} + 1\big)\,d\tau\Big)$$

for $t \in J(u^0) \cap [0,T]$ and $T \in J$. Thus the singular Gronwall inequality (e.g., [3, Corollary II.3.3.2]) entails that, given $t_0 \in \dot{J}(u^0)$,

$$\|u(t)\|_{\sigma,1} \le c\,\|u(t)\|_{\overline{\sigma},1} \le c(T) , \qquad t \in J(u^0) \cap [t_0,T] , \tag{8.5}$$

for every $T \in J$ with $T > t_0$. Now the assertion is a consequence of the last part of Theorem 5.1, since $E_\alpha = W_1^\sigma[\mathbb{F}]$ by the proof of Theorem 5.2.

(iii) By integrating (7.3) over Y with respect to the measure dy and using (3.2), (3.5), and the positivity of u it follows that $\overline{u} := \int_Y u\,dy$ satisfies the parabolic differential inequality

$$\partial_t \overline{u} + \mathcal{A}(t)\overline{u} \le \overline{h}(t) , \quad t \in \dot{J}(u^0) , \quad \overline{u}(0) = \int_Y u^0\,dy$$

on $\mathbb{R}^n$, where $\overline{h}(t) := \int_Y h(t)\,dy$. If $\overline{u} \in C_0^2(\mathbb{R}^n)$ then the maximum principle implies

$$\|u(t)\|_{L_\infty(\mathbb{R}^n,\mathbb{F}_0)} = \|\overline{u}(t)\|_{L_\infty(\mathbb{R}^n)} \le c(T)\ , \qquad t \in J(u^0) \cap [0,T]\ .$$

In the general case we obtain this estimate by an approximation argument similar to the one of the proof of Theorem 6.3. Hence Corollary 7.3 and (8.2) imply

$$\big\|R\big(t,u(t)\big)\big\|_{\overline{s},1} \le c(T)\ , \qquad t \in J(u^0) \cap [0,T]\ , \quad T \in J\ .$$

Thus (8.4) and [3, Corollary II.3.2.2] guarantee that (8.5) is true in this case also. □

Remark 8.2. Instead of assuming (iii) it suffices to presuppose that $\boldsymbol{a}$ is independent of $y \in Y$. This follows by an obvious modification of the above proof. □

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Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH–8057 Zürich
Switzerland