Function Spaces on Singular Manifolds

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It is shown that most of the well-known basic results for Sobolev-Slobodeckii and Bessel potential spaces, known to hold on bounded smooth domains in $\mathbb{R}^n$, continue to be valid on a wide class of Riemannian manifolds with singularities and boundary, provided suitable weights, which reflect the nature of the singularities, are introduced. These results are of importance for the study of partial differential equations on piece-wise smooth domains.

1 Introduction

It is our principal concern in this paper to develop a satisfactory theory of spaces of functions and tensor fields on Riemannian manifolds which may have a boundary and may be non-compact and non-complete. Such a theory has to extend the basic results known for function spaces on subdomains of $\mathbb{R}^n$ with smooth boundary to this more general setting, that is to say, embedding and interpolation properties, point-wise multiplier and trace theorems, duality characterizations and, last but not least, intrinsic local descriptions.

Our research is motivated by — and provides the basis for — the study of elliptic and parabolic boundary value problems on piece-wise smooth manifolds, on domains in $\mathbb{R}^n$ with piece-wise smooth boundary, in particular. Such domains occur in a wide variety of problems modeling physical, chemical, biological, and engineering processes by means of differential and pseudodifferential equations. In this connection Sobolev spaces play a predominant role, as is well-known from the theory of partial differential equations on smooth domains. In the presence of singularities, say edges on the boundary, solutions of differential equations lose their smoothness near these singularities. Since the seminal work of V.A. Kondrat′ev [22] on elliptic boundary value problems in domains with conical points it is known that an appropriate setting for the study of such problems is provided by Sobolev spaces with weights reflecting the nature of the singularity. This has since been exploited by numerous authors and there is a large number of papers and monographs devoted to elliptic problems on non-smooth domains. Besides of the early papers by V.G. Maz′ya and B.A. Plamenevskiĭ [26]–[28], the first successful approaches to this kind of problems, we cite only the following few books and refer the reader to the references therein for further information: P. Grisvard [19], M. Dauge [15], S.A. Nazarov and B.A. Plamenevskiĭ [30], V.A. Kozlov, V.G. Maz′ya, and J. Rossmann [23], V.G. Maz′ya, and J. Rossmann [29] (and many more papers and books by V.G. Maz′ya and coauthors), and the numerous contributions of B.-W. Schulze and co-workers on the L_2-theory of elliptic pseudo-differential boundary problems on singular manifolds for which [34] may stand representatively.

Weighted Sobolev spaces of a different type occur as solution spaces for degenerate elliptic equations. This fact has triggered a large amount of research on weighted Sobolev and related function spaces, e.g., A. Kufner [24], H. Triebel [37], H.-J. Schmeisser and H. Triebel [33], and the references therein. Since that work is not directly related to the subject of our paper we do not give more details or cite more recent references.

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In Section 2 we give a precise definition of our concept of a singular manifold M. It will be seen that, to a large extent, M is determined by a 'singularity function' $\rho \in C^\infty\bigl(M,(0,\infty)\bigr)$. The behavior of ρ at the 'singular ends' of M, that is, near that parts of M at which ρ gets either arbitrarily small or arbitrarily large, reflects the singular structure of M. It turns out that the basic building blocks for a useful theory of function spaces on singular manifolds are weighted Sobolev spaces based on the singularity function ρ. More precisely, we denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. Then, given $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and $p \in (1,\infty)$, the weighted Sobolev space $W_p^{k,\lambda}(M) = W_p^{k,\lambda}(M,\mathbb{K})$ is the completion of $\mathcal{D}(M)$, the space of smooth functions with compact support in M, in $L_{1,\mathrm{loc}}(M)$ with respect to the norm

$$u \mapsto \Bigl(\sum_{i=0}^{k} \bigl\|\rho^{\lambda+i}\,|\nabla^i u|_g\bigr\|_p^p\Bigr)^{1/p}. \tag{1.1}$$

Here ∇ denotes the Levi-Civita covariant derivative and $|\nabla^i u|_g$ is the 'length' of the covariant tensor field $\nabla^i u$ naturally derived from the Riemannian metric g of M. Of course, integration is carried out with respect to the volume measure of M. It turns out that $W_p^{k,\lambda}(M)$ is well-defined, independently — in the sense of equivalent norms — of the representation of the singularity structure of M by means of the particular singularity function.

A very special and simple example of a singular manifold is provided by a bounded smooth domain whose boundary contains a conical point. More precisely, suppose Ω is a bounded domain in $\mathbb{R}^m$ whose topological boundary, $\mathrm{bdry}(\Omega)$, contains the origin, and $\Gamma := \mathrm{bdry}(\Omega)\backslash\{0\}$ is a smooth $(m-1)$-dimensional submanifold of $\mathbb{R}^m$ lying locally on one side of Ω. Also suppose that $\Omega \cup \Gamma$ is near 0 diffeomorphic to a cone $\{\,ry\ ;\ 0 < r < 1,\ y \in B\,\}$, where B is a smooth compact submanifold of the unit sphere in $\mathbb{R}^m$. Then, endowing $M := \Omega \cup \Gamma$ with the Euclidean metric, we get a singular manifold with a single conical singularity, as considered in [30] and [23], for example. In this case the weighted norm (1.1) is equivalent to

$$u \mapsto \Bigl(\sum_{|\alpha|\le k} \|r^{\lambda+|\alpha|}\partial^\alpha u\|_{L_p(\Omega)}^p\Bigr)^{1/p},$$

where $r(x)$ is the Euclidean distance from $x \in M$ to the origin. Moreover, $W_p^{k,\lambda}(M)$ coincides with the space $V_{p,\lambda+k}^k(\Omega)$ employed by S.A. Nazarov and B.A. Plamenevskiĭ (cf. p. 319 of [30]) and, in the case $p = 2$, by V.A. Kozlov, V.G. Maz′ya, and J. Rossmann (see Section 6.2 of [23], for example).

As mentioned above, the theory of function spaces on singular manifolds is built on the weighted Sobolev spaces $W_p^{k,\lambda}(M)$. We define weighted Sobolev spaces of negative order by duality, and Bessel potential spaces, $H_p^{s,\lambda}(M)$, and Besov spaces, $B_{p,p}^{s,\lambda}(M)$, by complex and real interpolation, respectively. A basic result, which renders the theory useful, is the fact that these spaces can be characterized locally by their 'classical' non-weighted counterparts on $\mathbb{R}^m$ and on half-spaces. This implies, in particular, $H_p^{k,\lambda}(M) = W_p^{k,\lambda}(M)$ for $k \in \mathbb{N}$.

A linear differential operator on a Riemannian manifold is of the form $\sum_{i=0}^k a_i \cdot \nabla^i u$, where a_i is a contravariant tensor field of order i and $\cdot$ denotes complete contraction. In order to study continuity properties of such operators in the weighted function spaces under consideration we have to have at our disposal point-wise multiplier theorems for tensor fields. Thus it is mandatory to study spaces of tensor fields on singular manifolds.

In the particular case where we can choose the constant map 1 as singularity function, our spaces reduce to non-weighted Sobolev spaces $W_p^k(M)$, Bessel potential spaces $H_p^s(M)$, and Besov spaces $B_{p,p}^s(M)$, respectively. This is, for example, the case if M is a complete Riemannian manifold without boundary and with bounded geometry (that is, M has a positive injectivity radius and all covariant derivatives of the curvature tensor are bounded). To the best of our knowledge, this is the only class of Riemannian manifolds for which a general theory of function spaces has been developed so far. More precisely:

Integer order Sobolev spaces, with particular emphasis on the validity of Sobolev's embedding theorem, have been treated by Th. Aubin [12]–[14] in the case of compact manifolds with boundary, and for complete Riemannian manifolds without boundary, making essential use of curvature estimates and the positivity of the injectivity radius. Also see E. Hebey [20] and [21] for the case where M has no boundary.

Bessel potential spaces $H_p^s(M)$, $1 < p < \infty$, $s \in \mathbb{R}$, on complete Riemannian manifolds without boundary have been introduced and investigated by R.S. Strichartz [36] as domains of the fractional powers of $1 - \Delta_M$, where Δ_M is the Laplace-Beltrami operator. H. Triebel [38], [39] (see also [40]) established a general theory of Triebel-Lizorkin and Besov spaces on complete Riemannian manifolds without boundary and with bounded geometry. His work makes use of a distinguished coordinate system based on the exponential map and of mapping properties of the Laplace-Beltrami operator.

None of the above techniques is available in our situation, where M may be not complete or may not have bounded geometry. In particular, relevant properties of the Laplace-Beltrami operator are not at our disposal, even in the case where M has no boundary. Anyhow, they would not be helpful in the presence of a boundary.

B. Ammann, R. Lauter, and V. Nistor [8] introduce a class of complete non-compact Riemannian manifolds without boundary and with bounded geometry, called Lie manifolds. This class encompasses, in particular, manifolds with cylindrical ends and manifolds being Euclidean at infinity. In B. Ammann, A.D. Ionescu, and V. Nistor [7] Bessel potential spaces on suitable open subsets of Lie manifolds — called Sobolev spaces therein and denoted by $W^{s,p}$ — are being investigated to some extent. Lie manifolds are useful for the study of regularity properties of elliptic differential operators on polyhedral domains in which case the authors are led to introduce weighted Bessel potential spaces, the weight being equivalent to the distance to the non-smooth boundary points (also see [9], [10], and the references therein for related research). The results of the present paper apply to Lie manifolds and polyhedral domains as well and greatly extend and sharpen the investigations of these authors; in particular, as far as the trace theorem is concerned.

There seem to be only very few general results on spaces of tensor fields. J. Eichhorn [17] studies integer order Sobolev spaces of differential forms on complete Riemannian manifolds without boundary and with bounded geometry; also see [18]. Some results on Sobolev spaces of differential forms on compact manifolds with boundary can be found in G. Schwarz [35]. Of course, there are many 'ad hoc' results in the literature, predominantly on L_2-Sobolev spaces, for Riemannian manifolds (without boundary) possessing specific geometries.

Section 3 is of technical nature. There we review some concepts from differential geometry, mainly to fix notation. Then we prove basic estimates related to the singularity structure of the manifold. They are fundamental for the construction of universal retractions by which we can transplant the well-established theory of function spaces on $\mathbb{R}^m$ to the singular manifold. For this we first have to establish a localization procedure for tensor-field-valued distributions on M. This is done in Sections 4 and 5. In Section 6 we show that this localization procedure induces a corresponding retraction-coretraction system on Sobolev spaces. Then, by interpolation, we extend the retraction-coretraction theorem to Bessel potential and Besov spaces of positive order.

After having introduced weighted Hölder spaces in Section 8, we prove in Section 9 point-wise multiplier theorems. Section 10 is devoted to the trace theorem, and in the following section we characterize spaces with vanishing traces. This puts us in position to define, in Section 12, spaces of negative order by duality. All spaces under consideration possess the retraction-coretraction property induced from the localization procedure for tensor-field-valued sections constructed in Section 5. By means of this property we can then, in Sections 13 and 14, respectively, easily prove interpolation and embedding theorems for weighted spaces of tensor fields on singular manifolds.

Section 15 is concerned with spaces of differential forms. In particular, we establish mapping properties of the exterior differential and codifferential operators, and, as an application, of the gradient and divergence operators. These results are of importance in the study of differential operators on singular manifolds. Such investigations, which will be carried out elsewhere, rely fundamentally on the retraction-coretraction theorems established in this paper.

For simplicity, and being oriented towards differential equations, we restrict our considerations essentially to weighted Sobolev-Slobodeckii spaces. However, we include some brief remarks concerning possible extensions to spaces of Triebel-Lizorkin type.

2 Singular Manifolds

By a *manifold* we always mean a smooth, that is, C^∞ manifold with (possibly empty) boundary such that its underlying topological space is separable and metrizable. Thus, in the context of manifolds, we work in the smooth category. A manifold need not be connected, but all connected components are of the same dimension.

We denote by $\mathbb{H}^m$ the closed right half-space $\mathbb{R}^+ \times \mathbb{R}^{m-1}$ in $\mathbb{R}^m$, where $\mathbb{R}^0 = \{0\}$. We set $Q := (-1,1) \subset \mathbb{R}$. If κ is a local chart for an m-dimensional manifold M, then we write U_κ for the corresponding coordinate patch $\mathrm{dom}(\kappa)$. A local chart κ is *normalized* if $\kappa(U_\kappa) = Q^m$ whenever $U_\kappa \subset \mathring{M}$, the interior of M, whereas $\kappa(U_\kappa) = Q^m \cap \mathbb{H}^m$ if $U_\kappa \cap \partial M \neq \emptyset$. We put $Q^m_\kappa := \kappa(U_\kappa)$ if κ is normalized.

An atlas $\mathfrak{K}$ for M has *finite multiplicity* if there exists $k \in \mathbb{N}$ such that any intersection of more than k coordinate patches is empty. It is *uniformly shrinkable* if it consists of normalized charts and there exists $r \in (0,1)$ such that $\{\, \kappa^{-1}(rQ_\kappa^m)\ ;\ \kappa \in \mathfrak{K}\,\}$ is a cover of M.

Given an open subset X of $\mathbb{R}^m$ or $\mathbb{H}^m$ and a Banach space E over $\mathbb{K}$, we write $\|\cdot\|_{k,\infty}$ for the usual norm of $BC^k(X,E)$, the Banach space of all $u \in C^k(X,E)$ such that $|\partial^\alpha u|_E$ is uniformly bounded for $\alpha \in \mathbb{N}^m$ with $|\alpha| \le k$.

By c we denote constants ≥ 1 whose numerical value may vary from occurrence to occurrence; but c is always independent of the free variables in a given formula, unless an explicit dependence is indicated.

Let S be a nonempty set. On $\mathbb{R}^S$, the space of all real-valued functions on S, we introduce an equivalence relation $\sim$ by setting $f \sim g$ iff there exists $c \ge 1$ such that $f/c \le g \le cf$. By $\mathbf{1}$ we denote the constant function $s \mapsto 1$, whose domain will always be clear from the context.

The Euclidean metric on $\mathbb{R}^m$, $(dx^1)^2 + \cdots + (dx^m)^2$, is denoted by g_m. The same symbol is used for its restriction to an open subset U of $\mathbb{R}^m$ or $\mathbb{H}^m$, that is, for $\iota^* g_m$, where $\iota : U \hookrightarrow \mathbb{R}^m$ is the natural embedding. Here and below, we employ the standard notation for pull-back and push-forward operations.

Let $M = (M,g)$ be an m-dimensional Riemannian manifold. Suppose $\rho \in C^\infty\big(M,(0,\infty)\big)$ and $\mathfrak{K}$ is an atlas for M. Then $(\rho,\mathfrak{K})$ is a *singularity datum for M* if

$$
\begin{aligned}
&\text{(i)} && \mathfrak{K} \text{ is uniformly shrinkable, has finite multiplicity, and is orientation preserving if } M \text{ is oriented.}\\
&\text{(ii)} && \|\widetilde{\kappa} \circ \kappa^{-1}\|_{k,\infty} \le c(k), \ \ \kappa, \widetilde{\kappa} \in \mathfrak{K}, \ \ k \in \mathbb{N}.\\
&\text{(iii)} && \kappa_*(\rho^{-2} g) \sim g_m, \ \ \kappa \in \mathfrak{K}.\\
&\text{(iv)} && \|\kappa_*(\rho^{-2} g)\|_{k,\infty} \le c(k), \ \ \kappa \in \mathfrak{K}, \ \ k \in \mathbb{N}.\\
&\text{(v)} && \|\kappa_* \rho\|_{k,\infty} \le c(k) \rho_\kappa, \ \ \kappa \in \mathfrak{K}, \ \ k \in \mathbb{N}, \text{ where } \rho_\kappa := \kappa_*\rho(0) = \rho\big(\kappa^{-1}(0)\big).\\
&\text{(vi)} && 1/c \le \rho(p)/\rho_\kappa \le c, \ \ p \in U_\kappa, \ \ \kappa \in \mathfrak{K}.
\end{aligned}
\tag{2.1}
$$

In (ii) and in similar situations it is understood that only $\kappa, \widetilde{\kappa} \in \mathfrak{K}$ with $U_\kappa \cap U_{\widetilde{\kappa}} \ne \emptyset$ are being considered. Condition (iii) reads more explicitly:

$$\kappa_*\rho^2(x)\,|\xi|^2/c \le \kappa_* g(x)(\xi,\xi) \le c\kappa_*\rho^2(x)\,|\xi|^2, \qquad x \in Q_\kappa^m, \quad \xi \in \mathbb{R}^m, \quad \kappa \in \mathfrak{K}.$$

Note that the finite multiplicity of $\mathfrak{K}$ and the separability of M imply that $\mathfrak{K}$ is countable.

Let $(\rho,\mathfrak{K})$ and $(\widetilde{\rho},\widetilde{\mathfrak{K}})$ be singularity data for M. Set

$$\mathfrak{N}(\kappa) := \{\, \widetilde{\kappa} \in \widetilde{\mathfrak{K}}\ ;\ U_{\widetilde{\kappa}} \cap U_\kappa \ne \emptyset \,\}, \qquad \kappa \in \mathfrak{K}.$$

Then $(\rho,\mathfrak{K})$ and $(\widetilde{\rho},\widetilde{\mathfrak{K}})$ are *equivalent* if

$$
\begin{aligned}
&\text{(i)} && \rho \sim \widetilde{\rho};\\
&\text{(ii)} && \operatorname{card}\big(\mathfrak{N}(\kappa)\big) \le c, \ \ \kappa \in \mathfrak{K};\\
&\text{(iii)} && \|\widetilde{\kappa} \circ \kappa^{-1}\|_{k,\infty} \le c(k), \ \ \kappa \in \mathfrak{K}, \ \ \widetilde{\kappa} \in \widetilde{\mathfrak{K}}, \ \ k \in \mathbb{N}.
\end{aligned}
\tag{2.2}
$$

A *singularity structure*, $\mathfrak{S}(M)$, for M is a maximal family of equivalent singularity data. A *singularity function* for M is a function $\rho \in C^\infty\big(M,(0,\infty)\big)$ such that there exists an atlas $\mathfrak{K}$ with $(\rho,\mathfrak{K}) \in \mathfrak{S}(M)$. The set of all singularity functions is the *singularity type*, $\mathfrak{T}(M)$, of M. By a **singular manifold** we mean a Riemannian manifold M endowed with a singularity structure $\mathfrak{S}(M)$. Then M is said to be *singular of type* $\mathfrak{T}(M)$. If $\rho \in \mathfrak{T}(M)$, then it is convenient to set $[\![\rho]\!] := \mathfrak{T}(M)$. A singular manifold of type $[\![\mathbf{1}]\!]$ is called *uniformly regular*.

Let (M,g) be singular of type $[\![\rho]\!]$. It follows from (2.1)(i)–(iv) that then $(M,\rho^{-2}g)$ is uniformly regular. Suppose $\rho \not\sim \mathbf{1}$. Then either $\inf \rho = 0$ or $\sup \rho = \infty$, or both. Hence M is not compact but has singular ends. It follows from (2.1)(iii) that the diameter of the coordinate patches converges either to zero or to infinity near the singular ends in a manner controlled by the singularity type $\mathfrak{T}(M)$.

Examples 2.1 **(a)** Every compact Riemannian manifold is uniformly regular.

(b) Let M be an m-dimensional Riemannian submanifold of $\mathbb{R}^m$ possessing a compact boundary. Then M is uniformly regular.

(c) $\mathbb{R}^m = (\mathbb{R}^m, g_m)$ and $\mathbb{H}^m = (\mathbb{H}^m, g_m)$ are uniformly regular.

P r o o f. For $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$ and $z \in \mathbb{Z}^m \cap \mathbb{X}$ we set $Q_z^m := Q^m$ if either $\mathbb{X} = \mathbb{R}^m$ or $z \in \mathring{\mathbb{H}}^m$; otherwise we let $Q_z^m := Q^m \cap \mathbb{H}^m$. We put $U_z := z + Q_z^m$ and $\kappa_z(x) := x - z$ for $z \in \mathbb{Z}^m \cap \mathbb{X}$ and $x \in U_z$. Then $(\mathbf{1}, \mathfrak{K})$, where $\mathfrak{K} := \{\, \kappa_z \,;\, \mathbb{Z}^m \cap \mathbb{X} \,\}$, is a singularity datum for $\mathbb{X}$. □

(d) Let (M, g) be singular of type $[\![\rho]\!]$ and $\varphi :\ M \to N$ a diffeomorphism. Then $(N, \varphi_* g)$ is singular of type $[\![\varphi_* \rho]\!]$. Assume $(\rho, \mathfrak{K})$ is a singularity datum for M and set $\varphi_* \mathfrak{K} := \{\, \varphi_* \kappa \,;\, \kappa \in \mathfrak{K} \,\}$. Then $(\varphi_* \rho, \varphi_* \mathfrak{K})$ is a singularity datum for N.

(e) Let M be singular of type $[\![\rho]\!]$. Suppose $\partial M \neq \emptyset$. Denote by $\mathring{\imath} :\ \partial M \hookrightarrow M$ the natural injection and endow ∂M with the induced Riemannian metric $g_{\partial M} := \mathring{\imath}^* g$. Suppose $\kappa :\ U_\kappa \to \mathbb{R}^m$ is a local chart for M with $U_{\mathring{\kappa}} := \partial U_\kappa = U_\kappa \cap \partial M \neq \emptyset$. Put

$$\mathring{\kappa} := \iota_0 \circ (\mathring{\imath}^* \kappa) :\ U_{\mathring{\kappa}} \to \mathbb{R}^{m-1},$$

where $\iota_0 :\ \{0\} \times \mathbb{R}^{m-1} \to \mathbb{R}^{m-1},\ (0, x') \mapsto x'$. Let $\mathfrak{K}$ be a normalized atlas for M. Then a normalized atlas for ∂M is given by $\mathring{\mathfrak{K}} := \{\, \mathring{\kappa} \,;\, \kappa \in \mathfrak{K},\ \partial U_\kappa \neq \emptyset \,\}$, the one *induced by* $\mathfrak{K}$. Assume $(\rho, \mathfrak{K})$ is a singularity datum for M. Set $\mathring{\rho} := \mathring{\imath}^* \rho = \rho | \partial M$. Then $(\mathring{\rho}, \mathring{\mathfrak{K}})$ is a singularity datum for ∂M. Thus ∂M is singular of type $[\![\mathring{\rho}]\!]$.

(f) If M is a complete Riemannian manifold without boundary and with bounded geometry, then M is uniformly regular.

P r o o f. This follows from Lemma 2.2.6 in [13], for example. □

In order to describe nontrivial classes of singular manifolds we need some preparation. Let N be a complete Riemannian manifold without boundary and of dimension n. Suppose M is an m-dimensional submanifold of N. Denote by $\overline{M}$ the closure of M in N. Then $\mathcal{S}(M) := \overline{M} \backslash M$ is the *singularity set* of M (in N). Thus $\overline{M} = \mathring{M} \cup \partial M \cup \mathcal{S}(M)$ and $\mathcal{S}(M)$ is closed in N. In particular, M is not complete if $\mathcal{S}(M) \neq \emptyset$.

We assume now that M can be described, locally in the neighborhood of $\mathcal{S}(M)$, by model cusps and wedges over such cusps. More precisely: suppose $d \in \mathbb{N}^\times := \mathbb{N} \backslash \{0\}$ and B is a submanifold of $\mathbb{S}^{d-1}$, the unit sphere in $\mathbb{R}^d$. Then

$$K_1^d(B) := \{\, ry \in \mathbb{R}^d \,;\, 0 < r < 1,\ y \in B \,\},$$

where $y \in B$ is identified with its image in $\mathbb{R}^d$ under the natural embedding $\mathbb{S}^{d-1} \hookrightarrow \mathbb{R}^d$, is called *model cone over* B in $\mathbb{R}^d$.

Next, let $1 < \alpha < \infty$ and assume now that B is a submanifold of Q^{d-1}, where $d \geq 2$. Then

$$K_\alpha^d(B) := \big\{\, (r, r^\alpha y) \in \mathbb{R}^d \,;\, 0 < r < 1,\ y \in B \,\big\}$$

is the *model α-cusp* in $\mathbb{R}^d$. To allow for a unified treatment we call K_1^d, in abuse of language, model 1-cusp. Then, given $\alpha \in [1, \infty)$ and $\ell \in \mathbb{N}$,

$$K_\alpha^d(B, \ell) := K_\alpha^d(B) \times Q^\ell$$

is the *model (α, ℓ)-wedge over B* in $\mathbb{R}^{d+\ell}$. Here and below, all references to Q^ℓ have to be neglected if $\ell = 0$. Thus $K_\alpha^d(B, 0) = K_\alpha^d(B)$, and a model cusp is a specific instance of a model wedge.

If $b := \dim(B)$, then $K_\alpha^d(B, \ell)$ is a submanifold of $\mathbb{R}^{d+\ell}$ of dimension $b + 1 + \ell$ and boundary $K_\alpha^d(\partial B, \ell)$. Thus $\partial K_\alpha^d(B, \ell) = \emptyset$ if $\partial B = \emptyset$, which is the case, in particular, if $\alpha = 1$ and $B = \mathbb{S}^{d-1}$, or if $b = 0$.

Now we suppose $0 \leq \ell \leq m - 1$ and S is an ℓ-dimensional submanifold of N without boundary, contained in $\mathcal{S}(M)$. We also suppose $\alpha \in [1, \infty)$ and B is an $(m - \ell - 1)$-dimensional submanifold of $\mathbb{S}^{m-\ell-1}$ if $\alpha = 1$, or of $Q^{m-\ell-1}$ if $\alpha > 1$. Then S is called *(α, ℓ)-wedge of M over B* if for each $p \in S$ there exists a normalized local chart φ for N at p such that $\mathcal{S}(M) \cap U_\varphi = S \cap U_\varphi$,

$$\varphi(M \cap U_\varphi) = \big(K_\alpha^{m-\ell}(B, \ell) \times \{0\} \big) \cap Q^n,$$

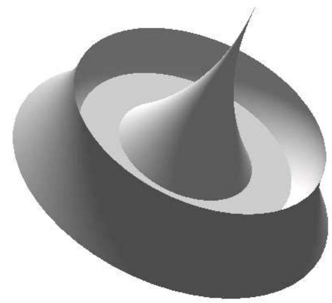
and

$$\varphi(S \cap U_\varphi) = \big(\{0\} \times Q^\ell \big) \times \{0\}.$$

Thus an (α, ℓ)-wedge of M over B looks locally like the model wedge $K_\alpha^{m-\ell}(B, \ell)$ in $\mathbb{R}^m$.

Finally, M is called *relatively compact (sub-) manifold (of N) with smooth cuspidal singularities* if $\overline{M}$ is compact, $\mathcal{S}(M) \neq \emptyset$, and for each connected component Γ of $\mathcal{S}(M)$ there exist $\alpha \in [1, \infty)$, $\ell \in \{0, \ldots, m - 1\}$, and a compact manifold B such that Γ is an (α, ℓ)-wedge of M over B.

In the adjacent figure we have depicted a three-dimensional relatively compact submanifold M of $\mathbb{R}^3$ with smooth cuspidal singularities. More precisely, $\mathcal{S}(M)$ consists of 5 connected components, namely of one 2.5-cusp, one $(2,1)$-wedge (the upper rim), and three $(1,1)$-wedges (one at the bottom of the figure and two on the inner plateau).



Let M be a relatively compact submanifold of N with smooth cuspidal singularities. Denote by $\mathbf{\Gamma}$ the set of connected components of $\mathcal{S}(M)$. Since $\mathcal{S}(M)$ is closed in $\overline{M}$, it is compact. Hence $\mathbf{\Gamma}$ is a finite set and each $\Gamma \in \mathbf{\Gamma}$ is a compact submanifold of N without boundary.

Given a nonempty subset S of $\mathcal{S}(M)$, we denote by $d_N(p,S)$ the Riemannian distance in N from $p \in N$ to S. For each $\Gamma \in \mathbf{\Gamma}$ we can find a relatively compact open neighborhood U_Γ in N such that $d_N\big(p,\mathcal{S}(M)\big) = d_N(p,\Gamma)$ for $p \in U_\Gamma$ and $d_N(\cdot,\Gamma)$ is smooth on U_Γ. Moreover, there exists a unique $\alpha_\Gamma \in [1,\infty)$ such that Γ is an $\big(\alpha_\Gamma, \dim(\Gamma)\big)$-wedge of M over some compact manifold B_Γ of dimension $m - \dim(\Gamma) - 1$.

Theorem 2.2 *Let M be a relatively compact manifold with smooth cuspidal singularities. Choose $\rho \in C^\infty\big(M,(0,1]\big)$ satisfying $\rho(p) \sim \big(d_N(p,\Gamma)\big)^{\alpha_\Gamma}$ for p near $\Gamma \in \mathbf{\Gamma}$. Then M is a singular manifold of type $[\![\rho]\!]$.*

P r o o f. H. Amann [4]. □

In the case of the manifold M depicted above, ρ behaves near $\mathcal{S}(M)$ like the power α of the Euclidean distance in $\mathbb{R}^3$ to $\mathcal{S}(M)$, where $\alpha = 2.5$ near the vertex of the cusp, $\alpha = 2$ near the upper rim, and $\alpha = 1$ near the remaining three wedges.

For manifolds with non-smooth cuspidal singularities we refer to [4]. There it is no longer assumed that B_Γ is a compact manifold, but B_Γ itself can have (non-) smooth cuspidal singularities. This covers the case of corners and intersecting wedges. In addition, in [4] we consider singular manifolds which are not relatively compact; for example: subdomains of $\mathbb{R}^m$ with 'outlets to infinity'.

3 Tensor Fields and Uniform Estimates

It is the purpose of this section to provide technical estimates on which much of what follows is based. First we prepare some results on tensor bundles and covariant derivatives. For general background information we refer to J. Dieudonné [16], for instance.

Let $M = (M,g)$ be an m-dimensional Riemannian manifold. We denote by TM and T^*M the (complexified, if $\mathbb{K} = \mathbb{C}$) tangent and cotangent bundle, respectively. Then, given $\sigma, \tau \in \mathbb{N}$,

$$T^\sigma_\tau M := TM^{\otimes\sigma} \otimes T^*M^{\otimes\tau}$$

is the (σ,τ)-tensor bundle of M, that is, the vector bundle of all tensors on M being contravariant of order σ and covariant of order τ. We use obvious conventions if $\sigma = 0$ or $\tau = 0$. In particular, $T^0_0 M = M \times \mathbb{K}$, a trivial vector bundle. We write $\mathcal{T}^\sigma_\tau M$ for the $C^\infty(M)$-module of all smooth sections of $T^\sigma_\tau M$, the smooth (σ,τ)-tensor fields on M. For abbreviation, $\mathcal{T}M := \mathcal{T}^1_0 M$ and $\mathcal{T}^*M := \mathcal{T}^0_1 M$.

For $\nu \in \mathbb{N}^\times$ we set $\mathbb{J}_\nu := \{1,\ldots,m\}^\nu$. Then, given local coordinates $\kappa = (x^1,\ldots,x^m)$ and setting

$$\frac{\partial}{\partial x^{(i)}} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}}, \quad dx^{(j)} := dx^{j_1} \otimes \cdots \otimes dx^{j_\tau}$$

for $(i) = (i_1,\ldots,i_\sigma) \in \mathbb{J}_\sigma$, $(j) \in \mathbb{J}_\tau$, the local representation of $a \in \mathcal{T}^\sigma_\tau M$ with respect to these coordinates is given by

$$a = a^{(i)}_{(j)} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \tag{3.1}$$

with $a^{(i)}_{(j)} \in C^\infty(U_\kappa)$. Here and below, we use the summation conventions whereby expressions are summed over all possible values of repeated indices.

We write $g_\flat : \mathcal{T}M \to \mathcal{T}^*M$ for the conjugate linear (fiber-wise defined) Riesz isomorphism. Thus

$$\langle g_\flat X, Y\rangle = g(Y,X), \qquad X,Y \in \mathcal{T}M, \tag{3.2}$$

where

$$\langle\cdot,\cdot\rangle : \mathcal{T}^*M \times \mathcal{T}M \to C^\infty(M) \tag{3.3}$$

is the (fiber-wise defined) duality pairing. The inverse of $g_\flat$, denoted by $g^\sharp$, satisfies

$$\langle \alpha, Y\rangle = g(Y, g^\sharp\alpha), \qquad \alpha \in \mathcal{T}^*M, \quad X \in \mathcal{T}M.$$

Denoting by g^* the adjoint Riemannian metric on T^*M it follows from (3.2) that

$$\langle \alpha, g^\sharp\beta\rangle = \langle g_\flat g^\sharp\alpha, g^\sharp\beta\rangle = g(g^\sharp\beta, g^\sharp\alpha) = g^*(\alpha,\beta), \qquad \alpha,\beta \in \mathcal{T}^*M. \tag{3.4}$$

From this we obtain, in local coordinates,

$$g_\flat X = g_{ij}\overline{X}^j\, dx^i, \quad g^\sharp\alpha = g^{ij}\overline{\alpha}_j \frac{\partial}{\partial x^i} \qquad \text{for} \quad X = X^i\frac{\partial}{\partial x^i}, \quad \alpha = \alpha_j\, dx^j, \tag{3.5}$$

where $g = g_{ij}\, dx^i \otimes dx^j$ and $[g^{ij}]$ is the inverse of the matrix $[g_{ij}]$.

We let

$$\langle\cdot,\cdot\rangle : \mathcal{T}^\sigma_\tau M \times \mathcal{T}^\tau_\sigma M \to C^\infty(M) \tag{3.6}$$

be the natural extension of (3.3). Thus, given $p \in M$, we write $(T^\tau_\sigma M)_p$ for the fiber of $T^\tau_\sigma M$ over p. Then, for decomposable tensors $u\otimes\alpha \in (T^\sigma_\tau M)_p$ and $v \otimes \beta \in (T^\tau_\sigma M)_p$,

$$\langle u\otimes\alpha, v\otimes\beta\rangle_p := \prod_{i=1}^{\sigma}\langle\beta_i, u_i\rangle_p \prod_{j=1}^{\tau}\langle\alpha_j, v_j\rangle_p,$$

where $u = u_1 \otimes\cdots\otimes u_\sigma \in (T^\sigma_0 M)_p$ and $\alpha = \alpha_1\otimes\cdots\otimes\alpha_\tau \in (T^0_\tau M)_p$, etc. Hence

$$(T^\sigma_\tau M)' = T^\tau_\sigma M$$

with respect to the 'tensor product duality pairing' (3.6). This is consistent with $(TM)' = T^*M$.

Suppose $\sigma+\tau \geq 1$. We put

$$(G^\tau_\sigma a)(\alpha_1,\ldots,\alpha_\tau, X_1,\ldots,X_\sigma) := a(g_\flat X_1,\ldots,g_\flat X_\sigma, g^\sharp\alpha_1,\ldots,g^\sharp\alpha_\tau) \tag{3.7}$$

for $a \in \mathcal{T}^\sigma_\tau M,\ \alpha_1,\ldots,\alpha_\tau \in \mathcal{T}^*M$, and $X_1,\ldots,X_\sigma \in \mathcal{T}M$. This induces a conjugate linear bijection

$$G^\tau_\sigma : T^\sigma_\tau M \to T^\tau_\sigma M, \quad (G^\tau_\sigma)^{-1} = G^\sigma_\tau.$$

Consequently,

$$(\cdot|\cdot)_g : T^\sigma_\tau M \times \mathcal{T}^\sigma_\tau M \to C^\infty(M), \quad (a,b) \mapsto \langle a, G^\tau_\sigma b\rangle \tag{3.8}$$

is an inner product (a vector bundle metric) on $T^\sigma_\tau M$, the *inner product induced by* g. It follows from (3.5) that, in local coordinates,

$$(a|b)_g = g_{(i)(j)}g^{(k)(\ell)}a^{(i)}_{(k)}\overline{b}^{(j)}_{(\ell)}, \qquad a,b \in \mathcal{T}^\sigma_\tau M, \tag{3.9}$$

where

$$g_{(i)(j)} := g_{i_1j_1}\cdots g_{i_\sigma j_\sigma}, \quad g^{(k)(\ell)} := g^{k_1\ell_1}\cdots g^{k_\tau\ell_\tau} \tag{3.10}$$

for $(i),(j) \in \mathbb{J}_\sigma$ and $(k),(\ell) \in \mathbb{J}_\tau$. Of course, $(a|b)_g = a\overline{b}$ for $a,b \in \mathcal{T}^0_0 M = C^\infty(M)$. Clearly,

$$|\cdot|_g : \mathcal{T}^\sigma_\tau M \to C(M), \quad a \mapsto \sqrt{(a|a)_g}$$

is called (vector bundle) *norm* induced by g. (We do not notationally indicate the dependence on (σ,τ). This will be clear from the context.) Note that $|a|_g^2 = g^*(a,a)$ for $a \in T_1^0 M$. For this reason we also write $|a|_{g^*}$ for $|a|_g$ if $a \in T_\tau^0 M$.

Let $\varphi : M \to N$ be a diffeomorphism onto some manifold N. Then one verifies

$$\varphi_*\big((a|b)_g\big) = (\varphi_* a\,|\,\varphi_* b)_{\varphi_* g}.$$

We denote by $\nabla = \nabla_g$ the (complexified, if $\mathbb{K} = \mathbb{C}$) Levi-Civita connection on TM. It has a unique extension over $\mathcal{T}_\tau^\sigma$ satisfying, for $X \in \mathcal{T}M$,

$$\begin{aligned} &\text{(i)} && \nabla_X f = \langle df, X\rangle, \quad f \in C^\infty(M);\\ &\text{(ii)} && \nabla_X (a \otimes b) = \nabla_X a \otimes b + a \otimes \nabla_X b, \quad a \in \mathcal{T}_{\tau_1}^{\sigma_1} M, \quad b \in \mathcal{T}_{\tau_2}^{\sigma_2} M;\\ &\text{(iii)} && \nabla_X \langle a, b\rangle = \langle \nabla_X a, b\rangle + \langle a, \nabla_X b\rangle, \quad a \in \mathcal{T}_\tau^\sigma M, \quad b \in \mathcal{T}_\sigma^\tau M. \end{aligned} \tag{3.11}$$

Then the covariant (Levi-Civita) derivative is the linear map

$$\nabla = \nabla_g : \mathcal{T}_\tau^\sigma M \to \mathcal{T}_{\tau+1}^\sigma M, \quad a \mapsto \nabla a,$$

defined by

$$\langle \nabla a, b \otimes X\rangle := \langle \nabla_X a, b\rangle, \qquad b \in \mathcal{T}_\sigma^\tau M, \quad X \in \mathcal{T}M.$$

Since it satisfies $\nabla g = 0$, it commutes with $g_\flat$ and $g^\sharp$. From this we infer

$$\nabla_X (a|b)_g = (\nabla_X a|b)_g + (a|\nabla_X b)_g, \qquad a, b \in \mathcal{T}_\tau^\sigma M, \quad X \in \mathcal{T}M. \tag{3.12}$$

Thus ∇ is a metric connection on $\mathcal{T}_\tau^\sigma M = \big(T_\tau^\sigma M, (\cdot|\cdot)_g\big)$.

Let $\varphi : M \to N$ be a diffeomorphism. The uniqueness of the Levi-Civita connection implies

$$\varphi_*(\nabla_g a) = \nabla_{\varphi_* g}(\varphi_* a), \qquad a \in \mathcal{T}_\tau^\sigma M.$$

For $k \in \mathbb{N}$ we define

$$\nabla^k : \mathcal{T}_\tau^\sigma M \to \mathcal{T}_{\tau+k}^\sigma M, \quad a \mapsto \nabla^k a$$

by $\nabla^0 a := a$ and $\nabla^{k+1} := \nabla \circ \nabla^k$.

Now we are ready for the proof of the needed estimates. In the following, dV_g denotes the Lebesgue volume measure for M. Furthermore, given $a \in \mathcal{T}_\tau^\sigma M$ and a local chart κ, we write $[\kappa_* a]$ for the $(m^\sigma \times m^\tau)$-matrix whose general entry equals $(\kappa_* a)_{(j)}^{(i)} = (a \circ \kappa^{-1})_{(j)}^{(i)}$, with $(i) \in \mathbb{J}_\sigma$ and $(j) \in \mathbb{J}_\tau$.

Lemma 3.1 *Let $(\rho, \mathfrak{K})$ be a singularity datum for (M,g). Then the following estimates hold uniformly with respect to $\kappa \in \mathfrak{K}$:*

(i) $\kappa_* g \sim \rho_\kappa^2 g_m$, $\kappa_* g^* \sim \rho_\kappa^{-2} g_m$.

(ii) $\rho_\kappa^{-2} \|\kappa_* g\|_{k,\infty} + \rho_\kappa^2 \|\kappa_* g^*\|_{k,\infty} \le c(k)$, $k \in \mathbb{N}$.

(iii) $\kappa_*(dV_g) \sim \rho_\kappa^m \, dV_{g_m}$.

(iv) *If $r, \sigma, \tau \in \mathbb{N}$, then* $\sum_{i=0}^r |\nabla_{\kappa_* g}^i (\kappa_* a)|_{g_m} \sim \sum_{|\alpha| \le r} |\partial^\alpha [\kappa_* a]|_{g_m}$ *for* $a \in \mathcal{T}_\tau^\sigma M$.

(v) *Given $\sigma, \tau \in \mathbb{N}$,*

$$\kappa_*(|a|_g) \sim \rho_\kappa^{\sigma-\tau} \, |\kappa_* a|_{g_m}, \qquad a \in \mathcal{T}_\tau^\sigma M,$$

and

$$|\kappa^* b|_g \sim \rho_\kappa^{\sigma-\tau} \kappa^*(|b|_{g_m}), \qquad b \in \mathcal{T}_\tau^\sigma Q_\kappa^m.$$

P r o o f. (1) The first part of claim (i) is immediate from (2.1)(iii) and (vi).

(2) By (i) and the symmetry of g the spectrum of the matrix $[\kappa_* g]$ is contained in an interval of the form $\rho_\kappa^2[1/c, c]$ for $\kappa \in \mathfrak{K}$. Hence $[\kappa_* g]^{-1}$ has its spectrum in $\rho_\kappa^{-2}[1/c, c]$ for $\kappa \in \mathfrak{K}$. This implies the second part of statement (i) and

$$\|\kappa_* g^*\|_\infty \le c\rho_\kappa^{-2}, \qquad \kappa \in \mathfrak{K}. \tag{3.13}$$

Furthermore,

$$\rho_\kappa^{-2}\kappa_* g = \Bigl(\frac{\kappa_*\rho}{\rho_\kappa}\Bigr)^2 \kappa_*(\rho^{-2}g). \tag{3.14}$$

Thus assertion (ii) follows from (2.1)(iv)–(vi), (3.13), (3.14), Leibniz' rule, and the formulas for derivatives of inverses (cf. Lemma 1.4.2 in H. Amann [3]).

(3) Writing, as usual, $\sqrt{g} := \sqrt{\det[g]}$, statement (iii) follows from (i) and $\kappa_*(dV_g) = \sqrt{\kappa_* g}\, dV_{g_m}$.

(4) Recall that, setting $\nabla_i := \nabla_{\partial_i}$ with $\partial_i = \partial/\partial x^i$,

$$\nabla_i X = (\partial_i X^k + \Gamma_{ij}^k X^j)\,\frac{\partial}{\partial x^k}, \qquad X = X^k\,\frac{\partial}{\partial x^k}, \tag{3.15}$$

where the Christoffel symbols Γ_{ij}^k are given by

$$2\Gamma_{ij}^k = g^{k\ell}(\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_\ell g_{ij}). \tag{3.16}$$

Suppose $a \in \mathcal{T}_\tau^\sigma M$ has the local representation (3.1). Correspondingly,

$$\nabla a = \nabla_k a_{(j)}^{(i)}\,\frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \otimes dx^k.$$

Then it follows from (3.11) and (3.15) that

$$\nabla_k a_{(j)}^{(i)} = \partial_k a_{(j)}^{(i)} + \sum_{s=1}^{\sigma} \Gamma_{k\ell}^{i_s} a_{(j)}^{(i_1,\ldots,\ell,\ldots,i_\sigma)} - \sum_{t=1}^{\tau} \Gamma_{kj_t}^{\ell} a_{(j_1,\ldots,\ell,\ldots,i_\tau)}^{(i)}, \tag{3.17}$$

where ℓ is at position s in the first sum and at position t in the second sum (and the terms are added up from $\ell = 1$ to $\ell = m$). We set $\nabla_{(k)} := \nabla_{k_r} \circ \cdots \circ \nabla_{k_1}$ and $\partial_{(k)} := \partial_{k_r} \circ \cdots \circ \partial_{k_1}$ for $(k) \in \mathbb{J}_r$ and $r \in \mathbb{N}^\times$. Then, writing $\nabla^r a = \bigl(\nabla_{(k)} a_{(j)}^{(i)}\bigr)\,\frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)} \otimes dx^{(k)}$, we obtain from (3.17)

$$\nabla_{(k)} a_{(j)}^{(i)} = \partial_{(k)} a_{(j)}^{(i)} + b_{(j)(k)}^{(i)}, \tag{3.18}$$

where $b_{(j)(k)}^{(i)}$ is a linear combination of the elements of

$$\bigl\{\, \partial^\alpha a_{(\widetilde{\jmath})}^{(\widetilde{\imath})}\,;\ |\alpha| \le r-1,\ (\widetilde{\imath}) \in \mathbb{J}_\sigma,\ (\widetilde{\jmath}) \in \mathbb{J}_\tau \,\bigr\},$$

the coefficients being polynomials in the derivatives of the Christoffel symbols of order at most $r - 1 - |\alpha|$.

We deduce from (ii) and (3.16)

$$\|\Gamma_{ij}^k \circ \kappa^{-1}\|_{\ell,\infty} \le c(\ell), \qquad 1 \le i,j,k \le m, \quad \kappa \in \mathfrak{K}, \quad \ell \in \mathbb{N}. \tag{3.19}$$

Hence (3.18) implies

$$\sum_{i=0}^{r} |\nabla_{\kappa_* g}^i (\kappa_* a)|_{g_m} \le c \sum_{|\alpha| \le r} |\partial^\alpha [\kappa_* a]|_{g_m}, \qquad a \in \mathcal{T}_\tau^\sigma M, \quad \kappa \in \mathfrak{K}.$$

By solving system (3.18) for $\partial^\alpha a_{(j)}^{(i)}$ we obtain an analogous expression for $\partial_{(k)} a_{(j)}^{(i)}$ in terms of $\nabla_{(\ell)}(\kappa_* a)$, $\ell \in \mathbb{J}_\sigma,\ 0 \le \sigma \le r-1$. Thus, invoking (3.19) once more, we get the second half of assertion (iv).

(5) The first part of (v) follows from (3.9), (3.10), and (ii). The second part is then deduced by applying this result to $a := \kappa^* b$. $\square$

From (2.1)(v) and (vi) and Lemma 3.1(ii) we find by the arguments of step (2)

$$\big\|\kappa_*\big((\rho^{-2}g)^*\big)\big\|_{k,\infty} \le c(k), \qquad \kappa\in\mathfrak{K}, \quad k\in\mathbb{N}. \tag{3.20}$$

This, in combination with (2.1)(iii) and (iv), is close to the statement that all covariant derivatives of the curvature tensor of $(M,\rho^{-2}g)$ are bounded. Note however that, taking (2.2) into consideration, (2.1)(iv) and (3.20) are only true for atlases in $\mathfrak{S}(M)$.

Let M be a manifold and $\mathfrak{K}$ an atlas for it consisting of normalized charts. A family $\big\{\,(\pi_\kappa,\chi_\kappa)\ ;\ \kappa\in\mathfrak{K}\,\big\}$ is a (uniform) *localization system subordinate to* $\mathfrak{K}$ if

(i) $\pi_\kappa\in\mathcal{D}\big(U_\kappa,[0,1]\big)$ and $\{\,\pi_\kappa^2\ ;\ \kappa\in\mathfrak{K}\,\}$ is a partition of unity subordinate to $\{\,U_\kappa\ ;\ \kappa\in\mathfrak{K}\,\}$;

(ii) $\chi_\kappa=\kappa^*\chi$ with $\chi\in\mathcal{D}\big(Q^m,[0,1]\big)$ and $\chi\,|\,\mathrm{supp}(\kappa_*\pi_\kappa)=\mathbf{1}$; (3.21)

(iii) $\|\kappa_*\pi_\kappa\|_{k,\infty}+\|\kappa_*\chi_\kappa\|_{k,\infty}\le c(k),\ \ \kappa\in\mathfrak{K},\ \ k\in\mathbb{N}$.

The crucial assumption, besides (i), is the uniform estimate (iii). Assumption (ii) will simplify some formulas. In principle, it would suffice to require that χ_κ be a cut-off function for $\mathrm{supp}(\pi_\kappa)$.

It should also be noted that, for the purpose of this paper, we could replace π_κ^2 in (3.21)(i) by π_κ. In fact, then some of the computations below would even become simpler. However, in applications to differential equations it will be important that we can use a partition of unity whose square root is smooth. For this reason we employ condition (3.21)(i).

Lemma 3.2 *Let $(\rho,\mathfrak{K})$ be a singularity datum for M. Then there exists a localization system subordinate to $\mathfrak{K}$.*

Proof. Fix $r\in(0,1)$ such that $r\mathfrak{U}:=\big\{\,\kappa^{-1}(rQ_\kappa^m)\ ;\ \kappa\in\mathfrak{K}\,\big\}$ is a cover of M. Choose $\widetilde{\pi}\in\mathcal{D}\big(Q^m,[0,1]\big)$ with $\widetilde{\pi}\,|\,rQ^m=\mathbf{1}$. Set $\widetilde{\pi}_\kappa:=\kappa^*\widetilde{\pi}$. Since $r\mathfrak{U}$ covers M and has finite multiplicity,

$$1\le\sum_\kappa\widetilde{\pi}_\kappa^2(p)\le c, \qquad p\in M.$$

Put $\pi_\kappa:=\widetilde{\pi}_\kappa\Big/\sqrt{\sum_{\widetilde{\kappa}}\widetilde{\pi}_{\widetilde{\kappa}}^2}$. Then $\pi_\kappa\in\mathcal{D}\big(U_\kappa,[0,1]\big)$ and $\sum_\kappa\pi_\kappa^2=\mathbf{1}$, where $\kappa_*(\pi_\kappa)$ has its support in $\mathrm{supp}(\widetilde{\pi})$. Fix $\chi\in\mathcal{D}\big(Q^m,[0,1]\big)$ with $\chi\,|\,\mathrm{supp}(\widetilde{\pi})=\mathbf{1}$. Set $\chi_\kappa:=\kappa^*\chi$. Then conditions (3.21)(i) and (ii) are satisfied. The validity of (3.21)(iii) is a consequence of (2.1)(ii). □

4 Distribution Sections

Given locally convex spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of continuous linear maps from $\mathcal{X}$ into $\mathcal{Y}$, and $\mathcal{L}(\mathcal{X}):=\mathcal{L}(\mathcal{X},\mathcal{X})$. By $\mathcal{L}\mathrm{is}(\mathcal{X},\mathcal{Y})$ we mean the set of all topological isomorphisms in $\mathcal{L}(\mathcal{X},\mathcal{Y})$. If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then $\mathcal{L}(\mathcal{X},\mathcal{Y})$ is endowed with the uniform operator norm. We write $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ for the duality pairing between $\mathcal{X}'$ and $\mathcal{X}$, that is, $\langle x',x\rangle_{\mathcal{X}}$ is the value of $x'\in\mathcal{X}'$ at $x\in\mathcal{X}$.

Let $M=(M,g)$ be a Riemannian manifold. Suppose $V=(V,\pi,M)$ is a $\mathbb{K}$-vector bundle over M. For a subset S of M we denote by V_S the restriction of V to S, that is, $V_S=\pi^{-1}(S)$. If $k\in\mathbb{N}\cup\{\infty\}$ and S is open in M, then $C^k(S,V)$ is the $C^k(S)$-module of C^k-sections over S.

We denote by $V'=V^*$ the dual vector bundle and by $\langle\cdot,\cdot\rangle$ the fiber-wise defined duality pairing between V' and V. We also assume that V is equipped with an inner product and write $|\cdot|_V$ for the corresponding vector bundle norm.

Given an open subset S of M and $q\in[1,\infty]$, the Lebesgue space $L_q(S,V)=\big(L_q(S,V),\|\cdot\|_q\big)$ is the Banach space of all (equivalence classes of measurable) sections v of V over S such that

$$\|v\|_q=\|v\|_{L_q(S,V)}:=\big\|\,|v|_V\big\|_{L_q(S)}<\infty,$$

where $L_q(S)=L_q(S,\mathbb{K};dV_g)$.

In the following, we write $U\subset\subset V$ to mean that U and V are open, U is relatively compact, and $\overline{U}\subset V$. Since M is locally compact, separable, and metrizable it is σ-compact. Thus there exists a sequence (M_j)

such that $M_j \subset\subset M_{j+1}$ and $\bigcup_j M_j = M$. Hence $L_{1,\mathrm{loc}}(M,V)$, the vector space of sections v of V such that $v\,|\,S \in L_1(S,V)$ for every $S \subset\subset M$, is a Fréchet space.

We denote by $\mathcal{D}(\mathring{M},V)$ and $\mathcal{D}(M,V)$ the spaces of smooth sections of V being compactly supported in $\mathring{M}$ and M, respectively. For $S \subset\subset \mathring{M}$, or $S \subset\subset M$, we write $\mathcal{D}_S(\mathring{M},V)$, respectively $\mathcal{D}_S(M,V)$, for the linear subspace of all $v \in \mathcal{D}(\mathring{M},V)$, respectively $v \in \mathcal{D}(M,V)$, with $\mathrm{supp}(v) \subset \overline{S}$. Then $\mathcal{D}_S(\mathring{M},V)$ and $\mathcal{D}_S(M,V)$ are Fréchet spaces (e.g., Section VII.2 of J. Dieudonné [16]). If $S \subset\subset S_1$, then $\mathcal{D}_S(\mathring{M},V) \subset \mathcal{D}_{S_1}(\mathring{M},V)$ and $\mathcal{D}_{S_1}(\mathring{M},V)$ induces on $\mathcal{D}_S(\mathring{M},V)$ its original topology. Hence we can endow $\mathcal{D}(\mathring{M},V)$ with the LF topology (the strict inductive limit topology) with respect to all such subspaces of $\mathcal{D}(\mathring{M},V)$. Similarly, $\mathcal{D}(M,V)$ is given the LF topology with respect to the subspaces $\mathcal{D}_S(M,V)$. Then

$$\mathcal{D}'(\mathring{M},V) := \mathcal{D}(\mathring{M},V')'_{w^*} \tag{4.1}$$

is the space of distribution sections on $\mathring{M}$, endowed with the weak* topology.

Given $v \in L_{1,\mathrm{loc}}(\mathring{M},V)$,

$$\Bigl(u \mapsto \langle v,u\rangle_{\mathcal{D}} := \int_M \langle v,u\rangle\, dV_g\Bigr) \in \mathcal{D}'(\mathring{M},V), \tag{4.2}$$

and the map

$$L_{1,\mathrm{loc}}(\mathring{M},V) \to \mathcal{D}'(\mathring{M},V), \quad v \mapsto \langle v,\cdot\rangle_{\mathcal{D}}$$

is linear, continuous, and injective. We identify $v \in L_{1,\mathrm{loc}}(\mathring{M},V)$ with the distribution section (4.2) and consider $L_{1,\mathrm{loc}}(\mathring{M},V)$ as a linear subspace of $\mathcal{D}'(\mathring{M},V)$. Then

$$\mathcal{D}(\mathring{M},V) \hookrightarrow \mathcal{D}(M,V) \overset{d}{\hookrightarrow} L_{1,\mathrm{loc}}(M,V) \overset{d}{\hookrightarrow} L_{1,\mathrm{loc}}(\mathring{M},V) \hookrightarrow \mathcal{D}'(\mathring{M},V), \tag{4.3}$$

where $\hookrightarrow$ means 'continuous' and $\overset{d}{\hookrightarrow}$ 'continuous and dense' embedding. Given $f \in C^\infty(M)$, the point-wise multiplication $u \mapsto fu$ belongs to $\mathcal{L}\bigl(\mathcal{D}(\mathring{M},V')\bigr)$. Hence, setting

$$(fT)(u) := T(fu), \qquad T \in \mathcal{D}'(\mathring{M},V), \quad u \in \mathcal{D}(\mathring{M},V'),$$

it follows $(T \mapsto fT) \in \mathcal{L}\bigl(\mathcal{D}'(\mathring{M},V)\bigr)$. We often identify f with this 'point-wise multiplication' operator.

Suppose $k,\ell \in \mathbb{N}$ satisfy $k+\ell \ge 1$ and $E = \bigl(\mathbb{K}^{k\times\ell}, (\cdot,\cdot)_{HS}\bigr)$, where

$$(\cdot,\cdot)_{HS} : \ E \times E \to \mathbb{K}, \quad (a,b) \mapsto \mathrm{trace}(b^*a)$$

is the Hilbert-Schmidt inner product, $b^* \in \mathbb{K}^{\ell\times k}$ being the conjugate matrix of b. Then

$$E \times E \to \mathbb{K}, \quad (a,b) \mapsto (a\,|\,\overline{b})_{HS} \tag{4.4}$$

is a separating bilinear form, the duality pairing of E, by which we identify E' with E.

Consider the trivial bundle $M \times E$. As usual, we write $\mathcal{D}(M,E)$ for $\mathcal{D}(M, M\times E)$ etc. By juxtaposition of the rows of a matrix $a \in \mathbb{K}^{k\times\ell}$ we fix an isomorphism from $\mathbb{K}^{k\times\ell}$ onto $\mathbb{K}^n$, where $n = k\ell$. By means of it we identify $\mathcal{D}(M,E)$ with $\mathcal{D}(M)^n$, etc. Then

$$T(u) = \sum_{i=1}^n T_i(u_i), \qquad (T,u) \in \mathcal{D}'(\mathring{M},E) \times \mathcal{D}(\mathring{M},E), \tag{4.5}$$

where $u = (u_1,\dots,u_n) \in \mathcal{D}(\mathring{M})^n$, etc.

Assume $\mathbb{X} = \bigl(\mathbb{X}, (\cdot|\cdot)_{g_m}\bigr)$ with $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$. Let $\mathcal{S}(\mathbb{X},E)$ be the Schwartz space of rapidly decreasing smooth E-valued functions on $\mathbb{X}$. Then $\mathcal{S}(\mathring{\mathbb{X}},E)$ is the closure of $\mathcal{D}(\mathring{\mathbb{X}},E)$ in $\mathcal{S}(\mathbb{X},E)$, and

$$\mathcal{S}'(\mathring{\mathbb{X}},E) := \mathcal{S}(\mathring{\mathbb{X}},E)'_{w^*}$$

is the space of E-valued tempered distributions on $\mathring{\mathbb{X}}$. Since $\mathring{\mathbb{X}} = \mathbb{R}^m$ if $\mathbb{X} = \mathbb{R}^m$, our notation is consistent with the well-known fact $\mathcal{D}(\mathbb{R}^m, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^m, E)$.

Set $V := \bigl(\mathbb{X} \times E, (\cdot|\cdot)_{HS}\bigr)$ and note that $\langle v, \cdot\rangle_{\mathcal{D}}$, defined by (4.2) and (4.4), is for each $v \in \mathcal{D}(M, V)$ continuous with respect to the topology induced by $\mathcal{S}(\mathbb{X}, E)$ on $\mathcal{D}(\mathring{\mathbb{X}}, E)$. From this it follows

$$\mathcal{D}(\mathring{\mathbb{X}}, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathring{\mathbb{X}}, E) \hookrightarrow \mathcal{S}(\mathbb{X}, E) \hookrightarrow \mathcal{S}'(\mathring{\mathbb{X}}, E) \hookrightarrow \mathcal{D}'(\mathring{\mathbb{X}}, E). \tag{4.6}$$

By mollifying we further obtain

$$\mathcal{D}(\mathring{\mathbb{X}}, E) \overset{d}{\hookrightarrow} \mathcal{D}'(\mathring{\mathbb{X}}, E). \tag{4.7}$$

For $u \in \mathcal{S}'(\mathbb{R}^m, E)$ we let r^+ be the restriction of u to $\mathring{\mathbb{H}}^m$ in the sense of distributions, that is,

$$\langle r^+ u, \varphi\rangle_{\mathcal{S}(\mathring{\mathbb{H}}^m, E)} = \langle u, \varphi\rangle_{\mathcal{S}(\mathbb{R}^m, E)}, \qquad \varphi \in \mathcal{S}(\mathring{\mathbb{H}}^m, E).$$

Then $r^+ \in \mathcal{L}\bigl(\mathcal{S}'(\mathbb{R}^m, E), \mathcal{S}'(\mathring{\mathbb{H}}^m, E)\bigr)$.

If no confusion seems likely we use the same symbol for a linear map and its restriction to a linear subspace of its domain. Furthermore, in a diagram arrows always represent continuous linear maps.

Recall that a *retraction* $\mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are locally convex spaces, is a continuous linear map possessing a continuous right inverse, a coretraction. Thus the following lemma guarantees that r^+ is a retraction.

Lemma 4.1 *There exists an extension operator* e^+ *such that the diagram*

$$\begin{array}{ccccc}
\mathcal{S}(\mathbb{H}^m, E) & \xrightarrow{\;e^+\;} & \mathcal{S}(\mathbb{R}^m, E) & \xrightarrow{\;r^+\;} & \mathcal{S}(\mathbb{H}^m, E) \\
\big\downarrow d & & \big\downarrow d & & \big\downarrow d \\
\mathcal{S}'(\mathring{\mathbb{H}}^m, E) & \xrightarrow{\;e^+\;} & \mathcal{S}'(\mathbb{R}^m, E) & \xrightarrow{\;r^+\;} & \mathcal{S}'(\mathring{\mathbb{H}}^m, E)
\end{array}$$

is commuting and $r^+ e^+ = \mathrm{id}$.

Proof. As in (4.5) we identify $\mathcal{S}(\mathbb{X}, E)$ with $\mathcal{S}(\mathbb{X})^n$ and $\mathcal{S}'(\mathring{\mathbb{X}}, E)$ with $\mathcal{S}'(\mathring{\mathbb{X}})^n$. Then the assertion follows from Theorems 4.2.2 and 4.2.4 in [3] (with $F := \mathbb{K}$). □

It is a consequence of this lemma, (4.3), (4.6), and (4.7) that

$$\mathcal{D}(\mathbb{X}, E) \hookrightarrow \mathcal{S}(\mathbb{X}, E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathring{\mathbb{X}}, E) \overset{d}{\hookrightarrow} \mathcal{D}'(\mathring{\mathbb{X}}, E)$$

and

$$\mathcal{D}(\mathbb{X}, E) \overset{d}{\hookrightarrow} \mathcal{D}'(\mathring{\mathbb{X}}, E), \tag{4.8}$$

due to $\mathcal{D}(\mathring{\mathbb{X}}, E) \subset \mathcal{D}(\mathbb{X}, E)$.

5 Localization of Distribution Sections

Let A be a countable index set. Suppose $\mathcal{X}_\alpha$ is for each $\alpha \in \mathsf{A}$ a locally convex space. We endow $\prod_\alpha \mathcal{X}_\alpha$ with the product topology, that is, the coarsest locally convex topology for which all projections $\mathrm{pr}_\beta : \prod_\alpha \mathcal{X}_\alpha \to \mathcal{X}_\beta$, $\boldsymbol{x} = (x_\alpha) \mapsto x_\beta$ are continuous. By $\bigoplus_\alpha \mathcal{X}_\alpha$ we mean the locally convex direct sum. Thus $\bigoplus_\alpha \mathcal{X}_\alpha$ is the vector subspace of $\prod_\alpha \mathcal{X}_\alpha$ consisting of all finitely supported $\boldsymbol{x} \in \prod_\alpha \mathcal{X}_\alpha$, equipped with the inductive topology, that is, the finest locally convex topology for which all injections $\mathcal{X}_\beta \to \bigoplus_\alpha \mathcal{X}_\alpha$ are continuous. Let $\langle\cdot,\cdot\rangle_\alpha$ be the $\mathcal{X}_\alpha$-duality pairing. Then

$$\boldsymbol{\langle}\cdot,\cdot\boldsymbol{\rangle} : \prod_\alpha \mathcal{X}'_\alpha \times \bigoplus_\alpha \mathcal{X}_\alpha \to \mathbb{K}, \quad (\boldsymbol{x}', \boldsymbol{x}) \mapsto \sum_\alpha \langle x'_\alpha, x_\alpha\rangle_\alpha$$

is a separating bilinear form, and (cf. Corollary 1 in Section IV.4.3 of H.H. Schaefer [32])

$$\Big(\bigoplus_{\alpha}\mathcal{X}_\alpha\Big)'_{w^*}=\prod_{\alpha}(\mathcal{X}_\alpha)'_{w^*} \tag{5.1}$$

with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$, (that is, $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is the $\bigoplus_\alpha \mathcal{X}_\alpha$-duality pairing).

Throughout the rest of this paper we assume

- $M=(M,g)$ is an m-dimensional singular manifold.
- $\rho\in\mathfrak{T}(M)$.
- $\sigma,\tau\in\mathbb{N}$ and $V=V^\sigma_\tau:=\big(T^\sigma_\tau M,(\cdot|\cdot)_g\big)$.

It follows that we can choose

- a singularity datum $(\rho,\mathfrak{K})$,
- a localization system $\{\,(\pi_\kappa,\chi_\kappa)\ ;\ \kappa\in\mathfrak{K}\,\}$ subordinate to $\mathfrak{K}$. (5.2)

For $K\subset M$ we put $\mathfrak{K}_K:=\{\,\kappa\in\mathfrak{K}\,;\,U_\kappa\cap K\neq\emptyset\,\}$. Then, given $\kappa\in\mathfrak{K}$,

$$\mathbb{X}_\kappa:=\begin{cases}\mathbb{R}^m & \text{if } \kappa\in\mathfrak{K}\backslash\mathfrak{K}_{\partial M},\\ \mathbb{H}^m & \text{otherwise,}\end{cases}$$

endowed with the Euclidean metric g_m.

We set

$$E=E^\sigma_\tau:=\big(\mathbb{K}^{m^\sigma\times m^\tau},(\cdot|\cdot)_{HS}\big)$$

and consider the trivial bundles $V_\kappa:=\big(\mathbb{X}_\kappa\times E,(\cdot|\cdot)_{g_m}\big)$ for $\kappa\in\mathfrak{K}$. For abbreviation,

$$\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E):=\bigoplus_{\kappa}\mathcal{D}(\mathring{\mathbb{X}}_\kappa,E),\quad \boldsymbol{\mathcal{D}}(\mathbb{X},E):=\bigoplus_{\kappa}\mathcal{D}(\mathbb{X}_\kappa,E),$$

and

$$\boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E):=\prod_{\kappa}\mathcal{D}'(\mathring{\mathbb{X}}_\kappa,E).$$

It follows from (5.1) that $\boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E)=\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E')'_{w^*}$, where $E'=E^\tau_\sigma$.

We introduce linear maps

$$\varphi_\kappa:\ \mathcal{D}(M,V)\to\mathcal{D}(\mathbb{X}_\kappa,E),\quad u\mapsto\kappa_*(\pi_\kappa u)$$

and

$$\psi_\kappa:\ \mathcal{D}(\mathbb{X}_\kappa,E)\to\mathcal{D}(M,V),\quad v_\kappa\mapsto\pi_\kappa\kappa^*v_\kappa$$

for $\kappa\in\mathfrak{K}$. Here and in similar situations it is understood that a partially defined and compactly supported section of a vector bundle is extended over the whole base manifold by identifying it with the zero section outside its original domain. Moreover,

$$\varphi:\ \mathcal{D}(M,V)\to\boldsymbol{\mathcal{D}}(\mathbb{X},E),\quad u\mapsto(\varphi_\kappa u)$$

and

$$\psi:\ \boldsymbol{\mathcal{D}}(\mathbb{X},E)\to\mathcal{D}(M,V),\quad \boldsymbol{v}\mapsto\textstyle\sum_\kappa\psi_\kappa v_\kappa.$$

The following retraction theorem shows, in particular, that these maps are well-defined and possess unique continuous linear extensions to distribution sections.

Theorem 5.1 *The diagram*

$$\begin{array}{ccccc}
\mathcal{D}(M,V) & \xrightarrow{\ \varphi\ } & \boldsymbol{\mathcal{D}}(\mathbb{X},E) & \xrightarrow{\ \psi\ } & \mathcal{D}(M,V) \\
\big\downarrow d & & \big\downarrow d & & \big\downarrow d \\
\mathcal{D}'(\mathring{M},V) & \xrightarrow{\ \varphi\ } & \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E) & \xrightarrow{\ \psi\ } & \mathcal{D}'(\mathring{M},V)
\end{array}$$

is commuting and $\psi\circ\varphi=\mathrm{id}$.

P r o o f. (1) We set

$$\mathring{\varphi}_\kappa u := \sqrt{\kappa_*g}\,\kappa_*(\pi_\kappa u), \qquad u\in\mathcal{D}(\mathring{M},V'), \quad \kappa\in\mathfrak{K}. \tag{5.3}$$

Suppose $K\subset\subset\mathring{M}$. Then $L_\kappa:=\kappa\big(K\cap\mathrm{dom}(\chi_\kappa)\big)\subset\subset\mathring{\mathbb{X}}_\kappa$. Assume $u\in\mathcal{D}_K(\mathring{M},V')$. Then $\kappa_*(\pi_\kappa u)$ belongs to $\mathcal{D}_{L_\kappa}(\mathring{\mathbb{X}}_\kappa,V'_\kappa)$. Since $\sqrt{\kappa_*g}\in C^\infty(Q^m_\kappa)$, it follows

$$\mathring{\varphi}_\kappa\in\mathcal{L}\big(\mathcal{D}_K(\mathring{M},V'),\mathcal{D}(\mathring{\mathbb{X}}_\kappa,V'_\kappa)\big), \qquad \kappa\in\mathfrak{K},$$

due to $\mathcal{D}_{L_\kappa}(\mathring{\mathbb{X}}_\kappa,V'_\kappa)\hookrightarrow\mathcal{D}(\mathring{\mathbb{X}}_\kappa,V')$. This being true for each $K\subset\subset\mathring{M}$, we obtain

$$\mathring{\varphi}_\kappa\in\mathcal{L}\big(\mathcal{D}(\mathring{M},V'),\mathcal{D}(\mathring{\mathbb{X}}_\kappa,V'_\kappa)\big), \qquad \kappa\in\mathfrak{K}.$$

(2) We put

$$\mathring{\psi}_\kappa v := \pi_\kappa\kappa^*\Big(\big(\sqrt{\kappa_*g}\big)^{-1}\chi v\Big), \qquad v\in\mathcal{D}(\mathring{\mathbb{X}}_\kappa,V'_\kappa), \quad \kappa\in\mathfrak{K}. \tag{5.4}$$

Suppose $L_\kappa\subset\subset\mathring{\mathbb{X}}_\kappa$ and set $K_\kappa:=\kappa^{-1}\big(L_\kappa\cap\mathrm{dom}(\chi)\big)$. Then $K_\kappa\subset\subset\mathring{M}$. Similarly as above, we find that $\mathring{\psi}_\kappa$ maps $\mathcal{D}_{L_\kappa}(\mathring{\mathbb{X}}_\kappa,V'_\kappa)$ continuously into $\mathcal{D}(\mathring{M},V')$. Consequently,

$$\mathring{\psi}_\kappa\in\mathcal{L}\big(\mathcal{D}(\mathring{\mathbb{X}}_\kappa,V'_\kappa),\mathcal{D}(\mathring{M},V')\big).$$

(3) Set

$$\mathring{\varphi}u:=(\mathring{\varphi}_\kappa u), \qquad u\in\mathcal{D}(\mathring{M},V').$$

Assume $K\subset\subset\mathring{M}$. Since $\mathfrak{K}$ is uniformly shrinkable there exist $r\in(0,1)$ and a finite subset $\mathfrak{L}_K$ of $\mathfrak{K}$ such that $\big\{\,\kappa^{-1}(rQ^m_\kappa)\ ;\ \kappa\in\mathfrak{L}_K\,\big\}$ is a cover of K. Put

$$\mathfrak{M}_K:=\{\,\kappa\in\mathfrak{K}\ ;\ \text{there exists }\widetilde{\kappa}\in\mathfrak{L}_K\text{ with }U_{\widetilde{\kappa}}\cap U_\kappa\neq\emptyset\,\}.$$

Then $\mathfrak{M}_K$ is a finite set, due to the finite multiplicity of $\mathfrak{K}$. Since $\mathring{\varphi}_\kappa u=0$ for $u\in\mathcal{D}_K(\mathring{M},V')$ and $\kappa\in\mathfrak{K}\backslash\mathfrak{M}_K$ it follows from step (1) that $\mathring{\varphi}$ maps $\mathcal{D}_K(\mathring{M},V')$ continuously into the closed linear subspace

$$\big\{\,\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E')\ ;\ v_\kappa=0\text{ for }\kappa\in\mathfrak{K}\backslash\mathfrak{M}_K\,\big\}$$

of $\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E')$, hence into $\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E')$. Since this is true for all $K\subset\subset\mathring{M}$,

$$\mathring{\varphi}\in\mathcal{L}\big(\mathcal{D}(\mathring{M},V'),\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E')\big). \tag{5.5}$$

(4) Put

$$\mathring{\psi}\boldsymbol{v}:=\sum_\kappa\mathring{\psi}_\kappa v_\kappa, \qquad \boldsymbol{v}=(v_\kappa)\in\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E').$$

Let $\mathfrak{L}$ be a finite subset of $\mathfrak{K}$ and put

$$\mathcal{X}_{\mathfrak{L}}:=\big\{\,\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}},E')\ ;\ v_\kappa=0\text{ if }\kappa\in\mathfrak{K}\backslash\mathfrak{L}\,\big\}.$$

Step (2) implies that $\mathring{\psi}$ maps $\mathcal{X}_{\mathfrak{L}}$ continuously into $\mathcal{D}(\mathring{M}, V')$. Thus, since this holds for all finite subset $\mathfrak{L}$ of $\mathfrak{K}$,

$$\mathring{\psi} \in \mathcal{L}\big(\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E'), \mathcal{D}(\mathring{M}, V')\big). \tag{5.6}$$

(5) For $u \in \mathcal{D}(\mathring{M}, V')$ and $\kappa \in \mathfrak{K}$ it follows from $\pi_\kappa \chi_\kappa = \pi_\kappa$ and $\chi_\kappa = \kappa^* \chi$ that $(\mathring{\psi}_\kappa \circ \mathring{\varphi}_\kappa)u = \pi_\kappa^2 u$. Hence $\sum_\kappa \pi_\kappa^2 = \mathbf{1}$ implies

$$(\mathring{\psi} \circ \mathring{\varphi})u = \sum_\kappa \psi_\kappa(\varphi_\kappa u) = \sum_\kappa \pi_\kappa^2 u = u, \qquad u \in \mathcal{D}(\mathring{M}, V').$$

Thus $\mathring{\psi}$ is a retraction from $\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E')$ onto $\mathcal{D}(\mathring{M}, V')$, and $\mathring{\varphi}$ is a coretraction.

(6) Steps (3) and (4) and relations (4.1) and (5.1) imply

$$\Psi := (\mathring{\varphi})' \in \mathcal{L}\big(\boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}}, E), \mathcal{D}'(\mathring{M}, V)\big)$$

and

$$\Phi := (\mathring{\psi})' \in \mathcal{L}\big(\mathcal{D}'(\mathring{M}, V), \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}}, E)\big).$$

By step (5),

$$\Psi \circ \Phi = (\mathring{\psi} \circ \mathring{\varphi})' = (\mathrm{id}_{\mathcal{D}(\mathring{M}, V')})' = \mathrm{id}_{\mathcal{D}'(\mathring{M}, V)}.$$

(7) Suppose $v \in \mathcal{D}(M, V)$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E')$. Then, see (4.2),

$$\begin{aligned}\boldsymbol{\langle} \Phi v, \boldsymbol{u} \boldsymbol{\rangle} = \langle v, \mathring{\psi} \boldsymbol{u} \rangle_{\mathcal{D}} = \sum_\kappa \langle v, \mathring{\psi}_\kappa u_\kappa \rangle_{\mathcal{D}} &= \sum_\kappa \int_M \pi_\kappa \big\langle v, \big(\sqrt{\kappa_* g}\,\big)^{-1} \kappa^*(\chi u_\kappa)\big\rangle \, dV_g \\ &= \sum_\kappa \int_{U_\kappa} \kappa^*\big(\langle \kappa_*(\pi_\kappa v), u_\kappa \rangle \, dV_{g_m}\big) = \sum_\kappa \int_{\mathbb{X}_\kappa} \langle \varphi_\kappa v, u_\kappa \rangle \, dV_{g_m} = \boldsymbol{\langle} \varphi v, \boldsymbol{u} \boldsymbol{\rangle}.\end{aligned}$$

This proves

$$\varphi = \Phi \,|\, \mathcal{D}(M, V).$$

By the arguments of steps (1) and (3), with $\mathring{M}$ replaced by M and $\mathring{\mathbb{X}}_\kappa$ by $\mathbb{X}_\kappa$, respectively, we find

$$\varphi \in \mathcal{L}\big(\mathcal{D}(M, V), \boldsymbol{\mathcal{D}}(\mathbb{X}, E)\big).$$

(8) Let $v \in \boldsymbol{\mathcal{D}}(\mathbb{X}, E)$ and $u \in \mathcal{D}(\mathring{M}, V')$. Then

$$\begin{aligned}\langle \Psi \boldsymbol{v}, u \rangle_{\mathcal{D}} = \boldsymbol{\langle} \boldsymbol{v}, \mathring{\varphi} u \boldsymbol{\rangle} &= \sum_\kappa \int_{\mathbb{X}_\kappa} \big\langle v_\kappa, \kappa_*(\pi_\kappa u) \big\rangle \sqrt{\kappa_* g} \, dV_{g_m} = \sum_\kappa \int_{Q_\kappa^m} \kappa_*\big(\langle \pi_\kappa \kappa^* v_\kappa, u \rangle \, dV_g\big) \\ &= \sum_\kappa \int_M \langle \psi_\kappa v_\kappa, u \rangle \, dV_g = \int_M \langle \psi \boldsymbol{v}, u \rangle \, dV_g = \langle \psi \boldsymbol{v}, u \rangle_{\mathcal{D}}.\end{aligned}$$

Consequently,

$$\psi = \Psi \,|\, \boldsymbol{\mathcal{D}}(\mathbb{X}, E).$$

Modifying the arguments of steps (2) and (4) in the obvious way gives $\Psi \in \mathcal{L}\big(\boldsymbol{\mathcal{D}}(\mathbb{X}, E), \mathcal{D}(M, V)\big)$.

(9) By collecting what has been proved so far we see that the diagram

$$\begin{array}{ccccc} \mathcal{D}(M, V) & \xrightarrow{\ \varphi\ } & \boldsymbol{\mathcal{D}}(\mathbb{X}, E) & \xrightarrow{\ \psi\ } & \mathcal{D}(M, V) \\ \big\downarrow & & \big\downarrow d & & \big\downarrow \\ \mathcal{D}'(\mathring{M}, V) & \xrightarrow{\ \Phi\ } & \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}}, E) & \xrightarrow{\ \Psi\ } & \mathcal{D}'(\mathring{M}, V) \end{array}$$

is commuting, where the embeddings symbolized by the vertical arrows follow from (4.3) and (4.8). Furthermore, Ψ is a retraction and Φ is a coretraction. Thus we read off this diagram that $\Psi\big(\boldsymbol{\mathcal{D}}(\mathbb{X},E)\big)$ is dense in $\mathcal{D}'(\mathring{M},V)$ (cf. Lemma 4.1.6 in [3]).

Let U be a neighborhood of 0 in $\mathcal{D}'(\mathring{M},V)$. Then there exists $\boldsymbol{u}\in\boldsymbol{\mathcal{D}}(\mathbb{X},E)$ such that $\Psi\boldsymbol{u}\in U$. Hence $\Psi\boldsymbol{u}=\psi\boldsymbol{u}\in\mathcal{D}(M,V)$ shows that $U\cap\mathcal{D}(M,V)\neq\emptyset$. This implies that $\mathcal{D}(M,V)$ is dense in $\mathcal{D}'(\mathring{M},V)$. Since Φ and Ψ are continuous linear extensions of φ and ψ, respectively, they are uniquely determined by the density of the 'vertical' embeddings in the above diagram. Thus we can denote Φ and Ψ also by φ and ψ, respectively, without fearing confusion. This establishes the theorem. □

6 Sobolev Spaces

Henceforth, we always assume

- $1<p<\infty, \quad \lambda\in\mathbb{R}.$

Suppose $k\in\mathbb{N}$. The weighted *Sobolev space* $W_p^{k,\lambda}(V;\rho)$ of (σ,τ)-tensor fields is the completion of $\mathcal{D}(M,V)$ in $L_{1,\mathrm{loc}}(M,V)$ with respect to the norm

$$u\mapsto\Big(\sum_{i=0}^{k}\big\|\rho^{\lambda+\tau-\sigma+i}\,|\nabla^i u|_g\big\|_p^p\Big)^{1/p}. \tag{6.1}$$

If $\rho'\in\mathfrak{T}(M)$, then $\rho'\sim\rho$ and we obtain an equivalent norm by replacing ρ in (6.1) by ρ'. Thus the topology of $W_p^{k,\lambda}(V;\rho)$ depends on the singularity type $\mathfrak{T}(M)$ only. Henceforth, we simply write $W_p^{k,\lambda}(V)$ for $W_p^{k,\lambda}(V;\rho)$ and denote the norm (6.1) by $\|\cdot\|_{k,p;\lambda}$. Moreover, $L_p^{\lambda}(V):=W_p^{0,\lambda}(V)$ and $\|\cdot\|_{p;\lambda}:=\|\cdot\|_{0,p;\lambda}$. If $\mathfrak{T}(M)=[\![\mathbf{1}]\!]$, then all these spaces are independent of λ and we obtain the 'standard' Sobolev spaces $W_p^k(V)$. The reader should be careful not to confuse $W_p^{k,0}(V)$ with $W_p^k(V)$.

We also define weighted spaces of bounded smooth (σ,τ)-tensor fields by

$$BC^{k,\lambda}(V):=\big(\{\,u\in C^k(M,V)\,;\ \|u\|_{k,\infty;\lambda}<\infty\,\},\ \|\cdot\|_{k,\infty;\lambda}\big),$$

where

$$\|u\|_{k,\infty;\lambda}:=\max_{0\le i\le k}\big\|\rho^{\lambda+\tau-\sigma+i}\,|\nabla^i u|_g\big\|_\infty.$$

The topology of $BC^{k,\lambda}(V)$ is independent of the particular choice of $\rho\in\mathfrak{T}(M)$.

The following basic retraction theorems show that these spaces can be characterized by means of local coordinates, similarly as in the case of function spaces on compact manifolds. Below we make free use, usually without further mention, of the theory of function spaces on $\mathbb{R}^m$ and $\mathbb{H}^m$. Everything for which we do not give specific references can be found in H. Triebel [37], for example.

Let E_α be a Banach space for each α in a countable index set. Then $\boldsymbol{E}:=\prod_\alpha E_\alpha$. For $1\le q\le\infty$ we denote by $\ell_q(\boldsymbol{E})$ the linear subspace of $\boldsymbol{E}$ consisting of all $\boldsymbol{x}=(x_\alpha)$ such that

$$\|\boldsymbol{x}\|_{\ell_q(\boldsymbol{E})}:=\begin{cases}\big(\sum_\alpha\|x_\alpha\|_{E_\alpha}^q\big)^{1/q}, & 1\le q<\infty,\\ \sup_\alpha\|x_\alpha\|_{E_\alpha}, & q=\infty,\end{cases}$$

is finite. Then $\ell_q(\boldsymbol{E})$ is a Banach space with norm $\|\cdot\|_{\ell_q(\boldsymbol{E})}$, and

$$\ell_p(\boldsymbol{E})\hookrightarrow\ell_q(\boldsymbol{E}),\qquad 1\le p<q\le\infty. \tag{6.2}$$

We also set $c_c(\boldsymbol{E}):=\bigoplus_\alpha E_\alpha$. Then

$$c_c(\boldsymbol{E})\hookrightarrow\ell_q(\boldsymbol{E}),\quad 1\le q\le\infty,\qquad c_c(\boldsymbol{E})\overset{d}{\hookrightarrow}\ell_q(\boldsymbol{E}),\quad q<\infty. \tag{6.3}$$

Furthermore, $c_0(\boldsymbol{E})$ is the closure of $c_c(\boldsymbol{E})$ in $\ell_\infty(\boldsymbol{E})$.

If each E_α is reflexive, then $\ell_p(\boldsymbol{E})$ is reflexive as well, and $\ell_p(\boldsymbol{E})' = \ell_{p'}(\boldsymbol{E}')$ with respect to the duality pairing $\boldsymbol{\langle}\cdot,\cdot\boldsymbol{\rangle} := \sum_\alpha \langle\cdot,\cdot\rangle_\alpha$. Of course, $p' := p/(p-1)$, $\boldsymbol{E}' := \prod_\alpha E'_\alpha$, and $\langle\cdot,\cdot\rangle_\alpha$ is the E_α-duality pairing.

Let (5.2) be chosen. For $1 \le q \le \infty$ we set

$$\varphi^\lambda_{q,\kappa} := \rho_\kappa^{\lambda+m/q}\varphi_\kappa, \quad \psi^\lambda_{q,\kappa} := \rho_\kappa^{-\lambda-m/q}\psi_\kappa, \qquad \kappa \in \mathfrak{K},$$

and

$$\varphi^\lambda_q u := (\varphi^\lambda_{q,\kappa} u), \quad \psi^\lambda_q \boldsymbol{v} := \sum_\kappa \psi^\lambda_{q,\kappa} v_\kappa$$

for $u \in \mathcal{D}'(\mathring{M},V)$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E)$. If the dependence on (σ,τ) is important, then we write $\varphi^\lambda_{q,(\sigma,\tau)}$, etc. Note $(\varphi^\lambda_{p,\kappa},\psi^\lambda_{p,\kappa}) = (\varphi_{p,\kappa},\psi_{p,\kappa})$ if $\rho = \mathbf{1}$.

Suppose $\mathfrak{F}$ is a symbol for one of the standard function spaces, say, Sobolev, Slobodeckii, Besov spaces, etc., on $\mathbb{R}^m$. Then we put $\boldsymbol{\mathfrak{F}} := \prod_\kappa \mathfrak{F}_\kappa$ and $\mathfrak{F}_\kappa := \mathfrak{F}(\mathbb{X}_\kappa,E)$. For example, $\boldsymbol{W}^k_p = \prod_\kappa W^k_{p,\kappa} = \prod_\kappa W^k_p(\mathbb{X}_\kappa,E)$.

Theorem 6.1 *Suppose $k \in \mathbb{N}$. The diagram*

$$\begin{array}{ccccc}
\mathcal{D}(M,V) & \xrightarrow{\ \varphi^\lambda_p\ } & \boldsymbol{\mathcal{D}}(\mathbb{X},E) & \xrightarrow{\ \psi^\lambda_p\ } & \mathcal{D}(M,V) \\
\big\downarrow d & & \big\downarrow d & & \big\downarrow d \\
W^{k,\lambda}_p(V) & \xrightarrow{\ \varphi^\lambda_p\ } & \ell_p(\boldsymbol{W}^k_p) & \xrightarrow{\ \psi^\lambda_p\ } & W^{k,\lambda}_p(V) \\
\big\downarrow d & & \big\downarrow d & & \big\downarrow d \\
\mathcal{D}'(\mathring{M},V) & \xrightarrow{\ \varphi^\lambda_p\ } & \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E) & \xrightarrow{\ \psi^\lambda_p\ } & \mathcal{D}'(\mathring{M},V)
\end{array}$$

is commuting and $\psi^\lambda_p \circ \varphi^\lambda_p = \mathrm{id}$.

Proof. (1) It is an obvious consequence of Theorem 5.1 that ψ^λ_p is a retraction from $\boldsymbol{\mathcal{D}}(\mathbb{X},E)$ onto $\mathcal{D}(M,V)$, and from $\mathcal{D}'(\mathring{\mathbb{X}},E)$ onto $\mathcal{D}'(\mathring{M},V)$, and that φ^λ_p is a coretraction in each case.

(2) Estimate (3.21)(iii), Leibniz' rule, and $\kappa_*(\pi_\kappa u) = (\kappa_*\pi_\kappa)\kappa_* u$ imply, due to $\chi_\kappa \,|\, \mathrm{supp}(\pi_\kappa) = \mathbf{1}$,

$$\|\kappa_*(\pi_\kappa u)\|_{W^k_{p,\kappa}} \le c\,\|\kappa_*(\chi_\kappa u)\|_{W^k_p(Q^m_\kappa,E)}, \qquad \kappa \in \mathfrak{K}. \tag{6.4}$$

From Lemma 3.1(iv) we deduce

$$\|\kappa_*(\chi_\kappa u)\|^p_{W^k_p(Q^m_\kappa,E)} = \int_{Q^m_\kappa} \chi \sum_{|\alpha|\le k} |\partial^\alpha(\kappa_* u)|^p_{g_m}\,dV_{g_m} \le \sum_{i=0}^k \int_{Q^m_\kappa} \chi\,|\nabla^i_{\kappa_* g}(\kappa_* u)|^p_{g_m}\,dV_{g_m}. \tag{6.5}$$

By part (v) of Lemma 3.1 we get, due to $\nabla^i u \in \mathcal{D}(M, T^\sigma_{\tau+i}M)$ for $u \in \mathcal{D}(M,V)$,

$$|\nabla^i_{\kappa_* g}(\kappa_* u)|_{g_m} \sim \kappa_*(\rho_\kappa^{\tau-\sigma+i}\,|\nabla^i u|_g), \qquad \kappa \in \mathfrak{K}.$$

Thus, observing Lemma 3.1(iii) and (2.1)(vi),

$$\begin{aligned}
\int_{Q^m_\kappa} \chi\,|\nabla^i_{\kappa_* g}(\kappa_* u)|^p_{g_m}\,dV_{g_m} &\sim \int_{\kappa(U_\kappa)} \kappa_*\Big(\big(\chi_\kappa \rho_\kappa^{\tau-\sigma+i-m/p}\,|\nabla^i u|_g\big)^p\,dV_g\Big) \\
&\sim \rho_\kappa^{-m} \int_M \chi_\kappa\big(\rho^{\tau-\sigma+i}\,|\nabla^i u|_g\big)^p\,dV_g
\end{aligned}$$

for $\kappa \in \mathfrak{K}$. Thus we get from (6.4) and (6.5)

$$\|\varphi^\lambda_{p,\kappa} u\|^p_{W^k_{p,\kappa}} \le c \sum_{i=0}^k \int_M \chi_\kappa\big(\rho^{\lambda+\tau-\sigma+i}\,|\nabla^i u|_g\big)^p\,dV_g.$$

The finite multiplicity of $\mathfrak{K}$ implies $0 \le \sum_\kappa \chi_\kappa \le c\mathbf{1}_M$. Consequently,

$$\|\varphi_p^\lambda u\|_{\ell_p(\boldsymbol{W}_p^k)} \le c\,\|u\|_{k,p;\lambda}, \qquad u \in \mathcal{D}(M,V). \tag{6.6}$$

Since $\mathcal{D}(M,V)$ is dense in $W_p^{k,\lambda}(V)$ it follows $\varphi_p^\lambda \in \mathcal{L}\big(W_p^{k,\lambda}(V), \ell_p(\boldsymbol{W}_p^k)\big)$.

(3) Similarly as in the preceding step we find

$$\|\psi_{p,\kappa}^\lambda v_\kappa\|_{k,p;\lambda} \le c\,\|v_\kappa\|_{W_{p,\kappa}^k}, \qquad \kappa \in \mathfrak{K}.$$

Since $\chi_\kappa \,|\, \operatorname{im}(\psi_{p,\kappa}^\lambda) = \mathbf{1}$ it follows from the finite multiplicity of $\mathfrak{K}$ and Hölder's inequality that

$$|\nabla^i(\psi_p^\lambda \boldsymbol{v})|_g^p = \Big|\sum_\kappa \chi_\kappa \nabla^i(\psi_{p,\kappa}^\lambda v_\kappa)\Big|_g^p \le c\sum_\kappa |\nabla^i(\psi_{p,\kappa}^\lambda v_\kappa)|_g^p.$$

Consequently,

$$\|\psi_p^\lambda \boldsymbol{v}\|_{k,p;\lambda} \le c\,\|\boldsymbol{v}\|_{\ell_p(\boldsymbol{W}_p^k)}, \qquad v \in \ell_p(\boldsymbol{W}_p^k).$$

Since $\psi_p^\lambda\varphi_p^\lambda u = u$ for $u \in W_p^{k,\lambda}(V)$ we have shown that ψ_p^λ is a retraction from $\ell_p(\boldsymbol{W}_p^k)$ onto $W_p^{k,\lambda}(V)$.

(4) For each $\kappa \in \mathfrak{K}$ it holds $\mathcal{S}(\mathbb{X}_\kappa, E) \overset{d}{\hookrightarrow} W_p^k(\mathbb{X}_\kappa, E)$. This is well-known if $\mathbb{X}_\kappa = \mathbb{R}^m$ (e.g., [37]) and follows from (4.4.3) in [3] if $\mathbb{X}_\kappa = \mathbb{H}^m$. Furthermore, $\mathcal{D}(\mathbb{X}_\kappa, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{X}_\kappa, E)$. In fact, this is standard knowledge if $\mathbb{X}_\kappa = \mathbb{R}^m$; otherwise it follows from Section 4.2 in [3]. Hence

$$\mathcal{D}(\mathbb{X}_\kappa, E) \overset{d}{\hookrightarrow} W_p^k(\mathbb{X}_\kappa, E), \qquad \kappa \in \mathfrak{K}. \tag{6.7}$$

Thus, since $c_c(\boldsymbol{W}_p^k)$ is dense in $\ell_p(\boldsymbol{W}_p^k)$, we obtain

$$\boldsymbol{\mathcal{D}}(\mathbb{X}, E) \overset{d}{\hookrightarrow} \ell_p(\boldsymbol{W}_p^k). \tag{6.8}$$

(5) Analogously we find $W_p^k(\mathbb{X}_\kappa, E) \hookrightarrow \mathcal{D}'(\mathring{\mathbb{X}}_\kappa, E)$ for $\kappa \in \mathfrak{K}$. From this and the definition of the product topology it follows

$$\ell_p(\boldsymbol{W}_p^k) \hookrightarrow \prod_\kappa W_p^k(\mathbb{X}_\kappa, E) \hookrightarrow \boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E).$$

Since $\boldsymbol{\mathcal{D}}(\mathbb{X}, E) \overset{d}{\hookrightarrow} \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}}, E)$ we thus obtain from (6.8) that $\ell_p(\boldsymbol{W}_p^k) \overset{d}{\hookrightarrow} \boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E)$. The theorem is proved. □

Corollary 6.2 *Suppose $M = \mathbb{R}^m$ or $M = \mathbb{H}^m$, and $V = M \times \mathbb{K}$. Then the above definition yields the usual Sobolev spaces.*

P r o o f. This follows from (6.7) and Example 2.1(c). □

Theorem 6.3 *Suppose $k \in \mathbb{N}$. The diagram*

$$\begin{array}{ccccc}
\mathcal{D}(M,V) & \xrightarrow{\ \varphi_\infty^\lambda\ } & \boldsymbol{\mathcal{D}}(\mathbb{X},E) & \xrightarrow{\ \psi_\infty^\lambda\ } & \mathcal{D}(M,V) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
BC^{k,\lambda}(V) & \xrightarrow{\ \varphi_\infty^\lambda\ } & \ell_\infty(\boldsymbol{BC}^k) & \xrightarrow{\ \psi_\infty^\lambda\ } & BC^{k,\lambda}(V) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{D}'(\mathring{M},V) & \xrightarrow{\ \varphi_\infty^\lambda\ } & \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E) & \xrightarrow{\ \psi_\infty^\lambda\ } & \mathcal{D}'(\mathring{M},V)
\end{array}$$

is commuting and $\psi_\infty^\lambda \circ \varphi_\infty^\lambda = \mathrm{id}$.

P r o o f. This is verified by modifying the preceding proof in the obvious way. □

Remark 6.4 Define $\widetilde{\varphi}_q^\lambda$ and $\widetilde{\psi}_q^\lambda$ by replacing π_κ in the definition of $(\varphi_q^\lambda, \psi_q^\lambda)$ by χ_κ. Then $\widetilde{\varphi}_q^\lambda$ and $\widetilde{\psi}_q^\lambda$ possess the same mapping properties as φ_q^λ and ψ_q^λ. (Of course, $\widetilde{\psi}_q^\lambda$ is not a retraction.)

P r o o f. This is clear from the preceding proofs. □

7 Sobolev-Slobodeckii and Bessel Potential Spaces

We denote by $[\cdot,\cdot]_\theta$ the complex and by $(\cdot,\cdot)_{\theta,q}$, $1 \le q \le \infty$, the real interpolation functor for $0<\theta<1$. Definitions and proofs of the results from interpolation theory which we use below without further mention can be found in [37]. (Also see Section I.2 of [1] for a summary.) We write $X \doteq Y$ if X and Y are Banach spaces which are equal, except for equivalent norms.

For $s \ge 0$ we define weighted *Bessel potential spaces* of (σ,τ)-tensor fields by

$$H_p^{s,\lambda} = H_p^{s,\lambda}(V) := \begin{cases} [W_p^{k,\lambda}, W_p^{k+1,\lambda}]_{s-k}, & k<s<k+1, \ k \in \mathbb{N}, \\ [W_p^{k-1,\lambda}, W_p^{k+1,\lambda}]_{1/2}, & s = k \in \mathbb{N}^\times, \\ L_p^\lambda, & s=0, \end{cases}$$

where $W_p^{k,\lambda} = W_p^{k,\lambda}(V)$. Similarly, weighted *Besov spaces* are defined for $s>0$ by

$$B_p^{s,\lambda} = B_p^{s,\lambda}(V) := \begin{cases} (W_p^{k,\lambda}, W_p^{k+1,\lambda})_{s-k,p}, & k<s<k+1, \ k \in \mathbb{N}, \\ (W_p^{k-1,\lambda}, W_p^{k+1,\lambda})_{1/2,p}, & s = k \in \mathbb{N}^\times. \end{cases}$$

In the remainder of this paper

- $\mathfrak{F} \in \{H, B\}$.

This allows us to develop the theory of Bessel potential and Besov spaces to a large extent in one and the same setting.

Theorem 7.1 *Let* (5.2) *be chosen and $s>0$. Then ψ_p^λ is a retraction from $\ell_p(\boldsymbol{\mathfrak{F}}_p^s)$ onto $\mathfrak{F}_p^{s,\lambda}$, and φ_p^λ is a coretraction.*

P r o o f. Suppose $k, \ell \in \mathbb{N}$ satisfy $k<\ell$. Theorem 6.1 implies that the diagram

$$\begin{array}{ccccc} W_p^{\ell,\lambda} & \xrightarrow{\ \varphi_p^\lambda\ } & \ell(\boldsymbol{W}_p^\ell) & \xrightarrow{\ \psi_p^\lambda\ } & W_p^{\ell,\lambda} \\ \big\downarrow d & & \big\downarrow d & & \big\downarrow d \\ W_p^{k,\lambda} & \xrightarrow{\ \varphi_p^\lambda\ } & \ell(\boldsymbol{W}_p^k) & \xrightarrow{\ \psi_p^\lambda\ } & W_p^{k,\lambda} \end{array}$$

is commuting, and $\psi_p^\lambda \circ \varphi_p^\lambda = \mathrm{id}$. From this it follows that ψ_p^λ is a retraction from $\big[\ell_p(\boldsymbol{W}_p^k), \ell_p(\boldsymbol{W}_p^\ell)\big]_\theta$ onto $[W_p^{k,\lambda}, W_p^{\ell,\lambda}]_\theta$ and from $\big(\ell_p(\boldsymbol{W}_p^k), \ell_p(\boldsymbol{W}_p^\ell)\big)_{\theta,p}$ onto $(W_p^{k,\lambda}, W_p^{\ell,\lambda})_{\theta,p}$ for $0<\theta<1$.

By Theorem 1.18.1 in [37] we obtain, using obvious notation,

$$\big[\ell_p(\boldsymbol{W}_p^k), \ell_p(\boldsymbol{W}_p^\ell)\big]_\theta = \ell_p\big([\boldsymbol{W}_p^k, \boldsymbol{W}_p^\ell]_\theta\big), \quad \big(\ell_p(\boldsymbol{W}_p^k), \ell_p(\boldsymbol{W}_p^\ell)\big)_{\theta,p} \doteq \ell_p\big((\boldsymbol{W}_p^k, \boldsymbol{W}_p^\ell)_{\theta,p}\big).$$

Since $[W_{p,\kappa}^k, W_{p,\kappa}^\ell]_\theta \doteq H_{p,\kappa}^{(1-\theta)k+\theta\ell}$, the assertion follows. □

For $\xi_0, \xi_1 \in \mathbb{R}$ and $0<\theta<1$ we set $\xi_\theta := (1-\theta)\xi_0 + \theta\xi_1$.

Corollary 7.2 (i) $H_p^{k,\lambda}(V) \doteq W_p^{k,\lambda}(V)$, $k \in \mathbb{N}$.

(ii) *Suppose* $0 \le s_0 < s_1 < \infty$ *and* $\theta \in (0,1)$. *Then*

$$[H_p^{s_0,\lambda}, H_p^{s_1,\lambda}]_\theta \doteq H_p^{s_\theta,\lambda}, \quad (B_p^{s_0,\lambda}, B_p^{s_1,\lambda})_{\theta,p} = B_p^{s_\theta,\lambda},$$

provided $s_0 > 0$ *in the latter case.*

Proof. (i) follows from $H_{p,\kappa}^k \doteq W_{p,\kappa}^k$ for $k \in \mathbb{N}$.
(ii) is a consequence of the reiteration theorems for the complex and real interpolation functors. □

The following theorem shows that weighted Bessel potential and Besov spaces can be characterized locally by intrinsic norms, since this is the case for the spaces $\mathfrak{F}_{p,\kappa}^s$. In particular, $B_{p,\kappa}^s \doteq W_{p,\kappa}^s$ for $s \notin \mathbb{N}$. For this reason we call

$$W_p^{s,\lambda} = W_p^{s,\lambda}(V) := B_p^{s,\lambda}, \qquad s \in \mathbb{R}^+ \backslash \mathbb{N},$$

weighted *Slobodeckii space*.

Theorem 7.3 *Let* (5.2) *be selected. Suppose* $s \ge 0$ *with* $s > 0$ *if* $\mathfrak{F} = B$. *Then* $u \in L_{1,\mathrm{loc}}(M,V)$ *belongs to* $\mathfrak{F}_p^{s,\lambda}(V)$ *iff* $\kappa_*(\pi_\kappa u) \in \mathfrak{F}_{p,\kappa}^s$ *and*

$$|||u|||_{\mathfrak{F}_p^{s,\lambda}} := \Big(\sum_\kappa \big(\rho_\kappa^{\lambda+m/p} \, \|\kappa_*(\pi_\kappa u)\|_{\mathfrak{F}_{p,\kappa}^s}\big)^p\Big)^{1/p} < \infty.$$

Moreover, $|||\cdot|||_{\mathfrak{F}_p^{s,\lambda}}$ *is a norm for* $\mathfrak{F}_p^{s,\lambda}$.

Proof. Let X and Y be Banach spaces, $r \in \mathcal{L}(X,Y)$ a retraction, and $e \in \mathcal{L}(Y,X)$ a coretraction. Then

$$\|ey\| \le \|e\| \, \|y\| = \|e\| \, \|rey\| \le \|e\| \, \|r\| \, \|ey\|, \qquad y \in Y,$$

implies $\|\cdot\|_Y \sim \|e\cdot\|_X$. Thus the assertion follows from Theorem 7.1, setting $e := \varphi_p^\lambda$. □

Of course, $|||\cdot|||_{\mathfrak{F}_p^{s,\lambda}}$ depends on the particular singularity datum $(\rho, \mathfrak{K})$ and on the chosen localization system subordinate to $\mathfrak{K}$. Since $\mathfrak{F}_p^{s,\lambda}$ has been invariantly defined it follows that another choice of these data results in an equivalent norm.

Theorem 7.4 $\mathfrak{F}_p^{s,\lambda}(V)$ *is a reflexive Banach space.*

Proof. Since $\mathfrak{F}_{p,\kappa}^s$ is reflexive (cf. Theorem 4.4.4 of [3] if $\mathbb{X}_\kappa = \mathbb{H}^m$), $\ell_p(\boldsymbol{\mathfrak{F}}_p^s)$ is reflexive. Theorem 7.1 implies that $\mathfrak{F}_p^{s,\lambda}(V)$ is isomorphic to a closed linear subspace of $\ell_p(\boldsymbol{\mathfrak{F}}_p^s)$ (e.g., Lemma I.2.3.1 in [1]). Hence $\mathfrak{F}_p^{s,\lambda}(V)$ is reflexive as well. □

The following theorem shows that the weighted Bessel potential and Besov spaces are natural with respect to ∇.

Theorem 7.5 *Suppose* $s \ge 0$ *with* $s > 0$ *if* $\mathfrak{F} = B$, *and* $k \in \mathbb{N}^\times$. *Then*

$$\nabla^k \in \mathcal{L}\big(\mathfrak{F}_p^{s+k,\lambda}(V_\tau^\sigma), \mathfrak{F}_p^{s,\lambda}(V_{\tau+k}^\sigma)\big).$$

Proof. Since $\nabla^k u$ is a $(\sigma, \tau+k)$-tensor field if u is a (σ,τ)-tensor field, it is obvious that

$$\nabla^k \in \mathcal{L}\big(W_p^{s+k,\lambda}(V_\tau^\sigma), W_p^{s,\lambda}(V_{\tau+k}^\sigma)\big)$$

for $s \in \mathbb{N}$. Now we obtain the assertion by interpolation, due to Corollary 7.2. □

Remarks 7.6 (a) We consider the simplest case: $M = (\mathbb{R}^m, g_m)$ and $V = M \times \mathbb{K}$ with $\mathfrak{T}(M) = [\![\mathbf{1}]\!]$. By the arguments of the proof of Lemma 3.2 we construct $\pi \in \mathcal{D}\big(Q^m, [0,1]\big)$ such that $\{\, \pi^2(\cdot + z) \,;\, z \in \mathbb{Z}^m \,\}$ is a partition of unity subordinate to the open covering $\{\, z + Q^m \,;\, z \in \mathbb{Z}^m \}$ of $\mathbb{R}^m$. Consequently, fixing $\chi \in \mathcal{D}\big(Q^m, [0,1]\big)$ with $\chi \,|\, \mathrm{supp}(\pi) = \mathbf{1}$, it follows that $\{\, \pi(\cdot + z), \chi(\cdot + z) \,;\, z \in \mathbb{Z} \,\}$ is a localization system subordinate to the 'translation atlas' constructed in the proof of Example 2.1(c). Hence Theorem 7.3 guarantees that

$$u \mapsto \Big(\sum_{z\in\mathbb{Z}^m} \|\pi u(\cdot + z)\|^p_{\mathfrak{F}^s_p(\mathbb{R}^m)} \Big)^{1/p} = \Big(\sum_{z\in\mathbb{Z}^m} \|\pi(\cdot - z)u\|^p_{\mathfrak{F}^s_p(\mathbb{R}^m)} \Big)^{1/p} \tag{7.1}$$

is an equivalent norm for $\mathfrak{F}^s_p(\mathbb{R}^m)$, where $s > 0$ if $\mathfrak{F} = B$. This assertion is equivalent to the 'localization principle' of Theorem 2.4.7 of [40] for the Bessel potential spaces $H^s_p(\mathbb{R}^m)$ with $s \geq 0$ and the Besov spaces $B^s_p(\mathbb{R}^m)$ with $s > 0$.

(b) Of course, it is natural to define $B^{s,\lambda}_{p,q}(V)$ with $1 \leq q \leq \infty$ by replacing $(\cdot,\cdot)_{\theta,p}$ in the definition of $B^{s,\lambda}_p(V)$ by $(\cdot,\cdot)_{\theta,q}$. However, in this case the proof of Theorem 7.1 does not apply. In fact, it follows from Theorem 2.4.7 in [40] that there is no characterization of $B^s_{p,q}(\mathbb{R}^m)$ analogous to (7.1) if $p \neq q$. For this reason the spaces $B^{s,\lambda}_{p,q}(V)$ with $q \neq p$ are less useful and we refrain from considering them here. □

In the case where $M = \mathbb{R}^m$, a retraction-coretraction pair (ψ_p, φ_p) based on a localization system equivalent to the one of Remark 7.6(a) has been introduced in H. Amann, M. Hieber, and G. Simonett [6]. In that paper, besides establishing the analogue of (7.1), it is shown that (ψ_p, φ_p) is useful to localize partial differential equations for deriving maximal regularity results. This localization technique has since been applied by several authors for the study of parabolic equations on $\mathbb{R}^m$ (eg., [25] and the references therein). An abstract formulation has been given by S. Angenent [11]. As mentioned in the introduction, the retraction-coretraction pair $(\psi^\lambda_p, \varphi^\lambda_p)$ is part of the fundament on which we build (elsewhere) a theory of parabolic equations on singular manifolds.

8 Hölder Spaces

Let (5.2) be chosen. For $k < s < k+1$ with $k \in \mathbb{N}$ we denote by $BC^s_\kappa := BC^s(\mathbb{X}_\kappa, E)$ the Banach space of all $u \in BC^k(\mathbb{X}_\kappa, E)$ such that $\partial^\alpha u$ is uniformly $(s-k)$-Hölder continuous for $|\alpha| = k$, endowed with one of its standard norms.

From $BC^{k+1}_\kappa \hookrightarrow BC^s_\kappa \hookrightarrow BC^k_\kappa$ and Theorem 6.3 it follows

$$\begin{array}{ccccc} \ell_\infty(\boldsymbol{BC}^{k+1}) & \hookrightarrow & \ell_\infty(\boldsymbol{BC}^{s}) & \hookrightarrow & \ell_\infty(\boldsymbol{BC}^{k}) \\ \downarrow{\scriptstyle \psi^\lambda_\infty} & & & & \downarrow{\scriptstyle \psi^\lambda_\infty} \\ BC^{k+1,\lambda} & & \hookrightarrow & & BC^{k,\lambda} \end{array}$$

Now we define $BC^{s,\lambda} := BC^{s,\lambda}(V)$, the weighted *space of* $(s$-*)Hölder continuous* (σ,τ)*-tensor fields*, to be the image space of $\psi^\lambda_\infty \,|\, \ell_\infty(\boldsymbol{BC}^s)$, so that the diagram

$$\begin{array}{ccccc} \ell_\infty(\boldsymbol{BC}^{k+1}) & \hookrightarrow & \ell_\infty(\boldsymbol{BC}^{s}) & \hookrightarrow & \ell_\infty(\boldsymbol{BC}^{k}) \\ \downarrow{\scriptstyle \psi^\lambda_\infty} & & \downarrow{\scriptstyle \psi^\lambda_\infty} & & \downarrow{\scriptstyle \psi^\lambda_\infty} \\ BC^{k+1,\lambda} & \hookrightarrow & BC^{s,\lambda} & \hookrightarrow & BC^{k,\lambda} \end{array}$$

is commuting. Of course, this definition depends on the choice of the singularity datum $(\rho, \mathfrak{K})$ and the localization system subordinate to $\mathfrak{K}$. The following theorem shows, however, that the topology of $BC^{s,\lambda}$ is determined by the singularity type $\mathfrak{T}(M)$ only.

Theorem 8.1 *Suppose* $k < s < k+1$ *with* $k \in \mathbb{N}$.

(i) ψ^λ_∞ *is a retraction onto* $BC^{s,\lambda}$ *and* φ^λ_∞ *is a coretraction.*

(ii) $BC^{s,\lambda}$ *is a Banach space and*

$$u \mapsto |||u|||_{s,\infty;\lambda} := \sup_\kappa \rho^\lambda_\kappa \, \|\kappa_*(\pi_\kappa u)\|_{BC^s_\kappa}$$

is a norm for it. Other choices of singularity data and localization systems lead to equivalent norms.

P r o o f. (1) Assertion (i) and the claim that $BC^{s,\lambda}$ is a Banach space and $|||\cdot|||_{s,\infty;\lambda}$ a norm are clear.

(2) Let $(\widetilde{\rho},\widetilde{\mathfrak{K}})$ be a singularity datum and $\bigl\{\,(\widetilde{\pi}_{\widetilde{\kappa}},\widetilde{\chi}_{\widetilde{\kappa}})\;;\;\widetilde{\kappa}\in\widetilde{\mathfrak{K}}\,\bigr\}$ a localization system subordinate to $\widetilde{\mathfrak{K}}$. Suppose $j\in\mathbb{N}$ and $w\in BC^j_{\widetilde{\kappa}}$. Then

$$\kappa_*\widetilde{\kappa}^*(\widetilde{\chi}w)=(\widetilde{\chi}w)\circ(\widetilde{\kappa}\circ\kappa^{-1})=(\widetilde{\kappa}\circ\kappa^{-1})^*(\widetilde{\chi}w).$$

Thus it follows from Leibniz' rule, (3.21), and (2.2)(iii) that

$$\|\kappa_*\widetilde{\kappa}^*(\widetilde{\chi}w)\|_{BC^j_\kappa}\le c(j)\,\|w\|_{BC^j_{\widetilde{\kappa}}}, \tag{8.1}$$

that is,

$$\bigl(w\mapsto\kappa_*\widetilde{\kappa}^*(\widetilde{\chi}w)\bigr)\in\mathcal{L}(BC^j_{\widetilde{\kappa}},BC^j_\kappa),\qquad j\in\mathbb{N}.$$

Since $BC^s_\kappa\doteq(BC^k_\kappa,BC^{k+1}_\kappa)_{s-k,\infty}$, we thus obtain

$$\bigl(w\mapsto\kappa_*\widetilde{\kappa}^*(\widetilde{\chi}w)\bigr)\in\mathcal{L}(BC^s_{\widetilde{\kappa}},BC^s_\kappa). \tag{8.2}$$

(3) Using $\sum_{\widetilde{\kappa}}\widetilde{\pi}^2_{\widetilde{\kappa}}=\mathbf{1}$ we find

$$\begin{aligned}\kappa_*(\rho^\lambda_\kappa\pi_\kappa u)&=\kappa_*\Bigl(\rho^\lambda_\kappa\pi_\kappa\sum_{\widetilde{\kappa}}\widetilde{\pi}^2_{\widetilde{\kappa}}u\Bigr)\\&=(\kappa_*\pi_\kappa)\sum_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)}(\rho_\kappa/\widetilde{\rho}_{\widetilde{\kappa}})^\lambda\bigl(\kappa_*\widetilde{\kappa}^*(\widetilde{\kappa}_*\widetilde{\pi}_{\widetilde{\kappa}})\bigr)\,\widetilde{\rho}^\lambda_{\widetilde{\kappa}}\bigl(\kappa_*\widetilde{\kappa}^*\bigl(\widetilde{\kappa}_*(\widetilde{\pi}_{\widetilde{\kappa}}\widetilde{\chi}u)\bigr)\bigr).\end{aligned} \tag{8.3}$$

From (2.1)(vi) and (2.2)(ii) it follows $\rho_\kappa\sim\widetilde{\rho}_{\widetilde{\kappa}}$ for $\kappa\in\mathfrak{K}$ and $\widetilde{\kappa}\in\mathfrak{N}(\kappa)$. Thus we infer from (3.21), (8.2), and (8.3) that

$$\|\rho^\lambda_\kappa\kappa_*(\pi_\kappa u)\|_{BC^s_\kappa}\le c\sum_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)}\|\widetilde{\rho}^\lambda_{\widetilde{\kappa}}\widetilde{\kappa}_*(\widetilde{\pi}_{\widetilde{\kappa}}u)\|_{BC^s_{\widetilde{\kappa}}},\qquad\kappa\in\mathfrak{K}.$$

This implies that the norm associated with $(\widetilde{\rho},\widetilde{\mathfrak{K}})$ and the corresponding localization system is stronger than the original one. Thus the last part of the assertion follows by interchanging the roles of the singularity data. □

We fix now any one of the equivalent norms for $BC^{s,\lambda}$. Then $\bigl[\,BC^{s,\lambda}(V)\;;\;s\ge0\,\bigr]$ is the weighted *Hölder scale* of (σ,τ)-tensor fields on M.

Remark 8.2 We expect

$$BC^{s,\lambda}\doteq(BC^{k,\lambda},BC^{k+1,\lambda})_{s-k,\infty},\qquad k<s<k+1,\quad k\in\mathbb{N}. \tag{8.4}$$

However, we cannot prove this relation since we do not know whether

$$\bigl(\ell_\infty(\boldsymbol{BC}^k),\ell_\infty(\boldsymbol{BC}^{k+1})\bigr)_{s-k,\infty}\doteq\ell_\infty\bigl((\boldsymbol{BC}^k,\boldsymbol{BC}^{k+1})_{s-k,\infty}\bigr).$$

Thus we leave (8.4) as an open problem. □

We denote by $C_0^{s,\lambda}(V)$ the closure of $\mathcal{D}(M,V)$ in $BC^{s,\lambda}(V)$ for $s\ge0$. Then $[\,C_0^{s,\lambda}(V)\;;\;s\ge0\,]$ is called weighted *small Hölder scale*. The small Hölder space $C_0^{s,\lambda}$ should not be confused with the *little* Hölder space $bc^{s,\lambda}$ which is the closure of $BC^{s+1,\lambda}$ in $BC^{s,\lambda}$. Of course, $bc^{s,\lambda}=C_0^{s,\lambda}$ if M is compact.

Theorem 8.3 *Suppose $s\ge0$. Then ψ^λ_∞ is a retraction from $c_0\bigl(\boldsymbol{C}^s_0(\mathbb{X},E)\bigr)$ onto $C_0^{s,\lambda}(V)$, and φ^λ_∞ is a coretraction.*

P r o o f. Since $\mathcal{D}(\mathbb{X}_\kappa,E)$ is dense in $C^s_{0,\kappa}:=C^s_0(\mathbb{X}_\kappa,E)$, it follows that $\boldsymbol{\mathcal{D}}(\mathbb{X},E)$ is dense in $c_0\bigl(\boldsymbol{C}^s_0(\mathbb{X},E)\bigr)$. By Theorems 5.1 and 8.1 the diagram

$$\begin{array}{ccccc}\boldsymbol{\mathcal{D}}(\mathbb{X},E) & \overset{d}{\hookrightarrow} & c_0(\boldsymbol{C}^s_0) & \hookrightarrow & \ell_\infty(\boldsymbol{BC}^s)\\ \downarrow{\scriptstyle\psi^\lambda_\infty} & & & & \downarrow{\scriptstyle\psi^\lambda_\infty}\\ \mathcal{D}(M,V) & \overset{d}{\hookrightarrow} & C_0^{s,\lambda}(V) & \hookrightarrow & BC^{s,\lambda}(V)\end{array}$$

is commuting. From this we read off that we can insert the missing vertical arrow. This gives the assertion. □

Corollary 8.4 *Suppose* $0 \le s_0 < s_1 < s_2 < \infty$. *Then*

$$C_0^{s_2,\lambda} \overset{d}{\hookrightarrow} C_0^{s_1,\lambda} \hookrightarrow BC^{s_1,\lambda} \hookrightarrow BC^{s_0,\lambda}.$$

Remarks 8.5 (a) Let (5.2) be chosen. For $q, r \in [1,\infty]$ and $s \in \mathbb{R}$ denote by $F^s_{q,r;\kappa}$ the E-valued Triebel-Lizorkin spaces on $\mathbb{X}_\kappa$. Define $F^{s,\lambda}_{q,r} = F^{s,\lambda}_{q,r}(V)$ by requiring that the diagram

$$\begin{array}{ccccc}
\boldsymbol{\mathcal{D}}(\mathbb{X},E) & \hookrightarrow & \ell_q(\boldsymbol{F}^s_{q,r}) & \hookrightarrow & \boldsymbol{\mathcal{D}}'(\mathring{\mathbb{X}},E) \\
\downarrow \psi^\lambda_p & & \downarrow \psi^\lambda_p & & \downarrow \psi^\lambda_p \\
\mathcal{D}(M,V) & \hookrightarrow & F^{s,\lambda}_{q,r}(V) & \hookrightarrow & \mathcal{D}'(\mathring{M},V)
\end{array}$$

be commuting. Then $F^{s,\lambda}_{q,r}(V)$ is a Banach space, a weighted Triebel-Lizorkin space of (σ,τ)-tensor fields on M, and

$$u \mapsto \|\varphi^\lambda_q u\|_{\ell_q(\boldsymbol{F}^s_{q,r})}$$

is a norm for it. The topology of $F^{s,\lambda}_{q,r}$ is independent of the particular choice of the singularity datum and the localization system. If $M = (\mathbb{R}^m, g_m)$ and $\mathfrak{T}(M) = [\![\mathbf{1}]\!]$, then we recover $F^s_{q,r}(\mathbb{R}^m)$.

P r o o f. The first part of the assertion follows by obvious modifications of the proof of Theorem 8.1 using the fact that $BC^k(\mathbb{R}^m)$ is a point-wise multiplier space for $F^s_{q,r}(\mathbb{R}^m)$, provided $k = k(s,q,r)$ is sufficiently large (cf. Theorem 6.1 in W. Yuan, W. Sickel, and D. Yang [41] or, if $q < \infty$, Theorem 4.2.2 in [40]). The last part is a consequence of the invariance of $F^s_{q,r}(\mathbb{R}^m)$ under diffeomorphisms (see Theorem 6.7 in [41]). □

(b) It is clear that we can replace in the above construction the Triebel-Lizorkin spaces $F^s_{q,r}(\mathbb{R}^m)$ by any scale of spaces for which a BC^k-point-wise multiplier and the diffeomorphism theorem are valid. Thus, due to Theorems 6.1 and 6.7 in [41], we can replace $F^s_{q,r}(\mathbb{R}^m)$ by the scales $F^{s,\tau}_{q,r}(\mathbb{R}^m)$ and $B^{s,\tau}_{q,r}(\mathbb{R}^m)$ of Triebel-Lizorkin and Besov type (see [41] for precise definitions). However, this has to be done with care. In fact, we could take, in particular, a scale $B^s_{p,q}(\mathbb{R}^m)$ with $q \ne p$. But then, due to Remark 7.6(b), the spaces $B^{s,\lambda}_{p,q}(V)$ constructed this way do not coincide with the Besov spaces obtained in Remark 7.6(b) by interpolation. □

9 Point-Wise Multipliers

Suppose $\sigma_i, \tau_i \in \mathbb{N}$ for $i = 0,1,2$. Then

$$V^{\sigma_1}_{\tau_1} \times V^{\sigma_2}_{\tau_2} \to V^{\sigma_0}_{\tau_0}, \quad (v_1, v_2) \mapsto v_1 \bullet v_2 \tag{9.1}$$

is called vector bundle *multiplication* if it is (fiber-wise) bilinear and satisfies

$$|v_1 \bullet v_2|_g \le c\, |v_1|_g\, |v_2|_g, \qquad v_i \in V^{\sigma_i}_{\tau_i}, \quad i = 1,2.$$

Examples 9.1 (a) The duality pairing $\langle\cdot,\cdot\rangle : V^\sigma_\tau \times V^\tau_\sigma \to V^0_0$ is a multiplication.
(b) The map $V^\sigma_\tau \times V^\sigma_\tau \to V^0_0,\ (u,v) \mapsto (u|\overline{v})_g$ is a multiplication.
(c) The tensor product $\otimes : V^{\sigma_1}_{\tau_1} \times V^{\sigma_2}_{\tau_2} \to V^{\sigma_1+\sigma_2}_{\tau_1+\tau_2}$ is a multiplication.
(d) Assume $1 \le i \le \sigma$ and $1 \le j \le \tau$. We denote by $\mathsf{C}^i_j : V^\sigma_\tau \to V^{\sigma-1}_{\tau-1},\ a \mapsto \mathsf{C}^i_j a$ the contraction with respect to positions i and j. Then $|\mathsf{C}^i_j a|_g \le |a|_g$ for $a \in V^\sigma_\tau$.
Suppose $1 \le i \le \sigma_1 + \sigma_2$ and $1 \le j \le \tau_1 + \tau_2$. Then

$$\mathsf{C}^i_j : V^{\sigma_1}_{\tau_1} \times V^{\sigma_2}_{\tau_2} \to V^{\sigma_1+\sigma_2-1}_{\tau_1+\tau_2-1}, \quad (a,b) \mapsto \mathsf{C}^i_j(a \otimes b)$$

is a multiplication, a *contraction*. □

In the following, we call the point-wise extension of (9.1) *point-wise multiplication induced* by (9.1) and denote it again by $\bullet$.

Theorem 9.2 *Let* (9.1) *be one of the multiplications of Examples 9.1. Suppose* $0 \le s \le t$, $\lambda_1, \lambda_2 \in \mathbb{R}$, *and* $\lambda_0 = \lambda_1 + \lambda_2$. *Then point-wise multiplication induced by* (9.1) *is a continuous bilinear map from*

$$BC^{t,\lambda_1}(V_{\tau_1}^{\sigma_1}) \times H_p^{s,\lambda_2}(V_{\tau_2}^{\sigma_2}) \text{ into } H_p^{s,\lambda_0}(V_{\tau_0}^{\sigma_0})$$

if either $s = t \in \mathbb{N}$ *or* $t > s$, *from*

$$BC^{t,\lambda_1}(V_{\tau_1}^{\sigma_1}) \times B_p^{s,\lambda_2}(V_{\tau_2}^{\sigma_2}) \text{ into } B_p^{s,\lambda_0}(V_{\tau_0}^{\sigma_0})$$

if $0 < s < t$, *and from*

$$BC^{s,\lambda_1}(V_{\tau_1}^{\sigma_1}) \times BC^{s,\lambda_2}(V_{\tau_2}^{\sigma_2}) \text{ into } BC^{s,\lambda_0}(V_{\tau_0}^{\sigma_0}).$$

Proof. Suppose $s > 0$ if $\mathfrak{F} = B$. Let assumption (5.2) be satisfied. Then, given $u \in BC^{t,\lambda_1}(V_{\tau_1}^{\sigma_1})$ and $v \in \mathcal{D}(M, V_{\tau_2}^{\sigma_2})$, it follows from $\sum_{\widetilde{\kappa}} \pi_{\widetilde{\kappa}}^2 = \mathbf{1}$ and the definition of $\mathfrak{N}(\kappa)$ that

$$\kappa_*\bigl(\pi_\kappa(u \bullet v)\bigr) = \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} \kappa_*(\pi_\kappa u) \bullet \kappa_*(\pi_{\widetilde{\kappa}}^2 v), \qquad \kappa \in \mathfrak{K}. \tag{9.2}$$

Hence the point-wise multiplier properties of the Hölder spaces $BC_\kappa^t = BC^t(\mathbb{X}_\kappa, E)$ (see, for example, Theorem 4.7.1 in Th. Runst and W. Sickel [31] for the case $t > s$; the case $s = t \in \mathbb{N}$ follows easily from Leibniz' rule) imply

$$\bigl\|\kappa_*\bigl(\pi_\kappa(u \bullet v)\bigr)\bigr\|_{\mathfrak{F}_{p,\kappa}^s} \le c\, \|\kappa_*(\pi_\kappa u)\|_{BC_\kappa^t} \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} \|\kappa_*(\pi_{\widetilde{\kappa}}^2 v)\|_{\mathfrak{F}_{p,\widetilde{\kappa}}^s} \tag{9.3}$$

for $\kappa \in \mathfrak{K}$. Note that

$$\operatorname{card}\bigl(\mathfrak{N}(\kappa)\bigr) \le c, \qquad \kappa \in \mathfrak{K}, \tag{9.4}$$

by the finite multiplicity of $\mathfrak{K}$.

It is a consequence of (2.1)(ii) and

$$\kappa_*(\pi_{\widetilde{\kappa}}^2 v) = \kappa_* \widetilde{\kappa}^* \widetilde{\kappa}_*(\pi_{\widetilde{\kappa}} v) = \bigl((\widetilde{\kappa}_* \pi_{\widetilde{\kappa}})(\widetilde{\kappa}_* v)\bigr) \circ (\widetilde{\kappa} \circ \kappa^{-1})$$

that (cf. (8.2) and (8.3))

$$\|\kappa_*(\pi_{\widetilde{\kappa}}^2 v)\|_{\mathfrak{F}_{p,\kappa}^s} \le c\, \|\widetilde{\kappa}_*(\pi_{\widetilde{\kappa}} v)\|_{\mathfrak{F}_{p,\widetilde{\kappa}}^s}, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa \in \mathfrak{K}. \tag{9.5}$$

Indeed, this follows from Leibniz' rule if $s \in \mathbb{N}$, and then, by interpolation if $s \notin \mathbb{N}$ (also see Theorem 4.3.2 in [40]). Thus we obtain from (9.3)–(9.5) and the density of $\mathcal{D}(M, V)$ in $\mathfrak{F}_p^{s,\lambda_2}$

$$|||u \bullet v|||_{\mathfrak{F}_p^{s,\lambda}} \le c\, |||u|||_{BC^{t,\lambda_1}}\, |||v|||_{\mathfrak{F}_p^{s,\lambda_2}}$$

for $u \in BC^{t,\lambda_1}(V_{\tau_1}^{\sigma_1})$ and $v \in \mathfrak{F}_p^{s,\lambda_2}(V_{\tau_2}^{\sigma_2})$. Now the first two assertions are implied by Theorems 7.3 and 8.1. The last one is a consequence of the fact that $BC^s(\mathbb{X}_\kappa)$ is a point-wise multiplication algebra. □

In applications this theorem is perhaps the most useful multiplier theorem. The next theorem is an extension of known multiplication algebra results to the present setting.

Theorem 9.3 *Suppose* $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_0 = \lambda_1 + \lambda_2$, *and* $s > m/p$. *Then point-wise multiplication induced by* (9.1) *is a continuous bilinear map from*

$$\mathfrak{F}_p^{s,\lambda_1}(V_{\tau_1}^{\sigma_1}) \times \mathfrak{F}_p^{s,\lambda_2}(V_{\tau_2}^{\sigma_2}) \text{ into } \mathfrak{F}^{s,\lambda_0+m/p}(V_{\tau_0}^{\sigma_0}).$$

Proof. Theorem 4.6.4 of [31] and standard extensions to the half-space case guarantee that $\mathfrak{F}_{p,\kappa}^s$ is a multiplication algebra. Hence we infer from (9.2) and(9.4)

$$\bigl\|\kappa_*\bigl(\pi_\kappa(u \bullet v)\bigr)\bigr\|_{\mathfrak{F}_{p,\kappa}^s}^p \le c\, \|\kappa_*(\pi_\kappa u)\|_{\mathfrak{F}_{p,\kappa}^s}^p \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} \|\widetilde{\kappa}(\pi_{\widetilde{\kappa}}^2 v)\|_{\mathfrak{F}_{p,\widetilde{\kappa}}^s}^p$$

for $\kappa \in \mathfrak{K}$. This implies, due to (9.5),

$$|||u \bullet v|||_{\mathfrak{F}_p^{s,\lambda_0+m/p}} \le c\, |||u|||_{\mathfrak{F}_p^{s,\lambda_1}}\, |||v|||_{\mathfrak{F}_p^{s,\lambda_2}},$$

hence the assertion. □

10 Traces

Throughout this section $\partial M\neq\emptyset$. We write $\overset{\bullet}{V}$ for the restriction $V_{\partial M}$ of V to ∂M.

Since $T(\partial M)$ is a subbundle of codimension 1 of the vector bundle $(TM)_{\partial M}$ over ∂M, there exists a unique vector field $\boldsymbol{n}$ in $(TM)_{\partial M}$ of length 1, orthogonal to $T(\partial M)$, and inward pointing (in any local chart meeting ∂M), the *inward pointing* unit *normal* vector field on ∂M. In local coordinates, $\kappa=(x^1,\ldots,x^m)$,

$$\boldsymbol{n}=\frac{1}{\sqrt{g_{11}\,|\,\partial U_\kappa}}\,\frac{\partial}{\partial x^1}.$$

Suppose $u\in\mathcal{D}(M,V)$ and $k\in\mathbb{N}$. The *trace of order* k of u *on* ∂M, $\gamma_k u\in\mathcal{D}(\partial M,\overset{\bullet}{V})$, is defined by

$$\langle\gamma_k u,a\rangle:=\bigl\langle\nabla^k u\,|\,\partial M,a\otimes\boldsymbol{n}^{\otimes k}\bigr\rangle,\qquad a\in\mathcal{D}(\partial M,V'_{\partial M}).$$

In local coordinates, where $u=u^{(i)}_{(j)}\frac{\partial}{\partial x^{(i)}}\otimes dx^{(j)}$, we infer from (3.18), writing

$$\gamma_k u=(\gamma_k u)^{(i)}_{(j)}\frac{\partial}{\partial x^{(i)}}\otimes dx^{(j)},$$

that

$$\Bigl(\sqrt{g_{11}\,|\,\partial U_\kappa}\Bigr)^k(\gamma_k u)^{(i)}_{(j)}=\biggl(\frac{\partial^k u^{(i)}_{(j)}}{(\partial x^1)^k}+\sum_{\ell=0}^{k-1}b^{(i)(\tilde{\jmath})}_{(j)(\tilde{\imath}),\ell}\,\frac{\partial^\ell u^{(\tilde{\imath})}_{(\tilde{\jmath})}}{(\partial x^1)^\ell}\biggr)\bigg|\,\partial U_\kappa,\qquad(10.1)$$

where $b^{(i)(\tilde{\jmath})}_{(j)(\tilde{\imath}),\ell}$ is a polynomial in the partial derivatives of the Christoffel symbols of order at most $k-\ell-1$. We write $\gamma=\gamma_0$ for the *trace operator on* ∂M.

In the next theorem, by a *universal* coretraction we mean a continuous linear map which is the unique continuous extension of its restriction to $\mathcal{D}(\partial M,\overset{\bullet}{V})$. In this sense it is independent of s and p.

Theorem 10.1 *Suppose $k\in\mathbb{N}$ and $s>k+1/p$. Then γ_k extends to a retraction from $\mathfrak{F}^{s,\lambda}_p(V)$ onto $B^{s-k-1/p,\lambda+k+1/p}_p(\overset{\bullet}{V})$. It possesses a universal coretraction γ^c_k satisfying $\gamma_i\circ\gamma^c_k=0$ for $0\le i\le k-1$.*

P r o o f. (1) Let (5.2) be chosen. It follows from Lemma 3.1(i) and (ii) and Lemma 1.4.2 in [3] that

$$\|\rho_\kappa^{-1}\sqrt{\kappa_*g_{11}}\|_{k,\infty}+\bigl\|\rho_\kappa\bigl(\sqrt{\kappa_*g_{11}}\bigr)^{-1}\bigr\|_{k,\infty}\le c,\qquad\kappa\in\mathfrak{K}.$$

(2) For $t>1/p$ we set

$$\overset{\bullet}{B}{}^{t-1/p}_{p,\kappa}:=\begin{cases}B^{t-1/p}_p(\mathbb{R}^{m-1},E), & \kappa\in\mathfrak{K}_{\partial M},\\ \{0\}, & \kappa\in\mathfrak{K}\backslash\mathfrak{K}_{\partial M},\end{cases}$$

with the convention $B^{t-1/p}_p(\mathbb{R}^0,E)=E$. We denote by $\gamma_\kappa:=\gamma_{\partial\mathbb{H}^m}$ the usual trace operator on $\partial\mathbb{H}^m$ if κ belongs to $\mathfrak{K}_{\partial M}$, and set $\gamma_\kappa:=0$ if $\kappa\in\mathfrak{K}\backslash\mathfrak{K}_{\partial M}$, where $\partial\mathbb{H}^m=\{0\}\times\mathbb{R}^{m-1}$ is identified with $\mathbb{R}^{m-1}$. Then we put

$$\gamma_{k,\kappa}:=\rho^k_\kappa\bigl(\sqrt{\gamma_\kappa(\kappa_*g_{11})}\bigr)^{-k}\gamma_\kappa\circ\partial^k_1,\qquad\kappa\in\mathfrak{K}.$$

Note $\rho_\kappa=\overset{\bullet}{\rho}_\kappa$ for $\kappa\in\mathfrak{K}\backslash\mathfrak{K}_{\partial M}$.

Theorems 4.6.2 and 4.6.3 of [3] imply that $\gamma_\kappa\circ\partial^k_1$ is a retraction from $\mathfrak{F}^s_{p,\kappa}$ onto $\overset{\bullet}{B}{}^{s-k-1/p}_{p,\kappa}$ and that there exists a universal coretraction $\widetilde{\gamma}^c_{k,\kappa}$ for it satisfying

$$(\gamma_\kappa\circ\partial^i_1)\circ\widetilde{\gamma}^c_{k,\kappa}=0,\qquad 0\le i\le k-1,\qquad(10.2)$$

(setting $\widetilde{\gamma}^c_{k,\kappa}:=0$ if $\kappa\in\mathfrak{K}\backslash\mathfrak{K}_{\partial M}$). We put

$$\gamma^c_{k,\kappa}:=\rho^{-k}_\kappa\bigl(\sqrt{\gamma_\kappa(\kappa_*g_{11})}\bigr)^k\,\widetilde{\gamma}^c_{k,\kappa},\qquad\kappa\in\mathfrak{K}.$$

It follows from step (1) that

$$\gamma_{k,\kappa} \in \mathcal{L}(\mathfrak{F}^s_{p,\kappa}, \overset{\bullet}{B}{}^{s-k-1/p}_{p,\kappa}), \quad \gamma^c_{k,\kappa} \in \mathcal{L}(\overset{\bullet}{B}{}^{s-k-1/p}_{p,\kappa}, \mathfrak{F}^s_{p,\kappa}) \tag{10.3}$$

and

$$\|\gamma_{k,\kappa}\| + \|\gamma^c_{k,\kappa}\| \le c, \qquad \kappa \in \mathfrak{K}.$$

From (10.2) and Leibniz' rule we infer

$$\gamma_{i,\kappa} \circ \gamma^c_{k,\kappa} = \delta_{ik}\mathrm{id}, \qquad 0 \le i \le k. \tag{10.4}$$

(3) We use the notation of Example 2.1(e) and set $(\overset{\bullet}{\pi}_{\overset{\bullet}{\kappa}}, \overset{\bullet}{\chi}_{\overset{\bullet}{\kappa}}) := (\pi_\kappa, \chi_\kappa) \,|\, U_{\overset{\bullet}{\kappa}}$ for $\overset{\bullet}{\kappa} \in \overset{\bullet}{\mathfrak{K}}$. Then it is verified that $\big\{ (\overset{\bullet}{\pi}_{\overset{\bullet}{\kappa}}, \overset{\bullet}{\chi}_{\overset{\bullet}{\kappa}}) \ ; \ \overset{\bullet}{\kappa} \in \overset{\bullet}{\mathfrak{K}} \big\}$ is a localization system subordinate to $\overset{\bullet}{\mathfrak{K}}$. We denote by

$$\overset{\bullet}{\psi}{}^\lambda_p : \ \ell_p(\overset{\bullet}{\boldsymbol{B}}{}^{s-k-1/p}_p) \to B^{s-k-1/p,\lambda}_p(\overset{\bullet}{V})$$

the 'boundary retraction' defined analogously to ψ^λ_p. Correspondingly, $\overset{\bullet}{\varphi}{}^\lambda_p$ is the 'boundary coretraction'.

We put

$$T_{k,\kappa} := \overset{\bullet}{\rho}{}^k_{\kappa} \overset{\bullet}{\kappa}_* \circ \gamma_k \circ \kappa^*, \qquad \kappa \in \mathfrak{K}.$$

It follows from (10.1) that

$$T_{k,\kappa} u_\kappa = \gamma_{k,\kappa} u_\kappa + \sum_{\ell=0}^{k-1} b_{\ell,\kappa} \gamma_{\ell,\kappa} u_\kappa, \qquad u_\kappa \in \mathcal{D}(\mathbb{X}_\kappa, E), \tag{10.5}$$

where, due to (3.19) and step (1), $\|b_{\ell,\kappa}\|_{k-1,\infty} \le c$ for $0 \le \ell \le k-1$ and $\kappa \in \mathfrak{K}$. Hence, using $\mathfrak{F}^s_{p,\kappa} \hookrightarrow \mathfrak{F}^{s-k+\ell}_{p,\kappa}$, we obtain

$$T_{k,\kappa} \in \mathcal{L}(\mathfrak{F}^s_{p,\kappa}, \overset{\bullet}{B}{}^{s-k-1/p}_{p,\kappa}), \quad \|T_{k,\kappa}\| \le c, \qquad \kappa \in \mathfrak{K}. \tag{10.6}$$

(4) For $\widetilde{\kappa} \in \mathfrak{N}(\kappa)$ with $\kappa, \widetilde{\kappa} \in \mathfrak{K}_{\partial M}$ we set $\overset{\bullet}{S}_{\kappa\widetilde{\kappa}} := (\overset{\bullet}{\kappa}_* \overset{\bullet}{\widetilde{\kappa}}{}^*)(\overset{\bullet}{\chi}_\kappa \cdot)$. It follows from (8.1) by interpolation that, given $t > 0$,

$$\overset{\bullet}{S}_{\kappa\widetilde{\kappa}} \in \mathcal{L}(\overset{\bullet}{B}{}^t_{p,\widetilde{\kappa}}, \overset{\bullet}{B}{}^t_{p,\kappa}), \quad \|\overset{\bullet}{S}_{\kappa\widetilde{\kappa}}\| \le c(t), \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa, \widetilde{\kappa} \in \mathfrak{K}_{\partial M}.$$

From this, (10.6), and $\overset{\bullet}{B}{}^{s-i-1/p}_{p,\kappa} \hookrightarrow \overset{\bullet}{B}{}^{s-k-1/p}_{p,\kappa}$ we infer

$$T_{i,\kappa\widetilde{\kappa}} := \overset{\bullet}{S}_{\kappa\widetilde{\kappa}} \circ T_{i,\widetilde{\kappa}} \in \mathcal{L}(\mathfrak{F}^s_{p,\widetilde{\kappa}}, \overset{\bullet}{B}{}^{s-k-1/p}_{p,\kappa}), \quad \|T_{i,\kappa\widetilde{\kappa}}\| \le c, \qquad \widetilde{\kappa} \in \mathfrak{N}(\kappa), \quad \kappa, \widetilde{\kappa} \in \mathfrak{K}_{\partial M}, \tag{10.7}$$

for $0 \le i \le k$.

The definition of γ_k implies

$$\overset{\bullet}{\pi}_{\overset{\bullet}{\kappa}} \gamma_k u = \gamma_k(\pi_\kappa u) - \sum_{j=0}^{k-1} \binom{k}{j} (\gamma_{k-j}\pi_\kappa) \gamma_j(\chi_\kappa u).$$

Since $\chi_\kappa u = \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} \pi^2_{\widetilde{\kappa}} u$ we thus get

$$\overset{\bullet}{\varphi}{}^{\lambda+k+1/p}_{p,\overset{\bullet}{\kappa}}(\gamma_k u) = T_{k,\kappa}(\varphi^\lambda_{p,\kappa} u) + \sum_{\widetilde{\kappa} \in \mathfrak{N}(\kappa)} R_{k-1,\kappa\widetilde{\kappa}}(\varphi^\lambda_p u), \tag{10.8}$$

where

$$R_{k-1,\kappa\widetilde{\kappa}} \boldsymbol{v} := \sum_{i=0}^{k-1} a_{i,\kappa\widetilde{\kappa}} T_{i,\kappa\widetilde{\kappa}}(\chi_{\widetilde{\kappa}} v_{\widetilde{\kappa}}), \qquad \boldsymbol{v} = (v_\kappa),$$

with

$$a_{i,\kappa\widetilde{\kappa}} := -\sum_{j=1}^{k-1} \binom{k}{j} \binom{j}{i} (\rho_\kappa/\rho_{\widetilde{\kappa}})^{\lambda+j+m/p} T_{k-j,\kappa}(\kappa_* \pi_\kappa) T_{j-i,\kappa\widetilde{\kappa}}(\widetilde{\kappa}_* \pi_{\widetilde{\kappa}})$$

for $\widetilde{\kappa} \in \mathfrak{N}(\kappa)$ with $\kappa, \widetilde{\kappa} \in \mathfrak{K}_{\partial M}$, and $a_{i,\kappa\widetilde{\kappa}} := 0$ otherwise.

It follows from (2.1)(vi), (3.21), (10.6), (10.7), and Leibniz' rule that

$$\|a_{i,\kappa\widetilde{\kappa}}\|_{BC^\ell(\partial\mathbb{X}_\kappa)} \le c(\ell), \qquad \kappa,\widetilde{\kappa}\in\mathfrak{K}, \quad 0\le i\le k, \quad \ell\in\mathbb{N}.$$

Hence, using (10.7) once more,

$$R_{k-1,\kappa\widetilde{\kappa}} \in \mathcal{L}(\boldsymbol{\mathfrak{F}}_p^s, \dot{B}_{p,\kappa}^{s-k-1/p}), \quad \|R_{k-1,\kappa\widetilde{\kappa}}\| \le c, \qquad \kappa,\widetilde{\kappa}\in\mathfrak{K}. \tag{10.9}$$

Lastly, we set

$$\boldsymbol{T}_{k,\kappa}\boldsymbol{v} := T_{k,\kappa}v_\kappa + \sum_{\widetilde{\kappa}\in\mathfrak{N}(\kappa)} R_{k-1,\kappa\widetilde{\kappa}}(\boldsymbol{v}) \tag{10.10}$$

and $\boldsymbol{T}_k\boldsymbol{v} := (\boldsymbol{T}_{k,\kappa}\boldsymbol{v})$. Then we deduce from (10.6), (10.9), and the finite multiplicity of $\mathfrak{K}$ that

$$\boldsymbol{T}_k \in \mathcal{L}(\boldsymbol{\mathfrak{F}}_p^s, \dot{\boldsymbol{B}}_p^{s-k-1/p}). \tag{10.11}$$

Moreover, (10.8) implies

$$\dot{\varphi}_p^{\lambda+k+1/p}\circ\gamma_k = \boldsymbol{T}_k\circ\varphi_p^\lambda.$$

Hence it follows from Theorem 7.1 and (10.11)

$$\gamma_k = \dot{\psi}_k^{\lambda+k+1/p}\circ\boldsymbol{T}_k\circ\varphi_p^\lambda \in \mathcal{L}\big(\mathfrak{F}_p^{s,\lambda}, \dot{B}_p^{s-k-1/p,\lambda+k+1/p}(\dot{V})\big).$$

(5) We set $\boldsymbol{\gamma}_k^c\boldsymbol{w} := (\gamma_{k,\kappa}^c w_\kappa)$. Then we get from (10.3)

$$\boldsymbol{\gamma}_k^c \in \mathcal{L}(\dot{\boldsymbol{B}}_p^{s-k-1/p}, \dot{\boldsymbol{\mathfrak{F}}}_p^s,).$$

Note that (10.4), (10.5), and (10.10) imply

$$\boldsymbol{T}_i\circ\boldsymbol{\gamma}_k^c = \delta_{ik}\mathrm{id}, \qquad 0\le i\le k. \tag{10.12}$$

Furthermore, given $\boldsymbol{v}\in\dot{\boldsymbol{\mathfrak{F}}}_p^s$,

$$\begin{aligned}
\gamma_k(\psi_p^\lambda\boldsymbol{v}) &= \sum_\kappa \rho_\kappa^{-(\lambda+m/p)}\gamma_k(\pi_\kappa\kappa^*v_\kappa)\\
&= \sum_\kappa \rho_\kappa^{-(\lambda+m/p)}\Big(\dot{\pi}_{\dot{\kappa}}\gamma_\kappa(\kappa^*v_\kappa) + \sum_{j=0}^{k-1}\binom{k}{j}(\gamma_{k-j}\pi_\kappa)\gamma_j(\kappa^*v_\kappa)\Big)\\
&= \sum_\kappa \rho_\kappa^{-(\lambda+k+m/p)}\Big(\dot{\pi}_{\dot{\kappa}}\dot{\kappa}^*T_{k,\kappa}v_\kappa + \dot{\kappa}^*\sum_{j=0}^{k-1}\binom{k}{j}T_{k-j,\kappa}(\kappa_*\pi_\kappa)T_{j,\kappa}v_\kappa\Big).
\end{aligned}$$

Thus we infer from (10.12)

$$\gamma_k(\psi_p^\lambda\boldsymbol{\gamma}_k^c\boldsymbol{w}) = \sum_\kappa \rho_\kappa^{-(\lambda+k+m/p)}\dot{\pi}_{\dot{\kappa}}\dot{\kappa}^*w_\kappa = \dot{\psi}_p^{\lambda+k+1/p}\boldsymbol{w}$$

for $\boldsymbol{w}\in\dot{\boldsymbol{B}}_p^{s-k-1/p}$. Hence, by Theorem 7.1,

$$\gamma_k^c := \psi_p^\lambda\circ\boldsymbol{\gamma}_k^c\circ\dot{\varphi}_p^{\lambda+k+1/p} \in \mathcal{L}\big(B_p^{s-k-1/p,\lambda+k+1/p}(\dot{V}), \mathfrak{F}_p^{s,\lambda}\big)$$

and $\gamma_k\circ\gamma_k^c = \mathrm{id}$. This proves the theorem. ☐

Corollary 10.2 *Suppose* $0\le j_1<\cdots<j_k$ *and* $s>j_k+1/p$. *Then*

$$(\gamma_{j_0},\dots,\gamma_{j_k}) : \mathfrak{F}_p^{s,\lambda}(V) \to \prod_{i=1}^k B_p^{s-j_i-1/p,\lambda+j_i+1/p}(\dot{V}) \tag{10.13}$$

is a retraction possessing a universal coretraction.

P r o o f. For $(v_1,\dots,v_k)$ belonging to the product space in (10.13) define u_i for $1\le i\le k$ inductively by $u_1 := \gamma_{j_1}^c v_1$ and $u_i := u_{i-1} + \gamma_{j_i}^c(v_i - \gamma_{j_i}u_{i-1})$ for $2\le i\le k$. Then γ^c, given by $\gamma^c(v_1,\dots,v_k) := u_k$, has the claimed properties. ☐

11 Spaces with Vanishing Boundary Values

Throughout this section we assume $\partial M \neq \emptyset$. We denote by $\mathring{\mathfrak{F}}_p^{s,\lambda} = \mathring{\mathfrak{F}}_p^{s,\lambda}(V)$ the closure of $\mathcal{D}(\mathring{M}, V)$ in $\mathfrak{F}_p^{s,\lambda}$.

Let (5.2) be chosen. Recalling definitions (5.3) and (5.4) we put

$$\mathring{\varphi}_{p,\kappa}^{\lambda} u := \rho_\kappa^{\lambda - m/p'} \mathring{\varphi}_\kappa u, \qquad u \in \mathcal{D}(M, V),$$

and

$$\mathring{\psi}_{p,\kappa}^{\lambda} v_\kappa := \rho_\kappa^{-\lambda + m/p'} \mathring{\psi}_\kappa v_\kappa, \qquad v_\kappa \in \mathcal{D}(\mathbb{X}_\kappa, E).$$

Furthermore,

$$\mathring{\varphi}_p^{\lambda} u := (\mathring{\varphi}_{p,\kappa}^{\lambda} u), \quad \mathring{\psi}_p^{\lambda} \boldsymbol{v} := \textstyle\sum_\kappa \mathring{\psi}_{p,\kappa}^{\lambda} v_\kappa$$

for $u \in \mathcal{D}(M, V)$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\mathbb{X}, E)$.

Theorem 11.1 *Suppose $s \in \mathbb{R}^+ \backslash (\mathbb{N} + 1/p)$ with $s > 0$ if $\mathfrak{F} = B$. Then the diagram*

$$\begin{array}{ccccc}
\mathcal{D}(\mathring{M}, V) & \xrightarrow{\ \mathring{\varphi}_p^{\lambda}\ } & \boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E) & \xrightarrow{\ \mathring{\psi}_p^{\lambda}\ } & \mathcal{D}(\mathring{M}, V) \\
\big\downarrow d & & \big\downarrow d & & \big\downarrow d \\
\mathring{\mathfrak{F}}_p^{s,\lambda}(V) & \xrightarrow{\ \mathring{\varphi}_p^{\lambda}\ } & \ell_p(\mathring{\boldsymbol{\mathfrak{F}}}_p^s) & \xrightarrow{\ \mathring{\psi}_p^{\lambda}\ } & \mathring{\mathfrak{F}}_p^{s,\lambda}(V) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathfrak{F}_p^{s,\lambda}(V) & \xrightarrow{\ \mathring{\varphi}_p^{\lambda}\ } & \ell_p(\boldsymbol{\mathfrak{F}}_p^s) & \xrightarrow{\ \mathring{\psi}_p^{\lambda}\ } & \mathfrak{F}_p^{s,\lambda}(V)
\end{array}$$

is commuting and $\mathring{\psi}_p^{\lambda} \circ \mathring{\varphi}_p^{\lambda} = \mathrm{id}$.

Proof. (1) It follows from (5.5) and (5.6) that the assertions concerning the first row of this diagram are valid and $\mathring{\psi}_p^{\lambda} \circ \mathring{\varphi}_p^{\lambda} = \mathrm{id}_{\mathcal{D}(\mathring{M}, V)}$.

(2) From Lemma 3.1(i) and (ii) and the rules for differentiating determinants we deduce

$$\sqrt{\kappa_* g} \sim \rho_\kappa^m, \quad \|\partial^\alpha \det(\kappa_* g)\|_\infty \leq c(\alpha) \rho_\kappa^{2m}, \qquad \alpha \in \mathbb{N}^m, \quad \kappa \in \mathfrak{K}.$$

For $\alpha, \beta \in \mathbb{N}^m$ with $\alpha = \beta + e_i$, where e_i is the i-th standard basis vector of $\mathbb{R}^m$, we get

$$\partial^\alpha (\sqrt{\kappa_* g}) = \partial^\beta \Big(\frac{1}{2\sqrt{\kappa_* g}} \, \partial_i \det(\kappa_* g) \Big).$$

From this, Leibniz' rule, and Lemma 1.4.2 in [3] we infer

$$\|\sqrt{\kappa_* g}\|_{k,\infty} \leq c(k) \rho_\kappa^m, \qquad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}.$$

This implies

$$\|\mathring{\varphi}_\kappa u\|_{W_{p,\kappa}^k} \leq c(k) \rho_\kappa^m \, \|\kappa_*(\chi_\kappa u)\|_{W_{p,\kappa}^k}, \qquad \kappa \in \mathfrak{K}, \quad k \in \mathbb{N}.$$

Now we obtain $\mathring{\varphi}_p^{\lambda} \in \mathcal{L}\big(W_p^{k,\lambda}, \ell_p(\boldsymbol{W}_p^k)\big)$ for $k \in \mathbb{N}$ from (6.5) and the arguments leading from there to (6.6). Analogously we find $\mathring{\psi}_p^{\lambda} \in \mathcal{L}\big(\ell_p(\boldsymbol{W}_p^k), W_p^{k,\lambda}\big)$ for $k \in \mathbb{N}$ by the arguments of step (3) of the proof of Theorem 6.1, as well as $\mathring{\psi}_p^{\lambda} \circ \mathring{\varphi}_p^{\lambda} = \mathrm{id}$.

(3) Since $\mathcal{D}(\mathring{\mathbb{X}}_\kappa, E) \overset{d}{\hookrightarrow} \mathring{W}_{p,\kappa}^k$ implies $\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E) \overset{d}{\hookrightarrow} c_c(\mathring{\boldsymbol{W}}_p^k)$, we deduce from (6.3) that $\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E)$ is dense in $\ell_p(\mathring{\boldsymbol{W}}_p^k)$. Clearly, $\mathring{\psi}_p^{\lambda}\big(\boldsymbol{\mathcal{D}}(\mathring{\mathbb{X}}, E)\big) \subset \mathcal{D}(\mathring{M}, V)$. Thus we infer $\mathring{\psi}_p^{\lambda} \in \mathcal{L}\big(\ell_p(\mathring{\boldsymbol{W}}_p^k), \mathring{W}_p^{k,\lambda}\big)$ for $k \in \mathbb{N}$ from steps (1) and (2). Similarly, we find $\mathring{\varphi}_p^{\lambda}\big(\mathcal{D}(\mathring{M}, V)\big) \subset \ell_p(\mathring{\boldsymbol{W}}_p^k)$, and thus $\mathring{\varphi}_p^{\lambda} \in \mathcal{L}\big(\mathring{W}_p^{k,\lambda}, \ell_p(\mathring{\boldsymbol{W}}_p^k)\big)$ for $k \in \mathbb{N}$. This proves the theorem if $s \in \mathbb{N}$.

(4) Suppose $s \in \mathbb{R}^+ \backslash (\mathbb{N} + 1/p)$. For $0 < \theta < 1$ set $(\cdot,\cdot)_\theta := [\cdot,\cdot]_\theta$ if $\mathfrak{F} = H$, and $(\cdot,\cdot)_\theta := (\cdot,\cdot)_{\theta,p}$ otherwise. Assume $0 < s < k$ with $k \in \mathbb{N}$. Then $s \notin \mathbb{N} + 1/p$ implies $\mathring{\mathfrak{F}}^s_{p,\kappa} \doteq (L_{p,\kappa}, \mathring{\mathfrak{F}}^p_{p,\kappa})_{s/k}$. Thus, cf. the proof of Theorem 7.1,

$$\ell_p(\mathring{\boldsymbol{\mathfrak{F}}}^s_p) = \bigl(\ell_p(\boldsymbol{L}_p), \ell_p(\mathring{\boldsymbol{\mathfrak{F}}}^k_p)\bigr)_{s/k}.$$

Now we infer from step (3) that r is a retraction from $\ell_p(\mathring{\boldsymbol{\mathfrak{F}}}^s_p)$ onto $(L^\lambda_p, \mathring{W}^{k,\lambda}_p)_{s/k} \doteq \mathring{\mathfrak{F}}^{s,\lambda}_p$, since the latter interpolation space is the closure of $\mathcal{D}(\mathring{M}, E)$ in $(L^\lambda_p, W^{k,\lambda}_p)_{s/k} \doteq \mathfrak{F}^{s,\lambda}_p$ by the density properties of $(\cdot,\cdot)_\theta$. □

Corollary 11.2 *Suppose* $0 \le s_0 < s_1 < \infty$ *and* $\theta \in (0,1)$. *If* $s_0, s_1, s_\theta \notin \mathbb{N} + 1/p$, *then*

$$[\mathring{H}^{s_0,\lambda}_p, \mathring{H}^{s_1,\lambda}_p]_\theta \doteq \mathring{H}^{s_\theta,\lambda}_p, \quad (\mathring{B}^{s_0,\lambda}_p, \mathring{B}^{s_1,\lambda}_p)_{\theta,p} \doteq \mathring{B}^{s_\theta,\lambda}_p,$$

provided $s_0 > 0$ *in the latter case.*

The next theorem characterizes the spaces $\mathring{\mathfrak{F}}^{s,\lambda}_p$ by means of trace operators.

Theorem 11.3 (i) *Suppose* $0 \le s < 1/p$ *with* $s > 0$ *if* $\mathfrak{F} = B$. *Then* $\mathring{\mathfrak{F}}^{s,\lambda}_p = \mathfrak{F}^{s,\lambda}_p$.

(ii) *Assume* $k \in \mathbb{N}$ *and* $k + 1/p < s < k + 1 + 1/p$. *Set* $\vec{\gamma}_k := (\gamma_0, \ldots \gamma_k)$. *Then*

$$\mathring{\mathfrak{F}}^{s,\lambda}_p = \{\, u \in \mathfrak{F}^{s,\lambda}_p \,;\, \vec{\gamma}_k u = 0 \,\}.$$

P r o o f. (i) follows from Theorem 11.1 and the corresponding properties of these spaces on $\mathbb{X}_\kappa$.

(ii) Let the assumptions of (ii) be satisfied. If $u \in \mathring{\mathfrak{F}}^{s,\lambda}_p$, then it is obvious by Corollary 10.2 that $\vec{\gamma}_k u = 0$.

Conversely, suppose $u \in \mathfrak{F}^{s,\lambda}_p$ and $\vec{\gamma}_k u = 0$. Then we infer from (10.1) that $(\gamma_\kappa \circ \partial^i_1)\kappa_*(\pi_\kappa u) = 0$ for $\kappa \in \mathfrak{K}$ and $0 \le i \le k$. Hence $\kappa_*(\pi_\kappa u) \in \mathring{\mathfrak{F}}^s_{p,\kappa}$ for $\kappa \in \mathfrak{K}_{\partial M}$ (cf. Theorem 2.9.4 in [37]). Consequently, $\mathring{\varphi}^\lambda_p u \in \ell_p(\mathring{\boldsymbol{\mathfrak{F}}}^s_p)$ and, by Theorem 11.1, $u = \mathring{\psi}^\lambda_p(\mathring{\varphi}^\lambda_p u) \in \mathring{\mathfrak{F}}^{s,\lambda}_p$. This proves assertion (ii). □

Theorem 11.4 *Suppose* $k \in \mathbb{N}$ *and* $k + 1/p < s < k + 1 + 1/p$. *Put*

$$\partial\mathfrak{F}^{s,\lambda}_p(\overset{\bullet}{V}) := \prod_{i=0}^{k} B^{s-i-1/p,\lambda+i+1/p}_p(\overset{\bullet}{V}).$$

Let $\vec{\gamma}^c_k$ *be a coretraction for* $\vec{\gamma}_k$. *Then* $\mathfrak{F}^{s,\lambda}_p(V) = \mathring{\mathfrak{F}}^{s,\lambda}_p(V) \oplus \vec{\gamma}^c_k\, \partial\mathfrak{F}^{s,\lambda}_p(\overset{\bullet}{V})$.

P r o o f. Let X and Y be Banach spaces, $r \in \mathcal{L}(X,Y)$ and $r^c \in \mathcal{L}(Y,X)$ with $r \circ r^c = \mathrm{id}$. Then $r^c \circ r$ is a projection in $\mathcal{L}(X)$ and

$$X = \ker(r^c \circ r) \oplus \mathrm{im}(r^c \circ r) = \ker(r) \oplus r^c Y,$$

where $r^c Y$ is the image space of Y in X, so that $r^c : Y \to r^c Y$ is an isometric isomorphism (cf. Lemma 4.1.5 in [3] or Lemma 2.3.1 in [1]). Hence the assertion follows from Corollary 10.2 and Theorem 11.3. □

12 Spaces of Negative Order

For $u \in \mathcal{D}(M, V')$ and $v \in \mathcal{D}(M, V)$ we put

$$\langle u, v\rangle_M := \int_M \langle u, v\rangle \, dV_g.$$

This bilinear form extends uniquely to a separating continuous bilinear form

$$\langle\cdot,\cdot\rangle_M : L^{-\lambda}_{p'}(V') \times L^\lambda_p(V) \to \mathbb{K}$$

by which we identify the dual Banach space of $L^\lambda_p(V)$ with $L^{-\lambda}_{p'}(V')$, that is,

$$L^\lambda_p(V)' = L^{-\lambda}_{p'}(V') \text{ by means of the duality pairing } \langle\cdot,\cdot\rangle_M. \tag{12.1}$$

It follows from Theorem 11.3(i) that

$$\mathcal{D}(\mathring{M},V) \overset{d}{\hookrightarrow} \mathring{\mathfrak{F}}_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} L_p^\lambda(V) \tag{12.2}$$

for $s \geq 0$, with $s > 0$ if $\mathfrak{F} = B$. Theorem 7.4 implies that $\mathring{\mathfrak{F}}_p^{s,\lambda}(V)$ is reflexive, being a closed linear subspace of a reflexive space. Thus we put, in accordance with (12.1),

$$\mathfrak{F}_p^{-s,\lambda}(V) := \big(\mathring{\mathfrak{F}}_{p'}^{s,-\lambda}(V')\big)', \qquad s > 0. \tag{12.3}$$

It is a consequence of (12.1), (12.2), and Theorem 7.1 that

$$\mathfrak{F}_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} L_p^\lambda(V) \overset{d}{\hookrightarrow} \mathfrak{F}_p^{-s,\lambda}(V) \overset{d}{\hookrightarrow} \mathcal{D}(\mathring{M},V), \qquad s > 0, \tag{12.4}$$

with respect to the duality pairing $\langle\cdot,\cdot\rangle_M$, that is,

$$\langle u,v\rangle_{\mathfrak{F}_p^{-s,\lambda}(V)} = \langle u,v\rangle_M, \qquad s > 0, \quad u \in \mathring{\mathfrak{F}}_{p'}^{s,-\lambda}(V'), \quad v \in L_p(V).$$

Finally, we define

$$B_p^{0,\lambda}(V) := \big(B_p^{-1,\lambda}(V), B_p^{1,\lambda}(V)\big)_{1/2,p}. \tag{12.5}$$

Theorem 12.1 *Suppose $s \in \mathbb{R}$ with $s \notin -\mathbb{N}^\times + 1/p$ if $\partial M \neq \emptyset$. Then ψ_p^λ is a retraction from $\ell_p(\boldsymbol{\mathfrak{F}}_p^s)$ onto $\mathfrak{F}_p^{s,\lambda}(V)$, and φ_p^λ is a coretraction.*

P r o o f. (1) If $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$, then this is a restatement of Theorem 7.1.

(2) Suppose $s < 0$, with $s \notin -\mathbb{N} + 1/p$ if $\partial M \neq \emptyset$. Then Theorem 11.1 guarantees that $\mathring{\psi}_{p'}^{-\lambda}$ is a retraction from $\ell_{p'}(\mathring{\boldsymbol{\mathfrak{F}}}_{p'}^{-s})$ onto $\mathfrak{F}_{p'}^{-s,-\lambda}(V')$ and $\mathring{\varphi}_{p'}^{-\lambda}$ is a coretraction. Since $(\mathring{\mathfrak{F}}_{p',\kappa}^{-s})' = \mathfrak{F}_{p,\kappa}^s$ with respect to the duality pairing $\langle\cdot,\cdot\rangle_\kappa := \langle\cdot,\cdot\rangle_{\mathbb{X}_\kappa}$, it follows

$$\big(\ell_{p'}(\mathring{\boldsymbol{\mathfrak{F}}}_{p'}^{-s})\big)' = \ell_p(\boldsymbol{\mathfrak{F}}_p^s)$$

with respect to $\boldsymbol{\langle}\cdot,\cdot\boldsymbol{\rangle}$. Using

$$\mathring{\varphi}_{p',\kappa}^{-\lambda} = \rho_\kappa^{-\lambda-m/p}\mathring{\varphi}_\kappa,$$

the proof of Theorem 5.1, and Theorem 7.1 we thus obtain

$$\psi_p^\lambda = (\mathring{\varphi}_{p'}^{-\lambda})' \in \mathcal{L}\big(\ell_p(\boldsymbol{\mathfrak{F}}_p^s), \mathfrak{F}_p^{s,\lambda}(V)\big)$$

and

$$\varphi_p^\lambda = (\mathring{\psi}_{p'}^{-\lambda})' \in \mathcal{L}\big(\mathfrak{F}_p^{s,\lambda}(V), \ell_p(\boldsymbol{\mathfrak{F}}_p^s)\big)$$

with $\psi_p^\lambda \circ \varphi_p^\lambda = \mathrm{id}$. This proves the assertion if $s < 0$.

(3) If $s = 0$, then the claim for $B_p^{0,\lambda}(V)$ follows by interpolation from (12.5) and steps (1) and (2). □

Corollary 12.2 *Suppose $s \in \mathbb{R}$ and $s \notin -\mathbb{N}^\times + 1/p$ if $\partial M \neq \emptyset$. Then $H_2^{s,\lambda}(V) \doteq B_2^{s,\lambda}(V)$.*

It is convenient to denote by $\mathring{\mathfrak{F}}_p^{s,\lambda}(V)$ *for each* $s \in \mathbb{R}$ the closure of $\mathcal{D}(\mathring{M},V)$ in $\mathfrak{F}_p^{s,\lambda}(V)$. Then

$$\mathring{\mathfrak{F}}_p^{s,\lambda}(V) = \mathfrak{F}_p^{s,\lambda}(V), \qquad s < 1/p.$$

In fact, this follows from Theorem 11.3(i) and (12.4).

Theorem 12.3 *The Banach spaces $\mathring{\mathfrak{F}}_p^{s,\lambda}(V)$ and $\mathfrak{F}_p^{s,\lambda}(V)$ are reflexive for $s \in \mathbb{R}$. Moreover,*

$$\big(\mathring{\mathfrak{F}}_p^{s,\lambda}(V)\big)' \doteq \mathring{\mathfrak{F}}_{p'}^{-s,-\lambda}(V'), \qquad s \in \mathbb{R}.$$

P r o o f. This follows from Theorem 7.4, the fact that closed linear subspaces and reflexive Banach spaces are reflexive, and the duality properties of the real interpolation functor $(\cdot,\cdot)_{1/2,p}$ (see (12.5)). □

Suppose $\partial M \neq \emptyset$. Since $\mathfrak{F}_p^{s,\lambda}(V)$ is reflexive and densely embedded in $L_p(V)$ for $s > 0$, we can define for $s > 0$

$$\breve{\mathfrak{F}}_p^{-s,\lambda}(V) := \big(\mathfrak{F}_{p'}^{s,-\lambda}(V')\big)'$$

with respect to the duality pairing $\langle \cdot, \cdot \rangle_M$. By Theorem 11.3(i)

$$\breve{\mathfrak{F}}_p^{s,\lambda}(V) := \mathfrak{F}_p^{s,\lambda}(V), \qquad -1 + 1/p < s < 0.$$

However, if $s < -1 + 1/p$, then $\breve{\mathfrak{F}}_p^{s,\lambda}(V)$ is no longer a space of distribution sections on $\mathring{M}$, but contains distribution sections supported on ∂M. This is made precise by the next theorem in which we use the notations of Theorem 11.4.

Theorem 12.4 *Suppose $\partial M \neq \emptyset$ and $-k - 2 + 1/p < s < -k - 1 + 1/p$ with $k \in \mathbb{N}$. Put*

$$\partial \mathfrak{F}_p^{s,\lambda}(\overset{\bullet}{V}) := \prod_{i=0}^{k} B_p^{s+i+1-1/p,\lambda-i-1+1/p}(\overset{\bullet}{V}).$$

Then

$$\breve{\mathfrak{F}}_p^{s,\lambda}(V) = \mathfrak{F}_p^{s,\lambda}(V) \oplus (\vec{\gamma}_k)' \, \partial \mathfrak{F}_p^{s,\lambda}(\overset{\bullet}{V}),$$

where $\vec{\gamma}_k$ maps $\mathfrak{F}_{p'}^{-s,-\lambda}(V)$ onto $\prod_{i=0}^{k} B_{p'}^{-s-i-1/p',-\lambda+i+1/p'}(\overset{\bullet}{V})$.

P r o o f. Since $\partial(\partial M) = \emptyset$ the statement follows from (12.3) and Theorem 11.4 by duality (cf. Section 2 of [2]). ☐

13 Interpolation

Now we can improve on the interpolation results already noted in Corollaries 7.2 and 11.2.

Theorem 13.1 *Suppose $-\infty < s_0 < s_1 < \infty,\ 0 < \theta < 1$, and $\lambda_0, \lambda_1 \in \mathbb{R}$.*

(i) *The following interpolation relations,*

$$\big[H_p^{s_0,\lambda_0}(V), H_p^{s_1,\lambda_1}(V)\big]_\theta \doteq H_p^{s_\theta,\lambda_\theta}(V), \quad \big(B_p^{s_0,\lambda_0}(V), B_p^{s_1,\lambda_1}(V)\big)_{\theta,p} \doteq B_p^{s_\theta,\lambda_\theta}(V),$$

are valid, provided $s_0, s_1, s_\theta \notin -\mathbb{N}^\times + 1/p$ if $\partial M \neq \emptyset$.

(ii) *Suppose $\partial M \neq \emptyset$ and $s_0, s_1, s_\theta \in \mathbb{R}^+ \backslash (N + 1/p)$. Then*

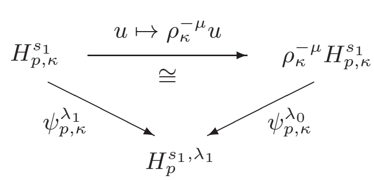
$$\big[\mathring{H}_p^{s_0,\lambda_0}(V), \mathring{H}_p^{s_1,\lambda_1}(V)\big]_\theta \doteq \mathring{H}_p^{s_\theta,\lambda_\theta}(V), \quad \big(\mathring{B}_p^{s_0,\lambda_0}(V), \mathring{B}_p^{s_1,\lambda_1}(V)\big)_{\theta,p} \doteq \mathring{B}_p^{s_\theta,\lambda_\theta}(V).$$

(iii) *If either $\partial M = \emptyset$ or $s_0, s_1, s_\theta \notin -\mathbb{N}^\times + 1/p$, then $\big(H_p^{s_0,\lambda_0}(V), H_p^{s_1,\lambda_1}(V)\big)_{\theta,p} \doteq B_p^{s_\theta,\lambda_\theta}(V)$.*

(iv) *Suppose $\partial M \neq \emptyset$ and $s_0, s_1, s_\theta \in \mathbb{R}^+ \backslash (\mathbb{N} + 1/p)$. Then $\big(\mathring{H}_p^{s_0,\lambda_0}(V), \mathring{H}_p^{s_1,\lambda_1}(V)\big)_{\theta,p} \doteq \mathring{B}_p^{s_\theta,\lambda_\theta}(V)$.*

P r o o f. Fix (5.2)

(1) Set $\mu := \lambda_1 - \lambda_0$. Denote by $\rho_\kappa^{-\mu} H_{p,\kappa}^{s_1}$ the image space of the self-map $u \mapsto \rho_\kappa^{-\mu} u$ of $H_{p,\kappa}^{s_1}$ so that this map is an isometric isomorphism from $H_{p,\kappa}^{s_1}$ onto $\rho_\kappa^{-\mu} H_{p,\kappa}^{s_1}$. Then Theorem 12.1 implies that the diagram



(13.1)

is commuting. Interpolation theory guarantees (cf. formula (7) in Section 3.4.1 of [37])

$$[H_{p,\kappa}^{s_0}, \rho_\kappa^{-\mu} H_{p,\kappa}^{s_1}]_\theta \doteq \rho_\kappa^{-\theta\mu} H_{p,\kappa}^{s_\theta}, \tag{13.2}$$

uniformly with respect to $\kappa \in \mathfrak{K}$. From Theorem 12.1 we infer that $\psi_p^{\lambda_0}$ is a retraction from $\ell_p(\boldsymbol{H}_p^{s_0})$ onto $H_p^{s_0,\lambda_0}$ and, due to (13.1), from $\ell_p(\boldsymbol{\rho}^{-\mu}\boldsymbol{H}_p^{s_1})$ onto $H_p^{s_1,\lambda_1}$, where $\boldsymbol{\rho}^{-\mu}\boldsymbol{H}_p^s := \prod_\kappa \rho_\kappa^\mu H_{p,\kappa}^s$. Thus, by (13.2) and interpolation, $\psi_p^{\lambda_0}$ is a retraction from $\ell_p(\boldsymbol{\rho}^{-\theta\mu}\boldsymbol{H}_p^{s_\theta})$ onto $[H_p^{s_0,\lambda_0}, H_p^{s_1,\lambda_1}]_\theta$. By replacing μ in (13.1) by $\theta\mu$ we see that $\psi_p^{\lambda_0}$ is a retraction from $\ell_p(\boldsymbol{\rho}^{-\theta\mu}\boldsymbol{H}_p^{s_\theta})$ onto $H_p^{s_\theta,\lambda_\theta}$. This implies the claim for $\mathfrak{F} = H$. The proof for $\mathfrak{F} = B$ is analogous.

(2) The assertions of (ii) follow by invoking in step (1) Theorem 11.1 instead of Theorem 12.1. The remaining statements are obtained by similar arguments from the corresponding results on $\mathbb{X}_\kappa$. □

14 Embedding Theorems

Weighted Bessel potential and Besov spaces on singular manifolds enjoy embedding properties similar to the ones known for the standard non-weighted spaces on $\mathbb{R}^m$.

Theorem 14.1 *Suppose* $s_0 < s < s_1$ *and* $\mu < \lambda$.

(i) *If* $\partial M \neq \emptyset$ *and* $s_0, s, s_1 \in \mathbb{R}^+ \backslash (\mathbb{N} + 1/p)$, *then*

$$\mathring{H}_p^{s_1,\lambda}(V) \overset{d}{\hookrightarrow} \mathring{B}_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} \mathring{H}_p^{s_0,\lambda}(V). \tag{14.1}$$

If, moreover, $\rho \leq 1$, *then* $\mathring{\mathfrak{F}}_p^{s,\mu}(V) \overset{d}{\hookrightarrow} \mathring{\mathfrak{F}}_p^{s,\lambda}(V)$, *whereas* $\mathring{\mathfrak{F}}_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} \mathring{\mathfrak{F}}_p^{s,\mu}(V)$ *if* $\rho \geq 1$.

(ii) *If either* $\partial M = \emptyset$ *or* $s_0, s, s_1 \notin -\mathbb{N}^\times + 1/p$, *then*

$$H_p^{s_1,\lambda}(V) \overset{d}{\hookrightarrow} B_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} H_p^{s_0,\lambda}(V). \tag{14.2}$$

Furthermore, $\mathfrak{F}_p^{s,\mu}(V) \overset{d}{\hookrightarrow} \mathfrak{F}_p^{s,\lambda}(V)$ *if* $\rho \leq 1$, *whereas* $\rho \geq 1$ *implies* $\mathfrak{F}_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} \mathfrak{F}_p^{s,\mu}(V)$.

Proof. Assertions (14.1) and (14.2) follow from Theorem 13.1(ii) and (i), respectively, and the general interrelations of the real and complex interpolation functors.

If $\rho \leq 1$, then it is obvious that

$$W_p^{k,\mu}(V) \overset{d}{\hookrightarrow} W_p^{k,\lambda}(V), \quad \mathring{W}_p^{k,\mu}(V) \overset{d}{\hookrightarrow} \mathring{W}_p^{k,\lambda}(V), \qquad k \in \mathbb{N}. \tag{14.3}$$

Thus, by duality,

$$H_p^{k,\mu}(V) \overset{d}{\hookrightarrow} H_p^{k,\lambda}(V), \qquad k \in -\mathbb{N}^\times. \tag{14.4}$$

From these embeddings we obtain, once more by interpolation, the second part of assertion (i) and assertion (ii), respectively, provided $\rho \leq 1$. If $\rho \geq 1$, then the embeddings in (14.3) and (14.4) are reversed. Thus the remaining statements are also clear. □

The next theorem concerns embedding theorems of Sobolev type.

Theorem 14.2 (i) *Suppose* $s_0 < s_1$ *and* $p_0, p_1 \in (1, \infty)$ *satisfy* $s_1 - m/p_1 = s_0 - m/p_0$. *Then*

$$\mathfrak{F}_{p_1}^{s_1,\lambda}(V) \overset{d}{\hookrightarrow} \mathfrak{F}_{p_0}^{s_0,\lambda+s_1-s_0}(V).$$

(ii) *Assume* $s \geq t + m/p$ *with* $t \geq 0$ *and* $s > t + m/p$ *if* $t \in \mathbb{N}$. *Then*

$$\mathfrak{F}_p^{s,\lambda}(V) \overset{d}{\hookrightarrow} C_0^{t,\lambda+m/p}(V).$$

P r o o f. (1) Let the assumptions of (i) be satisfied. Since $s_1 > s_0$ implies $p_1 < p_0$, it follows from the known embeddings $\mathfrak{F}^{s_1}_{p_1,\kappa} \hookrightarrow \mathfrak{F}^{s_0}_{p_0,\kappa}$ and from (6.2) that $\ell_{p_1}(\boldsymbol{\mathfrak{F}}^{s_1}_{p_1}) \hookrightarrow \ell_{p_0}(\boldsymbol{\mathfrak{F}}^{s_0}_{p_0})$. Moreover, $m/p_1 = m/p_0 + s_1 - s_0$ implies $\psi^{\lambda}_{p_1} = \psi^{\lambda+s_1-s_0}_{p_0}$. From this and Theorem 12.1 we infer that the diagram

$$\begin{array}{ccc}
\mathfrak{F}^{s_1,\lambda}_{p_1} & \xrightarrow{\ \varphi^{\lambda}_{p_1}\ } & \ell_{p_1}(\boldsymbol{\mathfrak{F}}^{s_1}_{p_1}) \\
\big\downarrow & & \big\downarrow \\
\mathfrak{F}^{s_0,\lambda+s_1-s_0}_{p} & \xleftarrow{\ \psi^{\lambda+s_1-s_0}_{p_0}\ } & \ell_{p_0}(\boldsymbol{\mathfrak{F}}^{s_0}_{p_0})
\end{array}$$

is commuting. Thus the assertions of (i) follow.

(2) Let s and t satisfy the hypotheses of (ii). Then the known embeddings $\mathfrak{F}^{s}_{p,\kappa} \hookrightarrow C^{t}_{0,\kappa}$ guarantee

$$c_c(\boldsymbol{\mathfrak{F}}^{s}_{p}) \subset c_c(\boldsymbol{C}^{t}_{0}) \hookrightarrow c_0(\boldsymbol{C}^{t}_{0}).$$

Thus $\ell_p(\boldsymbol{\mathfrak{F}}^{s}_{p}) \hookrightarrow c_0(\boldsymbol{C}^{t}_{0})$ since $c_0(\boldsymbol{C}^{t}_{0})$ is closed in $\ell_\infty(\boldsymbol{C}^{t}_{0})$. Hence, using $\psi^{\lambda}_{p} = \psi^{\lambda+m/p}_{\infty}$, it follows from Theorem 12.1 that the diagram

$$\begin{array}{ccc}
\mathfrak{F}^{s,\lambda}_{p} & \xrightarrow{\ \varphi^{\lambda}_{p}\ } & \ell_{p}(\boldsymbol{\mathfrak{F}}^{s}_{p}) \\
\big\downarrow & & \big\downarrow \\
C^{t,\lambda+m/p}_{0} & \xleftarrow{\ \psi^{\lambda+m/p}_{\infty}\ } & c_0(\boldsymbol{C}^{t}_{0})
\end{array}$$

is commuting. Thus claim (ii) is implied by the density of $\mathcal{D}(M,V)$ in each of the spaces. □

15 Differential Forms and Exterior Derivatives

Throughout this section

- M is oriented.

For $0 \le k \le m$ we consider the vector subbundle

$$\textstyle\bigwedge^k := \big(\bigwedge^k T^*M, (\cdot\,|\,\cdot)_{g^*}\big)$$

of $V^0_k = T^0_k M$, the k-fold exterior product of $V^0_1 = T^*M$, where $\bigwedge^0 = T^0_0 M = M \times \mathbb{K}$. The sections of $\bigwedge^k$ are the k-forms on M, that is, the differential forms of order k. We write $\Omega^k(M)$ for the $C^\infty(M)$-module of smooth k-forms, and we set $\Omega^k(M) := \{0\}$ for $k \notin \{0,1,\dots,m\}$.

We also consider the subbundle

$$\textstyle\bigwedge'^k := \big(\bigwedge^k TM, (\cdot\,|\,\cdot)_{g}\big)$$

of $V^k_0 = T^k_0 M$. Then $\bigwedge'^k = (\bigwedge^k)'$ with respect to the duality pairing $\langle\cdot,\cdot\rangle$ obtained by restriction from the V^0_k-pairing. It follows from (3.7) and the (vector bundle) conjugate linearity of $g^\sharp$ that

$$G^k : \textstyle\bigwedge^k \to \bigwedge'^k, \quad \alpha \mapsto G^k_0\,\overline{\alpha}$$

is a vector bundle isomorphism whose inverse is

$$G_k : \textstyle\bigwedge'^k \to \bigwedge^k, \quad v \mapsto G^0_k\,\overline{v}.$$

Let ω be the Riemannian volume form of M. The definition of the Hodge adjoint $*\beta \in \Omega^{m-k}(M)$ implies

$$(\alpha\,|\,\beta)_{g^*}\,\omega = \alpha \wedge *\overline{\beta}, \qquad \alpha,\beta \in \Omega^k(M), \tag{15.1}$$

(cf. Section XX.8 of [16] or Section XI.2 in [5]). By (3.8)

$$\langle v,\alpha\rangle=\langle \alpha,v\rangle=(\alpha\,|\,G_k v)_{g^*},\qquad \alpha\in{\textstyle\bigwedge}^k,\quad v\in{\textstyle\bigwedge}'^k.$$

Consequently,

$$\langle \alpha,v\rangle=\int_M\langle\alpha,v\rangle\,dV_g=\int_M(\alpha\,|\,G_k v)_{g^*}\,\omega=\int_M\alpha\wedge *G_k v \tag{15.2}$$

for $\alpha\in\Omega^k(M)$ and $v\in\mathcal{D}(M,{\textstyle\bigwedge}'^k)$.

Theorem 15.1 *All results obtained in the preceding sections for Bessel potential and Besov spaces of* (σ,τ)*-tensor fields remain valid for the corresponding spaces of* k*-forms, if* $(V_\tau^\sigma,V_\sigma^\tau)$ *is replaced by* $({\textstyle\bigwedge}^k,{\textstyle\bigwedge}'^k)$.

Proof. Obvious. □

Justified by this we refer in the following simply to the theorems and formulas of the preceding sections and it is understood that we mean the corresponding results for the spaces of differential forms.

The exterior derivative $d:\ \Omega^k(M)\to\Omega^{k+1}(M)$ is characterized by

$$\begin{aligned}d\alpha(X_0,X_1,\dots,X_k)=&\sum_{0\le i\le k}(-1)^i\,\nabla_{X_i}\bigl(\alpha(X_0,\dots,\widehat{X}_i,\dots,X_k)\bigr)\\&+\sum_{0\le i<j\le k}(-1)^{i+j}\,\alpha\bigl([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_k)\bigr)\end{aligned}$$

for $\alpha\in\Omega^k(M)$ and $X_0,\dots,X_k\in\mathcal{T}_0^1M$, where $[X_i,X_j]$ is the Lie bracket and $\widehat{\ }$ the usual omission symbol. Since ∇ is torsion free, that is, $\nabla_XY-\nabla_YX=[X,Y]$, it follows

$$d\alpha(X_0,\dots,X_k)=\sum_{0\le i\le k}(-1)^i(\nabla_{X_i}\alpha)(X_0,\dots,\widehat{X}_i,\dots,X_k). \tag{15.3}$$

The coderivative $\delta:\ \Omega^k(M)\to\Omega^{k-1}(M)$ is defined by

$$\delta\alpha:=(-1)^{m(k+1)+1}*d*\alpha,\qquad \alpha\in\Omega^k(M). \tag{15.4}$$

Recall

$$**\alpha=(-1)^{k(m-k)}\,\alpha,\qquad \alpha\in\Omega^k(M). \tag{15.5}$$

Suppose $\alpha\in\Omega^{k-1}(M)$ and $\beta\in\Omega^k(M)$. Then $\alpha\wedge*\beta\in\Omega^{m-1}(M)$ and $d*\beta\in\Omega^{m-k+1}(M)$. Note that (15.4) and (15.5) imply $*\delta\beta=(-1)^k\,d*\beta\in\Omega^k(M)$. From this we obtain

$$d(\alpha\wedge*\beta)=d\alpha\wedge*\beta+(-1)^{k-1}\,\alpha\wedge d*\beta=d\alpha\wedge*\beta-\alpha\wedge*\delta\beta.$$

Hence, setting $\beta=G_kv$ with $v\in\mathcal{D}(M,{\textstyle\bigwedge}'^k)$,

$$d(\alpha\wedge*G_kv)=d\alpha\wedge*G_kv-\alpha\wedge*G_{k-1}G^{k-1}\delta G_kv.$$

Thus Stoke's theorem implies, as is well-known, Green's formula which, due to (15.2), takes the form

$$\langle d\alpha,v\rangle_M-\langle\alpha,G^{k-1}\delta G_kv\rangle_M=\int_{\partial M}\overset{\bullet}{\imath}{}^*(\alpha\wedge*G_kv)$$

for $\alpha\in\Omega^{k-1}(M)$ and $v\in\mathcal{D}(M,{\textstyle\bigwedge}'^k)$. In particular,

$$\langle d\alpha,v\rangle_M=\langle\alpha,G^{k-1}\delta G_kv\rangle_M \tag{15.6}$$

if either α or v is compactly supported in $\mathring{M}$; thus, in particular, if $\partial M=\emptyset$. Similarly, using $\alpha\wedge*\beta=\beta\wedge*\alpha$ and setting $\alpha=G_{k-1}w$, we find

$$\langle\delta\beta,w\rangle_M=\langle\beta,G^kdG_{k-1}w\rangle_M,\qquad \beta\in\Omega^k(M),\quad w\in\mathcal{D}(M,{\textstyle\bigwedge}'^{k-1}), \tag{15.7}$$

if either β or w has compact support in $\mathring{M}$.

Now we can establish the fundamental mapping properties of the exterior differential and codifferential operators.

Theorem 15.2 *Suppose $s \in \mathbb{R}$.*

(i) *Assume either $\partial M = \emptyset$ or $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$. Then*

$$d \in \mathcal{L}\big(\mathfrak{F}_p^{s+1,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda}(\textstyle\bigwedge^{k+1})\big) \tag{15.8}$$

and

$$\delta \in \mathcal{L}\big(\mathfrak{F}_p^{s+1,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda+2}(\textstyle\bigwedge^{k-1})\big). \tag{15.9}$$

(ii) *Assume $\partial M \neq \emptyset$ and $s > -2 + 1/p$ with $s \neq -1 + 1/p$. Then*

$$d \in \mathcal{L}\big(\mathring{\mathfrak{F}}_p^{s+1,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda}(\textstyle\bigwedge^{k+1})\big) \tag{15.10}$$

and

$$\delta \in \mathcal{L}\big(\mathring{\mathfrak{F}}_p^{s+1,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda+2}(\textstyle\bigwedge^{k-1})\big). \tag{15.11}$$

Proof. (1) Suppose $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$. Then (15.8) is a consequence of (15.3) and Theorem 7.5.

(2) For $\alpha \in \Omega^k(M)$ it follows from (15.1) and (15.5) that

$$|{*\alpha}|_{g^*}^2 \omega = {*\alpha} \wedge {**\overline{\alpha}} = (-1)^{k(m-k)} {*\alpha} \wedge \overline{\alpha} = \overline{\alpha} \wedge {*\alpha} = |\alpha|_{g^*}^2 \omega.$$

Hence $\rho^{\lambda+2k-m+m-k} \, |{*\alpha}|_{g^*} = \rho^{\lambda+k} \, |\alpha|_{g^*}$. This implies

$$* \in \mathcal{L}\mathrm{is}\big(L_p^{\lambda}(\textstyle\bigwedge^k), L_p^{\lambda+2k-m}(\textstyle\bigwedge^{m-k})\big). \tag{15.12}$$

From (3.11)(ii) we infer for $X \in \mathcal{T}M$

$$\nabla_X(\alpha \wedge *\beta) = \nabla_X \alpha \wedge *\beta + \alpha \wedge \nabla_X(*\beta), \qquad \alpha, \beta \in \Omega^k(M). \tag{15.13}$$

Since $\nabla_X \omega = 0$ we obtain from (3.12)

$$\nabla_X\big((\alpha \,|\, \overline{\beta})_{g^*}\omega\big) = (\nabla_X \alpha \,|\, \overline{\beta})_{g^*}\omega + (\alpha \,|\, \nabla_X \overline{\beta})_{g^*}\omega.$$

Using this, (15.13), and (15.1) we deduce $\alpha \wedge \nabla_X(*\beta) = \alpha \wedge *\nabla_X \beta$ for $\alpha \in \Omega^k(M)$. Consequently,

$$\nabla_X(*\beta) = *(\nabla_X \beta), \qquad \beta \in \Omega^k(M), \quad X \in \mathcal{T}M.$$

By this and (15.12) we get

$$* \in \mathcal{L}\mathrm{is}\big(W_p^{j,\lambda}(\textstyle\bigwedge^k), W_p^{j,\lambda+2k-m}(\textstyle\bigwedge^{m-k})\big), \qquad j \in \mathbb{N}.$$

Hence, by interpolation,

$$* \in \mathcal{L}\mathrm{is}\big(\mathfrak{F}_p^{s,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda+2k-m}(\textstyle\bigwedge^{m-k})\big), \qquad s \in \mathbb{R}^+,$$

provided $s > 0$ if $\mathfrak{F} = B$. Now (15.9) follows from (15.4) and step (1), provided $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$.

(3) Definition (3.7) implies

$$|G^k \alpha|_g^2 = \langle G^k \alpha, G_k G^k \alpha \rangle = \langle \alpha, G^k \alpha \rangle = |\alpha|_{g^*}^2, \qquad \alpha \in \Omega^k(M).$$

Thus, since ∇ commutes with $g^\sharp$, hence with G^k,

$$\rho^{\lambda+2k+i-k} \, |\nabla^i G^k \alpha|_g = \rho^{\lambda+i+k} \, |G^k \nabla^i \alpha|_g = \rho^{\lambda+i+k} \, |\nabla^i \alpha|_{g^*}$$

for $i \in \mathbb{N}$. From this we deduce

$$G^k \in \mathcal{L}\mathrm{is}\big(W_p^{j,\lambda}(\textstyle\bigwedge^k), W_p^{j,\lambda+2k}(\textstyle\bigwedge'^k)\big), \qquad (G^k)^{-1} = G_k,$$

for $j \in \mathbb{N}$. Thus, by interpolation,

$$G^k \in \mathcal{L}\mathrm{is}\big(\mathfrak{F}_p^{s,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda+2k}(\textstyle\bigwedge'^k)\big), \quad (G^k)^{-1} = G_k, \tag{15.14}$$

for $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$.

The part of (15.9) which has already been shown and (15.14) imply

$$A := G^{k-1}\delta G_k \in \mathcal{L}\big(\mathfrak{F}_{p'}^{s+1,-\lambda}(\textstyle\bigwedge'^k), \mathfrak{F}_{p'}^{s,-\lambda}(\textstyle\bigwedge'^{k-1})\big). \tag{15.15}$$

(4) Suppose $\partial M = \emptyset$. Then (15.15) and Theorem 12.3 imply

$$A' \in \mathcal{L}\big(\mathfrak{F}_p^{-s,\lambda}(\textstyle\bigwedge^{k-1}), \mathfrak{F}_p^{-s-1,\lambda}(\textstyle\bigwedge^k)\big)$$

for $s \in \mathbb{R}^+$ with $s > 0$ if $\mathfrak{F} = B$. From this and (15.6) we infer, by density, that A' is the unique continuous extension of d. This proves (15.8) for all $s \in \mathbb{R}$ with the exception $s = 0$ if $\mathfrak{F} = B$. But now we close this gap by interpolation.

(5) Suppose $\partial M = \emptyset$ and $s > 0$. Then (15.8) and (15.14) imply

$$C := G^k d G_{k-1} \in \mathcal{L}\big(\mathfrak{F}_{p'}^{s+1,-\lambda-2}(\textstyle\bigwedge'^{k-1}), \mathfrak{F}_{p'}^{s,-\lambda}(\textstyle\bigwedge'^k)\big).$$

Hence

$$C' \in \mathcal{L}\big(\mathfrak{F}_p^{-s,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{-s-1,\lambda+2}(\textstyle\bigwedge^{k-1})\big).$$

Since (15.7) shows that C' is the unique continuous extension of δ over $\mathfrak{F}_p^{-s,\lambda}(\bigwedge^k)$ we get assertion (15.9) for $s < 0$. The case $\mathfrak{F} = B$ and $s = 0$ is once more covered by interpolation. Assertion (i) is thus proved.

(6) Suppose $\partial M \neq \emptyset$. If $s \geq 0$, then (15.10) and (15.11) are obvious by (i). Clearly, G^k maps $\mathcal{D}(\mathring{M}, \bigwedge^k)$ into $\mathcal{D}(\mathring{M}, \bigwedge'^k)$. Hence (15.14) implies

$$G^k \in \mathcal{L}\mathrm{is}\big(\mathring{\mathfrak{F}}_p^{s,\lambda}(\textstyle\bigwedge^k), \mathring{\mathfrak{F}}_p^{s,\lambda+2k}(\textstyle\bigwedge'^k)\big), \quad (G^k)^{-1} = G_k, \tag{15.16}$$

for $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$.

Suppose $-1 + 1/p < s < 0$, that is, $0 < -s < 1 - 1/p = 1/p'$. Then, by Theorem 11.3(i),

$$\mathfrak{F}_{p'}^{-s,-\lambda}(\textstyle\bigwedge'^{k-1}) = \mathring{\mathfrak{F}}_{p'}^{-s,-\lambda}(\textstyle\bigwedge'^{k-1}).$$

From this, (15.16), and the observation of the beginning of this step we infer

$$A \in \mathcal{L}\big(\mathring{\mathfrak{F}}_{p'}^{-s+1,-\lambda}(\textstyle\bigwedge'^k), \mathring{\mathfrak{F}}_{p'}^{-s,-\lambda}(\textstyle\bigwedge'^{k-1})\big).$$

Hence, by (12.3),

$$A' \in \mathcal{L}\big(\mathfrak{F}_p^{s,\lambda}(\textstyle\bigwedge^{k-1}), \mathfrak{F}_p^{s-1,\lambda}(\textstyle\bigwedge^k)\big).$$

Thus (15.6) implies

$$d \in \mathcal{L}\big(\mathfrak{F}_p^{s,\lambda}(\textstyle\bigwedge^{k-1}), \mathfrak{F}_p^{s-1,\lambda}(\textstyle\bigwedge^k)\big).$$

This proves claim (15.10) if $-2 + 1/p < s < -1 + 1/p$. Now we obtain assertion (15.10) for $-1 + 1/p < s < 0$ by interpolation, thanks to Theorem 13.1. The proof of statement (15.11) is similar. □

As an immediate consequence of this theorem we see that the Hodge Laplacian $\Delta_{\mathrm{Hodge}} := d\delta + \delta d$ satisfies

$$\Delta_{\mathrm{Hodge}} \in \mathcal{L}\big(\mathfrak{F}_p^{s+2,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda+2}(\textstyle\bigwedge^k)\big)$$

if either $s \in \mathbb{R}$ and $\partial M = \emptyset$, or $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$. If $\partial M \neq \emptyset$, then

$$\Delta_{\mathrm{Hodge}} \in \mathcal{L}\big(\mathring{\mathfrak{F}}_p^{s+2,\lambda}(\textstyle\bigwedge^k), \mathfrak{F}_p^{s,\lambda+2}(\textstyle\bigwedge^k)\big),$$

provided $s > -2 + 1/p$ with $s \neq -1 + 1/p$. Note that $\Delta_{\mathrm{Hodge}} = -\Delta_M$ if $k = 0$, where $\Delta_M = \operatorname{div}\operatorname{grad}$ is the Laplace-Beltrami operator of M.

Finally, we apply these results to derive the mapping properties of the basic differential operators of vector analysis. For this we recall that the gradient and the divergence operator can be represented (taking the complex case into account) by

$$\operatorname{grad} = G^1 \circ d : \ \mathcal{D}(M) \to \mathcal{D}(M, T_0^1 M) \tag{15.17}$$

and

$$\operatorname{div} = -\delta \circ G_1 : \ \mathcal{D}(M, T_0^1 M) \to \mathcal{D}(M), \tag{15.18}$$

respectively.

Theorem 15.3 *Suppose $s \in \mathbb{R}$.*

(i) *Assume either $\partial M = \emptyset$ or $s \geq 0$ with $s > 0$ if $\mathfrak{F} = B$. Then*

$$\operatorname{grad} \in \mathcal{L}\big(\mathfrak{F}_p^{s+1,\lambda}(M), \mathfrak{F}_p^{s,\lambda+2}(T_0^1 M)\big), \quad \operatorname{div} \in \mathcal{L}\big(\mathfrak{F}_p^{s+1,\lambda}(T_0^1 M), \mathfrak{F}_p^{s,\lambda}(M)\big).$$

(ii) *If $\partial M \neq \emptyset$ and $s > -2 + 1/p$ with $s \neq -1 + 1/p$, then*

$$\operatorname{grad} \in \mathcal{L}\big(\mathring{\mathfrak{F}}_p^{s+1,\lambda}(M), \mathfrak{F}_p^{s,\lambda+2}(T_0^1 M)\big), \quad \operatorname{div} \in \mathcal{L}\big(\mathring{\mathfrak{F}}_p^{s+1,\lambda}(T_0^1 M), \mathfrak{F}_p^{s,\lambda}(M)\big).$$

P r o o f. It follows from (3.4) that

$$\langle \alpha, G^1 \beta \rangle = \langle G^1 \alpha, \beta \rangle, \qquad \alpha, \beta \in \mathcal{D}(M, T_1^0 M).$$

From this and (15.14) we obtain by duality arguments similar to the ones used in the preceding proof that

$$G^1 \in \mathcal{L}\mathrm{is}\big(\mathfrak{F}_p^{s,\lambda}(T_1^0 M), \mathfrak{F}_p^{s,\lambda+2}(T_0^1 M)\big), \quad (G^1)^{-1} = G_1,$$

for all $s \in \mathbb{R}$ if $\partial M = \emptyset$. Similarly, (15.16) implies

$$G^1 \in \mathcal{L}\mathrm{is}\big(\mathring{\mathfrak{F}}_p^{s,\lambda}(T_1^0 M), \mathring{\mathfrak{F}}_p^{s,\lambda+2}(T_0^1 M)\big), \quad s \in \mathbb{R}.$$

Now the assertion follows from (15.17), (15.18), and Theorem 15.2. □

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