NONHOMOGENEOUS NAVIER-STOKES EQUATIONS IN SPACES OF LOW REGULARITY

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Dedicated to John G. Heywood on the occasion of his 60th birthday

Introduction

Throughout this paper $n \geq 2$ and Ω is a smooth domain in $\mathbb{R}^n$ having a nonempty compact boundary Γ, or a half space. If Ω is an exterior domain then we also assume that $n \geq 3$. We study the solvability of the nonstationary nonhomogeneous incompressible Navier-Stokes equations

$$\text{(0.1)} \qquad \begin{cases} \left.\begin{aligned} \nabla \cdot v &= 0 \\ \partial_t v + (v \cdot \nabla) v - \Delta v &= -\nabla \pi + f \end{aligned}\right\} & \text{in} \quad \Omega \times (0, \infty), \\ v = g & \text{on} \quad \Gamma \times (0, \infty), \\ v(\cdot, 0) = v^0 & \text{on} \quad \Omega. \end{cases}$$

Here f is the given exterior force, g the prescribed boundary velocity, v^0 the given initial and v the unknown velocity, and π the unknown pressure.

We are interested in (0.1) in a weak, that is, distributional, setting. In our recent work [4] we have already given a detailed discussion of the Navier-Stokes equations in spaces of low regularity under the assumption that $g = 0$. In this paper we extend those results to cover nonzero boundary conditions. For the sake of simplicity we do not give the most general theorems but restrict ourselves to an easy setting. We leave it to the reader to generalize the results of this paper by using the full strength of [4].

To give a precise meaning to a solution in our general setting and to describe our main results we need some preparation.

We use standard notation (explained in Section 1) and employ the following convention: If $\mathfrak{F}(\Omega, \mathbb{R}^n)$ is a vector space of $\mathbb{R}^n$-valued distributions on Ω then we simply denote it by $\mathfrak{F}$. If X is a subset of $\mathbb{R}^n$ different from Ω then we put $\mathfrak{F}(X) := \mathfrak{F}(X, \mathbb{R}^n)$. For example, $\mathcal{D}$, resp. $\mathcal{D}(\overline{\Omega})$, is the space of smooth

$\mathbb{R}^n$-valued functions having compact support in Ω, resp. $\overline{\Omega}$, and $C(\Gamma)$ is the Banach space of continuous $\mathbb{R}^n$-valued functions on Γ.

We always assume that $q \in (1,\infty)$. Then H_q^s are the usual Bessel potential spaces (of $\mathbb{R}^n$-valued distributions on Ω), where $s \in \mathbb{R}$. (See [4] for explanations. In particular, Theorem 2.1 of [4] gives useful intrinsic characterizations of H_q^s for $s \in [-2,0)$.) We also denote by $d\sigma$ the volume measure of Γ and set

$$\langle v,w\rangle := \int_\Omega v\cdot w\,dx, \quad v,w \in \mathcal{D}(\overline{\Omega}), \qquad \langle v,w\rangle_\Gamma := \int_\Gamma v\cdot w\,d\sigma, \quad v,w \in C(\Gamma).$$

In addition, we use $\langle\cdot,\cdot\rangle$ to denote the standard duality pairings between various spaces of (scalar-, vector-, or tensor-valued) distributions without fearing confusion. Similar conventions hold for $\langle\cdot,\cdot\rangle_\Gamma$. We write ∂_ν for the derivative on Γ with respect to the outer unit normal ν and denote by γ the trace operator and by γ_ν the normal trace, defined by $\gamma_\nu u := \nu\cdot\gamma u$.

We set

$$\boldsymbol{H}_q^s := \begin{cases} \{\, u \in H_q^s \ ;\ \gamma u = 0 \,\}, & 1/q < s \le 2, \\ \{\, u \in H_q^{1/q}(\mathbb{R}^n,\mathbb{R}^n) \ ;\ \mathrm{supp}(u) \subset \overline{\Omega} \,\}, & s = 1/q, \\ H_q^s, & 0 \le s < 1/q, \\ (\boldsymbol{H}_{q'}^{-s})', & -2 \le s < 0, \end{cases}$$

where the dual space is determined by means of the duality pairing $\langle\cdot,\cdot\rangle$.

It follows that

$$\boldsymbol{H}_q^s = H_q^s, \qquad -2+1/q < s < 1/q, \quad s \ne -1+1/q. \tag{0.2}$$

Furthermore,

$$\boldsymbol{H}_q^s \overset{d}{\hookrightarrow} \boldsymbol{H}_q^t, \qquad -2 \le t < s \le 2. \tag{0.3}$$

We also put

$$\mathbb{H}_q := \{\, u \in L_q \ ;\ \nabla\cdot u = 0,\ \gamma_\nu u = 0 \,\}$$

and

$$\mathbb{H}_q^s := \begin{cases} \boldsymbol{H}_q^s \cap \mathbb{H}_q, & 0 \le s \le 2, \\ (\mathbb{H}_{q'}^{-s})', & -2 \le s < 0, \end{cases}$$

by means of the duality pairing $\langle\cdot,\cdot\rangle_\sigma$, obtained by restricting $\langle\cdot,\cdot\rangle$ to $\mathbb{H}_{q'} \times \mathbb{H}_q$. Then

$$\mathbb{H}_q^s \overset{d}{\hookrightarrow} \mathbb{H}_q^t, \qquad -2 \le t < s \le 2. \tag{0.4}$$

In addition,

$$\mathbb{G}_q := \{\, v \in L_q \ ;\ v = \nabla p,\ p \in L_{q,\mathrm{loc}}(\overline{\Omega},\mathbb{R}) \,\},$$

and

$$\mathbb{G}_q^s \text{ is the closure of } \mathbb{G}_q \text{ in } \boldsymbol{H}_q^s \text{ for } -2 \le s < 0. \tag{0.5}$$

Proofs of (0.2)–(0.4) are given below. (In [4] the spaces $\boldsymbol{H}_q^s$ and $\mathbb{H}_q^s$ are denoted by $H_{q,0}^s$ and $H_{q,0,\sigma}^s$, respectively.) Lastly,

$$\mathcal{D}_0(\overline{\Omega}) := \{\, u \in \mathcal{D}(\overline{\Omega}) \ ; \ \gamma u = 0 \,\}.$$

Now we suppose that

$$q > n, \quad 0 \le s < \tau < 1/q < \sigma \le 2 \tag{0.6}$$

and that

$$\big(v^0, (f,g)\big) \in \mathbb{H}_q^s \times C\big(\mathbb{R}^+, H_q^{\sigma-2} \times W_q^{\tau-1/q}(\Gamma)\big). \tag{0.7}$$

Let J be a subinterval of $\mathbb{R}^+$ containing 0 such that $\dot{J} := J \backslash \{0\} \ne \emptyset$. Then (u,w) is said to be a **very weak H_q^s-solution** of the Navier-Stokes equations on J, provided

$$(u,w) \in C(J, \mathbb{H}_q^s \times \mathbb{G}_q^{s-2}) \tag{0.8}$$

and

$$\begin{aligned} &\int_J \big\{ \langle (\partial_t + \Delta)\varphi, u \rangle + \langle \nabla\varphi, u \otimes u \rangle \big\}\, dt \\ &= \int_J \big\{ -\langle \varphi, w \rangle - \langle f, \varphi \rangle + \langle g, \partial_\nu \varphi \rangle_\Gamma \big\}\, dt - \big\langle \varphi(0), v^0 \big\rangle \end{aligned} \tag{0.9}$$

for all $\varphi \in \mathcal{D}\big(J, \mathcal{D}_0(\overline{\Omega})\big)$. A very weak H_q^s-solution is **maximal** if there does not exist such a solution being a proper extension of it.

Clearly, (0.9) is formally obtained from the differential equations in (0.1) by multiplying the second one by φ, integrating by parts, and using Green's formula, the boundary, and the initial data, setting $w := -\nabla\pi$, of course. The terms $\langle \nabla\varphi, u \otimes u \rangle$ and $\langle \varphi, w \rangle$ in (0.9) have to be interpreted with care. In fact, since $s < 1/q$, the space $\boldsymbol{H}_q^{s-2}$ is not a space of distributions on Ω but contains distributions supported on Γ as well. Indeed, Theorem 1.1 shows that

$$\boldsymbol{H}_q^{t-2} \cong H_q^{t-2} \times W_q^{t-1/q}(\Gamma), \qquad 0 \le t < 1/q, \tag{0.10}$$

where this isomorphism is specified in Corollary 1.2. This is the key result for our treatment of nonhomogeneous boundary conditions. The expressions $\langle \nabla\varphi, u \otimes u \rangle$ and $\langle \varphi, w \rangle$ are well-defined if $u \otimes u$ and w belong to $\mathbb{H}_q^2$ and $\boldsymbol{H}_q^2$, respectively. In the general case these quantities are defined by continuous extension on the basis of (0.3)–(0.5).

After these preparations we can formulate the main result of this paper.

Theorem. *Let assumptions* (0.6) *and* (0.7) *be satisfied. Then:*

(i) *There exists a unique maximal H_q^s-solution (v,ω) of the Navier-Stokes equations, and $J^+ := \mathrm{dom}(v,\omega)$ is an open interval in $\mathbb{R}^+$.*
(ii) *It depends continuously on the data.*
(iii) *If Ω is bounded then it is global, that is, $J^+ = \mathbb{R}^+$, provided the data are sufficiently small.*

Remarks. (a) Observe that there is no compatibility condition for g of the form $\nu \cdot g = 0$, even though $u \in \mathbb{H}_q^s$ implies $\gamma_\nu u = 0$. This is explained by the fact that, due to (0.10), the (generalized) pressure gradient ω is not a distribution on Ω but can contain a part supported on Γ, thus compensating in the weak formulation (0.9) of (0.1) for a possible normal component of g.

(b) The proof of the Theorem shows that the velocity component v possesses more regularity, namely

$$v \in \mathcal{C}^1\big(J^+, (\mathbb{H}_q^{s-2}, \mathbb{H}_q^s)\big) := C(J^+, \mathbb{H}_q^s) \cap C^1(J^+, \mathbb{H}_q^{s-2}).$$

(c) The solution (v,ω) is independent of $s \in [0, 1/q)$ and of q in the following sense: Suppose, in addition to (0.6) and (0.7), that $\overline{q} > q$ and $0 \leq \overline{s} < 1/\overline{q}$, and suppose that $v^0 \in \mathbb{H}_{\overline{q}}^{\overline{s}}$. Let $(\overline{v}, \overline{\omega})$ be the unique maximal very weak $H_{\overline{q}}^{\overline{s}}$-solution of the Navier-Stokes equations. Then $(\overline{v}, \overline{\omega}) \supset (v,\omega)$, that is, the interval $\overline{J}^+$ of existence of $(\overline{v}, \overline{\omega})$ contains J^+ and $(\overline{v}, \overline{\omega}) \,|\, J^+ = (v,\omega)$. □

Example. *Suppose that $p, q, r \in (1,\infty)$ satisfy $q > n$,*

$$1/q \leq 1/p < (n-1)/nq + 2/n, \quad 1/q < 1/r < n/(n-1)q.$$

Then the Navier-Stokes equations possess for each

$$\big(v^0, (f,g)\big) \in L_q \times C\big(\mathbb{R}^+, L_p \times L_r(\Gamma)\big)$$

with

$$\nabla \cdot v^0 = 0, \quad \gamma_\nu v^0 = 0$$

a unique maximal very weak L_q-solution.

Proof. Set $\sigma := 2 - n(1/p - 1/q)$ and $\tau := 1/q - (n-1)(1/r - 1/q)$. It follows that $1/q < \sigma \leq 2$ and $0 < \tau < 1/q$. Thus, setting $s := 0$, assumption (0.6) is satisfied. Then $\boldsymbol{H}_{q'}^{2-\sigma} \overset{d}{\hookrightarrow} L_{p'}$ and $W_{q'}^{1/q-\tau}(\Gamma) \overset{d}{\hookrightarrow} L_{r'}(\Gamma)$ by Sobolev's embedding theorem so that $L_p \hookrightarrow \boldsymbol{H}_q^{\sigma-2} = H_q^{\sigma-2}$ and $L_r(\Gamma) \hookrightarrow W_q^{\tau-1/q}(\Gamma)$ by duality. Hence assumption (0.7) is satisfied. □

Navier-Stokes equations with nonhomogeneous boundary data for Dirichlet (and other) boundary conditions have been intensively studied by G. Grubb and V.A. Solonnikov [9] and, in particular, by G. Grubb (see [6], [7], [8]). These authors use techniques from the theory of pseudodifferential operators and work in scales of anisotropic Bessel potential and Besov spaces. In her most recent paper on this subject, Grubb [8] extended her earlier results for problem (0.1) to cover the case of low regularity data. To be more precise, we fix $T > 0$, set $Q_T := \Omega \times (0, T)$ and $\Sigma_T := \Gamma \times (0, T)$, and write $H_q^{(\tau)}(Q_T)$ for the anisotropic spaces $H_q^{\tau,\tau/2}(Q_T)$, etc. In the main result, that is, Theorem 3.4 of [8] it is assumed that

$$2/q < \tau < 1 + 1/q, \quad \tau > -1 + n/q + 2/q, \tag{0.11}$$

and

$$(v^0, f, g) \in B_{q,q}^{\tau-2/q} \times H_q^{(\tau-2)}(Q_T) \times B_{q,q}^{(\tau-1/q)}(\Sigma_T) \tag{0.12}$$

with

$$(1 - P)v^0 = 0, \quad (1 - P)f = 0, \quad \nu \cdot g = 0,$$

where P denotes the Helmholtz projection. (In fact, f and g have to comply to the additional restriction that they are suitably 'extensible' distributions. Also, if $v^0 = 0$ then the restriction $\tau > 2/q$ can be slightly weakened, provided the data have sufficiently small norm. For precise definitions and statements we refer to [8].) Then it is shown that there exists $b > 0$ such that (0.1) possesses a unique solution (v, p) such that

$$(v, \nabla p) \in H_q^{(\tau)}(Q_b) \times H_q^{\tau-2}(Q_b),$$

where p is appropriately normalized if Ω is bounded. (We again refer to [8] for the precise meaning of 'solution' in this case.)

These results are difficult to compare to ours. One difference is that the elements in anisotropic spaces have less time regularity than the ones of our setting. However, as far a space regularity is concerned, (0.11) and (0.12) imply that v^0 and g have to belong to spaces of *positive* regularity, whereas we can allow data (v^0, g) having *zero* regularity as the Example shows.

In the next section we prove the basic isomorphism theorem (0.10) and collect some facts on Stokes scales. In Section 2 we derive an important direct sum decomposition of the spaces $\boldsymbol{H}_q^s$ extending the Helmholtz decomposition to all values of $s \in [-2, 2]$. The last section contains the proofs of the Theorem and of Remark (c).

1. Dirichlet scales

First we prove the fundamental isomorphism theorem (0.10). As usual, we denote by $\mathring{H}_q^s$ the closure of $\mathcal{D}$ in H_q^s.

Given Banach spaces E and F, we write $\mathcal{L}(E,F)$ for the Banach space of all bounded linear operators from E into F, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. Furthermore, $\mathcal{L}\mathrm{is}(E,F)$ is the set of all isomorphisms in $\mathcal{L}(E,F)$. If E is continuously injected in F then we denote this by $E \hookrightarrow F$, and $E \stackrel{d}{\hookrightarrow} F$ means that E is also dense in F. Finally, we write $A \in \mathcal{H}(E_1, E_0)$ iff $A \in \mathcal{L}(E_1, E_0)$ and $-A$, considered as a linear operator in E_0 with domain E_1, generates an analytic semigroup on E_0.

Next we recall some elementary facts about direct sum decompositions. The duality considerations will be of particular importance in the next section. Let X be a Banach space and $P \in \mathcal{L}(X)$ a projection, that is, $P^2 = P$. Then

$$X = Y \oplus Z, \quad Y := \mathrm{im}(P), \quad Z := \ker(P),$$

which means that

$$\big((y,z) \mapsto y+z\big) \in \mathcal{L}\mathrm{is}(Y \times Z, X).$$

In particular, Y and Z are closed linear subspaces of X, hence Banach spaces. Furthermore, $P' \in \mathcal{L}(X')$ is also a projection, so that

$$X' = Y' \oplus Z', \quad Y' := \mathrm{im}(P'), \quad Z' := \ker(P').$$

It is easily verified that

$$\langle x', x\rangle = \langle y' \oplus z', y \oplus z\rangle = \langle y', y\rangle + \langle z', z\rangle \tag{1.1}$$

for $x' = y' \oplus z' \in X'$ and $x = y \oplus z \in X$, where $y \oplus z := y + z$ with $y \in Y$ and $z \in Z$. Here

$$\langle \cdot,\cdot\rangle_X := \langle \cdot,\cdot\rangle : \ X' \times X \to \mathbb{R}$$

is the 'X-duality pairing'. Denoting by $\langle \cdot,\cdot\rangle_Y$ and $\langle \cdot,\cdot\rangle_Z$ the restriction of $\langle \cdot,\cdot\rangle$ to $Y' \times Y$ and $Z' \times Z$, respectively, (1.1) takes the form

$$\langle \cdot,\cdot\rangle_X = \langle \cdot,\cdot\rangle_Y + \langle \cdot,\cdot\rangle_Z. \tag{1.2}$$

Consequently, Y' (resp. Z') is the dual space of Y (resp. Z) with respect to the duality pairing $\langle \cdot,\cdot\rangle_Y$ (resp. $\langle \cdot,\cdot\rangle_Z$). We say that $\langle \cdot,\cdot\rangle_Y$ and $\langle \cdot,\cdot\rangle_Z$ are the duality pairings induced by $\langle \cdot,\cdot\rangle$ through the direct sum decomposition $X = Y \oplus Z$. Note that, although Y' is the dual of Y with respect to $\langle \cdot,\cdot\rangle_Y$, it is the annihilator of Z with respect to the duality pairing $\langle \cdot,\cdot\rangle$. If X is reflexive then it is obvious from (1.2) that Y and Z are reflexive as well (with respect to $\langle \cdot,\cdot\rangle_Y$ and $\langle \cdot,\cdot\rangle_Z$, of course).

Theorem 1.1. *Suppose that* $-2 \le s < -2 + 1/q$. *Then*

$$\boldsymbol{H}_q^s \cong H_q^s \times W_q^{s+2-1/q}(\Gamma).$$

Proof. By the trace theorem,

$$\partial_\nu \in \mathcal{L}\big(H_{q'}^{-s}, W_{q'}^{-s-2+1/q}(\Gamma)\big),$$

and it is a retraction. In fact (cf. (the proof of) Theorem B.3 in [1]), there exists

$$\mathcal{R} \in \mathcal{L}\big(W_{q'}^{-s-2+1/q}(\Gamma), H_{q'}^{-s}\big)$$

satisfying $\partial_\nu \circ \mathcal{R} = 1$ and $\gamma \circ \mathcal{R} = 0$ (where 1 is the identity on $W_{q'}^{-s-2+1/q}(\Gamma)$, of course). Thus, setting

$$\delta := -\partial_\nu \,|\, \boldsymbol{H}_{q'}^{-s}, \quad \delta^c := -\mathcal{R},$$

it follows that

$$\delta \in \mathcal{L}\big(\boldsymbol{H}_{q'}^{-s}, W_{q'}^{-s-2+1/q}(\Gamma)\big), \quad \delta^c \in \mathcal{L}\big(W_{q'}^{-s-2+1/q}(\Gamma), \boldsymbol{H}_{q'}^{-s}\big), \tag{1.3}$$

and $\delta\delta^c = 1$. Note that

$$\ker(\delta) = \{\, v \in H_{q'}^{-s} \ ; \ \gamma v = 0,\ \partial_\nu v = 0 \,\} = \mathring{H}_{q'}^{-s}.$$

Thus, since δ is a retraction, [3, Lemma I.2.3.1]) implies that

$$\boldsymbol{H}_{q'}^{-s} = \mathrm{im}(\delta^c) \oplus \mathring{H}_{q'}^{-s}$$

and that

$$p := p_\Omega := 1 - \delta^c\delta \in \mathcal{L}(\boldsymbol{H}_{q'}^{-s})$$

is the projection onto $\mathring{H}_{q'}^{-s}$ parallel to $\mathrm{im}(\delta^c)$. From (1.3) and $\delta\delta^c = 1$ we infer by duality that

$$r := r_\Gamma := (\delta^c)' \in \mathcal{L}\big(\boldsymbol{H}_q^s, W_q^{s+2-1/q}(\Gamma)\big)$$

is a retraction and

$$r^c := \delta' \in \mathcal{L}\big(W_q^{s+2-1/q}(\Gamma), \boldsymbol{H}_q^s\big)$$

is a coretraction. Hence

$$\boldsymbol{H}_q^s = \ker(r) \oplus \mathrm{im}(r^c) \cong \ker(r) \times W_q^{s+2-1/q}(\Gamma) \tag{1.4}$$

since r^c is an isomorphism from $W_q^{s+2-1/q}(\Gamma)$ onto its image. Note that $u \in \boldsymbol{H}_q^s$ belongs to $\ker(r)$ iff $\langle u, \delta^c g\rangle = 0$ for all $g \in W_{q'}^{-s-2+1/q}(\Gamma)$, that is, iff

$$\langle u, v\rangle = \big\langle u, (1-\delta^c\delta)v\big\rangle = \langle u, pv\rangle, \qquad v \in \boldsymbol{H}_{q'}^{-s}.$$

Since $\mathrm{im}(p) = \mathring{H}_{q'}^{-s}$ and $p^2 = p$ we see that $u \in \ker(r)$ iff $u = r_\Omega u$, where

$$r_\Omega \in \mathcal{L}(\boldsymbol{H}_q^s, H_q^s)$$

is the restriction map

$$u \mapsto r_\Omega u := u \,|\, \mathring{H}_{q'}^{-s}, \qquad u \in \boldsymbol{H}_q^s.$$

Thus $\ker(r) = H_q^s$ which, thanks to (1.4), proves the assertion. □

Corollary 1.2. *Suppose that* $-2 \le s < -2 + 1/q$ *and set*

$$R(f,g) := p'_\Omega f - (\partial_\nu)' g.$$

Then

$$R \in \mathcal{L}\mathrm{is}\big(H_q^s \times W_q^{s+2-1/q}(\Gamma), \boldsymbol{H}_q^s\big)$$

and $u \mapsto (r_\Omega u, r_\Gamma u)$ *is its inverse.*

Remark 1.3. Assume that $-2 \le s < -2 + 1/q < \sigma \le 0$ and $f \in H_q^\sigma$. Then $f \supset p'_\Omega f \in \boldsymbol{H}_q^s$.

Proof. Since $-\sigma < 1 + 1/q'$ it follows that $\mathcal{D}$ is dense in $\boldsymbol{H}_{q'}^{-\sigma}$. From $pv = v$ for $v \in \mathcal{D}$ it follows that

$$\langle f, v\rangle = \langle f, pv\rangle = \langle p'f, v\rangle, \qquad v \in \mathcal{D}.$$

Thus $p'f \in \boldsymbol{H}_q^s$ has a unique continuous extension over $\boldsymbol{H}_{q'}^{-\sigma}$, namely f. □

We set $\boldsymbol{A} := -\Delta_{D,q}$, where $\Delta_{D,q} := \Delta \,|\, \boldsymbol{H}_q^2$ is the Dirichlet-Laplacian, considered as an unbounded linear operator in L_q. Then $\boldsymbol{A} \in \mathcal{H}(\boldsymbol{H}_q^2, L_q)$ and $\big[(\boldsymbol{E}_\alpha, \boldsymbol{A}_\alpha)\ ;\ \alpha \in \mathbb{R}\big]$ denotes the interpolation-extrapolation scale generated by $(\boldsymbol{E}_0, \boldsymbol{A}_0) := (L_q, \boldsymbol{A})$ and the complex interpolation functors $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. (We refer to Section 2 in [4] for more details on Dirichlet scales. We also use the opportunity to point out that in Proposition 2.4 of [4] the restriction $s \ne -1 + 1/q$ is missing). We denote by $\big[(\boldsymbol{E}_\alpha^\sharp, \boldsymbol{A}_\alpha^\sharp)\ ;\ \alpha \in \mathbb{R}\big]$ the dual scale generated by $(\boldsymbol{E}_0^\sharp, \boldsymbol{A}_0^\sharp) := (L_{q'}, -\Delta_{D,q'})$ and $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. Then

$$\boldsymbol{E}_\alpha \doteq \boldsymbol{H}_q^{2\alpha}, \quad (\boldsymbol{E}_\alpha)' \doteq \boldsymbol{E}_{-\alpha}^\sharp \doteq \boldsymbol{H}_{q'}^{-2\alpha}, \qquad |\alpha| \le 1, \tag{1.5}$$

where $\doteq$ means 'equivalent norms', and (0.3) is true. It is known that $\boldsymbol{A}_0$ has bounded imaginary powers (e.g., [10]). Hence it follows from Theorem V.1.5.4 in [3] that

$$[\boldsymbol{H}_q^{s_0}, \boldsymbol{H}_q^{s_1}]_\theta \doteq \boldsymbol{H}_q^{(1-\theta)s_0+\theta s_1}, \qquad -2 \le s_0 < s_1 \le 2, \quad 0 < \theta < 1. \tag{1.6}$$

In the next lemma we give explicit representations of the extrapolated operators $\boldsymbol{A}_{\alpha-1}$ for $0 \le \alpha < 1$, where $\langle\cdot,\cdot\rangle_\alpha$ is the $\boldsymbol{E}_\alpha$-duality pairing for $\alpha \in \mathbb{R}$.

Lemma 1.4. (i) *If* $1 + 1/q < 2\alpha \le 2$ *then* $\boldsymbol{A}_{\alpha-1} = -\Delta \,|\, \boldsymbol{H}_q^{2\alpha}$.

(ii) *If* $1/q < 2\alpha < 1 + 1/q$ *then*

$$\langle v, \boldsymbol{A}_{\alpha-1} u\rangle_{\alpha-1} = \langle \nabla v, \nabla u\rangle, \qquad (v,u) \in \boldsymbol{H}_{q'}^{2-2\alpha} \times \boldsymbol{H}_q^{2\alpha}.$$

(iii) *If* $0 \le 2\alpha < 1/q$ *then*

$$\langle v, \boldsymbol{A}_{\alpha-1} u\rangle_{\alpha-1} = \langle -\Delta v, u\rangle, \qquad (v,u) \in \boldsymbol{H}_{q'}^{2-2\alpha} \times H_q^{2\alpha}.$$

Proof. This follows from [2, Theorem 8.3]. □

Remark 1.5. The space $\mathcal{D}_0(\overline{\Omega})$ is dense in $\boldsymbol{H}_q^s$ for $|s| \leq 2$.

Proof. Thanks to $\boldsymbol{H}_q^2 \overset{d}{\hookrightarrow} \boldsymbol{H}_q^s$ for $-2 \leq s < 2$, it suffices to show that $\mathcal{D}_0(\overline{\Omega})$ is dense in $\boldsymbol{H}_q^2$. Since $1 + \boldsymbol{A} \in \mathcal{L}\mathrm{is}(\boldsymbol{H}_q^2, \boldsymbol{H}_q^0)$ and $\mathcal{D}$ is dense in $\boldsymbol{H}_q^0$ we see that $\boldsymbol{M} := (1 + \boldsymbol{A})^{-1}(\mathcal{D})$ is dense in $\boldsymbol{H}_q^2$. Since $\boldsymbol{M} \subset C^\infty(\overline{\Omega})$, the assertion follows if Ω is bounded. Otherwise, we obtain the desired result by multiplying the elements of $\boldsymbol{M}$ by suitable smooth cut-off functions. □

2. Stokes scales

Recall that $\mathbb{H}_q$ is the closure of

$$\mathcal{D}_\sigma := \{\, u \in \mathcal{D} \ ; \ \nabla \cdot u = 0 \,\}$$

in L_q and that the Helmholtz decomposition

$$\boldsymbol{H}_q^0 = L_q = \mathbb{H}_q \oplus \mathbb{G}_q \tag{2.1}$$

is valid. The Helmholtz projection, $P := P_q$, is the projection of L_q onto $\mathbb{H}_q$ parallel to $\mathbb{G}_q$ (cf. [4] for references).

We denote by

$$\mathbb{A} := -P_q \Delta \,|\, \mathbb{H}_q^2, \quad \mathbb{A}^\sharp := -P_{q'} \Delta \,|\, \mathbb{H}_{q'}^2$$

the Stokes operators in L_q and $L_{q'}$, respectively. Then $\mathbb{A} \in \mathcal{H}(\mathbb{H}_q^2, \mathbb{H}_q)$, and we write $\big[(\mathbb{E}_\alpha, \mathbb{A}_\alpha) \ ; \ \alpha \in \mathbb{R} \big]$ and $\big[(\mathbb{E}_\alpha^\sharp, \mathbb{A}_\alpha^\sharp) \ ; \ \alpha \in \mathbb{R} \big]$ for the interpolation-extrapolation scales induced by $(\mathbb{E}_0, \mathbb{A}_0) := (\mathbb{H}_q, \mathbb{A})$ and $(\mathbb{E}_0^\sharp, \mathbb{A}_0^\sharp) := (\mathbb{H}_{q'}, \mathbb{A}^\sharp)$, respectively, and by complex interpolation. It follows that

$$\mathbb{E}_\alpha \doteq \mathbb{H}_q^{2\alpha}, \quad (\mathbb{E}_\alpha)' \doteq \mathbb{E}_{-\alpha}^\sharp \doteq \mathbb{H}_{q'}^{-2\alpha}, \qquad |\alpha| \leq 1, \tag{2.2}$$

and also that (0.4) is true. It is known that $\mathbb{A}$ has bounded imaginary powers (see [4, Remark 8.1]). From this we deduce that

$$[\mathbb{H}_q^{s_0}, \mathbb{H}_q^{s_1}]_\theta \doteq \mathbb{H}_q^{(1-\theta)s_0 + \theta s_1}, \qquad -2 \leq s_0 < s_1 \leq 2, \quad 0 < \theta < 1, \tag{2.3}$$

(cf. [3, Theorem V.1.5.4]).

Lemma 2.1. *Suppose that $-2 \leq s < 0$. Then $\mathbb{H}_q^s$ is a closed linear subspace of $\boldsymbol{H}_q^s$.*

Proof. Set

$$Q_1^\sharp := (1 + \mathbb{A}_0^\sharp)^{-1} P^\sharp (1 + \boldsymbol{A}_0^\sharp) \in \mathcal{L}(\boldsymbol{E}_1^\sharp),$$

where $P^\sharp := P_{q'}$. In the proof of [4, Lemma 3.2] it has been shown that $Q_1^\sharp$ is a projection onto $\mathbb{E}_1^\sharp$ which extends uniquely to a projection $Q_0^\sharp \in \mathcal{L}(\boldsymbol{E}_0^\sharp)$ onto $\mathbb{E}_0^\sharp$.

By interpolation we obtain from (1.6), (2.3), and [4, Lemma 3.2] that $Q_0^\sharp$ restricts to $Q_\alpha^\sharp \in \mathcal{L}(\boldsymbol{E}_\alpha^\sharp)$ and that $Q_\alpha^\sharp$ is a projection onto $\mathbb{E}_\alpha^\sharp$ for $0 < \alpha < 1$. Consequently,

$$\boldsymbol{E}_\alpha^\sharp = \mathbb{E}_\alpha^\sharp \oplus F_\alpha^\sharp, \quad F_\alpha^\sharp := \ker(Q_\alpha^\sharp), \qquad 0 \le \alpha \le 1.$$

Hence, putting $F_{-\alpha} := (F_\alpha^\sharp)'$, we infer from (1.5) and (2.2) that

$$\boldsymbol{E}_{-\alpha} \doteq \mathbb{E}_{-\alpha} \oplus F_{-\alpha}, \qquad 0 < \alpha \le 1.$$

This implies the assertion. □

Theorem 2.2. *Suppose that* $-2 \le s \le 2$. *Then*

$$\boldsymbol{H}_q^s = \mathbb{H}_q^s \oplus \mathbb{G}_q^s, \tag{2.4}$$

and $\mathbb{G}_q^s$ *is the closure of* $\mathbb{G}_q$ *in* $\boldsymbol{H}_q^s$ *for* $-2 \le s < 0$. *Denote by* $\boldsymbol{P}_s := \boldsymbol{P}_{q,s}$ *the projection of* $\boldsymbol{H}_q^s$ *onto* $\mathbb{H}_q^s$ *parallel to* $\mathbb{G}_q^s$. *Then*

$$\boldsymbol{P}_{q,s} = (\boldsymbol{P}_{q',-s})' \tag{2.5}$$

and $\boldsymbol{P}_{-t}$ *is the unique continuous extension of the Helmholtz projection* P *over* $\boldsymbol{H}_q^{-t}$ *for* $0 < t \le 2$. *Furthermore,* $\mathbb{H}_{q'}^s$ *is the annihilator of* $\mathbb{G}_q^{-s}$ *(with respect to* $\langle \cdot, \cdot \rangle$*).*

Proof. (i) From [4, Lemma 3.3] we know that there exists a unique extension $P_{-\alpha} \in \mathcal{L}(\boldsymbol{E}_{-\alpha}, \mathbb{E}_{-\alpha})$ of P for $0 < \alpha \le 1$. Hence $P_{-\alpha} \in \mathcal{L}(\boldsymbol{E}_{-\alpha})$ by Lemma 2.1. Thus $(P_{-\alpha})^2$ is well-defined. Since $P^2 = P$ and $P_{-\alpha} \supset P$, the density of $\boldsymbol{E}_0$ in $\boldsymbol{E}_{-\alpha}$ implies $(P_{-\alpha})^2 = P_{-\alpha}$, that is, $P_{-\alpha}$ is a projection of $\boldsymbol{E}_{-\alpha}$ onto $\mathbb{E}_{-\alpha}$. Consequently,

$$\boldsymbol{E}_{-\alpha} = \mathbb{E}_{-\alpha} \oplus \ker(P_{-\alpha}), \qquad 0 \le \alpha \le 1. \tag{2.6}$$

From $\ker(P) = \mathbb{G}_q$ we see that $\mathbb{G}_q \subset \ker(P_{-\alpha})$. Suppose $u \in \ker(P_{-\alpha})$. Then $\boldsymbol{E}_0 \overset{d}{\hookrightarrow} \boldsymbol{E}_{-\alpha}$ implies the existence of a sequence (u_j) in $\boldsymbol{E}_0$ such that $u_j \to u$ in $\boldsymbol{E}_{-\alpha}$. Hence $Pu_j = P_{-\alpha} u_j \to P_{-\alpha} u = 0$ in $\boldsymbol{E}_{-\alpha}$. Thus $v_j := (1-P)u_j \to u$ in $\boldsymbol{E}_{-\alpha}$ and $v_j \in \ker(P) = \mathbb{G}_q$. This shows that

$$\ker(P_{-\alpha}) = \mathbb{G}_q^{-2\alpha}, \qquad 0 \le \alpha \le 1.$$

Thus, thanks to (2.6), the direct sum decomposition (2.4) holds for $-2 \le s < 0$.

(ii) From (i) it follows that

$$\boldsymbol{H}_{q'}^{-s} = \mathbb{H}_{q'}^{-s} \oplus \mathbb{G}_{q'}^{-s}, \qquad 0 < s \le 2.$$

Thus, by duality and thanks to (1.5) and (2.2),

$$\boldsymbol{H}_q^s = \mathbb{H}_q^s \oplus \mathbb{G}_q^s, \qquad 0 < s \le 2,$$

where

$$\mathbb{G}_q^s := \ker\big[(\boldsymbol{P}_{q',-s})'\big].$$

The Helmholtz decomposition (2.1) guarantees that $(\mathbb{G}_{q'})' = \mathbb{G}_q$ with respect to $\langle\cdot,\cdot\rangle_\pi$, where $\langle\cdot,\cdot\rangle_\sigma$ and $\langle\cdot,\cdot\rangle_\pi$ are the duality pairings induced by (2.1) so that $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\sigma + \langle\cdot,\cdot\rangle_\pi$. It also implies $\boldsymbol{P}_{q,0} = P_q = (P_{q'})' = (\boldsymbol{P}_{q',0})'$. From this (2.5) follows by density and reflexivity. The last assertion is now clear by the general facts on direct sums. □

Corollary 2.3. *If $-2 \le t < s \le 2$ then $\mathbb{G}_q^s$ is dense in $\mathbb{G}_q^t$.*

Henceforth, we simply write $\boldsymbol{P}$ for $\boldsymbol{P}_s$ if no confusion seems likely.

It is known by results of G. Grubb and V.A. Solonnikov (see [5, Example 3.14], [9], [7]) that the Helmholtz projection P has a pseudodifferential representation which implies that $P \in \mathcal{L}\big(H_q^s(\Omega)\big)$ for $-1+1/q < s < \infty$. From this and Theorem 2.2 it follows that $P = \boldsymbol{P}_s$ for $-1+1/q < s < 1/q$ since $H_q^s = \boldsymbol{H}_q^s$ in this range. However, $P \ne \boldsymbol{P}_s$ for $1/q \le s \le 2$ since $\boldsymbol{H}_q^s$ involves homogeneous Dirichlet boundary conditions.

3. The proof

Let assumptions (0.6) and (0.7) be satisfied. Put

$$B(u,v) := -\nabla\cdot(u\otimes v) = -(u\cdot\nabla)v, \qquad u,v\in\mathbb{H}_q^2.$$

An obvious modification of the proof of [4, Theorem 4.2] (replace in step (iii) the map b by B and the image spaces $\mathbb{H}_q^{t_j-1-n/p}$ by $\boldsymbol{H}_q^{t_j-1-n/p}$) guarantees that

$$B \in \mathcal{L}^2(\mathbb{H}_q^s, \boldsymbol{H}_q^{2s-1-n/q}), \tag{3.1}$$

where $\mathcal{L}^2(E,F)$ is the Banach space of all continuous bilinear maps from E into F.

Set

$$2\gamma := (\tau\wedge\sigma - s)\wedge\big(1-(n-1)/q\big). \tag{3.2}$$

Note that $0<\gamma<1/2$. Also observe that

$$L_q(\Gamma)\hookrightarrow W_q^{\tau-1/q}(\Gamma)\hookrightarrow W_q^{s+2\gamma-1/q}(\Gamma)$$

and, thanks to (0.2), (0.3), and (0.6),

$$H_q^{\sigma-2} = \boldsymbol{H}_q^{\sigma-2}\hookrightarrow \boldsymbol{H}_q^{s-2+2\gamma}.$$

Hence Corollary 1.2 and (0.6) and (0.7) imply

$$h := R(f,g)\in C(\mathbb{R}^+, \boldsymbol{H}_q^{s-2+2\gamma}). \tag{3.3}$$

Let J be an open subinterval of $\mathbb{R}^+$ containing 0 and set $B(u) := B(u,u)$. Suppose that

$$u \in C^1\big(J, (\mathbb{H}_q^{s-2}, \mathbb{H}_q^s)\big), \quad w \in C(J, \mathbb{G}_q^{s-2}) \tag{3.4}$$

satisfy

$$\dot{u} + \boldsymbol{A}_{(s-2)/2} u = w + B(u) + h \quad \text{in } \dot{J}, \qquad u(0) = v^0. \tag{3.5}$$

Then (u, w) is said to be a strict solution on J of the initial value problem (3.5). It is maximal if there does not exist such a solution being a proper extension of it.

Recall that $\boldsymbol{A}_{(s-2)/2} \in \mathcal{H}(\boldsymbol{H}_q^s, \boldsymbol{H}_q^{s-2})$. From this and (3.1)–(3.4) it follows that (3.5) is a well-defined equation in $\boldsymbol{H}_q^{s-2}$. Theorem 2.2 and the definition of the Stokes operator imply that (3.5) is equivalent to the system

$$\begin{aligned} \dot{u} + \mathbb{A}_{(s-2)/2} u &= \boldsymbol{P} B(u) + \boldsymbol{P} h \quad \text{in } \dot{J}, \qquad u(0) = v^0, \\ w &= (1 - \boldsymbol{P})(\boldsymbol{A}_{(s-2)/2} u - B(u) - h). \end{aligned} \tag{3.6}$$

Set $(E_0, E_1) := (\mathbb{H}_q^{s-2}, \mathbb{H}_q^s)$ and $A := \mathbb{A}_{(s-2)/2}$. Also put $E_\theta := [E_0, E_1]_\theta$ for $0 < \theta < 1$. Then we obtain from (2.3) that $E_\gamma \doteq \mathbb{H}_q^{s-2+2\gamma}$. Hence we infer from (0.3), Theorem 2.2, and (3.1)–(3.3) that

$$b := \boldsymbol{P} B \in \mathcal{L}^2(E_1, E_\gamma), \quad k := \boldsymbol{P} h \in C(\mathbb{R}^+, E_\gamma).$$

Now we can apply [4, Theorem 5.6] with $\alpha = 0$ to deduce that

$$\dot{u} + Au = b(u) + k \quad \text{in } \dot{J}, \qquad u(0) = v^0 \tag{3.7}$$

possesses a unique maximal strict solution v, and $J^+ := \operatorname{dom}(v)$ is an open interval of $\mathbb{R}^+$. Consequently,

$$\omega := (1 - \boldsymbol{P})\big(\boldsymbol{A}_{(s-2)/2} v - B(v) - h\big) \in C(J^+, \boldsymbol{H}_q^{s-2}). \tag{3.8}$$

By the equivalence of (3.5) and (3.6) this proves that (3.5) possesses a unique maximal strong solution, namely (v, ω), defined on J^+.

From [4, Remark 5.7(a) and Theorem 7.2] and Remark 1.3 it follows that u is a strict solution of (3.7) on J iff

$$u \in C(J, \mathbb{H}_q^s) \tag{3.9}$$

and

$$\begin{aligned} &\int_J \big\{ \big\langle (\partial_t + \Delta)\psi, u \big\rangle + \langle \nabla\psi, u \otimes u \rangle \big\}\, dt \\ &\qquad = \int_J \big\{ -\langle f, \psi \rangle + \langle g, \partial_\nu \psi \rangle \big\}\, dt - \big\langle v^0, \psi(0) \big\rangle \end{aligned} \tag{3.10}$$

for all

$$\psi \in \mathcal{W}_1^1\big(J, (\mathbb{H}_{q'}^{-s}, \mathbb{H}_{q'}^{2-s})\big) := L_1(J, \mathbb{H}_{q'}^{2-s}) \cap W_1^1(J, \mathbb{H}_{q'}^{-s})$$

vanishing near the right endpoint of J. Remark 1.5 and Theorem 2.2 imply that $\boldsymbol{P}\mathcal{D}_0(\overline{\Omega})$ is dense in $\mathbb{H}_{q'}^{2-s}$ and $(1-\boldsymbol{P})\mathcal{D}_0(\overline{\Omega})$ is dense in $\mathbb{G}_{q'}^{2-s}$. This implies that $\mathcal{D}\big(J, \boldsymbol{P}\mathcal{D}_0(\overline{\Omega})\big)$ is dense in $\mathcal{W}_1^1\big(J, (\mathbb{H}_{q'}^{-s}, \mathbb{H}_{q'}^{2-s})\big)$ and $\mathcal{D}\big(J, (1-\boldsymbol{P})\mathcal{D}_0(\overline{\Omega})\big)$ is dense in $L_1(J, \mathbb{G}_{q'}^{s-2})$. Since the left-hand side of (3.10) is continuous with respect to $u \in \mathcal{W}_1^1\big(J, (\mathbb{H}_q^{s-2}, \mathbb{H}_q^s)\big)$ it follows that u is a strict solution of (3.7) on J iff u satisfies (3.9) and (3.10) for all $\psi \in \mathcal{D}\big(J, \boldsymbol{P}\mathcal{D}_0(\overline{\Omega})\big)$. If

$$(u, w) \in C(J, \mathbb{H}_q^s \times \mathbb{G}_q^{s-2}) \tag{3.11}$$

then the second equality of (3.6) is equivalent to

$$\int_J \langle \chi, w\rangle\, dt = \int_J \big\langle \chi, \boldsymbol{A}_{(s-2)/2} u - B(u) - R(f,g)\big\rangle\, dt \tag{3.12}$$

for all $\chi \in \mathcal{D}(J, \mathbb{G}_{q'}^{2-s})$. This is the case iff (3.12) holds for all χ belonging to the space $\mathcal{D}\big(J, (1-\boldsymbol{P})\mathcal{D}_0(\overline{\Omega})\big)$. From Lemma 1.4 and (3.1) we infer, by a density argument, that (3.12) equals

$$\int_J \langle \chi, w\rangle\, dt = \int_J \big\{\langle -\Delta\chi, u\rangle + \langle \nabla\chi, u\otimes u\rangle - \langle f, \chi\rangle + \langle g, \partial_\nu \chi\rangle_\Gamma\big\}\, dt. \tag{3.13}$$

Thus, setting

$$\varphi := \psi + \chi \in \mathcal{D}\big(J, \boldsymbol{P}\mathcal{D}_0(\overline{\Omega})\big) + \mathcal{D}\big(J, (1-\boldsymbol{P})\mathcal{D}_0(\overline{\Omega})\big) = \mathcal{D}\big(J, \mathcal{D}_0(\overline{\Omega})\big),$$

we obtain from (3.5)–(3.13) that (u, w) is a strict solution of (3.5) on J iff it is a very weak H_q^s-solution of the Navier-Stokes equations on J. This proves part (i) of the Theorem and Remark (b).

The assertions of part (ii) follow by applying Remark 5.7(b) in [4] to (3.7).

Similarly, part (iii) is obtained by applying [4, Theorem 5.8] to (3.7) which is possible thanks to Remark 3.1 of [4].

The assertions of Remark (c) follow by the arguments of the proof of Theorem 6.1 and by Proposition 6.5 of [4], by employing once more the equivalence of (0.8) and (0.9) with system (3.6). □

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