Uniformly Regular and Singular Riemannian Manifolds

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Abstract A detailed study of uniformly regular Riemannian manifolds and manifolds with singular ends is carried out in this paper. Such classes of manifolds are of fundamental importance for a Sobolev space solution theory for parabolic evolution equations on non-compact Riemannian manifolds with and without boundary. Besides pointing out this connection in some detail we present large families of uniformly regular and singular manifolds which are admissible for this analysis.

1 Introduction

The principal object of our concern is an in-depth study of evolution equations on non-compact Riemannian manifolds. We are particularly interested in establishing an optimal local existence theory for quasilinear parabolic initial boundary value problems in a Sobolev space framework. For this we need, in the first instance, a good understanding of fractional order L_p-Sobolev spaces, including sharp embedding and trace theorems, etc. Although fractional order Sobolev spaces can be invariantly defined on any non-compact Riemannian manifold, it is not possible to establish embedding and trace theorems in this generality. For these to hold one has to impose restrictions on the underlying manifold near infinity.

In our paper [2] we have introduced the class of uniformly regular Riemannian manifolds and shown, in particular, that fractional order Sobolev spaces on such manifolds possess all the properties alluded to above. (Also see [1] for complements and extensions to anisotropic settings.) This class encompasses the well-studied case of complete Riemannian manifolds without boundary and bounded geometry. Of course, in the study of boundary value problems manifolds with boundary are indispensable. In our previous papers [1], [2], and [3] we have presented examples of manifolds with boundary which are uniformly regular. Yet proofs have not been

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included. The reason being that it needs quite a bit of argumentation to establish these claims. It is the purpose of this paper to close this gap and carry out a detailed study of uniformly regular Riemannian manifolds. Some of the main results and their ramifications are explained in the following.

Let (M,g) be a smooth m-dimensional Riemannian manifold with boundary (which may be empty). Unless explicitly stated otherwise, $m \in \mathbb{N}^{\times} := \mathbb{N} \backslash \{0\}$. An atlas $\mathfrak{K}$ for M is said to be uniformly regular if it consists of normalized charts[1], has finite multiplicity, all coordinate changes have uniformly bounded derivatives of all orders, and if it is shrinkable. By the latter we mean that there is a uniform shrinking of all chart domains such that the result is an atlas as well. The normalization of the local charts also means that they are well adapted to the boundary in a natural precise sense. The shrinkability assumption is the most restrictive one. For example, the open unit ball in $\mathbb{R}^m$, endowed with the Euclidean metric $|dx|^2$, does not possess a uniformly regular atlas.

Let M be equipped with a uniformly regular atlas $\mathfrak{K}$. The metric g is called uniformly regular if its local representation $\kappa_* g$ is equivalent to the Euclidean metric of $\mathbb{R}^m$ and has bounded derivatives of all orders, uniformly with respect to $\kappa \in \mathfrak{K}$. Then (M,g) is said to be *uniformly regular* if it possesses a uniformly regular atlas $\mathfrak{K}$ and g is uniformly regular. Loosely speaking, this means that M has an atlas whose coordinate patches are all 'of approximately the same size'. The concept of uniform regularity is independent of the particular choice of the atlas $\mathfrak{K}$ in a natural sense (made precise in (2.5)). Fortunately, in practice a specific atlas is rarely needed. It suffices to know that there exists one.

We denote by c constants ≥ 1 whose actual value may vary from occurrence to occurrence; but c is always independent of the free variables in a given formula, unless a dependence is explicitly indicated.

On the set of all nonnegative functions, defined on some nonempty set S, whose specific realization will be clear in any given situation, we introduce an equivalence relation $\sim$ by writing $f \sim g$ iff there exists c such that $f/c \leq g \leq cf$. Here inequalities between symmetric bilinear forms are understood as inequalities between the corresponding polar forms. By $\mathbf{1}$, more precisely $\mathbf{1}_S$, we denote the constant function $S \to \mathbb{R}, \ s \mapsto 1$.

Now we present some examples to illustrate the extent of the concept of uniform regularity.

Examples 1.1. (a) $\mathbb{R}^m = (\mathbb{R}^m, |dx|^2)$ and closed half-spaces thereof are uniformly regular.

(b) Every compact Riemannian manifold is uniformly regular.

(c) If (M,g) is a Riemannian submanifold with compact boundary of a uniformly regular Riemannian manifold, then it is uniformly regular also.

(d) Complete Riemannian manifolds without boundary and bounded geometry are uniformly regular.

(e) Products of uniformly regular manifolds are uniformly regular.

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[1] Precise definitions of all concepts used in this introduction without further explanation are found in the main body of this paper—in Section 2, in particular.

(f) If (M_1, g_1) and (M_2, g_2) are isometric, then (M_1, g_1) is uniformly regular iff (M_2, g_2) is so.

Proofs. For (a) see (3.3). Statements (b)–(d) are proved in Section 4. Assertion (e) is a particular instance of Theorem 3.1. Claim (f) follows from Lemma 3.4.   □

There are also Riemannian manifolds with singular ends which are uniformly regular. To explain this in more detail we need some preparation. We fix $d \geq m - 1$ and suppose that (B, g_B) is an $(m-1)$-dimensional Riemannian submanifold of $\mathbb{R}^d$. Then, given $\alpha \geq 0$,

$$\left\{ (t, t^{\alpha} y) \; ; \; t > 1, \; y \in B \right\} \subset \mathbb{R} \times \mathbb{R}^d = \mathbb{R}^{1+d} \tag{1.1}$$

is the *infinite model α-funnel over B*. We denote it by $F_{\alpha}(B)$. It is an infinite cylinder if $\alpha = 0$, and an infinite (blunt) cone if $\alpha = 1$. Note that $F_{\alpha}(B)$ is an m-dimensional submanifold of $\mathbb{R}^{1+d}$. If $\partial B \neq \emptyset$, then $\partial F_{\alpha}(B) = F_{\alpha}(\partial B)$.

In Fig. 1 there is depicted part of a (rotated) three-dimensional model funnel $F_{1/2}(B)$ with a compact base B having two connected components and three boundary components.
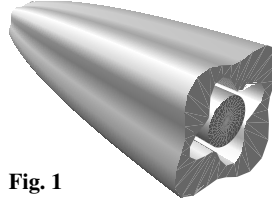
Fig. 1

For $0 \leq \alpha \leq 1$ we endow $F_{\alpha}(B)$ with the metric $g_{F_{\alpha}(B)}$ induced by the embedding $F_{\alpha}(B) \hookrightarrow \mathbb{R}^{1+d}$ so that $\left(F_{\alpha}(B), g_{F_{\alpha}(B)} \right)$ is a Riemannian submanifold of $(\mathbb{R}^{1+d}, |dx|^2)$. An open subset V of M is an *infinite α-funnel over B* if (V, g) is isometric to $\left(F_{\alpha}(B), g_{F_{\alpha}(B)} \right)$. It is a *tame end of* (M, g) if it is an infinite α-funnel with α belonging to $[0, 1]$ and B being compact.

Suppose $\{V_0, V_1, \ldots, V_k\}$ is a finite open covering of M with $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$ such that V_i is a tame end for $1 \leq i \leq k$ and V_0, $V_0 \cap V_1, \ldots, V_0 \cap V_k$ are relatively compact in M. Then (M, g) is said to have (finitely many) *tame ends*.

Theorem 1.2. *If (M, g) has tame ends, then it is uniformly regular.*

Proof. Section 8.   □

As mentioned above, our motivation for the study of uniformly regular Riemannian manifolds stems from the theory of parabolic equations. To explain their role in the present environment we consider a simple model problem. We set

$$\mathscr{A} u := -\operatorname{div}(a \cdot \operatorname{grad} u), \tag{1.2}$$

with a being a symmetric positive definite $(1,1)$-tensor field on (M, g) which is bounded and has bounded and continuous first order (covariant) derivatives. This is expressed by saying that $\mathscr{A}$ is a regular uniformly strongly elliptic differential operator. This low regularity assumption for a is of basic importance for treating quasilinear problems in which a depends on u.

We assume that $\partial_0 M$ is open and closed in ∂M and $\partial_1 M := \partial M \setminus \partial_0 M$. Then we put

$$\mathscr{B}_0 u := u \text{ on } \partial_0 M , \quad \mathscr{B}_1 u := (\nu \,|\, a \cdot \operatorname{grad} u) \text{ on } \partial_1 M ,$$

where these operators are understood in the sense of traces and ν is the inward pointing unit normal vector field on $\partial_1 M$. Thus $\mathscr{B} := (\mathscr{B}_0, \mathscr{B}_1)$ is the Dirichlet boundary operator on $\partial_0 M$ and the Neumann operator on $\partial_1 M$.

Suppose $0 < T < \infty$. We write $M_T := M \times [0, T]$ for the space time cylinder. Moreover, $\partial = \partial_t$ is the 'time derivative', $\Sigma_T := \partial M \times [0, T]$ the lateral boundary, and $M_0 = M \times \{0\}$ the 'initial surface' of M_T. Then we consider the problem

$$\partial u + \mathscr{A} u = f \text{ on } M_T , \quad \mathscr{B} u = 0 \text{ on } \Sigma_T , \quad u = u_0 \text{ on } M_0. \tag{1.3}$$

The last equation is to be understood as $\gamma_0 u = u_0$ with the 'initial trace' operator γ_0 .

Of course, $\partial_0 M$ or $\partial_1 M$ or both may be empty. In such a situation obvious interpretations and modifications are to be applied.

We are interested in an optimal L_p-theory for (1.3). To describe it we have to introduce Sobolev-Slobodeckii spaces. We always assume that $1 < p < \infty$. The Sobolev space $W_p^k(M)$ is then defined for $k \in \mathbb{N}$ to be the completion of $\mathscr{D}(M)$, the space of smooth functions with compact support, in $L_{1,\mathrm{loc}}(M)$ with respect to the norm

$$u \mapsto \left(\sum_{j=0}^{k} \big\| \, |\nabla^j u|_{g_0^j} \big\|_{L_p(M)}^p \right)^{1/p} . \tag{1.4}$$

Here $\nabla = \nabla_g$ is the Levi-Civita covariant derivative and $|\cdot|_{g_0^j}$ the $(0, j)$-tensor norm naturally induced by g. Thus $W_p^0(M) = L_p(M)$. If $s \in \mathbb{R}^+ \setminus \mathbb{N}$, then the Slobodeckii space $W_p^s(M)$ is defined by real interpolation:

$$W_p^s(M) := \big(W_p^k(M), W_p^{k+1}(M) \big)_{s-k,p} , \qquad k < s < k+1 , \quad k \in \mathbb{N} .$$

Although these definitions are meaningful on any Riemannian manifold, they are not too useful in such a general setting since they may lack basic Sobolev type embedding properties, for example. The situation is different if we restrict ourselves to uniformly regular Riemannian manifolds. The following theorem is a consequence of the results of our paper [2] to which we direct the reader for details, proofs, and many more facts.

Theorem 1.3. *Let (M, g) be uniformly regular. Then the Sobolev-Slobodeckii spaces $W_p^s(M)$, $s \geq 0$, possess the same embedding, interpolation, and trace properties as in the classical Euclidean case. They can be characterized by means of local coordinates.*

We denote by $W_{p,\mathscr{B}}^s(M)$ the closed linear subspace of all $u \in W_p^s(M)$ satisfying $\mathscr{B} u = 0$ whenever s is such that this condition is well-defined (cf. [3, (1.4)]).

We set

$$A := \mathscr{A} \,|\, W_{p,\mathscr{B}}^2(M) ,$$

considered as an unbounded linear operator in $L_p(M)$ with domain $W^2_{p,\mathscr{B}}(M)$. Then (1.3) can be expressed as an initial value problem for the evolution equation

$$\dot{u} + Au = f \text{ on } [0,T] , \quad u(0) = u_0$$

in $L_p(M)$.

Now we are ready to formulate the basic well-posedness result in the present model setting.

Theorem 1.4. *Let (M,g) be uniformly regular and $p \notin \{3/2,\ 3\}$. Suppose that $\mathscr{A}$ is regularly uniformly strongly elliptic. Then (1.3) has for each*

$$(f,u_0) \in L_p\big([0,T],L_p(M)\big) \times W^{2-2/p}_{p,\mathscr{B}}(M)$$

a unique solution

$$u \in L_p\big([0,T],W^2_{p,\mathscr{B}}(M)\big) \cap W^1_p\big([0,T],L_p(M)\big).$$

The map $(f,u_0) \mapsto u$ is linear and continuous.
Equivalently: $-A$ generates an analytic semigroup on $L_p(M)$ and has the property of maximal regularity.

For this theorem we refer to [3] where non-homogeneous boundary conditions and lower order terms are treated as well and further references are given. Analogous theorems apply to higher order problems and parabolic equations operating on sections of uniformly regular vector bundles.

On the surface, Theorem 1.4 looks exactly the same as the very classical existence and uniqueness theorem for second order parabolic equations on open subsets of $\mathbb{R}^m$ with smooth compact boundary (e.g., O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'ceva [23, Chapter IV] and R. Denk, M. Hieber, and J. Prüss [9]). However, it is, in fact, a rather deep-rooted vast generalization thereof since it applies to any uniformly regular Riemannian manifold.

In this connection we have to mention the work of G. Grubb [15] who established a general L_p theory for parabolic pseudo-differential boundary value problems (also see Section IV.4.1 in [16]). It applies to a class of noncompact manifolds, called 'admissible', introduced in G. Grubb and N.J. Kokholm [17]. It is a subclass of the above manifolds with tame ends, namely a family of manifolds with conical ends. Earlier investigations of pseudo-differential operators on manifolds with conical ends are due to E. Schrohe [27] who employs weighted Sobolev spaces.

Recently, a maximal regularity theory for parabolic differential equations on Riemannian manifolds without boundary and cylindrical ends has been presented by Th. Krainer [22]. This author uses a compactification technique to 'reduce' the problem to a compact Riemannian manifold $(\tilde{M},\tilde{g})$, where $\tilde{g}$ is the cusp metric $dt^2/t^4 + g_Y$ in a collar neighborhood of the boundary Y of $\tilde{M}$. Then the theory of cusp pseudo-differential operators is applied in conjunction with the general $\mathscr{R}$-boundedness theory of maximal regularity for parabolic evolution equations. The final result is then formulated in the Sobolev space setting for $(\tilde{M},\tilde{g})$ which involves

rather complicated weighted norms. In contrast, our result Theorem 1.4 uses the Sobolev space setting of (M,g) only. Due to Theorem 1.2, it applies to manifolds with cylindrical ends, in particular.

There is a tremendous amount of literature on heat equations on complete Riemannian manifolds without boundary and bounded geometry. Most of it concerns heat kernel estimates and spectral theory (see, for example, E.B. Davies [8] or A. Grigor'yan [13] and the references therein). By imposing further structural conditions, as the assumption of non-negative Ricci curvature, for instance, heat kernel estimates lead to maximal regularity results for the Laplace-Beltrami operator (e.g. M. Hieber and J. Prüss [18], A.L. Mazzucato and V. Nistor [24]. Also see A. Grigor'yan and L. Saloff-Coste [14] and L. Saloff-Coste [26]). Due to Example 1.1(d) our Theorem 1.4 applies in this setting without any additional restriction on the geometry of (M,g).

Let now (M,g) be a Riemannian manifold which is not uniformly regular. It is said to be *singular of type* ρ if $\rho \in C^\infty\big(M,(0,\infty)\big)$ and $(M,g/\rho^2)$ is uniformly regular. Any such ρ is a *singularity function* for (M,g). We assume that ρ is bounded from above. Then $\inf \rho = 0$ and (M,g) is said to be *singular near* $\rho = 0$. In order for $\rho \in C^\infty\big(M,(0,\infty)\big)$ to qualify as a singularity function it has to satisfy structural conditions naturally associated with (M,g) (see (2.7)). Below we describe a large class of singularity functions which are closely related to the geometric structure near the 'singular ends' of (M,g), that is, the behavior of (M,g) 'near infinity'.

Suppose (M,g) is singular of type ρ. We set $\hat{g} := g/\rho^2$ and $(\hat{M},\hat{g}) := (M,g/\rho^2)$. Then we can apply the preceding results to the uniformly regular Riemannian manifold $(\hat{M},\hat{g})$. Since $\hat{g}$ is conformally equivalent to g (and ρ satisfies appropriate structural conditions) we can express the Sobolev-Slobodeckii spaces $W_p^s(\hat{M})$, which are constructed by means of $\nabla_{\hat{g}}$, in terms of weighted Sobolev-Slobodeckii spaces on M. More precisely, we define $W_p^{k,\lambda}(M;\rho)$ for $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ by replacing (1.4) in the definition of $W_p^k(M)$ by

$$u \mapsto \Big(\sum_{j=0}^{k} \big\| \rho^{\lambda+j} |\nabla^j u|_{g_0^j} \big\|_{L_p(M)}^p \Big)^{1/p} .$$

Furthermore,

$$W_p^{s,\lambda}(M;\rho) := \big(W_p^{k,\lambda}(M;\rho), W_p^{k+1,\lambda}(M;\rho)\big)_{s-k,p} , \qquad k < s < k+1 , \quad k \in \mathbb{N} ,$$

and $L_p^\lambda(M;\rho) := W_p^{0,\lambda}(M;\rho)$. Then, see [4],

$$W_p^s(\hat{M}) \doteq W_p^{s,-m/p}(M;\rho) , \qquad s \geq 0 ,$$

where $\doteq$ means: equal except for equivalent norms. In [4] it is also shown that

$$W_p^s(\hat{M}) \to W_p^{s,\lambda}(M;\rho) , \quad u \mapsto \rho^{-\lambda+m/p} u$$

is an isomorphism. With its help we can transfer all properties enjoyed by the Sobolev-Slobodeckii spaces $W_p^s(\hat{M})$ to the weighted spaces $W_p^{s,\lambda}(M;\rho)$ (direct proofs, not using this isomorphism, are given in [2]).

There are also simple relations between the differential operators div and grad on (M,g) and $\mathrm{div}_{\hat{g}}$ and $\mathrm{grad}_{\hat{g}}$ on $(\hat{M},\hat{g})$, respectively. In fact, setting $\hat{a} := \rho^{-2}a$ we find (cf. [3, (5.19)])

$$\mathrm{div}(a\,\mathrm{grad}\,u) = \mathrm{div}_{\hat{g}}(\hat{a}\cdot\mathrm{grad}_{\hat{g}}\,u) + (u\hat{a}\cdot\rho^{-1}\,\mathrm{grad}_{\hat{g}}\rho\,|\,\mathrm{grad}_{\hat{g}}\,u)_{\hat{g}}\,.$$

Note that Theorem 1.4 applies to the operator

$$\hat{\mathscr{A}}u := -\,\mathrm{div}_{\hat{g}}(\hat{a}\cdot\mathrm{grad}_{\hat{g}}\,u)$$

provided it is regularly uniformly strongly elliptic on $(\hat{M},\hat{g})$. This is equivalent to the assumption that (1.2) be *regularly uniformly strongly ρ-elliptic*. By this we mean that the following conditions are satisfied:

(i) $\qquad \big(a(q)\cdot X\,|\,X\big)_{g(q)} \sim \rho^2(q)\,|X|_{g(q)}^2,\ \ X \in T_qM,\ \ q \in M\,.$

(ii) $\qquad |\nabla a|_{g_1^2} \le c\rho\,.$

An elaboration of these facts leads to the following optimal well-posedness result for degenerate parabolic equations on singular manifolds. It is a special case of Theorem 5.2 of [3].

Theorem 1.5. *Let (M,g) be singular of type ρ and $p \notin \{3/2,\,3\}$. Suppose $\mathscr{A}$ is regularly uniformly strongly ρ-elliptic and $\lambda \in \mathbb{R}$. Then problem (1.3) has for each*

$$(f,u_0) \in L_p\big([0,T],L_p^\lambda(M;\rho)\big) \times W_{p,\mathscr{B}}^{2-2/p,\lambda}(M;\rho)$$

a unique solution

$$u \in L_p\big([0,T],W_{p,\mathscr{B}}^{2,\lambda}(M;\rho)\big) \cap W_p^1\big([0,T],L_p^\lambda(M;\rho)\big)\,.$$

The map $(f,u_0) \mapsto u$ is linear and continuous.

Equivalently: let

$$A^\lambda := \mathscr{A}\,|\,W_{p,\mathscr{B}}^{2,\lambda}(M;\rho)\,.$$

Then $-A^\lambda$ generates a strongly continuous analytic semigroup on $L_p^\lambda(M;\rho)$ and has the property of maximal regularity.

In order to render this theorem useful we have to provide sufficiently large and interesting classes of singular manifolds. This is the aim of the following considerations.

Let (B,g_B) be as in definition (1.1). If we choose there $\alpha < 0$, then we call the resulting Riemannian submanifold of $\mathbb{R}^{1+d}$ *infinite model α-cusp over B* and denote it by $C_{\infty,\alpha}(B)$ and its metric by $g_{C_{\infty,\alpha}(B)}$. Similarly as for funnels, an open subset V of M is an *infinite α-cusp over B* of (M,g) if (V,g) is isometric to an infinite model α-cusp $\big(C_{\infty,\alpha}(B),g_{C_{\infty,\alpha}(B)}\big)$. It is *smooth* if B is compact.

We consider the following conditions:

- (i) $(\mathcal{M},g)$ is an m-dimensional Riemannian manifold .
- (ii) $\mathbf{V}$ is a finite set of pairwise disjoint infinite smooth α-cusps V of $(\mathcal{M},g)$.
- (iii) V_0 is an open subset of $\mathcal{M}$ such that $\{V_0\}\cup\mathbf{V}$ is a covering of $\mathcal{M}$ and $(\mathcal{M},g)$ is uniformly regular on[2] V_0 .
- (iv) $\boldsymbol{\Gamma}$ is a finite set of pairwise nonintersecting compact connected Riemannian submanifolds Γ of $\mathcal{M}$ without boundary and codimension at least 1 such that $\Gamma\subset\partial\mathcal{M}$ if $\Gamma\cap\partial\mathcal{M}\neq\emptyset$ and $\Gamma\cap V=\emptyset$ for $\Gamma\in\boldsymbol{\Gamma}$ and $V\in\mathbf{V}$.
- (v) $\beta_\Gamma\geq 1$ for $\Gamma\in\boldsymbol{\Gamma}$.

$$(1.5)$$

We set

$$\boldsymbol{\alpha}:=\{\,\alpha_V\;;\;V\in\mathbf{V}\,\}\,,\quad \boldsymbol{\beta}:=\{\beta_\Gamma\;;\;\Gamma\in\boldsymbol{\Gamma}\,\}\,,\quad \mathcal{S}:=\bigcup\nolimits_{\Gamma\in\boldsymbol{\Gamma}}\Gamma\,,$$

and

$$(M,g):=\bigl(\mathcal{M}\backslash\mathcal{S},\; g\,|\,(\mathcal{M}\backslash\mathcal{S})\bigr)\,.$$

Then $\boldsymbol{\alpha}$, resp. $\boldsymbol{\beta}$, is the *cuspidal weight* (vector) for $\mathbf{V}$, resp. $\boldsymbol{\Gamma}$, and $\mathcal{S}$ the (*compact*) *singularity set* of M. Furthermore, (M,g) is said to be a *Riemannian manifold with smooth cuspidal singularities of type* $[\mathbf{V},\boldsymbol{\alpha},\boldsymbol{\Gamma},\boldsymbol{\beta}]$.

If $\mathbf{V}=\emptyset$ and $\boldsymbol{\Gamma}=\emptyset$, then $\mathcal{M}=V_0$ and $(\mathcal{M},g)=(M,g)$ is uniformly regular. Thus we assume henceforth that $\mathbf{V}\cup\boldsymbol{\Gamma}\neq\emptyset$. If $\mathbf{V}=\emptyset$, then $V_0=\mathcal{M}$ and $(\mathcal{M},g)$ is uniformly regular. By the preceding results this is the case, in particular, if $\mathcal{M}$ is compact or $(\mathcal{M},g)$ has tame ends. However, (M,g) is not uniformly regular.

By its definition, (M,g) is a Riemannian submanifold of the ambient manifold $(\mathcal{M},g)$. In turn, the latter is obtained from (M,g) by setting $\mathcal{M}:=M\cup\mathcal{S}$ and defining $g_{\mathcal{M}}$ by smooth extension of g. The crucial point of this procedure is that $(\mathcal{M},g_{\mathcal{M}})$ is a Riemannian manifold as well. To avoid technical subtleties we prefer to take $(\mathcal{M},g)$ as initial object. Due to the intimate connection between $(\mathcal{M},g)$ and (M,g) there is often no need to mention $(\mathcal{M},g)$ explicitly.

For $V\in\mathbf{V}$ we fix $q=q_V\in\bar{V}\backslash V$ and set

$$\delta_V=\delta_{V,q}:=1+\mathrm{dist}_{\mathcal{M}}(\cdot,q)\colon V\to[1,\infty) \qquad\qquad (1.6)$$

where $\mathrm{dist}_{\mathcal{M}}$ is the Riemannian distance in $(\mathcal{M},g)$. Note that $\sup\delta_V=\infty$ and $\mathrm{dist}_{\mathcal{M}}(\cdot,q)=\mathrm{dist}_M(\cdot,q)$ on V.

For $\Gamma\in\boldsymbol{\Gamma}$ there exists an open neighborhood $\mathcal{U}_\Gamma$ of Γ in $\mathcal{M}$ with $\mathcal{U}_\Gamma\cap\tilde{\Gamma}=\emptyset$ for $\tilde{\Gamma}\in\boldsymbol{\Gamma}$ satisfying $\tilde{\Gamma}\neq\Gamma$, and such that $\mathrm{dist}_{\mathcal{M}}(\cdot,\Gamma)$ is a well-defined smooth function. Then $U_\Gamma:=\mathcal{U}_\Gamma\backslash\Gamma$ is open in M and the restriction δ_Γ of $\mathrm{dist}_{\mathcal{M}}(\cdot,\Gamma)$ to U_Γ is smooth and everywhere positive.

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[2] Cf. the localized definitions in Section 2.

The following theorem is the main result of this paper as far as singular manifolds are concerned. Its proof is given in Section 8. Here and in similar situations obvious interpretations have to be used if either $\mathbf{V}$ or $\boldsymbol{\Gamma}$ is empty.

Theorem 1.6. *Let (M,g) be a Riemannian manifold with smooth cuspidal singularities of type $[\mathbf{V},\boldsymbol{\alpha},\boldsymbol{\Gamma},\boldsymbol{\beta}]$. Fix $\rho \in C^\infty\big(M,(0,1]\big)$ such that $\rho \sim \mathbf{1}$ on V_0, $\rho \sim \delta_V^{\alpha_V}$ on $V \in \mathbf{V}$, and $\rho \sim \delta_\Gamma^{\beta_\Gamma}$ near $\Gamma \in \boldsymbol{\Gamma}$. Then (M,g) is singular of type ρ.*

Corollary 1.7. *Theorem 1.5 applies whenever (M,g) is a Riemannian manifold with cuspidal singularities.*

It follows from the considerations in the main body of this paper that the special choice of ρ is of no importance. In fact: if ρ is replaced by $\tilde{\rho}$ with $\tilde{\rho} \sim \rho$, then $(M,\, g/\rho^2)$ and $(M,\, g/\tilde{\rho}^2)$ are equivalent in the sense defined in Section 2. In particular, the Sobolev-Slobodeckii spaces $W_p^s(M;\rho)$ and $W_p^s(M;\tilde{\rho})$ differ only by equivalent norms. Thus merely the behavior of ρ 'near infinity along V', for $V \in \mathbf{V}$, and near $\Gamma \in \boldsymbol{\Gamma}$, where ρ approaches zero, does matter. Notably, this shows that the choice of $q_V \in \bar{V}\setminus V$, as well as the special form of ρ on compact subsets of M, is irrelevant.

It remains to explain why the naming 'manifold with smooth cuspidal singularities' has been chosen. This is clear if $\boldsymbol{\Gamma} = \emptyset$, but needs elucidation otherwise. The following considerations contribute to it. But first we introduce some notation.

For $d \in \mathbb{N}^\times$ we denote by $\mathbb{B}^d$ the open unit ball in $\mathbb{R}^d$, by $\mathbb{S}^{d-1}$ its boundary, the unit sphere, and by $\mathbb{H}^d := \mathbb{R}^+ \times \mathbb{R}^{d-1}$ the closed right half-space, where $\mathbb{R}^0 := \{0\}$. Then $\mathbb{B}_+^d := \mathbb{B}^d \cap \mathbb{H}^d$ and $\mathbb{S}_+^{d-1} := \mathbb{S}^{d-1} \cap \mathbb{H}^d$ are the right half-ball and half-sphere, respectively. Note that $\partial \mathbb{B}_+^d = \{0\} \times \mathbb{B}^{d-1} \cong \mathbb{B}^{d-1}$ and $\partial \mathbb{S}_+^{d-1} = \{0\} \times \mathbb{S}^{d-2}$ if $d \geq 2$ and $\partial \mathbb{S}^0 = \emptyset$. Lastly, $\dot{\mathbb{B}} := \mathbb{B}\setminus\{0\}$ for $\mathbb{B} \in \{\mathbb{B}^d, \mathbb{B}_+^d\}$.

Suppose $1 \leq \ell \leq m$ and $\mathbb{S} \in \{\mathbb{S}^{\ell-1}, \mathbb{S}_+^{\ell-1}\}$. Given $\alpha \geq 1$,

$$C_\alpha(\mathbb{S}) = C_{\alpha,\ell}(\mathbb{S}) := \big\{ (t,t^\alpha y) \,;\, 0 < t < 1,\, y \in \mathbb{S} \big\} \subset \mathbb{R}^{1+\ell} \qquad (1.7)$$

is an ℓ-dimensional submanifold of $\mathbb{R}^{1+\ell}$ and

$$\varphi_\alpha \colon C_\alpha(\mathbb{S}) \to (0,1) \times \mathbb{S}, \quad (t,t^\alpha y) \mapsto (t,y)$$

is the 'canonical stretching diffeomorphism'. Observe that $\partial C_\alpha(\mathbb{S}) = \emptyset$ if $\mathbb{S} = \mathbb{S}^{\ell-1}$ or $\ell = 1$, and $\partial C_\alpha(\mathbb{S}_+^{\ell-1}) = C_{\alpha,\ell-1}(\mathbb{S}^{\ell-2})$ otherwise.

$C_\alpha(\mathbb{S})$ is a (blunt) *model α-cusp*, respectively *cone* if $\alpha = 1$, which is *spherical* if $\mathbb{S} = \mathbb{S}^{\ell-1}$ and *semi-spherical* otherwise. In Fig. 2 there is depicted a (rotated) semi-circular model 2-cusp in $\mathbb{R}^3$. Its boundary consists of two disjoint one-dimensional generators.

We endow $C_\alpha = C_\alpha(\mathbb{S})$ with the Riemannian metric g_{C_α} induced by the natural embedding $C_\alpha \hookrightarrow \mathbb{R}^{1+\ell}$. Then g_{C_α} is equivalent to the pull-back by φ_α of the metric $dt^2 + t^{2\alpha} g_{\mathbb{S}}$ of $(0,1) \times \mathbb{S}$, where $g_{\mathbb{S}}$ is the standard metric induced by $\mathbb{S} \hookrightarrow \mathbb{R}^\ell$.
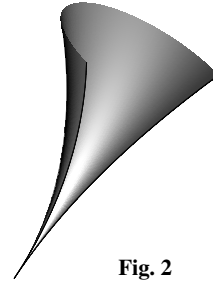

Fig. 2

Assume (Γ, g_Γ) is an $(m-\ell)$-dimensional compact connected Riemannian manifold without boundary. Then $W_\alpha := C_\alpha \times \Gamma$, whose metric is $g_{W_\alpha} := g_{C_\alpha} + g_\Gamma$, is a *model (α, Γ)-wedge* which is also called *spherical* if C_α is so, and *semi-spherical* otherwise. If $m = \ell$, then Γ is a one-point space, W_α is naturally identified with C_α, and all references to and occurrences of Γ are to be disregarded. Thus every cusp is a wedge also.

Let U be open in M. Then (U, g), more loosely: U, is a *spherical*, resp. *semi-spherical*, *cuspidal end of type* (α, Γ) *of* (M, g) if there exists an isometry Φ_α from (U, g) onto a spherical, resp. semi-spherical, model (α, Γ)-wedge (W_α, g_{W_α}). In this case U is *represented by* $[\Phi_\alpha, W_\alpha, g_{W_\alpha}]$ or, simply, by Φ_α.

Now we return to the setting of Theorem 1.6 and consider a particular simple constellation. Namely, we assume that M is obtained from a three-dimensional ellipsoid $\mathscr{M}$ in $\mathbb{R}^3$ by removing an equator Γ. Its metric g is induced by the natural embedding $\mathscr{M} \hookrightarrow \mathbb{R}^3$. In this case $\mathbf{V} = \emptyset$ and $\boldsymbol{\Gamma} = \{\Gamma\}$.
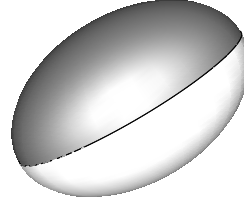

Fig. 3

On one component, $\partial_0 M$, of the boundary of M we put Dirichlet conditions (e.g. on the dark side of Fig. 3) and Neumann conditions on the other one, $\partial_1 M$. Note that $\partial_0 M$ and $\partial_1 M$ meet in $\mathscr{M}$ along Γ, but 'do not see each other' in M. In other words, $\partial_0 M$ and $\partial_1 M$ are both open and closed in ∂M.

We consider a tubular neighborhood U of Γ in M and represent it as $\mathring{\mathbb{B}}_+^2 \times \Gamma$ by means of the tubular diffeomorphism $\tau \colon U \to \mathring{\mathbb{B}}_+^2 \times \Gamma$ (see Section 8 for details). A part of it is depicted in Fig. 4 in which the curve along the flat side represents Γ $(= \{0\} \times \Gamma)$, which does not belong to $\tau(U)$, however.

Fig. 4

Let
$$\pi \colon \mathring{\mathbb{B}}_+^2 \to (0,1) \times \mathbb{S}_+^1, \quad x \mapsto (|x|,\, x/|x|)$$

be the polar coordinate diffeomorphism. Then, given $\alpha \geq 1$, the composition

$$U \xrightarrow{\ \tau\ } \mathring{\mathbb{B}}_+^2 \times \Gamma \xrightarrow{\ \pi \times \mathrm{id}_\Gamma\ } (0,1) \times \mathbb{S}_+^1 \times \Gamma \xrightarrow{\ \varphi_\alpha^{-1} \times \mathrm{id}_\Gamma\ } C_\alpha(\mathbb{S}_+^1) \times \Gamma \qquad (1.8)$$

defines a diffeomorphism Φ_α from U onto the semi-circular model (α, Γ)-wedge $W_\alpha = C_\alpha(\mathbb{S}_+^1) \times \Gamma$. We equip $C_\alpha(\mathbb{S}_+^1)$ with the equivalent metric $\varphi_\alpha^*(\mathrm{d}t^2 + t^{2\alpha} g_{\mathbb{S}_+^1})$ and give U the pull-back metric $\Phi_\alpha^* g_{W_\alpha}$.

Let $\mathfrak{g}$ be a Riemannian metric for M such that $\mathfrak{g} = \Phi_\alpha^* g_{W_\alpha}$ on U. Then U is a semi-circular (α, Γ)-end of $(M, \mathfrak{g})$. In Section 8 it is shown that $\Phi_\alpha^* g_{W_\alpha} \sim g/\delta_\Gamma^{2\alpha}$ on U. Thus, if we fix any $\rho \in C^\infty\big(M, (0,1]\big)$ with $\rho \sim \delta_\Gamma^{2\alpha}$ on U and $\rho \sim \mathbf{1}$ on $M \backslash U$, it follows from Theorem 1.6 that $(M,\, g/\rho^2)$ is uniformly regular.

These considerations and Corollary 1.7 show that the Zaremba problem on $\mathcal{M}$ for (1.3), in which Dirichlet boundary conditions are assigned on one half of the boundary of the ellipsoid $\mathcal{M}$ and Neumann conditions on the other half, is well-posed provided $\mathcal{A}$ is regularly uniformly strongly ρ-elliptic where $\rho \sim \delta_\Gamma^{2\alpha}$ near Γ

Fig. 5

and $\rho \sim \mathbf{1}$ away from Γ. They also show that (M,g) can be visualized as a manifold with a cuspidal end of type (α,Γ). This is illustrated by Fig. 5 for the case where $\alpha = 1$.
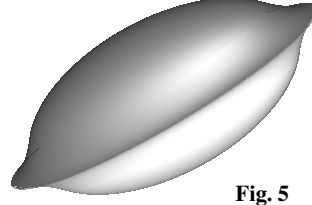
The arguments used in this simple case extend to the general setting. This leads in Section 8 to the proof of the following proposition which clarifies our choice of the name for (1.5).

Proposition 1.8. *Let (M,g) have smooth cuspidal singularities of type $[\mathbf{V},\boldsymbol{\alpha},\boldsymbol{\Gamma},\boldsymbol{\beta}]$ and let $\beta = \beta_\Gamma$ be the cuspidal weight for $\Gamma \in \boldsymbol{\Gamma}$. Then there exists an open neighborhood $\mathcal{U}$ of Γ in $\mathcal{M}$ such that $U := \mathcal{U} \setminus \Gamma$ is a (β,Γ)-cuspidal end of (M,g).*

The preceding treatment indicates that there are two possible ways of looking at these problems. In the first one we put forward the differential equation setting. Then the singular manifold has an inferior position and it is only the singularity function ρ which comes into play. In the second approach the geometric appearance of the singular manifold is relevant. In this case we start off with a singular manifold (M,g) which may not be obtained from a uniformly regular ambient manifold by cutting out lower-dimensional submanifolds. Instead, (M,g) can have more general singular ends U; namely such that U is isometric to a model (α,Γ)-wedge over (B,g_B), where (B,g_B) is as in (1.1), and B replaces $\mathbb{S}$ in definition (1.7). Theorem 8.1 and Proposition 8.2(i) guarantee then the existence of singularity functions ρ, modeling again the geometric structure of (M,g), such that (M,g) is singular of type ρ. Consequently, we can obtain well-posedness theorems for degenerate parabolic equations on singular manifolds by applying Theorem 1.5.

Up to now we have considered the case in which we introduce a conformal metric g/ρ^2 on M in order to render it uniformly regular. This means that we restrict ourselves to differential operators with isotropic degenerations. However, other choices are possible also. For example, in the setting (1.8) we can endow $(0,1) \times \mathbb{S}_+^1 \times \Gamma$ with the metric $t^{-2\alpha}dt^2 + g_{\mathbb{S}_+^1} + g_\Gamma$ instead of $dt^2 + t^{2\alpha}g_{\mathbb{S}_+^1} + g_\Gamma$ as above. This is a consequence of the next theorem which is also proved in Section 8. For simplicity, we consider the case where M has only one singular end. The extension to the general case is straightforward. Moreover,

$$(0,1) \times \mathbb{S} \times \Gamma \tag{1.9}$$

is the canonical representation of a tubular neighborhood U of Γ in (M,g) in the sense made precise later in this paper.

Theorem 1.9. *Let (1.5) be satisfied with $\mathbf{V} = \emptyset$ and $\boldsymbol{\Gamma} = \{\Gamma\}$, and fix $\alpha > 0$. Let U be a tubular neighborhood of Γ in (M,g). Suppose $\mathfrak{g}$ is a metric for M which*

coincides on M\U with g and equals near Γ

$$t^{-2}\mathrm{d}t^2 + t^{-2\alpha}(g_{\mathbb{S}} + g_{\Gamma}) \quad \textit{if } 0 < \alpha \le 1 \,,$$

respectively

$$t^{-2(\alpha+1)}\mathrm{d}t^2 + g_{\mathbb{S}} + g_{\Gamma} \quad \textit{if } \alpha > 1 \,, \tag{1.10}$$

in the canonical representation (1.9) of U. Then $(M, \mathfrak{g})$ *is uniformly regular.*

Recall that $g_{\mathbb{S}}$, resp. g_{Γ}, is absent if $\ell = m$, resp. $\ell = 1$.

By applying Theorem 1.4 to the setting of Theorem 1.9 we obtain well-posedness results for parabolic problems with anisotropic degeneration. To indicate the inherent potential of such applications we consider the particularly interesting setting in which Γ is a compact connected component of the boundary of $\mathscr{M}$. We also suppose, for simplicity, that $\mathscr{A}$ is the negative Laplace-Beltrami operator $-\Delta$ of (M, g) and assume $\alpha > 1$. Then it follows from (1.10) that the (interior) flux vector field satisfies in a collar neighborhood of Γ

$$\mathrm{grad} \sim (\delta^{2\alpha}\partial_{\nu}, \mathrm{grad}_{\Gamma}) \,.$$

Hence it degenerates in the normal direction only and there is no degeneration at all in tangential directions. This is in contrast to the isotropic case in which Corollary 1.7 applies and, in the present setting, gives

$$\mathrm{grad} \sim \delta^{2\alpha}(\partial_{\nu}, \mathrm{grad}_{\Gamma})$$

near Γ.

There has been done an enormous amount of research on *elliptic* equations on singular manifolds. All of it is related, in one way or another, to the seminal paper by V.A. Kondrat′ev [20]. It is virtually impossible to review this work here and to do justice to the many authors who contributed. It may suffice to mention the three most active groups and some of their principal exponents. First, there is the Russian school which builds directly on Kondrat′ev's work and is also strongly application-oriented (see the numerous papers and books by V.G. Maz′ya, S.A. Nazarov, and their coauthors, for example). Second, the group gathering around B.-W. Schulze has constructed an elaborate calculus of pseudo-differential algebras on manifolds with singularities, mainly of conical and cuspidal type. For a lucid presentation of some of its aspects in the simplest setting of manifolds with cuspidal points and wedges we refer to the book of V.E. Nazaikinskii, A.Yu. Savin, B.-W.-Schulze, and B.Yu. Sternin [25]. Third, another general approach to pseudo-differential operators on manifolds with singularities has been developed by R. Melrose and his coworkers. A brief explanation, stressing the differences of the techniques used by the latter two groups, is found in the section 'Bibliographical Remarks' of [25]. Henceforth, we call these methods 'classical' for easy reference.

To explain to which extent our point of view differs from the classical approach we consider the simplest case, namely, a manifold with one conical singularity. By means of the stretching diffeomorphism the model cone $C_1(\mathbb{S})$ is represented by the

'stretched manifold' $(0,1) \times \mathbb{S}$ whose metric is

$$g = \mathrm{d}t^2 + t^2 g_{\mathbb{S}} = t^2\big((\mathrm{d}t/t)^2 + g_{\mathbb{S}}\big) \ .$$

Thus the corresponding Laplace-Beltrami operator is given by $t^{-2}\big((t\partial_t)^2 + \Delta_{\mathbb{S}}\big)$. More generally, in the classical theories there are considered differential operators which, on the stretched manifold, are (in the second order case) of the form $t^{-2}L$, where L is a uniformly elliptic operator generated by the vector fields $t\partial_t, \partial_{\theta^1}, \ldots, \partial_{\theta^{m-1}}$ with $(\theta^1, \ldots, \theta^{m-1})$ being local coordinates for $\mathbb{S}$.

Instead, our approach is based on the metric

$$\hat{g} = g/t^2 = (\mathrm{d}t/t)^2 + g_{\mathbb{S}}$$

whose Laplacian is $(t\partial_t)^2 + \Delta_{\mathbb{S}}$. Hence our theory addresses operators of type L. As has been shown in [3], and explained above, this amounts to the study of degenerate differential operators in the original setting. (Let us mention, in passing, that the variable transformation $t = e^{-s}$ carries $\big((0,1) \times \mathbb{S}, \ \mathrm{d}t^2/t^2 + g_{\mathbb{S}}\big)$ onto $\big((1,\infty) \times \mathbb{S}, \ \mathrm{d}s^2 + g_{\mathbb{S}}\big)$ whose Laplacian is $\partial_s^2 + \Delta_{\mathbb{S}}$. The latter Riemannian manifold is easily seen to be uniformly regular 'near infinity', that is, cofinally uniformly regular as defined in Section 6. These trivial observations form part of the basis of this paper.)

The factor t^{-2} multiplying L in the classical approach does not play a decisive role for the proof of many results in the elliptic theory since it can be 'moved to the right-hand side'. However, the situation changes drastically if a spectral parameter is included since $t^{-2}L + \lambda = t^{-2}(L + \lambda t^2)$ is no longer of the same type as L. This is the reason why—at least up to now—there is no general theory of 'classical' parabolic equations on singular manifolds.

All singular manifolds discussed so far belong to the class of manifolds with 'smooth singularities'. By this we mean that the bases of the cusps themselves do not have singularities. If they are also singular, we model manifolds with cuspidal corners and more complicated higher order singularities. For the sake of simplicity we do not consider such cases in this paper. However, all definitions and theorems presented below have been 'localized' so that an extension to 'corner manifolds' can be built directly on the present work.

In the next section, besides fixing our basic notation, we give precise (localized) definitions of Riemannian manifolds which are uniformly regular, respectively singular of type ρ. All subsequent considerations are given for the latter class. Corresponding assertions for uniformly regular manifolds are obtained by setting $\rho = \mathbf{1}$.

Section 3 contains preliminary technical results and, in particular, the proof of (an extended version of) Example 1.1(e). As a first application of these investigations we present, in Section 4, some easy examples of uniformly regular Riemannian manifolds.

In Section 5 we introduce a general class of 'cusp characteristics' which provides us with ample families of singularity functions ρ. It is a consequence of Example 5.1(b) that our results do not only apply to manifolds with cuspidal singularities,

but also to manifolds with 'exponential' cusps and wedges, or in more general situations (see Example 5.1(b) and Lemma 8.4).

In the proximate section we introduce model wedges and explore their singularity behavior under various Riemannian metrics. The case of the 'natural' metric, induced by the embedding in the ambient Euclidean space, is treated in Section 7. The last section contains the main results and the proofs left out in the introduction.

2 Notations and Definitions

By a *manifold* we always mean a smooth, that is, C^∞ manifold with (possibly empty) boundary such that its underlying topological space is separable and metrizable. Thus we work in the smooth category. A manifold does not need to be connected, but all connected components are of the same dimension.

Let M be a submanifold of some manifold N. Then $\iota\colon M \hookrightarrow N$, or simply $M \hookrightarrow N$, denotes the natural embedding $p \mapsto p$, for which we also write ι_M. (The meaning of N will always be clear from the context.) This embedding induces the natural (fiber-wise linear) embedding $\iota\colon TM \hookrightarrow TN$ of the tangent bundle of M into the one of N.

Let (N,h) be a Riemannian manifold. Then $\iota^* h$ denotes the restriction of h to $M \hookrightarrow N$, that is, $(\iota^* h)(p)(X,Y) = h(p)(X,Y)$ for $p \in M$ and $X,Y \in T_p M \hookrightarrow T_p N$. If g is a Riemannian metric for M, then (M,g) is a *Riemannian submanifold* of (N,h), in symbols: $(M,g) \hookrightarrow (N,h)$, if $g = \iota^* h$. If M has codimension 0, then we write again h for $\iota^* h$.

The Euclidean metric

$$|\mathrm{d}x|^2 = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^m)^2$$

of $\mathbb{R}^m$ is also denoted by g_m. Unless explicitly stated otherwise, we identify $\mathbb{R}^m$ with $(\mathbb{R}^m, g_m)$.

Given a finite-dimensional normed vector space $E = (E, |\cdot|)$ and an open subset V of $\mathbb{R}^m$ or $\mathbb{H}^m$, we write $\|\cdot\|_{k,\infty}$ for the usual norm of $BC^k(V,E)$, the Banach space of all $v \in C^k(V,E)$ such that $|\partial^\alpha v|$ is uniformly bounded for $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq k$. (We use standard multi-index notation.) As usual, $C^k(V) = C^k(V,\mathbb{R})$ etc., and $\|\cdot\|_\infty = \|\cdot\|_{\infty,0}$.

Suppose M and N are manifolds and $\varphi\colon M \to N$ is a diffeomorphism. By φ^* we denote the pull-back by φ (of general tensor fields) and $\varphi_* := (\varphi^{-1})^*$ is the corresponding push-forward. Thus $\varphi^* v = v \circ \varphi$ for a function v on N. Recall that the pull-back $\varphi^* h$ of a Riemannian metric h on N is given by

$$(\varphi^* h)(X,Y) = \varphi^*\big(h(\varphi_* X, \varphi_* Y)\big) \tag{2.1}$$

for all vector fields X and Y on M.

As usual, $(\partial/\partial x^1,\dots,\partial/\partial x^m)$ is the coordinate frame for $T_{U_\kappa}M$ associated with the local coordinates $\kappa = (x^1,\dots,x^m)$ on $U_\kappa := \mathrm{dom}(\kappa)$. Here $T_{U_\kappa}M$ denotes the restriction of TM to $U_\kappa \hookrightarrow M$. Thus $\kappa_*(\partial/\partial x^i) = e_i$, where $(e_1,\dots,e_m)$ is the standard basis for $\mathbb{R}^m$. The basis for $T^*_{U_\kappa}M$, dual to $(\partial/\partial x^1,\dots,\partial/\partial x^m)$, is $(\mathrm{d}x^1,\dots,\mathrm{d}x^m)$ with $\mathrm{d}x^i$ being the differential of the coordinate function x^i.

Let g be a Riemannian metric on M. For a local chart $\kappa = (x^1,\dots,x^m)$ the local representation for g with respect to these coordinates is given by

$$g = g_{ij}\mathrm{d}x^i\,\mathrm{d}x^j\,, \qquad g_{ij} := g\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\,.$$

Here and below, we employ the standard summation convention. Then, given vector fields $\xi = \xi^i e_i$ and $\eta = \eta^j e_j$ on $\kappa(U_\kappa)$, it follows from (2.1) that

$$\kappa_* g(\xi,\eta) = \kappa_*\big(g(\kappa^*\xi,\kappa^*\eta)\big) = \kappa_*\big(g(\xi^i\partial/\partial x^i,\eta^j\partial/\partial x^j)\big)$$
$$= \kappa_*(g_{ij}\xi^i\eta^j) = \kappa_* g_{ij}\xi^i\eta^j = (g_{ij}\circ\kappa^{-1})\xi^i\eta^j\,.$$

Thus $\kappa_* g(x)$ is for each $x \in \kappa(U_\kappa)$ a positive definite symmetric bilinear form. Hence there exists $c(x) \geq 1$ such that

$$|\xi|^2/c(x) \leq \kappa_* g(x)(\xi,\xi) \leq c(x)|\xi|^2\,, \qquad \xi \in \mathbb{R}^m\,, \quad x \in \kappa(U_\kappa)\,, \qquad (2.2)$$

where $|\xi| := \sqrt{g_m(\xi,\xi)} = \sqrt{(\xi\,|\,\xi)}$ is the Euclidean norm of $\xi \in \mathbb{R}^m$. In other words,

$$g_m/c(x) \leq \kappa_* g(x) \leq c(x)g_m\,, \qquad x \in \kappa(U_\kappa)\,.$$

We set $Q := (-1,1) \subset \mathbb{R}$. If κ is a local chart for an m-dimensional manifold M, then it is *normalized* (at p) if $\kappa(U_\kappa) = Q^m$ whenever $U_\kappa \subset \mathring{M}$, the interior of M, whereas $\kappa(U_\kappa) = Q^m \cap \mathbb{H}^m$ if U_κ has a nonempty intersection with the boundary ∂M of M (and $\kappa(p) = 0$). We put $Q^m_\kappa := \kappa(U_\kappa)$ if κ is normalized. (We find it convenient to use normalization by cubes. Of course, we could equally well normalize by employing Euclidean balls.)

Let M be an m-dimensional manifold and S a (nonempty) subset thereof. Given an atlas $\mathfrak{K}$ for M, we set

$$\mathfrak{K}_S := \big\{\,\kappa \in \mathfrak{K}\,;\,U_\kappa \cap S \neq \emptyset\,\big\}\,.$$

Then $\mathfrak{K}_S$ has *finite multiplicity* or: $\mathfrak{K}$ has *finite multiplicity on* S, if there exists $k \in \mathbb{N}$ such that any intersection of more than k coordinate patches U_κ with $\kappa \in \mathfrak{K}_S$ is empty. The least such k is then the multiplicity, $\mathrm{mult}(\mathfrak{K}_S)$, of $\mathfrak{K}_S$. The atlas $\mathfrak{K}$ is *shrinkable on* S, or: $\mathfrak{K}_S$ is *shrinkable*, if $\mathfrak{K}_S$ consists of normalized charts and there exists $r \in (0,1)$ such that

$$\big\{\,\kappa^{-1}(rQ^m_\kappa)\,;\,\kappa \in \mathfrak{K}_S\,\big\} \qquad\qquad (2.3)$$

is a cover of S. It is *shrinkable on S to* $r_0 \in (0,1)$ if (2.3) holds for each $r \in (r_0,1)$.

An atlas $\mathfrak{K}$ for M is *uniformly regular on S* if

(i) $\mathfrak{K}_S$ is shrinkable and has finite multiplicity ;

(ii) $\| \tilde{\kappa} \circ \kappa^{-1} \|_{k,\infty} \leq c(k)$, $\kappa, \tilde{\kappa} \in \mathfrak{K}_S$, $k \in \mathbb{N}$.   (2.4)

In (ii) and in similar situations it is understood that only $\kappa, \tilde{\kappa} \in \mathfrak{K}_S$ with $U_\kappa \cap U_{\tilde{\kappa}} \neq \emptyset$ are being considered. Two atlases $\mathfrak{K}$ and $\tilde{\mathfrak{K}}$ for M, which are uniformly regular on S, are *equivalent on S*, in symbols: $\mathfrak{K} \underset{S}{\approx} \tilde{\mathfrak{K}}$, if

(i) $\mathrm{card}\{ \tilde{\kappa} \in \tilde{\mathfrak{K}}_S ,\ U_{\tilde{\kappa}} \cap U_\kappa \neq \emptyset \} \leq c$, $\kappa \in \mathfrak{K}_S$;

(ii) $\| \tilde{\kappa} \circ \kappa^{-1} \|_{k,\infty} + \| \kappa \circ \tilde{\kappa}^{-1} \|_{k,\infty} \leq c(k)$, $\kappa \in \mathfrak{K}_S$, $\tilde{\kappa} \in \tilde{\mathfrak{K}}_S$, $k \in \mathbb{N}$.   (2.5)

This defines an equivalence relation on the class of all atlases for M which are uniformly regular on S. Each equivalence class is a *structure of uniform regularity on S*. We write $[\![\mathfrak{K}]\!]_S$ for it to indicate that it is *generated* by $\mathfrak{K}$, that is, contains $\mathfrak{K}$ as a representative. If M is endowed with a structure $[\![\mathfrak{K}]\!]_S$ of uniform regularity on S, then $\big(M, [\![\mathfrak{K}]\!]_S \big)$ is a *uniformly regular manifold on S*.

Let $\big(M, [\![\mathfrak{K}]\!]_S \big)$ be a uniformly regular manifold on S and let g be a Riemannian metric for M. Suppose

(i) $\kappa_* g \sim g_m$, $\kappa \in \mathfrak{K}_S$.

(ii) $\| \kappa_* g \|_{k,\infty} \leq c(k)$, $\kappa \in \mathfrak{K}_S$, $k \in \mathbb{N}$.   (2.6)

It follows from (2.5) that (2.6) prevails if $\mathfrak{K}_S$ is replaced by any $\tilde{\mathfrak{K}}_S$ with $\tilde{\mathfrak{K}} \underset{S}{\approx} \mathfrak{K}$. Thus it is meaningful to say that g is a *Riemannian metric for* $\big(M, [\![\mathfrak{K}]\!]_S \big)$ which is *uniformly regular on S* if (2.6) applies to some, hence every, representative of $[\![\mathfrak{K}]\!]_S$. We also say that two such metrics g and $\bar{g}$ are *equivalent on S*, $g \underset{S}{\sim} \bar{g}$, if $g\,|\,S \sim \bar{g}\,|\,S$. This defines an equivalence relation on the class of all Riemannian metrics for $\big(M, [\![\mathfrak{K}]\!]_S \big)$ which are uniformly regular on S. Similarly as above, $[\![g]\!]_S$ is the equivalence class containing the representative g.

By a **uniformly regular Riemannian manifold** *on S*, written as $\big(M, [\![\mathfrak{K}]\!]_S, [\![g]\!]_S \big)$, we mean a uniformly regular manifold $\big(M, [\![\mathfrak{K}]\!]_S \big)$ on S equipped with an equivalence class of uniformly regular Riemannian metrics on S. It is a convenient abuse of language to say instead that $(M, \mathfrak{K}, g)$ is a Riemannian manifold which is uniformly regular on S. Even more loosely, (M, g) is (a manifold which is) **uniformly regular** *on S*, if there exists an atlas $\mathfrak{K}$ which is uniformly regular on S such that $\big(M, [\![\mathfrak{K}]\!]_S, [\![g]\!]_S \big)$ is a uniformly regular Riemannian manifold on S.

Suppose $\rho \in C^\infty \big(M, (0, \infty) \big)$ and let g be a Riemannian metric for M. Then ρ is a *singularity function for* (M, g) *on S*, if there exists an atlas $\mathfrak{K}$ which is uniformly regular on S such that $(M, \mathfrak{K}, g/\rho^2)$ is a Riemannian manifold which is uniformly regular on S. Two singularity functions are *equivalent on S*, $\rho \underset{S}{\approx} \tilde{\rho}$, if $\mathfrak{K} \underset{S}{\approx} \tilde{\mathfrak{K}}$ and $g/\rho^2 \underset{S}{\sim} g/\tilde{\rho}^2$. We denote by $[\![\rho]\!]_S$ the equivalence class of singularity functions containing the representative ρ, the *singularity type* of (M, g) on S. Finally, the **Riemannian manifold** (M, g) **is singular of type** $[\![\rho]\!]_S$—more loosely: **of type** ρ *on S*—if

$(M,g/\rho^2)$ is uniformly regular on S. Clearly, (M,g) is singular of type $[\![\mathbf{1}]\!]_S$ iff it is uniformly regular on S.

A pair $(\rho,\mathfrak{K})$ is a *singularity datum* for (M,g) on S if

$$
\begin{aligned}
&\text{(i)} && \rho \in C^\infty\big((M,(0,\infty))\big)\,. \\
&\text{(ii)} && \mathfrak{K} \text{ is an atlas which is uniformly regular on } S\,. \\
&\text{(iii)} && \|\kappa_*\rho\|_{k,\infty} \le c(k)\rho_\kappa\,, \quad \kappa \in \mathfrak{K}_S\,, \quad k \in \mathbb{N}\,, \\
& && \text{where } \rho_\kappa := \kappa_*\rho(0) = \rho\big(\kappa^{-1}(0)\big)\,. && (2.7) \\
&\text{(iv)} && \rho\,|\,U_\kappa \sim \rho_\kappa\,, \quad \kappa \in \mathfrak{K}_S\,. \\
&\text{(v)} && \kappa_*g \sim \rho_\kappa^2 g_m\,, \quad \kappa \in \mathfrak{K}_S\,. \\
&\text{(vi)} && \|\kappa_*g\|_{k,\infty} \le c(k)\rho_\kappa^2\,, \quad \kappa \in \mathfrak{K}_S\,, \quad k \ge 0\,.
\end{aligned}
$$

It is easily verified that $(M,\mathfrak{K},g/\rho^2)$ is uniformly regular on S if $(\rho,\mathfrak{K})$ is a singularity datum for (M,g) on S. Thus ρ is a singularity function for (M,g) if $(\rho,\mathfrak{K})$ is a singularity datum for it.

The 'localization' of all these quantities 'to S' is introduced for technical reasons. Our principal interest concerns the choice $S = M$. In this case the qualifiers 'on S' and the symbol S are omitted, of course.

3 Preliminaries

Let (M,g) be a Riemannian manifold and $X \subset M$. For $p,q \in X$ we denote by $d_X(p,q) = d_{g,X}(p,q)$ the distance between p and q *in X*. Thus $d_X(p,q)$ is the infimum of the lengths of all piece-wise smooth paths of M joining p to q within X. If p and q lie in different connected components, then $d_X(p,q) := \infty$.

We suppose $\mathbb{X} \in \{\mathbb{R}^m, \mathbb{H}^m\}$, X is open in $\mathbb{X}$, and $S \subset X$. We denote by δ_S the distance in X from S to $\mathbb{X}\backslash X$, that is, $\delta_S := \inf_{p \in S} d_X(p, \mathbb{X}\backslash X)$, where $d_X(p,\emptyset) := \infty$. Then we assume $0 < \delta \le \delta_S/\sqrt{m}$ and set

$$
Z_{\delta,X} := \big\{\, z \in \mathbb{Z}^m \cap \mathbb{X}\ ;\ \delta(z+Q_z) \cap X \ne \emptyset \,\big\}\,,
$$

where $Q_z := Q^m$ if $z \in \mathring{X}$ and $Q_z := Q^m \cap \mathbb{H}^m$ otherwise. Given $z \in Z_{\delta,X}$,

$$
\lambda_{\delta,z}(x) := -z + x/\delta\,, \qquad x \in \delta(z+Q_z) \cap X\,. \tag{3.1}
$$

Then

$$
\mathfrak{L} = \mathfrak{L}(\delta,X) := \{\, \lambda_{\delta,z}\ ;\ z \in Z_{\delta,X} \,\}
$$

is an atlas for X of multiplicity 2^m. Since $\mathrm{diam}\big(\delta(z+Q_z)\big) = \sqrt{m}\delta \le \delta_S$ we see that $\mathfrak{L}_S$ is normalized and shrinkable to $1/2$. Given $\lambda, \tilde{\lambda} \in \mathfrak{L}_S$ with $\lambda = \lambda_{\delta,z}$ and $\tilde{\lambda} = \lambda_{\delta,\tilde{z}}$,

$$
\tilde{\lambda} \circ \lambda^{-1}(y) = z - \tilde{z} + y\,, \qquad y \in \lambda(U_\lambda \cap U_{\tilde\lambda})\,.
$$

This shows that $\mathfrak{L}$ is uniformly regular on S. Furthermore, denoting by ∂ the Fréchet derivative,

$$\partial \lambda^{-1} = \delta 1_m , \quad \lambda_* g_X = \delta^2 g_m , \qquad \lambda \in \mathfrak{L}_S , \tag{3.2}$$

where $g_X = \iota_X^* g_m$ and 1_m is the identity in $\mathbb{R}^{m \times m}$. In particular, setting $X := \mathbb{X}$ it follows that

$$\mathbb{R}^m \text{ and } \mathbb{H}^m \text{ are uniformly regular Riemannian manifolds .} \tag{3.3}$$

Let M be an m-dimensional manifold and $S \subset M$. Suppose $\mathfrak{K}$ is an atlas for M which is uniformly regular on S. Then there exists $r \in (0,1)$ such that (2.3) is a cover of S. Given $\kappa \in \mathfrak{K}_S$, we fix $\delta \in \big(0, (1-r)/\sqrt{m}\big)$ and put $\mathfrak{L}_\kappa := \mathfrak{L}(\delta, Q_\kappa^m)$. By the above $\mathfrak{L}_\kappa$ is an atlas for Q_κ^m of multiplicity 2^m which is uniformly regular on rQ_κ^m and shrinkable to $1/2$ on rQ_κ^m. Hence

$$\mathfrak{M} = \mathfrak{M}(\delta, \mathfrak{K}) := \{\, \lambda \circ \kappa \; ; \; \kappa \in \mathfrak{K}_S, \, \lambda \in \mathfrak{L}_\kappa \} \cup (\mathfrak{K} \backslash \mathfrak{K}_S) \tag{3.4}$$

is an atlas for M such that

$$U_{\lambda \circ \kappa} = \kappa^{-1}(U_\lambda) \subset U_\kappa , \qquad \kappa \in \mathfrak{K}_S , \quad \lambda \in \mathfrak{L}_\kappa . \tag{3.5}$$

It has multiplicity at most $2^m \mathrm{mult}(\mathfrak{K}_S)$ on S and is shrinkable to $1/2$ on S. For $\mu, \tilde{\mu} \in \mathfrak{M}_S$ with $\mu = \lambda \circ \kappa$ and $\tilde{\mu} = \tilde{\lambda} \circ \tilde{\kappa}$ we get from (3.1) and (3.2)

$$\|\partial^\alpha (\tilde{\mu} \circ \mu^{-1}\|_\infty \le \delta^{-1} \delta^{|\alpha|} \|\partial^\alpha (\tilde{\kappa} \circ \kappa^{-1})\|_\infty , \qquad \alpha \in \mathbb{N}^m \backslash \{0\} . \tag{3.6}$$

Note that $\lambda \circ \kappa \in \mathfrak{M}_S$ implies $\kappa \in \mathfrak{K}_S$. Thus, since $\mathfrak{K}_S$ is uniformly regular and $\delta \le 1$,

$$\|\partial^\alpha (\tilde{\mu} \circ \mu^{-1})\|_\infty \le c(\alpha)$$

for $\mu, \tilde{\mu} \in \mathfrak{M}_S$ with $\mu = \lambda \circ \kappa$ and $\tilde{\mu} = \tilde{\lambda} \circ \tilde{\kappa}$ and $\alpha \in \mathbb{N}^m \backslash \{0\}$. Hence

$$\mathfrak{M} \text{ is uniformly regular on } S . \tag{3.7}$$

Let g be a Riemannian metric for M. Then (3.2) implies

$$\mu_* g = \lambda_* \kappa_* g = \delta^2 \kappa_* g , \qquad \mu = \lambda \circ \kappa \in \mathfrak{M}_S . \tag{3.8}$$

Consequently,

$$\|\partial^\alpha (\mu_* g)\|_\infty \le c(\alpha) \delta^2 \|\partial^\alpha (\kappa_* g)\|_\infty , \qquad \mu \in \mathfrak{M}_S , \quad \alpha \in \mathbb{N}^m . \tag{3.9}$$

Suppose $(\rho, \mathfrak{K})$ is a singularity datum for M on S. Then we infer from (2.7)(iii) and (iv) and from (3.5)

$$\mu_* \rho = (\kappa_* \rho) \circ \lambda^{-1} \sim (\kappa_* \rho)(0) = \rho_\kappa \sim \rho_\mu \tag{3.10}$$

and, using $\delta \leq 1$ once more,

$$\|\partial^\alpha(\mu_*\rho)\|_\infty \leq \delta^{|\alpha|} \|\partial^\alpha(\mu_*\rho)\|_\infty \leq c(\alpha)\rho_\kappa \leq c(\alpha)\rho_\mu \qquad (3.11)$$

for $\mu = \lambda \circ \kappa \in \mathfrak{M}_S$ and $\alpha \in \mathbb{N}^m$.

These considerations show, in particular, that a uniformly regular Riemannian manifold possesses a uniformly regular atlas consisting of arbitrarily small charts; also see Lemma 3.2.

Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold without boundary. Then we endow the product manifold $M \times \tilde{M}$ with the product metric, denoted (slightly loosely) by $g + \tilde{g}$.

Theorem 3.1. *Suppose ρ is a bounded singularity function for (M, g) on $S \subset M$ and $\tilde{\rho}$ is one for $(\tilde{M}, \tilde{g})$ on $\tilde{S} \subset \tilde{M}$. Then $\rho \otimes \tilde{\rho}$ is a singularity function for $(M \times \tilde{M}, g + \tilde{g})$ on $S \times \tilde{S}$.*

Proof. (1) We choose $0 < \bar{r} < r < 1$ and an atlas $\mathfrak{K}$ for M, resp. $\tilde{\mathfrak{K}}$ for $\tilde{M}$, such that $\mathfrak{K}$, resp. $\tilde{\mathfrak{K}}$, is shrinkable to $\bar{r}$ on S, resp. $\tilde{S}$, and $(\rho, \mathfrak{K})$, resp. $(\tilde{\rho}, \tilde{\mathfrak{K}})$, is a singularity datum for (M, g) on S, resp. $(\tilde{M}, \tilde{g})$ on $\tilde{S}$. Denoting by m, resp. $\tilde{m}$, the dimension of M, resp. $\tilde{M}$, we set $\delta := (1 - r)/\sqrt{m + \tilde{m}}$. Given $\kappa \in \mathfrak{K}_S$ and $\tilde{\kappa} \in \mathfrak{K}_{\tilde{S}}$, we put

$$\delta_{\tilde{\kappa}} := \min\{\tilde{\rho}_{\tilde{\kappa}}, \delta\}, \quad \tilde{\delta}_\kappa := \min\{\rho_\kappa, \delta\}. \qquad (3.12)$$

We set

$$\mathfrak{M}' := \big\{ (\lambda \circ \kappa) \times (\tilde{\lambda} \circ \tilde{\kappa}) \; ; \; \kappa \in \mathfrak{K}_S, \; \tilde{\kappa} \in \tilde{\mathfrak{K}}_{\tilde{S}}, \; \lambda \in \mathfrak{L}(\delta_{\tilde{\kappa}}, Q_\kappa^m), \; \tilde{\lambda} \in \mathfrak{L}(\tilde{\delta}_\kappa, Q^{\tilde{m}}) \big\}$$

and

$$\mathfrak{M}'' := \big\{ \kappa \times \tilde{\kappa} \; ; \; \text{either } \kappa \in \mathfrak{K} \backslash \mathfrak{K}_S \text{ or } \tilde{\kappa} \in \tilde{\mathfrak{K}} \backslash \tilde{\mathfrak{K}}_{\tilde{S}} \big\}.$$

Then $\mathfrak{M} := \mathfrak{M}' \cup \mathfrak{M}''$ is an atlas for $M \times \tilde{M}$ and a refinement of the product atlas $\mathfrak{K} \otimes \tilde{\mathfrak{K}}$ in the sense that for each $\mu \in \mathfrak{M}$ there exists $\kappa \times \tilde{\kappa} \in \mathfrak{K} \otimes \tilde{\mathfrak{K}}$ such that $U_\mu \subset U_{\kappa \times \tilde{\kappa}}$. Moreover,

$$\mathfrak{M}_{S \times \tilde{S}} \subset \mathfrak{M}'. \qquad (3.13)$$

Note that $\mathfrak{M}$ is normalized on $S \times \tilde{S}$ and has finite multiplicity thereon.

Suppose $\mu_i = (\lambda_i \circ \kappa_i) \times (\tilde{\lambda}_i \circ \tilde{\kappa}_i) \in \mathfrak{M}'$ for $i = 1, 2$, and $U_{\mu_1} \cap U_{\mu_2} \neq \emptyset$. Then both $U_{\kappa_1} \cap U_{\kappa_2}$ and $U_{\tilde{\kappa}_1} \cap U_{\tilde{\kappa}_2}$ are nonempty. Hence $\tilde{\rho}_{\tilde{\kappa}_1} \sim \tilde{\rho}_{\tilde{\kappa}_2}$ and $\rho_{\kappa_1} \sim \rho_{\kappa_2}$. From this, $\delta_{\tilde{\kappa}_i} \leq \tilde{\rho}_{\tilde{\kappa}_i}$, and the boundedness of $\tilde{\rho}$ we infer $\delta_{\tilde{\kappa}_1}/\delta_{\tilde{\kappa}_2} \leq c$ and, analogously, $\tilde{\delta}_{\kappa_1}/\tilde{\delta}_{\kappa_2} \leq c$. Thus, using (3.1), (3.2), the finite multiplicity of $\mathfrak{M}_{S \times \tilde{S}}$ and the fact that κ_i and $\tilde{\kappa}_i$ are normalized, we obtain (cf. (3.6))

$$\|\mu_1 \circ \mu_2^{-1}\|_{k,\infty} \leq c \big(\|\kappa_1 \circ \kappa_2^{-1}\|_{k,\infty} + \|\tilde{\kappa}_1 \circ \tilde{\kappa}_2^{-1}\|_{k,\infty} \big) \leq c(k)$$

for $\mu_1, \mu_2 \in \mathfrak{M}_{S \times \tilde{S}}$ and $k \in \mathbb{N}$. This proves that $\mathfrak{M}$ is uniformly regular on $S \times \tilde{S}$.

(2) By adapting (3.10) and (3.11) to the present setting we find, due to (3.13),

$$\mu_*(\rho \otimes \tilde{\rho}) \sim (\rho \otimes \tilde{\rho})_\mu \sim \rho_\kappa \tilde{\rho}_{\tilde{\kappa}} \qquad (3.14)$$

for $\mu = (\lambda \circ \kappa) \times (\tilde{\lambda} \circ \tilde{\kappa}) \in \mathfrak{M}_{S \times \tilde{S}}$ and

$$\|\mu_*(\rho \otimes \tilde{\rho})\|_{k,\infty} \leq c(k)(\rho \otimes \tilde{\rho})_\mu , \qquad \mu \in \mathfrak{M}_{S \times \tilde{S}} , \quad k \in \mathbb{N} .$$

(3) For $\mu = (\lambda \circ \kappa) \times (\tilde{\lambda} \circ \tilde{\kappa}) \in \mathfrak{M}'$ we find by (3.8)

$$\begin{aligned}
\mu_*(g + \tilde{g}) &= (\lambda \circ \kappa)_* g + (\tilde{\lambda} \circ \tilde{\kappa})_* \tilde{g} \\
&\sim \delta_{\tilde{\kappa}}^2 \kappa_* g + \tilde{\delta}_\kappa^2 \tilde{\kappa}_* \tilde{g} \sim \delta_{\tilde{\kappa}}^2 \rho_\kappa^2 g_m + \tilde{\delta}_\kappa^2 \tilde{\rho}_{\tilde{\kappa}}^2 g_{\tilde{m}} ,
\end{aligned} \tag{3.15}$$

uniformly with respect to $\mu \in \mathfrak{M}_{S \times \tilde{S}}$. Definition (3.12) and the boundedness of ρ and $\tilde{\rho}$ imply $\delta_{\tilde{\kappa}} \sim \tilde{\rho}_{\tilde{\kappa}}$ and $\tilde{\delta}_\kappa \sim \rho_\kappa$. Using this and (3.14) we get from (3.15)

$$\mu_*(g + \tilde{g}) \sim \rho_\kappa^2 \tilde{\rho}_{\tilde{\kappa}}^2 (g_m + g_{\tilde{m}}) \sim (\rho \otimes \tilde{\rho})_\mu^2 g_{m+\tilde{m}} , \qquad \mu \in \mathfrak{M}_{S \times \tilde{S}} .$$

Lastly, we infer from (3.9) and (3.14)

$$\begin{aligned}
\|\mu_*(g + \tilde{g})\|_{k,\infty} &\leq c(k)\big(\delta_{\tilde{\kappa}}^2 \|\kappa_* g\|_{k,\infty} + \tilde{\delta}_\kappa^2 \|\tilde{\kappa}_* \tilde{g}\|_{k,\infty}\big) \\
&\leq c(k)\rho_\kappa^2 \tilde{\rho}_{\tilde{\kappa}}^2 \leq c(k)(\rho \otimes \tilde{\rho})_\mu^2
\end{aligned}$$

for $\mu \in \mathfrak{M}_{S \times \tilde{S}}$ and $k \in \mathbb{N}$. This proves the assertion.   □

Our next considerations exploit the 'localization to S'.

Lemma 3.2. *Let (M, g) be uniformly regular on $S \subset M$. Suppose V is open in M and*

$$d_{\bar{V}}(S, M \backslash V) > 0 . \tag{3.16}$$

Then there exists an atlas $\mathfrak{M}$ for M belonging to the structure of uniform regularity on S such that $U_\mu \subset V$ for $\mu \in \mathfrak{M}_S$.

Proof. Let $\mathfrak{K}$ be an atlas belonging to the structure of uniform regularity on S. Choose $r \in (0, 1)$ such that (2.3) is a cover of S. Fix $\delta \in \big(0, (1-r)/\sqrt{m}\big)$ and set $\mathfrak{M} := \mathfrak{M}(\delta, \mathfrak{K})$. Then $\mathfrak{M}_S$ is uniformly regular by (3.7).

It follows from $\kappa_* g \sim g_m$ for $\kappa \in \mathfrak{K}_S$, (3.8), and (3.16) that we can choose δ so small that $\operatorname{diam}(U_\mu) < d_{\bar{V}}(S, M \backslash V)$ for $\mu \in \mathfrak{M}_S$.

Lastly, we infer from (2.4), (3.1), and (3.2) that

$$\|\kappa \circ \mu^{-1}\|_{k,\infty} + \|\mu \circ \kappa^{-1}\|_{k,\infty} \leq c(k) , \qquad \kappa \in \mathfrak{K}_S , \quad \mu \in \mathfrak{M}_S .$$

Thus $\mathfrak{M} \underset{S}{\approx} \mathfrak{K}$, which proves the claim.   □

The following lemma will be fundamental for the construction of singular Riemannian manifolds by 'patching together simpler pieces'.

Lemma 3.3. *Suppose:*

(i) $\{V_\alpha ; \alpha \in \mathsf{A}\}$ *is a finite family of open subsets of M.*
(ii) $S_\alpha \subset V_\alpha$ *and* $\{S_\alpha ; \alpha \in \mathsf{A}\}$ *is a covering of M.*

(iii) $(\rho_\alpha, \mathfrak{K}_\alpha)$ *is a singularity datum for* (V_α, g) *on* S_α.
(iv) $\rho_\alpha \,|\, V_\alpha \cap V_{\tilde\alpha} \sim \rho_{\tilde\alpha} \,|\, V_\alpha \cap V_{\tilde\alpha}$, $\alpha, \tilde\alpha \in \mathsf{A}$.
 (v) $\|\kappa_{\tilde\alpha} \circ \kappa_\alpha^{-1}\|_{k,\infty} + \|\kappa_\alpha \circ \kappa_{\tilde\alpha}^{-1}\|_{k,\infty} \le c(k)$ *for*
     $(\kappa_\alpha, \kappa_{\tilde\alpha}) \in \mathfrak{K}_{\alpha,S_\alpha} \times \mathfrak{K}_{\tilde\alpha,S_{\tilde\alpha}}$, $\alpha, \tilde\alpha \in \mathsf{A}$, $\alpha \ne \tilde\alpha$, $k \in \mathbb{N}$.

Then $\mathfrak{K} := \bigcup_\alpha \mathfrak{K}_{\alpha,S_\alpha}$ *is a uniformly regular atlas for* M *and there exists* ρ *belonging to* $C^\infty\big(M, (0, \infty)\big)$ *and satisfying*

$$\rho \,|\, S_\alpha \sim \rho_\alpha \,, \qquad \alpha \in \mathsf{A} \,, \tag{3.17}$$

such that $(\rho, \mathfrak{K})$ *is a singularity datum for* (M, g).

Proof. (1) It is a consequence of (i)–(iii) and (v) that $\mathfrak{K}$ is a uniformly regular atlas for M.

(2) Since M is locally compact, separable, and metrizable the same applies to V_α. Thus V_α is paracompact. Hence there exists a smooth partition of unity $\{\chi_{\alpha,\beta} \; ; \; \beta \in \mathfrak{K}_\alpha\}$ on V_α subordinate to $\{U_\beta \; ; \; \beta \in \mathfrak{K}_\alpha\}$ (e.g., [7]). We extend each $\chi_{\alpha,\beta}$ over M by setting it equal to 0 outside V_α and set $\psi_\alpha := \sum_{\beta \in \mathfrak{K}_{\alpha,S_\alpha}} \chi_{\alpha,\beta}$. Then $\psi_\alpha \in C^\infty\big(M, [0,1]\big)$ with $\psi_\alpha \,|\, S_\alpha = \mathbf{1}$. We put $\varphi_\alpha := \psi_\alpha \big/ \sum_{\tilde\alpha \in \mathsf{A}} \psi_{\tilde\alpha}$. Assumptions (i) and (ii) guarantee that $\{\varphi_\alpha \; ; \; \alpha \in \mathsf{A}\}$ is a smooth partition of unity on M subordinate to the open cover $\{V_\alpha \; ; \; \alpha \in \mathsf{A}\}$ of M.

We put $\rho := \sum_\alpha \varphi_\alpha \rho_\alpha$. Then, given $\alpha \in \mathsf{A}$ and $x \in V_\alpha$, we infer from (iv)

$$\begin{aligned}
\rho(x) &= \sum_{V_\beta \cap V_\alpha \ne \emptyset} \varphi_\beta(x) \rho_\beta(x) \sim \rho_\alpha(x) \sum_{V_\beta \cap V_\alpha \ne \emptyset} \varphi_\beta(x) \\
&= \rho_\alpha(x) \sum_\beta \varphi_\beta(x) = \rho_\alpha(x) \,.
\end{aligned} \tag{3.18}$$

This proves (3.17).

(3) By (iii)
$$\kappa_*(g/\rho_\alpha^2) \sim g_m \,, \quad \|\kappa_*(g/\rho_\alpha^2)\|_{k,\infty} \le c(k)$$

for $\kappa \in \mathfrak{K}_{\alpha,S_\alpha}$, $\alpha \in \mathsf{A}$, and $k \in \mathbb{N}$. We deduce from (3.18)

$$\kappa_*(g/\rho^2) = \kappa_* g / \kappa_* \rho^2 \sim \kappa_* g / \kappa_* \rho_\alpha^2 = \kappa_*(g/\rho_\alpha^2) \sim g_m \tag{3.19}$$

for $\kappa \in \mathfrak{K}_{\alpha,S_\alpha}$ and $\alpha \in \mathsf{A}$, that is, for $\kappa \in \mathfrak{K}$. The definition of ρ implies

$$\kappa_* \rho = \sum_\alpha (\kappa_* \varphi_\alpha)(\kappa \circ \kappa_\alpha^{-1})_* \kappa_{\alpha*} \rho_\alpha \,, \qquad \kappa \in \mathfrak{K} \,.$$

From this, (iii), the chain rule, and the uniform regularity of $\mathfrak{K}$ we deduce the estimate $\|\kappa_* \rho\|_{k,\infty} \le c(k)$ for $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}$. Consequently, we infer from the chain rule and (3.19)

$$\|\kappa_*(g/\rho^2)\|_{k,\infty} \le c(k) \,, \qquad \kappa \in \mathfrak{K} \,, \quad k \in \mathbb{N} \,.$$

This proves the last part of the assertion. $\qquad \square$

The following (almost trivial) lemma shows that the class of singular manifolds is invariant under Riemannian isometries.

Lemma 3.4. *Let $f\colon \tilde{M} \to M$ be a diffeomorphism of manifolds. Suppose g is a Riemannian metric for M and ρ is a singularity function for (M,g) on $S \subset M$. Then $f^*\rho$ is a singularity function for $(\tilde{M}, f^*g)$ on $f^{-1}(S)$.*

Proof. Let $\mathfrak{K}$ be an atlas which is uniformly regular on S. It is easily verified that $f^*\mathfrak{K} := \{ f^*\kappa \; ; \; \kappa \in \mathfrak{K} \}$ is an atlas for $\tilde{M}$ which is uniformly regular on $f^{-1}(S)$. Note

$$(f^*\kappa)_* f^*\rho = (\rho \circ f) \circ (\kappa \circ f)^{-1} = \rho \circ \kappa = \kappa_* \rho$$

and

$$(f^*\kappa)_*(f^*g) = (\kappa \circ f)_*(f^{-1})_* g = \big((\kappa \circ f) \circ f^{-1}\big)_* g = \kappa_* g$$

for $\kappa \in \mathfrak{K}$. From this it is obvious that conditions (2.7) carry over from ρ, $\mathfrak{K}$, and g to $f^*\rho$, $f^*\mathfrak{K}$, and f^*g. $\qquad\square$

Suppose $\partial M \neq \emptyset$ and let $\imath\colon \partial M \hookrightarrow M$ be the natural embedding. Let g be a Riemannian metric for M. Then $\dot{g} := \imath^*g$ is the Riemannian metric for ∂M induced by g. Given a local chart κ for M with $\partial U_\kappa = U_\kappa \cap \partial M \neq \emptyset$, we set $U_{\dot{\kappa}} := \partial U_\kappa$ and $\dot{\kappa} := \iota_0 \circ (\imath^*\kappa)\colon U_{\dot{\kappa}} \to \mathbb{R}^{m-1}$, where $\iota_0\colon \{0\} \times \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$, $(0,x') \mapsto x'$. Moreover, $\dot{\rho} := \imath^*\rho = \rho \,|\, \partial M$ for $\rho\colon M \to \mathbb{R}$.

Lemma 3.5. *Let $\mathfrak{K}$ be an atlas for M which is uniformly regular on S. Then*

$$\dot{\mathfrak{K}} := \{ \dot{\kappa} \; ; \; \kappa \in \mathfrak{K}_{\partial M} \}$$

is one for ∂M and it is uniformly regular on $\partial M \cap S$. If $(\rho, \mathfrak{K})$ is a singularity datum for (M,g) on S, then $(\dot{\rho}, \dot{\mathfrak{K}})$ is one for $(\partial M, \dot{g})$ on $\partial M \cap S$.

Proof. Obvious. $\qquad\square$

In this lemma it is implicitly assumed that $m \geq 2$. However, calling—in abuse of language—every 0-dimensional manifold uniformly regular, Lemma 3.5 holds for $m = 1$ also, employing obvious interpretations and adaptions.

4 Uniformly Regular Riemannian Manifolds

On the basis of the preceding considerations we now provide proofs for some of the claims made in Example 1.1.

Let (M,g) be a Riemannian manifold. It has *bounded geometry* if it has an empty boundary, is complete, has a positive injectivity radius, and all covariant derivatives of the curvature tensor are bounded.

Theorem 4.1. *If (M,g) has bounded geometry, then it is uniformly regular.*

Proof. This follows from Th. Aubin [6, Lemma 2.2.6] and J. Eichhorn [11] (also see M.A. Shubin [28]). □

A uniformly regular Riemannian manifold without boundary is complete (cf. M. Disconzi, Y. Shao, and G. Simonett [10]). It has been shown by R.E. Greene [12] that every manifold M without boundary admits a Riemannian metric g such that (M, g) has bounded geometry. However, in view of applications to differential equations which we have in mind, this result is of restricted interest, in general. Indeed, the metric is then given a priori and is closely related to the differential operators under consideration.

Although Theorem 4.1 is very general it has the disadvantage that it applies only to manifolds without boundary. The following results do not require ∂M to be empty.

Lemma 4.2. *Let (M, g) be a Riemannian manifold and suppose $S \subset M$ is compact. Then there exists a unique uniformly regular structure for M on S, and (M, g) is uniformly regular on S.*

Proof. (1) For each $p \in M$ there exists a local chart $\tilde{\kappa}_p$ of M with $p \in U_{\tilde{\kappa}}$. We set $W_p := Q^m$ if $p \in \mathring{M}$, and $W_p := Q^m \cap \mathbb{H}^m$ for $p \in \partial M$. Then we can fix $\delta_p > 0$ such that $\overline{\tilde{\kappa}_p(p) + \delta_p W_p} \subset \kappa_p(U_{\kappa_p})$. From this it follows that, by translation and dilation, we find for each pair $p, q \in M$ local charts κ_p and κ_q, normalized at p and q, respectively, such that $\|\kappa_p \circ \kappa_q^{-1}\|_{k,\infty} \leq c(p, q, k)$ for $k \in \mathbb{N}$.

By the compactness of S we can determine a finite subset Σ of S such that $\{ \kappa_p^{-1}(2^{-1}Q^m_{\kappa_p}) ; p \in \Sigma \}$ is an open cover of S. Let $\mathfrak{N}$ be an atlas for the open submanifold $M \setminus S$ of M. Then

$$\mathfrak{K} := \{ \kappa_p ; p \in \Sigma \} \cup \mathfrak{N}$$

is an atlas for M, and $\mathfrak{K}_S = \{ \kappa_p ; p \in \Sigma \}$. Since Σ is finite $\mathfrak{K}$ is uniformly regular on S and (cf. (2.2)) condition (2.6) is satisfied.

(2) Let $\mathfrak{L}$ be an atlas for M which is uniformly regular on S. By the compactness of S we find a subatlas $\mathfrak{M}$ of $\mathfrak{L}$ such that $\mathfrak{M}_S$ is a finite subset of $\mathfrak{L}_S$. It is obvious that $\mathfrak{M}$ can be chosen such that $\mathfrak{M} \underset{S}{\approx} \mathfrak{L}$. Since $\mathfrak{K}_S$ and $\mathfrak{M}_S$ are both finite, $\mathfrak{M} \underset{S}{\approx} \mathfrak{K}$. Consequently, $\mathfrak{L} \underset{S}{\approx} \mathfrak{K}$. This proves the uniqueness assertion. □

Corollary 4.3. *Every compact Riemannian manifold is uniformly regular.*

The next theorem concerns submanifolds of codimension 0 of uniformly regular Riemannian manifolds.

Theorem 4.4. *Let (N, g) be an m-dimensional uniformly regular Riemannian manifold and (M, g) an m-dimensional Riemannian submanifold with compact boundary. Then (M, g) is uniformly regular.*

Proof. By the preceding corollary we can assume $\partial M \neq \emptyset$.

Since M is locally compact and ∂M is compact there exist relatively compact open neighborhoods W_1 and W_2 of ∂M in M with $W_1 \subset \bar{W}_1 \subset W_2$. We set $V_1 := W_2$

and $S_1 := \bar{W}_1$ as well as $V_2 := \mathring{M}$ and $S_2 := M \backslash W_1$. Then V_i is open in M, $S_i \subset V_i$, and $S_1 \cup S_2 = M$.

The compactness of S_1 in M and $\mathrm{d}_M(S_1, M \backslash W_2) > 0$ imply, due to Lemmas 3.2 and 4.2, that there exists an atlas $\mathfrak{K}_1$ for M such that $(\mathbf{1}, \mathfrak{K}_1)$ is a singularity datum for V_1 on S_1.

Note that $\mathrm{d}_M(S_2, \partial M) > 0$. Hence Lemma 3.2 and the uniform regularity of (N, g) imply the existence of an atlas $\mathfrak{K}_2$ for $\mathring{M}$ such that $(\mathbf{1}, \mathfrak{K}_2)$ is a singularity datum for V_2 on S_2.

Since $S := S_1 \cap S_2 = \bar{W}_1 \backslash W_1$ is compact we can assume that $\mathfrak{K}_{1,S}$ and $\mathfrak{K}_{2,S}$ are finite. Hence it is obvious that condition (v) of Lemma 3.3 is satisfied. Thus that lemma guarantees the validity of the claim.   □

Corollary 4.5. *Let M be an m-dimensional Euclidean submanifold of $\mathbb{R}^m$ with compact boundary. Then M is a uniformly regular Riemannian manifold.*

Proof. Set $N := \mathbb{R}^m$ and recall (3.3).   □

5 Characteristics

We write $J_0 := (0,1]$, $J_\infty := [1,\infty)$, and assume throughout that $J \in \{J_0, J_\infty\}$. A sub-interval I of J is *cofinal* if $1 \notin I$, and $J \backslash \mathring{I}$ is a compact interval.

We denote by $\mathscr{R}(J)$ the set of all $R \in C^\infty\big(J, (0,\infty)\big)$ satisfying $R(1) = 1$, such that $R(\omega) := \lim_{t \to \omega} R(t)$ exists in $[0,\infty]$ if $J = J_\omega$. Then we write $R \in \mathscr{C}(J)$ if

$$
\begin{aligned}
&\text{(i)} &&R \in \mathscr{R}(J) \text{ and } R(\infty) = 0 \text{ if } J = J_\infty\,; \\
&\text{(ii)} &&\int_J \mathrm{d}t / R(t) = \infty\,; &&(5.1)\\
&\text{(iii)} &&\|\partial^k R\|_\infty < \infty\,,\ k \geq 1\,.
\end{aligned}
$$

The elements of $\mathscr{C}(J)$ are called *cusp characteristics* on J.

On J_∞ we introduce, in addition, the set $\mathscr{F}(J_\infty)$ of *funnel characteristics* by: $R \in \mathscr{F}(J_\infty)$ if

$$
\begin{aligned}
&\text{(i)} &&R \in \mathscr{R}(J_\infty) \text{ and } R(\infty) > 0\,; &&(5.2)\\
&\text{(ii)} &&\|\partial^k R\|_\infty < \infty\,,\ k \geq 1\,.
\end{aligned}
$$

Examples 5.1. (a) We set $\mathsf{R}_\alpha(t) := t^\alpha$ for $\alpha \in \mathbb{R}$. Then $\mathsf{R}_\alpha \in \mathscr{C}(J_0)$ if $\alpha \geq 1$, $\mathsf{R}_\alpha \in \mathscr{C}(J_\infty)$ if $\alpha < 0$, and $\mathsf{R}_\alpha \in \mathscr{F}(J_\infty)$ if $0 \leq \alpha \leq 1$.

(b) Suppose $\beta > 0$ and $\gamma \in \mathbb{R}$. Put $R(t) := \mathrm{e}^{\beta(1 - t^\gamma)}$. Then $R \in \mathscr{C}(J_0)$ if $\gamma < 0$, whereas $R \in \mathscr{C}(J_\infty)$ for $\gamma > 0$.

(c) For $\alpha \geq -2/\pi$ and $\beta > 0$ we put $R_{\mathrm{arctan},\alpha,\beta}(t) := 1 + \alpha \arctan\big(\beta(t-1)\big)$. Then $R_{\mathrm{arctan},-2/\pi,\beta} \in \mathscr{C}(J_\infty)$ and $R_{\mathrm{arctan},\alpha,\beta} \in \mathscr{F}(J_\infty)$ if $\alpha > -2/\pi$.   □

Let $R \in \mathscr{C}(J)$, resp. $R \in \mathscr{F}(J_\infty)$. Then the *R-gauge diffeomorphism*

$$
\sigma = \sigma[R] : J \to \mathbb{R}^+
$$

is defined by

$$\sigma(t) := \begin{cases} \operatorname{sign}(t-1) \int_1^t d\tau/R & \text{if } R \in \mathscr{C}(J), \\ \int_1^t \sqrt{1+\dot{R}^2}\, d\tau & \text{if } R \in \mathscr{F}(J_\infty). \end{cases}$$

Note that $\sigma(J) = \mathbb{R}^+$ and $\dot{\sigma}(t) \neq 0$ for $t \in J$. Hence σ is indeed a diffeomorphism whose inverse is written $\tau = \tau[R] := \sigma^{-1} \colon \mathbb{R}^+ \to J$. We define the *R-sequence* (t_j) by $t_j = t_j[R] := \tau(j)$ for $j \in \mathbb{N}$. Then (t_j) is strictly increasing to ∞ if $J = J_\infty$, whereas it strictly decreases to 0 otherwise. For $k \geq 1$ we put

$$I_k = I_k[R] := \begin{cases} (0, t_k] & \text{if } J = J_0, \\ [k, \infty) & \text{if } J = J_\infty. \end{cases}$$

Thus I_k is a cofinal interval of J.

Lemma 5.2. *Suppose $R \in \mathscr{C}(J)$ or[3] $R \in \mathscr{F}(J_\infty)$. Set*

$$r = r[R] := \begin{cases} R & \text{if } R \in \mathscr{C}(J), \\ 1 & \text{if } R \in \mathscr{F}(J_\infty). \end{cases} \tag{5.3}$$

Then r is a singularity function for $(\mathring{J}, dt^2)$ on I_2.

Proof. (1) We set

$$J_j = J_j[R] := \begin{cases} (t_{j+1}, t_{j-1}) & \text{if } J = J_0, \\ (t_{j-1}, t_{j+1}) & \text{if } J = J_\infty. \end{cases}$$

Then J_j is a nonempty open subinterval of $\mathring{J}$ for $j \geq 1$, and $\{J_j \, ; \, j \geq 1\}$ is a covering of $\mathring{J}$ of multiplicity 2. We let

$$\sigma_j := \sigma \,|\, J_j - j, \qquad j \geq 1. \tag{5.4}$$

Then $\mathfrak{S} = \mathfrak{S}[R] := \{\sigma_j \, ; \, j \geq 1\}$ is a normalized atlas, the *R-atlas*, for $\mathring{J}$ of multiplicity 2 which is shrinkable to $1/2$. Note that $\tau_j = \tau_j[R] := \sigma_j^{-1}$ satisfies

$$\tau_j(s) = \tau(s+j), \qquad s \in Q, \quad j \geq 1. \tag{5.5}$$

By (5.4) and (5.5) we see that $\sigma_j \circ \tau_k(s) = s + k - j \in Q$ if $s \in Q$ and $\tau_k(s) \in J_j$. This proves that $\mathfrak{S}$ is uniformly regular on I.

(2) We set $\rho := R \circ \tau = \tau^* R$. Then

$$\dot{\rho} = (\tau^* \dot{R}) \dot{\tau}. \tag{5.6}$$

---

[3] More precisely: $J = J_\infty$ and $R \in \mathscr{F}(J)$.

Furthermore, $\sigma \circ \tau = \mathrm{id}$ implies

$$\dot{\tau} = 1/\tau^* \dot{\sigma} \, . \tag{5.7}$$

(3) Assume $R \in \mathscr{C}(J)$. If $J = J_0$, then $R(0) = 0$ by (5.1)(ii). Thus, for each choice of J,

$$0 < \rho \le c \, . \tag{5.8}$$

Since $\dot{\sigma}(t) = \mathrm{sign}(t-1)/R(t)$ we get from (5.7)

$$\dot{\tau} = \mathrm{sign}(\tau - 1)\rho \, . \tag{5.9}$$

Hence, by (5.6) and setting $\varepsilon := \mathrm{sign}(\tau - 1)$,

$$\dot{\rho} = b_1 \rho \, , \qquad b_1 := \varepsilon \tau^* \dot{R} \in BC(\mathbb{R}^+) \, . \tag{5.10}$$

Furthermore,

$$\dot{b}_1 = \varepsilon(\tau^* \ddot{R})\dot{\tau} = (\tau^* \ddot{R})\rho \in BC(\mathbb{R}^+) \, , \tag{5.11}$$

due to (5.9) and (5.1)(iii). Consequently, we obtain from (5.10)

$$\ddot{\rho} = b_2 \rho \, , \qquad b_2 := \dot{b}_1 + b_1^2 \in BR(\mathbb{R}) \, .$$

By induction

$$\partial^k \rho = b_k \rho \, , \qquad b_k := \dot{b}_{k-1} + b_{k-1} b_1 \, , \quad k \ge 2 \, . \tag{5.12}$$

Thus b_k is a polynomial function in the variables $b_1, \dot{b}_1, \dots, \partial^{k-1} b_1$ with coefficients in $\mathbb{Z}$.

From (5.9)–(5.11) we get

$$\ddot{b}_1 = \varepsilon(\tau^* \partial^3 R)\rho^2 + \tau^*(\ddot{R})(\tau^* \dot{R})\rho \, .$$

Hence we find, once more inductively, that $\partial^\ell b_1$ is a polynomial function in the variables $\rho, \tau^* \partial R, \dots, \tau^* \partial^{\ell+1} R$ with coefficients in $\mathbb{Z}$. Consequently, b_k is a polynomial function in the variables $\rho, \tau^* \partial R, \dots, \tau^* \partial^{k+1} R$. Hence $b_k \in BC(\mathbb{R})$ by (5.8) and (5.1)(iii). Thus we obtain from (5.12)

$$|\partial^k \rho| \le c(k)\rho \, , \qquad k \ge 1 \, . \tag{5.13}$$

It follows from $\partial \log \rho = \dot{\rho}/\rho$ and the last estimate that $\beta := \|\partial \log \rho\|_\infty < \infty$. Hence, by the mean-value theorem,

$$\left| \log\big(\rho(s)/\rho(t)\big) \right| = |\log \rho(s) - \log \rho(t)| \le \beta |s - t| \, , \qquad s, t \ge 0 \, .$$

This implies $\mathrm{e}^{-\beta} \le \rho(s)/\rho(t) \le \mathrm{e}^{\beta}$ for $|s - t| \le 1$, that is,

$$\rho(s) \sim \rho(t) \, , \qquad s, t \in \mathbb{R}^+ \, , \quad |s - t| \le 1 \, . \tag{5.14}$$

Since $\rho_j := \tau_j^* R = \rho(\cdot + j)$ we deduce from (5.14)

$$\rho_j \sim \rho_j(0) , \qquad j \geq 1 . \tag{5.15}$$

Furthermore, since $\partial \rho_j = (\partial \rho)(\cdot + j)$, we obtain from (5.13) and (5.15)

$$\|\partial^k \rho_j\|_\infty \leq c(k)\rho_j(0) , \qquad j \geq 1 , \quad k \geq 0 . \tag{5.16}$$

Due to $R = r$ and $\tau_j^* r = \kappa_{j*} r$ we see from (5.15) and (5.16) that $r \in C\big(\mathring{J}, (0, \infty)\big)$ satisfies (2.7)(iii), (iv) with $\mathfrak{K} = \mathfrak{S}$ and $S = I_2$.

(4) Suppose $R \in \mathscr{F}(J_\infty)$. Since $\dot{\sigma} = (1 + \dot{R}^2)^{1/2}$ we get from (5.7)

$$\dot{\tau} = \big(1 + (\tau^* \dot{R})^2\big)^{-1/2} . \tag{5.17}$$

Using this and $\|\dot{R}\|_\infty < \infty$ we obtain

$$1/c \leq \dot{\tau} \leq 1 . \tag{5.18}$$

From (5.17), (5.18), and (5.2)(ii) we deduce inductively

$$\|\partial^k \tau\|_\infty < \infty , \qquad k \geq 1 . \tag{5.19}$$

Hence (5.5) implies

$$\dot{\tau}_j \sim \mathbf{1} , \qquad \|\partial^k \tau_j\|_\infty \leq c(k) , \qquad j \geq 1 , \quad k \geq 1 . \tag{5.20}$$

Thus $r = \mathbf{1}$ satisfies (2.7)(iv) with $\mathfrak{K} = \mathfrak{S}$ and $S = I_2$, and (2.7)(iii) is trivially true.

(5) Again we assume $R \in \mathscr{C}(J)$ or $R \in \mathscr{F}(J_\infty)$. Then

$$\sigma_{j*} dt^2 = \tau_j^* dt^2 = d\tau_j^2 = \dot{\tau}_j^2 ds^2 . \tag{5.21}$$

If $R \in \mathscr{C}(J)$, then we get $\dot{\tau}_j^2 = \rho_j^2$ from (5.9). Hence

$$\sigma_{j*} dt^2 = \rho_j^2 ds^2 = (r_{\sigma_j})^2 ds^2 , \qquad j \geq 1 .$$

If $R \in \mathscr{F}(J_\infty)$, then we obtain from (5.20) and (5.21)

$$\sigma_{j*} dt^2 \sim ds^2 = (r_{\sigma_j})^2 ds^2 , \qquad j \geq 1 .$$

Hence (2.7)(v) applies to r and $g = dt^2$ with $\mathfrak{K} = \mathfrak{S}$ and $S = I_2$ as well.

(6) Using (5.21), we infer from (5.9) and (5.16) if $R \in \mathscr{C}(J)$, respectively from (5.9) and (5.19) if $R \in \mathscr{F}(J_\infty)$, that

$$\|\partial^k(\sigma_* dt^2)\|_\infty \leq c(k) r_\sigma^2 , \qquad \sigma \in \mathfrak{S}_I , \quad k \geq 0 .$$

Thus (2.7)(vi) is also satisfied. This proves the assertion. $\qquad \square$

6 Model Cusps and Funnels

We suppose $R \in \mathscr{R}(J)$, $0 \le d \le \bar{d}$, and B is a d-dimensional submanifold of $\mathbb{R}^{\bar{d}}$. Let I be an open cofinal subinterval of $\mathring{J}$. We set

$$P(R,B;I) := \left\{ \, \big(t,R(t)y\big) \; ; \; t \in \mathring{J}, \; y \in B \right\} \subset \mathbb{R} \times \mathbb{R}^{\bar{d}} = \mathbb{R}^{1+\bar{d}} \, .$$

Then $P = P(R,B) = P(R,B;\mathring{J})$ is a $(1+d)$-dimensional submanifold of $\mathbb{R}^{1+\bar{d}}$, the (*model*) (R,B)-*pipe on* J, also called (*model*) R-*pipe over* (*the basis*) B *on* J. Note

$$\partial P(R,B) = P(R,\partial B) \, ,$$

where $P(R,\emptyset) := \emptyset$. An R-pipe is an R-*cusp* if $R(\omega) = 0$, where $\omega \in \{0,\infty\}$ and $J = J_\omega$, and an R-*funnel* otherwise. The map

$$\varphi = \varphi[R] \colon P \to \mathring{J} \times B \, , \quad \big(t,R(t)y\big) \mapsto (t,y) \tag{6.1}$$

is a diffeomorphism, the *canonical stretching diffeomorphism* of P.

If $d = 0$, then B is a countable discrete subset of $\mathbb{R}^{\bar{d}}$. In abuse of language and for a unified presentation we call it uniformly regular Riemannian manifold as well and write formally (B,g_B) for B, although g_B has no proper meaning. In this case g_B has to be replaced by 0 in the formulas below.

Suppose $p \in C^\infty\big(P,(0,\infty)\big)$ and g_P is a Riemannian metric for P. Then p is a *cofinal singularity function for* (P,g_P) *on* $S \subset B$ if there exists a cofinal subinterval I of J such that p is a singularity function for (P,g_P) on $\varphi^{-1}(I \times S)$. It follows from Lemma 4.2 that this is then true for every cofinal subinterval of J. In related situations the qualifier 'cofinal' has similar (obvious) meanings.

We consider the following assumption:

$$\begin{aligned} &g_B \text{ is a Riemannian metric for } B, \; S \subset B, \text{ and} \\ &b \text{ is a bounded singularity function for } (B,g_B) \text{ on } S. \end{aligned} \tag{6.2}$$

Lemma 6.1. *Let condition (6.2) apply. Suppose* $a \in C^\infty\big(J,(0,\infty)\big)$ *and* r *is a bounded singularity function for* $(\mathring{J},a\,dt^2)$ *on some cofinal subinterval of* J. *Let* $R \in \mathscr{R}(J)$ *and set*

$$g := \varphi^*(a\,dt^2 + g_B) \, , \quad p := \varphi^*(r \otimes b) \, . \tag{6.3}$$

Then p *is a cofinal singularity function for* (P,g) *on* S.

Proof. By Theorem 3.1 $r \otimes b$ is a cofinal singularity function for $(\mathring{J} \times B, \; a\,dt^2 + g_B)$ on S. Hence the assertion follows from (6.1) and Lemma 3.4. $\qquad\square$

Corollary 6.2. *Put*

$$\hat{g} := \varphi^*\big((a\,dt^2 + g_B)/(r \otimes b)^2\big) \, . \tag{6.4}$$

Then $(P,\hat{g})$ *is cofinally uniformly regular on* S.

The following two propositions are cornerstones for the construction of wide classes of singular manifolds.

Proposition 6.3. *Let (6.2) be satisfied and suppose $R \in \mathscr{C}(J)$ or $R \in \mathscr{F}(J_\infty)$. Set $a := \mathbf{1}$. Define r by (5.3) and g by (6.3). Then p is a cofinal singularity function for (P,g) on S.*

Proof. Lemmas 5.2 and 6.1. $\qquad\qquad\square$

We write $\psi \in \mathscr{R}_0(J,\hat{J})$ if $\psi \in \mathscr{R}(J)$ with $\psi(J) = \hat{J} \in \{J_0, J_\infty\}$ and $\dot{\psi}(t) \neq 0$ for $t \in J$. Thus ψ is a diffeomorphism from J onto $\hat{J}$.

Proposition 6.4. *Suppose (6.2) applies, $\psi \in \mathscr{R}_0(J,\hat{J})$, and $\hat{R} \in \mathscr{C}(\hat{J})$, or $\hat{J} = J_\infty$ and $\hat{R} \in \mathscr{F}(J_\infty)$. Set*

$$R := \psi^* \hat{R}\,, \quad \varphi := \varphi[R]\,, \quad g := \varphi^*(\dot{\psi}^2\, dt^2 + g_B)\,,$$

and

$$r = r[R] := \begin{cases} R & \text{if } \hat{R} \in \mathscr{C}(\hat{J})\,, \\ \mathbf{1} & \text{if } \hat{R} \in \mathscr{F}(J_\infty)\,. \end{cases}$$

Then $p := \varphi^*(r \otimes b)$ is a cofinal singularity function for (P,g) on S.

Proof. We write $\hat{P} := P(\hat{R}, B)$, $\hat{\varphi} := \varphi[\hat{R}]$, and $\hat{g} := \hat{\varphi}^*(ds^2 + g_B)$. Then

$$\Phi := \hat{\varphi}^{-1} \circ (\psi \times \mathrm{id}_B) \circ \varphi : P \to \hat{P} \tag{6.5}$$

is a diffeomorphism and

$$\Phi^* \hat{g} = \varphi^*(\psi^* \times \mathrm{id}_B)\hat{\varphi}_* \hat{g} = \varphi^*(\psi^* \times \mathrm{id}_B)(ds^2 + g_B) = \varphi^*(\dot{\psi}^2\, dt^2 + g_B) = g\,. \tag{6.6}$$

Furthermore, setting $\hat{r} := r[\hat{R}]$ and $\hat{p} := \hat{\varphi}^*(\hat{r} \otimes b)$,

$$\Phi^* \hat{p} = \varphi^*(\psi^* \times \mathrm{id}_B)(\hat{r} \otimes b) = \varphi^*(r \otimes b) = p\,. \tag{6.7}$$

Proposition 6.3 guarantees that $\hat{p}$ is a cofinal singularity function for $(\hat{P}, \hat{g})$ on S. Hence the assertion follows from (6.5)–(6.7) and Lemma 3.4. $\qquad\square$

Now we provide some examples. The most important ones concern α-*pipes*, that is, R_α-pipes over B on J. We write $P_\alpha = P_\alpha(B) := P(\mathsf{R}_\alpha, B)$ and $\varphi_\alpha := \varphi[\mathsf{R}_\alpha]$ for $\alpha \in \mathbb{R}$.

Examples 6.5. Let (6.2) be satisfied.
   (a) Set $g_\alpha := \varphi_\alpha^*(dt^2 + g_B)$,

$$p_\alpha := \varphi_\alpha^*(\mathsf{R}_\alpha \otimes b) \text{ if either } J = J_0 \text{ and } \alpha \geq 1\,, \text{ or } J = J_\infty \text{ and } \alpha < 0\,,$$

and

$$p_\alpha := \varphi_\alpha^*(\mathbf{1} \otimes b) \text{ if } J = J_\infty \text{ and } 0 \leq \alpha \leq 1\,.$$

Then p_α is a cofinal singularity function for (P_α, g_α) on S.

Proof. Example 5.1(a) and Proposition 6.3.   □

   (b) We put
$$p_\alpha := \varphi_\alpha^*(\mathsf{R}_\alpha \otimes b) \text{ if } J = J_0 \text{ and } 0 < \alpha \leq 1 \,,$$

and
$$p_\alpha := \varphi_\alpha^*(\mathbf{1} \otimes b) \text{ if either } J = J_0 \text{ and } \alpha \leq 0 \,, \text{ or } J = J_\infty \text{ and } \alpha \geq 1 \,.$$

We also fix $\beta \neq 0$ such that $0 < \beta \leq \alpha$ if $J = J_0$ and $0 < \alpha \leq 1$, $\beta \geq \alpha$ if $J = J_\infty$ and $\alpha > 1$, and $\beta \leq \alpha$ if $J = J_0$ and $\alpha \leq 0$. Then p_α is a cofinal singularity function for $(P_\alpha, g_{\alpha,\beta})$ on S, where $g_{\alpha,\beta} := \varphi_\alpha^*(t^{2(\beta-1)}\,dt^2 + g_B)$.

Proof. Note that $\mathsf{R}_\beta \in \mathscr{R}_0(J,\hat{J})$ with $\hat{J} = J$ if $\beta > 0$, and $\hat{J} = J_\infty$ if $J = J_0$ and $\beta < 0$. Moreover, $\mathsf{R}_\beta^* \mathsf{R}_\gamma = \mathsf{R}_{\beta\gamma}$ for $\gamma \in \mathbb{R}$.

   We put $\psi := \mathsf{R}_\beta$ and $\hat{R} := \mathsf{R}_{\alpha/\beta}$ so that $\psi^*\hat{R} = \mathsf{R}_\alpha$. It follows from Example 5.1(a) that $\hat{R} \in \mathscr{C}(J_0)$ if $J = J_0$ and $0 < \alpha \leq 1$, and $\hat{R} \in \mathscr{F}(J_\infty)$ otherwise. Moreover, $\dot{\psi} = \dot{\mathsf{R}}_\beta \sim \mathsf{R}_{\beta-1}$. Now the claim follows from Proposition 6.4.   □

   (c) Suppose $J = J_0$ and $R(t) := 1 - \alpha\arctan(1 - 1/t)$ with $\alpha \geq -2/\pi$. Set
$$p := \varphi^*(\mathbf{1} \otimes b) \text{ if } \alpha > -2/\pi \,, \quad p := \varphi^*(R \otimes b) \text{ if } \alpha = -2/\pi \,,$$

and $g := \varphi^*(t^{-4}\,dt^2 + g_B)$. Then p is a cofinal singularity function for (P,g) on S.

Proof. Put $\hat{R} := R_{\arctan,\alpha,1}$ (see Example 5.1(c)) and $\psi := \mathsf{R}_{-1}$. Then $R = \psi^*\hat{R}$ and $\dot{\psi} \sim \mathsf{R}_{-2}$. Hence Example 5.1(c) and Proposition 6.4 imply the assertion.   □

7 Submanifolds of Euclidean Spaces

Now we consider the case where (M,g) is a Riemannian submanifold of $(\mathbb{R}^n, g_n)$ for some $n \in \mathbb{N}^\times$. In other words, we assume

$$(M,g) \hookrightarrow (\mathbb{R}^n, g_n) \,.$$

By Nash's theorem this is no restriction of generality. It is now natural and convenient to describe M by local parametrizations. Hereby, given a local chart κ for M, the map

$$i_\kappa := \iota_M \circ \kappa^{-1} \in C^\infty\big(\kappa(U_\kappa), \mathbb{R}^n\big)$$

is the *local parametrization associated with* κ. The following lemma provides a useful tool for establishing that a given function ρ on M is a singularity function for (M,g).

By a parametrization-regular (p-r) singularity datum for (M,g) on $S \subset M$ we mean a pair $(\rho, \mathfrak{K})$ with the following properties:

$$
\begin{aligned}
&\text{(i)} && \mathfrak{K} \text{ is an atlas for } M \text{ such that } \mathfrak{K}_S \text{ is shrinkable} \\
& && \text{and has finite multiplicity .} \\
&\text{(ii)} && \rho \in C^\infty\big((M,(0,\infty))\big) \text{ satisfies (2.7)(iii) and (iv) .} \\
&\text{(iii)} && \kappa_* g \geq \rho_\kappa^2 g_m/c , \quad \kappa \in \mathfrak{K}_S . \\
&\text{(iv)} && \|\partial^k i_\kappa\|_\infty \leq c(k)\rho_\kappa , \quad \kappa \in \mathfrak{K}_S , \ k \geq 1 ,
\end{aligned}
\tag{7.1}
$$

where ∂ denotes the Fréchet derivative. Clearly, ρ is a *p-r singularity function for* (M,g) *on* S if there exists an atlas $\mathfrak{K}$ such that $(\rho,\mathfrak{K})$ is a p-r singularity datum for (M,g) on S.

Lemma 7.1. *Suppose* $(\rho,\mathfrak{K})$ *is a p-r singularity datum for* (M,g) *on S. Then it is a singularity datum for* (M,g) *on S.*

Proof. (1) In the following, we identify a linear map $a\colon \mathbb{R}^m \to \mathbb{R}^n$ with its representation matrix $[a] \in \mathbb{R}^{n \times m}$ with respect to the standard bases. Then

$$
\kappa_* g = \kappa_*(\iota_M^* g_n) = i_\kappa^* g_n = (\partial i_\kappa)^\top \partial i_\kappa , \qquad \kappa \in \mathfrak{K} .
\tag{7.2}
$$

From this and (7.1)(iii) and (iv) it follows

$$
\kappa_* g \sim \rho_\kappa^2 g_m , \quad \|\kappa_* g\|_{k,\infty} \leq c(\kappa)\rho_\kappa^2 , \qquad \kappa \in \mathfrak{K}_S , \quad k \in \mathbb{N} .
\tag{7.3}
$$

Hence $[\kappa_* g]$ has its spectrum in $[\rho_\kappa^2/c,\ c\rho_\kappa^2] \subset \mathbb{R}$ for $\kappa \in \mathfrak{K}_S$. Consequently, the spectrum of $[\kappa_* g]^{-1}$ is contained in $[\rho_\kappa^{-2}/c,\ c\rho_\kappa^{-2}]$ for $\kappa \in \mathfrak{K}_S$. This implies

$$
\|[\kappa_* g]^{-1}\|_\infty \leq c/\rho_\kappa^2 , \qquad \kappa \in \mathfrak{K}_S .
$$

Thus, by the chain rule and (7.3), it follows

$$
\|[\kappa_* g]^{-1}\|_{k,\infty} \leq c(k)\rho_\kappa^{-2} , \qquad \kappa \in \mathfrak{K}_S , \quad k \in \mathbb{N} .
\tag{7.4}
$$

(2) We set

$$
\Lambda_\kappa(x) := [\kappa_* g]^{-1}(\partial i_\kappa)^\top \in \mathbb{R}^{m \times n} , \qquad x \in Q_\kappa^m , \quad \kappa \in \mathfrak{K}_S .
$$

Then (7.1)(iv) and (7.4) imply

$$
\|\Lambda_\kappa\|_{k,\infty} \leq c(k)\rho_\kappa^{-1} , \qquad \kappa \in \mathfrak{K}_S , \quad k \in \mathbb{N} .
\tag{7.5}
$$

Given $\kappa \in \mathfrak{K}_S$ and $p \in U_\kappa$,

$$
T_p M = \{p\} \times \partial i_\kappa\big((\kappa(p))(\mathbb{R}^m)\big) \hookrightarrow \{p\} \times \mathbb{R}^n = T_p\mathbb{R}^n .
\tag{7.6}
$$

We read off (7.2) that Λ_κ is a left inverse for ∂i_κ. Furthermore,

$$
\ker(\Lambda_\kappa) = \ker\big((\partial i_\kappa)^\top\big) = \big(\mathrm{im}(\partial i_\kappa)\big)^\perp .
$$

It follows from this and (7.6) that $T_p\kappa$, the tangential of κ at p, is given by

$$T_p\kappa\colon T_pM \to T_{\kappa(p)}\mathbb{R}^m\,,\quad (p,\xi)\mapsto \big(\kappa(p),\Lambda_\kappa(\kappa(p))\xi\big)$$

(cf. [5, Remark 10.3(d)]). Thus we find $\partial(\tilde\kappa\circ\kappa^{-1}) = \Lambda_{\tilde\kappa}\partial i_\kappa$ for $\kappa,\tilde\kappa \in \mathfrak{K}_S$ with $U_\kappa \cap U_{\tilde\kappa} \neq \emptyset$. Hence (7.1)(iv) and (7.5) imply

$$\|\tilde\kappa\circ\kappa^{-1}\|_{k,\infty} \leq c(k)\,,\qquad \kappa,\tilde\kappa \in \mathfrak{K}_S\,,\quad k\in\mathbb{N}\,,$$

due to $\mathrm{im}(\tilde\kappa\circ\kappa^{-1}) \subset Q^m$. Thus, recalling (7.1)(i), we see that $\mathfrak{K}$ is uniformly regular on S. This proves the claim. $\qquad\square$

In the next lemma we consider a particularly simple, but important, p-r regular singularity datum. In this special situation it is the converse of the preceding lemma.

Lemma 7.2. *Suppose $S \subset M$ and S is compact in $\mathbb{R}^n$. If $(M,\mathfrak{K},g)$ is uniformly regular on S, then $(\mathbf{1},\mathfrak{K})$ is a p-r singularity datum for (M,g) on S.*

Proof. Due to the hypotheses, conditions (7.1)(i)–(iii) are trivially satisfied with $\rho = \mathbf{1}$.

For each $p \in S$ there is a normalized local chart φ_p for $\mathbb{R}^n$ such that $\varphi_p(p) = 0$,

$$\|\varphi_p^{-1}\|_{k,\infty} \leq c(k,p)\,,\qquad k\in\mathbb{N}\,,\tag{7.7}$$

and $\kappa_p := \varphi_p\,|\,(M\cap U_{\varphi_p})$ is a normalized local chart for M with $\kappa_p(p) = 0 \in \mathbb{R}^m$. By the compactness of S in $\mathbb{R}^n$ there exists a finite subset P of S such that $\big\{\, U_p := \mathrm{dom}(\kappa_p)\,;\ p\in P\,\big\}$ is an open covering of S in M. We set $\hat{\mathfrak{K}} := \{\, \kappa_p\,;\ p\in P\,\}$ and $\tilde{\mathfrak{K}} := \hat{\mathfrak{K}}\cup(\mathfrak{K}\backslash\mathfrak{K}_S)$. Then $\tilde{\mathfrak{K}}$ is an atlas for M and $\tilde{\mathfrak{K}}_S = \hat{\mathfrak{K}}$. For $p\in P$ we define $f_p := \varphi_p^{-1}\colon Q^n_{\varphi_p}\to\mathbb{R}^n$. Then $i_{\kappa_p} = f_p\,|\,Q_{\kappa_p}$, where $\mathbb{R}^m$ is identified with the subspace $\mathbb{R}^m\times\{0\}$ of $\mathbb{R}^n$, of course. Since $\tilde{\mathfrak{K}}_S$ is finite, it is obvious from (7.7) that

$$\|\partial^k i_\kappa\|_\infty \leq c(k)\,,\qquad \kappa\in\tilde{\mathfrak{K}}_S\,,\quad k\geq 1\,.\tag{7.8}$$

By the same reason, and since $\mathfrak{K}$ has finite multiplicity on S, we see that $\tilde{\mathfrak{K}}\underset{S}{\approx}\mathfrak{K}$. Hence (7.8) holds for $\mathfrak{K}_S$ as well, that is, condition (7.1)(iv) is valid also. $\qquad\square$

Now we return to the setting of the preceding section. It follows from Corollary 6.2 and Proposition 6.3 that, given $R\in\mathscr{C}(J)$ or $R\in\mathscr{F}(J_\infty)$, the R-pipe $P = P(R,B)$ can be equipped with countably many nonequivalent metrics which make it into a cofinally uniformly regular Riemannian manifold. However, since $\iota_P\colon P\hookrightarrow\mathbb{R}^{1+\bar d}$, it is most natural to endow P with the metric $g_P := \iota_P^*g_{1+\bar d}$. The following proposition gives sufficient conditions guaranteeing that g in Proposition 6.3 can be replaced by g_P.

Proposition 7.3. *Suppose (B,g_B) is a d-dimensional bounded Riemannian submanifold of $(\mathbb{R}^{\bar d},g_{\bar d})$ and b is a p-r singularity function for (B,g_B) on $S\subset B$. Also suppose $R\in\mathscr{C}(J)$ or $R\in\mathscr{F}(J_\infty)$ and define r by (5.3). Then $p = \varphi^*(R\otimes b)$ is a cofinal p-r singularity function for (P,g_P) on S.*

Proof. (1) Let $\mathfrak{B}$ be an atlas for B such that $(b, \mathfrak{B})$ is a p-r singularity datum for (B, g) on S, and let $\mathfrak{S}$ be the R-atlas for $\mathring{J}$. We write

$$f := \iota_P \circ \varphi^{-1} \colon \mathring{J} \times B \to \mathbb{R}^{1+\bar{d}}, \quad Y_\beta := i_\beta,$$

and use the notations of Section 5. Then

$$f_{j\beta} := f \circ (\tau_j \times \beta^{-1}) = (\tau_j, \rho_j Y_\beta) \colon Q \times \beta(U_\beta) \to \mathbb{R}^{1+\bar{d}}$$

is a diffeomorphism onto an open subset $U_{j\beta}$ of P. We denote by ϖ the permutation $\mathbb{R}^{1+d} \to \mathbb{R}^{d+1}$, $(t, y) \mapsto (y, t)$ (which is only needed if $\partial B = \emptyset$). Then, see (6.1),

$$\kappa_{j\beta} := \varpi \circ f_{j\beta}^{-1}(\beta, \tau_j) \colon U_{j\beta} \to \beta(U_\beta) \times Q$$

is a local chart for P and $f_{j\beta} = i_{\kappa_{j\beta}}$. We set

$$\mathfrak{K} := \{ \kappa_{j\beta} \; ; \; j \geq 1, \; \beta \in \mathfrak{B} \} = \varphi^*\big(\varpi \circ (\mathfrak{B} \otimes \mathfrak{S})\big)$$

where $\mathfrak{B} \otimes \mathfrak{S}$ is the product atlas on $B \times \mathring{J}$ and

$$\varpi \circ (\mathfrak{B} \otimes \mathfrak{S}) := \{ \sigma \times \beta \; ; \; \beta \in \mathfrak{B}, \; \sigma \in \mathfrak{S} \} .$$

By Lemma 5.2 we know that $\mathfrak{S}$ is uniformly regular on $I := I_2[R]$. Hence $\mathfrak{B} \otimes \mathfrak{S}$ is uniformly regular on $S \times I$ by Theorem 3.1. From this and Lemma 3.4 it follows that $\mathfrak{K}$ is a uniformly regular atlas for P on $V := \varphi^{-1}(I \times S)$.

(2) Given $\kappa = \kappa_{j\beta} \in \mathfrak{K}$,

$$\begin{aligned}
\kappa_* g_P &= \kappa_* \iota_P^* g_{1+\bar{d}} = \big(\iota_P \circ \kappa^{-1}\big)^* g_{1+\bar{d}} \\
&= f_{j\beta}^* (\mathrm{d}t^2 + |\mathrm{d}y|^2) = \mathrm{d}\tau_j^2 + |\mathrm{d}(\rho_j Y_\beta)|^2 .
\end{aligned}$$

Hence $\mathrm{d}(\rho_j Y_\beta) = \dot{\rho}_j \, \mathrm{d}s \, Y_\beta + \rho_j \, \mathrm{d}Y_\beta$ implies

$$\kappa_* g_P = (\dot{\tau}_j^2 + \dot{\rho}_j^2 |Y_\beta|^2) \, \mathrm{d}s^2 + 2\rho_j \dot{\rho}_j \, \mathrm{d}s \, (Y_\beta \,|\, \mathrm{d}Y_\beta) + \rho_j^2 \, |\mathrm{d}Y_\beta|^2 .$$

Using $|\mathrm{d}Y_\beta|^2 = \beta_* g_B$ and estimating the next to the last term by the Cauchy-Schwarz inequality gives

$$\kappa_* g_P \geq (\dot{\tau}_j^2 + (1 - 1/\varepsilon)\dot{\rho}_j^2 |Y_\beta|^2) \, \mathrm{d}s^2 + (1 - \varepsilon)\rho_j^2 \beta_* g_B$$

for $0 < \varepsilon < 1$, $j \geq 1$, and $\beta \in \mathfrak{B}$.

(3) Suppose $R \in \mathscr{C}(J)$. Then

$$\dot{\tau}_j^2 = \rho_j^2 \tag{7.9}$$

by (5.9), and $\dot{\rho}_j^2 \leq c\rho_j^2$ by (5.15) and (5.16). Thus the boundedness of B in $(\mathbb{R}^{\bar{d}}, g_{\bar{d}})$ implies that we can choose ε sufficiently close to 1 such that

$$\kappa_* g_B \geq \rho_j^2 (\mathrm{d}s^2 + \beta_* g_B)/c \geq \rho_j^2 (\mathrm{d}s^2 + b_\beta^2 g_d)/c \tag{7.10}$$

for $j \geq 1$, $\beta \in \mathfrak{B}_S$, and $\kappa = \kappa_{j\beta}$, where the last inequality holds since $(b, \mathfrak{K})$ is a p-r singularity datum for (B, g_B) on S.

(4) Assume $R \in \mathscr{F}(J_\infty)$. Then (5.18) implies

$$\dot{\tau}_j \sim \mathbf{1}, \qquad j \geq 1.$$

From this and

$$\dot{\rho}_j = (\dot{R} \circ \tau_j)\dot{\tau}_j, \qquad \|\dot{R}\|_{k,\infty} < \infty, \quad k \in \mathbb{N},$$

we get

$$\|\dot{\rho}_j\|_{k,\infty} \leq c(k), \qquad j \geq 1, \quad k \in \mathbb{N}. \tag{7.11}$$

Thus, similarly as above,

$$\kappa_* g_B \geq \left(\mathrm{d}s^2 + \rho_j^2(0)b_\beta^2 g_d\right)\big/c, \qquad j \geq 1, \quad \beta \in \mathfrak{B}_S, \quad \kappa = \kappa_{j\beta}. \tag{7.12}$$

(5) Now we proceed analogously to the proof of Theorem 3.1. Recalling that $\mathfrak{S}$ is shrinkable to $1/2$ on I we fix $r \in (1/2, 1)$ such that $\left\{\kappa^{-1}(rQ_\kappa^{d+1}) \, ; \, \kappa \in \mathfrak{K}_V\right\}$ is a covering of V. Then we set $\delta := (1-r)/\sqrt{d+1}$,

$$\delta_\beta := \min\{b_\beta, \delta\}, \quad \delta_j := \min\{1/R(t_j), \delta\},$$

and

$$\mathfrak{L}_\beta := \mathfrak{L}(\delta_\beta, Q), \quad \mathfrak{L}_j := \mathfrak{L}(\delta_j, Q_\beta^d)$$

for $\beta \in \mathfrak{B}_S$ and $j \geq 1$. Note that the boundedness of b implies

$$\delta_\beta \sim b_\beta, \qquad \beta \in \mathfrak{K}_S. \tag{7.13}$$

Furthermore,

$$\delta_j \sim 1/\rho_j(0), \quad j \geq 1, \quad \text{if } r \in \mathscr{F}(J_\infty), \tag{7.14}$$

since $R(t_j) = \rho_j(0)$ and $1/R \leq c$ in this case.

Given $\kappa = \kappa_{j\beta} \in \mathfrak{K}_V$, we define

$$\mathfrak{N}_\kappa := \begin{cases} \{\mu \times \lambda \, ; \, \lambda \in \mathfrak{L}_\beta, \, \mu := \mathrm{id}_{Q_\beta^d}\} & \text{if } R \in \mathscr{C}(J), \\[2mm] \{\mu \times \lambda \, ; \, \lambda \in \mathfrak{L}_\beta, \, \mu \in \mathfrak{L}_j\} & \text{if } R \in \mathscr{F}(J_\infty). \end{cases}$$

Then $\mathfrak{N}_\kappa$ is an atlas for Q_κ^{d+1} which is uniformly regular on rQ_κ^{d+1}. Consequently, cf. (3.4),

$$\mathfrak{P} := \{\nu \circ \kappa \, ; \, \kappa \in \mathfrak{K}_V, \, \nu \in \mathfrak{N}_\kappa\} \cup (\mathfrak{K} \setminus \mathfrak{K}_V)$$

is an atlas for P which is uniformly regular on V. Observe

$$\mathfrak{P}_V \subset \{\nu \circ \kappa \, ; \, \kappa \in \mathfrak{K}_V, \, \nu \in \mathfrak{N}_\kappa\}.$$

Hence condition (7.1)(i) is satisfied.

(6) By the assumption on $(b, \mathfrak{B})$

$$\|\beta_* b\|_{k,\infty} \leq c(k) b_\beta , \quad b \,|\, U_\beta \sim b_\beta , \qquad \beta \in \mathfrak{B}_S , \quad k \geq 0 . \tag{7.15}$$

Furthermore,

$$\|\rho_j\|_{k,\infty} \leq c(k)\rho_j(0) , \quad \rho_j \,|\, J_j \sim \rho_j(0) , \qquad j \geq 1 , \quad k \geq 0 . \tag{7.16}$$

Indeed, if $R \in \mathscr{C}(J)$, then this is a consequence of (5.16) and (5.15), respectively. If $R \in \mathscr{F}(J_\infty)$, then $\rho_j(0) = R(t_j) \geq 1/c$ for $j \geq 1$. Hence (7.16) follows from (7.11).

We deduce from (3.10) that

$$b_\beta = \beta_* b(0) = \kappa_* b(0) \sim (\nu \circ \kappa)_* b(0) = \pi_* b(0) \tag{7.17}$$

and

$$\rho_j(0) = (\sigma_j)_* R(0) = \kappa_* R(0) \sim (\nu \circ \kappa)_* R(0) = \pi_* R(0) \tag{7.18}$$

for $\pi = \nu \circ \kappa \in \mathfrak{P}_V$ with $\kappa = \kappa_{j\beta} \in \mathfrak{K}_V$ and $\nu \in \mathfrak{L}_\kappa$. From (7.15)–(7.18) we derive

$$\|\pi_* p\|_{k,\infty} \leq c(k) p_\pi , \quad p \,|\, U_\pi \sim p_\pi , \qquad \pi \in \mathfrak{P}_V , \quad k \geq 0 .$$

Thus condition (7.1)(ii) applies.

(7) From (7.10), (7.12), (7.17), (7.18), and (3.8) we get

$$\pi_* g_P \geq \rho_j^2(0)(\delta_\beta^2 \, ds^2 + b_\beta^2 g_d)/c \quad \text{if } R \in \mathscr{C}(J) ,$$

respectively

$$\pi_* g_P \geq \left(\delta_\beta^2 \, ds^2 + \rho_j^2(0)\delta_j^2 b_\beta^2 g_d \right)/c \quad \text{if } R \in \mathscr{F}(J_\infty) ,$$

for $\pi = \nu \circ \kappa \in \mathfrak{P}_V$ with $\kappa = \kappa_{j\beta}$ and $\nu \in \mathfrak{N}_\kappa$. From this, (7.13), and (7.14) we obtain in either case $\pi_* g_P \geq p_\pi^2 g_{1+d}/c$ for $\pi \in \mathfrak{P}_V$. Thus condition (7.1)(iii) is fulfilled.

(8) By the assumption on $(p, \mathfrak{B})$

$$\|\partial^\alpha Y_\beta\|_\infty \leq c(\alpha) b_\beta , \qquad \beta \in \mathfrak{B}_S , \quad \alpha \in \mathbb{N}^d \setminus \{0\} . \tag{7.19}$$

Given $\pi = \nu \circ \kappa \in \mathfrak{P}_V$ with $\kappa = \kappa_{j\beta} \in \mathfrak{K}_V$ and $\nu = \mu \times \lambda \in \mathfrak{N}_{\kappa_{j\beta}}$,

$$i_\pi = i_\kappa \circ \nu^{-1} = \left(\lambda_* \tau_j, (\lambda_* \rho_j) \mu_* Y_\beta \right) . \tag{7.20}$$

Suppose $R \in \mathscr{C}(J)$. Then we get from (7.9) and (3.2)

$$\|\partial^k(\lambda_* \tau_j)\|_\infty = \delta_\beta \, \|\partial^{k-1}(\lambda_* \rho_j)\|_\infty = \delta_\beta^k \, \|\lambda_*(\partial^{k-1}\rho_j)\|_\infty \leq c(k) b_\beta \rho_j(0) \tag{7.21}$$

for $j, k \geq 1$ and $\beta \in \mathfrak{B}_S$, due to $0 < \delta_\beta \leq 1$ and (7.13). By means of (7.17)–(7.21) and $\mu = \mathrm{id}$ we deduce

$$\|\partial^\alpha i_\pi\|_\infty \leq c(\alpha) p_\pi , \qquad \pi \in \mathfrak{P}_V , \quad \alpha \in \mathbb{N}^{1+d} \setminus \{0\} , \tag{7.22}$$

if $R \in \mathscr{C}(J)$.

Assume $R \in \mathscr{F}(J_\infty)$. Then (5.20) and the definition of r imply, similarly as above,

$$\|\partial^k(\lambda_* \tau_j)\|_\infty \leq c(k) b_\beta \leq c(k) p_\pi$$

for $\pi = \nu \circ \kappa \in \mathfrak{P}_V$, $\kappa = \kappa_{jp}$, and $\nu \in \mathfrak{N}_\kappa$. Analogously, we get from (7.11)

$$\|\partial^k(\lambda_* \rho_j)\|_\infty \leq c(k) p_\pi , \qquad \pi = \nu \circ \kappa \in \mathfrak{P}_V , \quad \kappa = \kappa_{j\beta} , \quad \nu \in \mathfrak{N}_\kappa , \tag{7.23}$$

for $k \geq 1$. Finally, similar arguments invoking (7.19) lead to

$$\|\partial^\alpha(\mu_* Y_\beta)\|_\infty \leq c(\alpha) \delta_j p_\pi , \qquad \alpha \in \mathbb{N}^{1+d} \setminus \{0\} , \tag{7.24}$$

for $\pi = \nu \circ \kappa \in \mathfrak{P}_V$ with $\kappa = \kappa_{j\beta}$ and $\nu \in \mathfrak{N}_\kappa$. By (7.11) $|\rho_j(s) - \rho_j(0)| \leq c$ for $s \in Q$ and $j \geq 1$. Hence

$$1 - c/\rho_j(0) \leq \rho_j(s)/\rho_j(0) \leq 1 + c/\rho_j(0) , \qquad s \in Q , \quad j \geq 1 . \tag{7.25}$$

Assume $R(\infty) < \infty$. Then $1/c \leq \rho_j(s) \leq c$ for $s \in Q$ and $j \geq 1$. In this case it is obvious that

$$\rho_j \sim \rho_j(0) , \qquad j \geq 1 . \tag{7.26}$$

If, however, $R(\infty) = \infty$, then we see from (7.25) that there exists j_0 such that (7.26) holds for $j \geq j_0$. As above, we observe that (7.26) applies for $1 \leq j \leq j_0$ also. Thus (7.26) is true in general. Using this we infer from (7.14) and (7.24) that

$$\|(\lambda_* \rho_j)\partial^\alpha(\mu_* Y_\beta)\|_\infty \leq c(\alpha) p_\pi , \qquad \alpha \in \mathbb{N}^d \setminus \{0\} ,$$

for $\pi = \nu \circ \kappa \in \mathfrak{P}_V$ with $\kappa = \kappa_{j\beta}$ and $\nu \in \mathfrak{N}_\kappa$. Moreover, (7.23), (7.24), $0 < \delta_j \leq 1$, and the boundedness of b guarantee

$$\|\partial^k(\lambda_* \rho_j)\partial^\alpha(\mu_* Y_\beta)\|_\infty \leq c(k, \alpha) p_\pi , \qquad k \geq 1 , \quad \alpha \in \mathbb{N}^d ,$$

for $\pi = \nu \circ \kappa \in \mathfrak{P}_V$ with $\kappa = \kappa_{j\beta}$ and $\nu \in \mathfrak{N}_\kappa$. Here we also use the boundedness of B in $\mathbb{R}^{\bar{d}}$ if $\alpha = 0$. This implies that estimate (7.22) holds in this case as well. Hence condition (7.1)(iv) is also satisfied. This proves the assertion.   □

Remark 7.4. Let the hypotheses of Proposition 7.3 be satisfied with $R \in \mathscr{C}(J_0)$. Set $(B_1, g_{B_1}) := (P, g_P)$, $\bar{d}_1 := 1 + \bar{d}$, $b_1 := p$, and $S_1 := V = \varphi^{-1}(I \times S)$. Then (B_1, g_{B_1}) is a bounded Riemannian submanifold of $(\mathbb{R}^{\bar{d}_1}, g_{\bar{d}_1})$ and b_1 is a bounded p-r singularity function for (B_1, g_{B_1}) on S_1.

Fix $J_1 \in \{J_0, J_\infty\}$ and $R_1 \in \mathscr{C}(J_1)$, resp. $R_1 \in \mathscr{F}(J_\infty)$. Set $r_1 := R_1$ if $R_1 \in \mathscr{C}(J_1)$, resp. $r_1 := \mathbf{1}$ if $R_1 \in \mathscr{F}(J_1)$. Denote by $\varphi_1 : P_1 = P(R_1, B_1) \to \mathring{J}_1 \times B_1$ the canonical

stretching diffeomorphism of P_1 and set $g_{P_1} := \iota_{P_1}^* g_{1+\bar{d}_1}$. Then Proposition 7.3 applies to guarantee that $p_1 := \varphi_1^*(r_1 \otimes b_1)$ is a cofinal singularity function for (P_1, g_{P_1}) on S_1. In particular, $(P_1, g_{P_1}/p_1^2)$ is cofinally uniformly regular and, given cofinal subintervals I_1 of J_1 and I of J, resp.,

$$\varphi_{1*}(g_{P_1}/p_1^2) \underset{I_1 \times I \times S}{\sim} (r_1 \otimes R \otimes b_1)^{-2}(\mathrm{d}s_1^2 + \mathrm{d}s^2 + g_B)$$

and $\varphi_1(P_1) = \mathring{J}_1 \times \mathring{J} \times B$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This remark shows that we can iterate Proposition 7.3 to handle 'higher order' singularities, e.g. cuspidal corners or funnels with edges.

8 Singular Ends

Throughout this section, (M, g) is an m-dimensional Riemannian manifold and $J \in \{J_0, J_\infty\}$.

Suppose:

(i) $R \in \mathscr{C}(J)$, $\ell \in \{1, \ldots, m\}$, $\bar{\ell} \geq \ell$.

(ii) (B, g_B) is a compact $(\ell - 1)$-dimensional

       Riemannian submanifold of $\mathbb{R}^{\bar{\ell}}$.

(iii) (Γ, g_Γ) is a compact connected $(m - \ell)$-dimensional

       Riemannian manifold without boundary .

Then

$$W = W(R, B, \Gamma) := P(R, B) \times \Gamma$$

is the *smooth model Γ-wedge over the (R, B)-pipe $P = P(R, B)$*. It is a submanifold of $\mathbb{R}^{1+\bar{\ell}} \times \Gamma$ of dimension m, and $\partial W = \partial P \times \Gamma$. If $\ell = m$, then Γ is a one-point space and W is naturally identified with P (equivalently: there is no (Γ, g_Γ)). Thus every pipe is also a wedge. This convention allows for a uniform language by speaking, in what follows, of wedges only. Given a cofinal subinterval I of J, we set

$$W[I] := P(R, B; I) \times \Gamma .$$

We fix a Riemannian metric h_P for P and set $g_W := h_P + g_\Gamma$.

Let V be open in M. Then (V, g), more loosely: V, is a *smooth wedge of type* (W, g_W) in (M, g) if it is isometric to (W, g_W). More precisely, (Vg) is said to be *modeled* by $[\Phi, W, g_W]$ if Φ is an isometry from (V, g) onto (W, g_W), a *modeling isometry* for (V, g).

Assume

$\{V_0, V_1, \ldots, V_k\}$ is a finite open covering of M such that

$$
\begin{array}{lll}
\text{(i)} & V_i \cap V_j = \emptyset, \ 1 \le i < j \le k\,; & \\
\text{(ii)} & V_0 \cap V_i \text{ is a relatively compact for } 1 \le i \le k\,; & (8.1) \\
\text{(iii)} & (V_i, g) \text{ is a smooth wedge in } (M, g) \text{ for } 1 \le i \le k\,. &
\end{array}
$$

Then (M, g) is a Riemannian *manifold with* (finitely many) *smooth singularities*.

The following theorem is the main result of this paper. It is shown thereafter that we can derive from it all results stated in the introduction—and many more—by appropriate choices of the modeling data.

Theorem 8.1. *Suppose (M, g) is a Riemannian manifold with smooth singularities. Let ρ_0 be a singularity function for (M, g) on V_0 and assume that ρ_i is a cofinal singularity function for (V_i, g), $1 \le i \le k$. Then there exists a singularity function ρ for (M, g) such that $\rho \sim \rho_j$ on V_j for $0 \le j \le k$. Thus $(M, g/\rho^2)$ is uniformly regular.*

Proof. Suppose (V_i, g) is modeled by $[\Phi_i, W_i, g_i]$ for $1 \le i \le k$, where we write W_i for $W(R_i, B_i, \Gamma_i)$ with $R_i \in \mathscr{R}(J_i)$ and $g_i := g_{W_i}$. Given a cofinal subinterval I_i of J_i, we set $S_i := \Phi_i^{-1}\big(W_i[I_i]\big)$. By the relative compactness of $V_0 \cap V_i$ we can find a closed subset S_0 of V_0 such that $S_0 \supset V_0 \setminus \bigcup_{i=1}^{k} V_i$ and $\mathrm{dist}(S_0 \cap V_i, V_i \setminus V_0) > 0$ as well as closed cofinal subintervals I_i of J_i, $1 \le i \le k$, such that $\{S_0, S_1, \ldots, S_k\}$ is a covering of M. By the assumptions on ρ_j, $0 \le j \le k$, we can find atlases $\mathfrak{K}_j$, $0 \le j \le k$, such that $(\rho_j, \mathfrak{K}_j)$ is a singularity datum for (V_j, g_j) on S_j. Since $V_0 \cap V_i$ is relatively compact it follows that $\rho_0 \sim \mathbf{1}$ and $\rho_i \sim \mathbf{1}$ on $V_0 \cap V_i$ for $1 \le i \le k$. Thus $\rho_i \,|\, (V_i \cap V_j) \sim \rho_j \,|\, (V_i \cap V_j)$ for $0 \le i < j \le k$, due to (8.1)(i). Note that $S_0 \cap S_i$ is relatively compact in $V_0 \cap V_i$. Hence we can assume that $\mathfrak{K}_{i, \, S_0 \cap S_i}$ is finite for $1 \le i \le k$. From this and (8.1)(i) it is clear that condition (v) of Lemma 3.3 is satisfied. Hence that lemma guarantees the validity of the assertion. $\qquad\square$

Let (V, g) be a smooth wedge in (M, g) modeled by $[\Phi, W, g_W]$. Then $W = P \times \Gamma$ with $P = P[R, B]$, and $\varphi = \varphi[R]$ is the canonical stretching isometry from (P, h_P) onto $(J \times B, \ \varphi_* h_P)$. Hence

$$
\Psi := (\varphi \times \mathrm{id}_\Gamma) \circ \Phi \colon (V, g) \mapsto (J \times B \times \Gamma, \ \varphi_* h_P + g_\Gamma) \tag{8.2}
$$

is a modeling isometry for (V, g). Since B and Γ are compact, $\mathbf{1}_B$ and $\mathbf{1}_\Gamma$ are singularity functions for B and Γ, respectively. Suppose $r \in C^\infty\big(J, (0, \infty)\big)$. Then $r \otimes \mathbf{1}_B \otimes \mathbf{1}_\Gamma$ is the 'constant extension' of r over $J \times B \times \Gamma$. It satisfies

$$
(\varphi \times \mathrm{id}_\Gamma)^*(r \otimes \mathbf{1}_B \otimes \mathbf{1}_\Gamma)(t, y, z) = r(t)\,, \qquad (t, y, z) \in J \times B \times \Gamma\,.
$$

Thus, in abuse of notation, we set

$$
\Phi^* r := \Psi^*(r \otimes \mathbf{1}_B \otimes \mathbf{1}_\Gamma) \tag{8.3}
$$

without fearing confusion. In other words: we identify r with its point-wise extension over $P \times \Gamma$.

Proposition 8.2. *Let (V,g) be a smooth wedge in (M,g) modeled by $[\Phi,W,g_W]$. Assume that one of the following conditions is satisfied:*

(i) *$R \in \mathscr{C}(J)$ or $R \in \mathscr{F}(J_\infty)$, $h_P = g_P$, and $r := R$ if $R \in \mathscr{C}(J)$, whereas $r = \mathbf{1}$ otherwise.*

(ii) (α) *$J = J_0$, $\alpha \in (-\infty,1]$, and $R = \mathsf{R}_\alpha$.*
  (β) *$\beta \neq 0$ and satisfies $\beta \leq \alpha$ with $\beta > 0$ if $\alpha > 0$.*
  (γ) *$h_P = \varphi_\alpha^*(t^{2(\beta-1)}\mathrm{d}t^2 + g_B)$.*
  (δ) *$r := \mathsf{R}_\alpha$ if $0 < \alpha \leq 1$ and $r := \mathbf{1}$ otherwise.*

Then $\rho := \Phi^*r$ is a cofinal singularity function for (V,g).

Proof. Suppose p is a cofinal singularity function for (P,h_P). Then $p \otimes \mathbf{1}_\Gamma$ is one for $W = P \times \Gamma$, due to Theorem 3.1.

If (i) is satisfied, then Lemma 7.1 and Proposition 7.3 guarantee that $\varphi^*(R \otimes \mathbf{1}_B)$ is a cofinal singularity function for (P,g_P).

Let (ii) apply. Then it follows from Example 6.5(b) that $\varphi_\alpha^*(r \otimes \mathbf{1}_B)$ is a cofinal singularity function for (P_α,h_P). Now the considerations preceding the proposition imply the claims. $\square$

For the next lemma we recall definition (1.6) where now $\mathscr{M}$ is replaced by M.

Lemma 8.3. *Suppose $R \in \mathscr{C}(J_\infty) \cup \mathscr{F}(J_\infty)$. Let (V,g) be a smooth wedge in (M,g) modeled by $[\Phi,P(R,B),g_{P(R,B)}]$. If $R \in \mathscr{C}(J_\infty)$, then there exists a cofinal singularity function ρ for (V,g) satisfying $\rho \sim R \circ \delta_V$. If $R \in \mathscr{F}(J_\infty)$, then (V,g) is cofinally uniformly regular.*

Proof. Suppose $R \in \mathscr{C}(J_\infty)$. Then Φ^*R is a cofinal singularity function for (V,g) by Proposition 8.2(i). Since Φ is an isometry it follows $\Phi^*R \sim R \circ \delta_V$. This implies the assertion in the present case. If $R \in \mathscr{F}(J_\infty)$, then the claim follows also from the cited proposition. $\square$

Proof of Theorem 1.2. The foregoing lemma shows that a tame end is cofinally uniformly regular. Let $\{V_0,V_1,\ldots,V_k\}$ be an open covering of (M,g) as in the definition preceding Theorem 1.2. Then we can shrink V_0 slightly to $\tilde{V}_0$ such that $\{\tilde{V}_0,V_1,\ldots,V_k\}$ is still an open covering and Lemma 4.2 applies to guarantee that (M,g) is uniformly regular on $\tilde{V}_0$. Now the assertion follows from Theorem 8.1. $\square$

With the help of Theorem 8.1 and Proposition 8.2 it is easy to construct uniformly regular Riemannian metrics in a great variety of geometric constellations. We leave this to the reader and proceed to study manifolds with smooth cuspidal singularities. For this we suppose:

(i) $(\mathscr{M},g)$ is an m-dimensional Riemannian manifold .
(ii) (Γ,g_Γ) is a compact connected Riemannian submanifold
       of $(\mathscr{M},g)$ without boundary and codimension $\ell \geq 1$.   (8.4)
(iii) $\Gamma \subset \partial \mathscr{M}$ if $\Gamma \cap \partial \mathscr{M} \neq \emptyset$.

In the following, we use the notation preceding definition (1.7). First we assume $\Gamma \subset \mathring{\mathscr{M}}$. Then there exists a uniform open tubular neighborhood $\mathscr{U}$ of Γ in $\mathring{\mathscr{M}}$ (e.g.

M.W. Hirsch [19] or A.A. Kosinski [21]). More precisely, there exist $\varepsilon \in (0,1)$, an open subset $\mathscr{U} = \mathscr{U}_\varepsilon$ of $\mathscr{M}$ with $\mathscr{U} \cap \Gamma = \Gamma$, and a 'tubular' diffeomorphism $\tau \colon \mathscr{U} \to \mathbb{B}^\ell \times \Gamma$ such that $\tau(\Gamma) = \{0\} \times \Gamma$, the tangential $T\tau$ of τ equals on $T\Gamma$ the identity multiplied with the factor ε, and

$$\tau_* g \sim g_{\mathbb{B}^\ell} + g_\Gamma \ . \tag{8.5}$$

Let $T^\perp \Gamma$ be the normal bundle of Γ. For $\xi \in \mathbb{S}^{\ell-1}$ and $q \in \Gamma$ there exists a unique $\nu_\xi(q) \in T_q^\perp \Gamma$ satisfying

$$(T_q \tau)\nu_\xi(q) = \big((0,\xi),q\big) \in T_0 \mathbb{R}^\ell \times \Gamma \ .$$

Let $\gamma_{\nu,q} \colon [0,\varepsilon] \to \mathscr{M}$ be the geodesic emanating from q in direction $\nu \in T_q^\perp \Gamma$. Then

$$p = p(t,\xi,q) := \tau^{-1}(t,\xi,q) = \gamma_{\varepsilon\nu_\xi(q),q}(t) \ , \qquad (t,\xi,q) \in [0,1) \times \mathbb{S}^{\ell-1} \times \Gamma \ .$$

From this we infer

$$t \sim \delta_{\mathscr{U}}\big(p(t,\xi,q),\Gamma\big) \ , \qquad (t,\xi,q) \in [0,1) \times \mathbb{S}^{\ell-1} \times \Gamma \ . \tag{8.6}$$

Next we suppose $\Gamma \subset \partial \mathscr{M}$. Let $\mathscr{U}^\bullet = \mathscr{U}_\varepsilon^\bullet$ be an open tubular neighborhood of Γ in $\partial \mathscr{M}$ with associated tubular diffeomorphism

$$\tau^\bullet \colon \mathscr{U}^\bullet \to \mathbb{B}^{\ell-1} \times \Gamma \ . \tag{8.7}$$

Furthermore, there exists a uniform collar $\mathscr{V} = \mathscr{V}_\varepsilon$ for $\partial \mathscr{M}$ over $\mathscr{U}^\bullet$. That is to say: by making ε smaller, if necessary, we can assume that $\mathscr{V}$ is an open subset of $\mathscr{M}$ such that $\mathscr{V} \cap \partial \mathscr{M} = \mathscr{U}^\bullet$ and there exists a diffeomorphism $\tau^+ \colon \mathscr{V} \to [0,1) \times \mathscr{U}^\bullet$ with $\tau^+(\mathscr{U}^\bullet) = \{0\} \times \mathscr{U}^\bullet$, $T\tau^+$ equals the identity in $T_\Gamma \partial \mathscr{M}$ multiplied by ε, and

$$\tau_*^+ g \sim \mathrm{d}t^2 + g_{\partial \mathscr{M}} \ . \tag{8.8}$$

Note that $\mathbb{B}_+^\ell \subset [0,1) \times \mathbb{B}^{\ell-1}$. Hence it follows from (8.7) that there exists an open subset $\mathscr{U} = \mathscr{U}_\varepsilon$ of $\mathscr{W}$ such that $\mathscr{U} \cap \partial \mathscr{M} = \mathscr{U}^\bullet$ and

$$\tau := (\mathrm{id}_{[0,1)} \times \tau^\bullet) \circ \tau^+ \colon \mathscr{U} \to \mathbb{B}_+^\ell \times \Gamma \tag{8.9}$$

is a diffeomorphism satisfying

$$\tau(\mathscr{U}^\bullet) = \{0\} \times \mathbb{B}^{\ell-1} \times \Gamma \ , \quad \tau(\Gamma) = \{0\} \times \Gamma \ .$$

By (8.5) and (8.8) we find

$$\tau_* g \sim \mathrm{d}t^2 + g_{\mathbb{B}^{\ell-1}} \times g_\Gamma \sim g_{\mathbb{B}_+^\ell} \times g_\Gamma \ .$$

We let $\gamma_{\nu^\bullet,q}^\bullet$ be the geodesic in $\partial \mathscr{M}$ emanating from $q \in \Gamma$ in direction $\nu^\bullet \in T_{\partial \mathscr{M}}^\perp \Gamma$, where $T_{\partial \mathscr{M}}^\perp \Gamma$ is the orthogonal complement of $T_q \Gamma$ in $T_q \partial \mathscr{M}$. Suppose $\xi = (s,\eta)$

belongs to $\mathbb{S}^{\ell-1}_+$ with $s \in [0,1)$ and $\eta \in \mathbb{R}^{\ell-2}$, $0 \leq t \leq 1$, and $q \in \Gamma$. Define $\nu^\bullet_\eta(q)$ in $T^\perp_{\partial\mathscr{M},q}\Gamma$ by $(T^\bullet_q\tau^\bullet)\nu^\bullet_\eta(q) = \big((0,\eta),q\big) \in T_0\mathbb{R}^{\ell-2} \times \Gamma$, where $T^\bullet\tau^\bullet$ is the tangential of $\tau^\bullet$ in $\partial\mathscr{M}$. Set $r = r(t,\eta,q) := \gamma^\bullet_{\varepsilon\nu^\bullet_\eta(q),q}(t) \in \mathscr{U}^\bullet$. Analogously, let $\mu_s(r)$ in $T^\perp_r\partial\mathscr{M}$ be given by $(T_r\tau^+)\mu_s(r) = \big((0,s),r\big) \in T_0\mathbb{R} \times \mathscr{U}^\bullet$. Then

$$p = p(t,\xi,q) := \tau^{-1}(t\xi,q) = \gamma_{\varepsilon\mu_s(r(t,\eta,q)),r(t,\eta,q)}(t) \in \mathscr{U} \ .$$

This means that we reach p from $q \in \Gamma$ in two steps. First we go from q to $r \in \mathscr{U}^\bullet$ by following during the time interval $[0,t]$ the geodesic in $\mathscr{U}^\bullet$ which emanates from q in direction $\varepsilon\nu^\bullet_\eta(q)$. Second, we follow during the time interval $[0,t]$ the geodesic in $\mathscr{U}$ emanating from r in direction $\varepsilon\mu_s(r)$ to arrive at p. Observe

$$\mathrm{dist}_{\mathscr{U}}(p,r) = \mathrm{dist}_{\mathscr{U}}(p,\mathscr{U}^\bullet) = \delta_{\mathscr{U}}(p,\mathscr{U}^\bullet) \ .$$

Hence

$$t \sim \mathrm{dist}_{\mathscr{U}}(p,r) \leq \mathrm{dist}_{\mathscr{U}}(p,q) \leq \mathrm{dist}_{\mathscr{U}}(p,r) + \mathrm{dist}_{\mathscr{U}^\bullet}(r,q) \leq 2t \ .$$

From this we infer

$$t \sim \delta_{\mathscr{U}}\big(p(t,\xi,q),\Gamma\big) \ , \qquad (t,\xi,q) \in [0,1) \times \mathbb{S}^{\ell-1} \times \Gamma \ . \tag{8.10}$$

Henceforth, $\mathbb{B} := \mathbb{B}^\ell$ and $\mathbb{S} := \mathbb{S}^{\ell-1}$ if $\Gamma \in \mathring{\mathscr{M}}$, whereas $\mathbb{B} := \mathbb{B}^\ell_+$ and $\mathbb{S} := \mathbb{S}^{\ell-1}_+$ otherwise. Then $U = U_\Gamma := \mathscr{U} \setminus \Gamma$ is, in either case, a *tubular neighborhood of* Γ *in* (M,g) and $\tau = \tau|U : U \to \dot{\mathbb{B}} \times \Gamma$ is the *(associated) tubular diffeomorphism*, defined by (8.9) if $\Gamma \in \partial\mathscr{M}$. By δ_Γ we denote the restriction of $\mathrm{dist}_{\mathscr{U}}(\cdot,\Gamma)$ to U.

Let $R \in \mathscr{C}(J_0)$ and $\varphi = \varphi[R]$. With the (ℓ-dimensional) polar coordinate diffeomorphism π the composition

$$U \xrightarrow{\ \tau\ } \dot{\mathbb{B}} \times \Gamma \xrightarrow{\ \pi \times \mathrm{id}_\Gamma\ } (0,1) \times \mathbb{S} \times \Gamma \xrightarrow{\ \varphi^{-1} \times \mathrm{id}_\Gamma\ } W(R,\mathbb{S},\Gamma) \tag{8.11}$$

defines a diffeomorphism Φ from U onto the model Γ-wedge $W = W(R,\mathbb{S},\Gamma)$ over the spherical, resp. semi-spherical, R-cusp $P = P(R,\mathbb{S})$. We call U *smooth singular end of* (M,g) *of type* (R,Γ) if Φ is an isometry from (U,g) onto (W,g_W), where $h_P := g_P$.

Lemma 8.4. *Let U be a smooth singular end of (M,g) of type (R,Γ). Then there exists a cofinal singularity function ρ for (U,g) satisfying $\rho \sim R \circ \delta_\Gamma$.*

Proof. It is a consequence of Proposition 7.3, Lemma 7.1, and Lemma 3.4 that $\rho := \Phi^* R$ is a cofinal singularity function for (U,g). From (8.11) and (8.2) we deduce $\Psi = (\pi \otimes \mathrm{id}_\Gamma) \circ \tau$. Moreover, $\Psi\big(p(t,\xi,q)\big) = (t,\xi,q)$ for (t,ξ,q) belonging to $(0,1) \times \mathbb{S} \times \Gamma$. Hence $(\Phi^* R)\big(p(t,\xi,q)\big) = R(t)$ by (8.3). Now the claim is implied by (8.6), respectively (8.10). $\qquad\qquad\square$

It is clear that the assertion of this lemma is independent of the particular choice of U, that is, of ε.

Proof of Theorem 1.6 and Proposition 1.8. The statements follow directly from Lemma 8.4 with $R = \mathsf{R}_\alpha$, Lemma 8.3, and Theorem 8.1. □

Proof of Theorem 1.9. We set $R := \mathsf{R}_\alpha$ if $0 < \alpha \leq 1$, and $R := \mathsf{R}_{-\alpha}$ for $\alpha > 1$. It follows from Example 6.5(b) (setting $\beta := \alpha$ if $\alpha \leq 1$ and $\beta := -\alpha$ otherwise) and Theorem 3.1 that

$$g_W := \varphi^*\big(t^{-2\alpha}(t^{2(\alpha-1)}dt^2 + g_{\mathbb{S}})\big) + t^{-2\alpha}g_\Gamma = \varphi^*(t^{-2}dt^2 + t^{-2\alpha}g_{\mathbb{S}}) + t^{-2\alpha}g_\Gamma$$

is a cofinally uniformly regular metric for $W = W(R, \mathbb{S}, \Gamma)$ if $0 < \alpha \leq 1$, whereas

$$g_W := \varphi^*(t^{-2(\alpha+1)}dt^2 + g_{\mathbb{S}}) + g_\Gamma$$

is one if $\alpha > 1$. Thus Φ, defined by (8.11), is an isometry from $(U, \mathfrak{g})$ onto (W, g_W). Hence the claim follows once more from Lemma 3.4. □

Lastly, we mention that there occur interesting and important singular manifolds if assumption (8.4)(iii) is dropped, that is, if Γ intersects $\mathring{\mathcal{M}}$ as well as $\partial\mathcal{M}$. Then Γ is no longer a smooth singular end but has cuspidal corners, for example. Such cases are not considered here although the technical means for their study have been provided in the preceding sections.

References

1. H. Amann. Anisotropic function spaces on singular manifolds, 2012. arXiv:1204.0606.
2. H. Amann. Function spaces on singular manifolds. *Math. Nachr.*, **286** (2012), 436–475.
3. H. Amann. Parabolic equations on uniformly regular Riemannian manifolds and degenerate initial boundary value problems, 2014. arXiv:1403.2418.
4. H. Amann. Parabolic equations on noncompact Riemannian manifolds. In preparation.
5. H. Amann, J. Escher. *Analysis II*. Birkhäuser, Basel, 2006. English Translation.
6. Th. Aubin. *Nonlinear Analysis on Manifolds. Monge-Ampère equations*. Springer-Verlag, New York, 1982.
7. L. Conlon. *Differentiable manifolds*. Birkhäuser, Boston, MA, 2001.
8. E.B. Davies. *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge, 1989.
9. R. Denk, M. Hieber, J. Prüss. $\mathscr{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, **166**(788) (2003).
10. M. Disconzi, Y. Shao, G. Simonett. Remarks on uniformly regular Riemannian manifolds, 2014. Preprint.
11. J. Eichhorn. The boundedness of connection coefficients and their derivatives. *Math. Nachr.*, **152** (1991), 145–158.
12. R.E. Greene. Complete metrics of bounded curvature on noncompact manifolds. *Arch. Math. (Basel)*, **31**(1) (1978/79), 89–95.
13. A. Grigor'yan. *Heat Kernel and Analysis on Manifolds*. Amer. Math. Soc., 2009.
14. A. Grigor'yan, L. Saloff-Coste. Heat kernel on manifolds with ends. *Ann. Inst. Fourier (Grenoble)*, **59**(5) (2009), 1917–1997.
15. G. Grubb. Parameter-elliptic and parabolic pseudodifferential boundary problems in global L_p Sobolev spaces. *Math. Z.*, **218** (1995), 43–90.
16. G. Grubb. *Functional Calculus of Pseudodifferential Boundary Problems*. Birkhäuser, Boston, MA, 1996.

17. G. Grubb, N.J. Kokholm. A global calculus of parameter-dependent pseudodifferential boundary problems in L_p Sobolev spaces. *Acta Math.*, **171** (1993), 165–229.

18. M. Hieber, J. Prüss. Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Comm. Partial Differential Equations*, **22**(9-10) (1997), 1647–1669.

19. M.W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

20. V.A. Kondrat′ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obšč.*, **16** (1967), 209–292.

21. A.A. Kosinski. *Differential manifolds*. Academic Press Inc., Boston, MA, 1993.

22. Th. Krainer. Maximal L^p-L^q regularity for parabolic partial differential equations on manifolds with cylindrical ends. *Integral Equations Operator Theory*, **63**(4) (2009), 521–531.

23. O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*. Amer. Math. Soc., Transl. Math. Monographs, Providence, R.I., 1968.

24. A.L. Mazzucato, V. Nistor. Mapping properties of heat kernels, maximal regularity, and semilinear parabolic equations on noncompact manifolds. *J. Hyperbolic Differ. Equ.*, **3**(4) (2006), 599–629.

25. V.E. Nazaikinskii, A.Yu. Savin, B.-W. Schulze, B.Yu. Sternin. *Elliptic theory on singular manifolds*. Chapman & Hall/CRC, Boca Raton, FL, 2006.

26. L. Saloff-Coste. The heat kernel and its estimates. In *Probabilistic approach to geometry*, volume 57 of *Adv. Stud. Pure Math.*, pages 405–436. Math. Soc. Japan, Tokyo, 2010.

27. E. Schrohe. Spaces of weighted symbols and weighted Sobolev spaces on manifolds. In *Pseudodifferential operators (Oberwolfach, 1986)*, volume 1256 of *Lecture Notes in Math.*, pages 360–377. Springer, Berlin, 1987.

28. M.A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, (207) (1992), 5, 35–108. Méthodes semi-classiques, Vol. 1 (Nantes, 1991).