Efficient automatic quadrature in 3–d Galerkin BEM *

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Abstract

We present cubature methods approximating the surface integrals arising by Galerkin discretization of boundary integral equations on surfaces in $\mathbb{R}^3$. This numerical integrator does not depend on the explicit form of the kernel function, the trial and test space, or the surface parametrization. Thus, it is possible to generate the system matrix for a broad class of integral equations just by replacing the subroutine for evaluating the kernel function. We will present formulae to determine the minimal order of the cubature methods for a required accuracy. Emphasize is laid on numerical experiments confirming the theoretical results.

Introduction

In this paper, we will consider Fredholm integral equations on two dimensional surfaces in $\mathbb{R}^3$, which typically arise by applying the boundary integral method to boundary value problems. For the discretization of the integral equation we use the Galerkin method. For the generation of the system matrix, one has to compute integrals over pairs of surface panels, where the integrals have singular or near-singular behaviour depending on the form of the kernel functions.

Our approach will be fully discrete, i.e., the integrators depend neither on the explicit form of the kernel function nor on the choice of the trial and test space, and the surface parametrization. For implementation, one has only to replace the subroutine for evaluating the kernel, not the numerical integrator itself. The idea of the presented techniques is to transform the singular integrals by some suitable regularizing coordinate transform such that the integrands are analytic and standard cubature formulae can be used. These coordinate transforms were first introduced in [8] and [2] where they were combined with semi–analytic optimization techniques. A fully implicit treatment of such integrals for special applications as wavelets and hp–methods were presented in [12] and [7]. An analysis of consistency and stability for such integrals in the farfield case is presented in [5]. Here, emphasize is laid on numerical experiments confirming the theoretical results.

In the first section, we will give a short introduction to the Galerkin BEM. The second section contains the cubature techniques – especially the coordinate transforms to render the singularities analytically. The formulae for determining the minimal orders of the cubature methods to achieve a desired accuracy are given in Section 3. They are based on the analysis of consistency and stability of the discussed method. In the last section, we will show some numerical results.

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1 The Boundary Element Method

Let Γ be a piecewise analytic, orientable Lipschitz surface of a bounded domain $\Omega \subset \mathbb{R}^3$. In many cases, by using the integral equation method, elliptic boundary value problems can be transformed into integral equations on the boundary of the domain (see [1]).

We consider the space $L^2(\Gamma)$ of functions $u : \Gamma \to \mathbb{R}$ which are square integrable with respect to the surface measure $d\Gamma$. Furthermore, for $0 < s < \frac{3}{2}$, we define the Sobolev space $H^s(\Gamma)$ as the trace space of $H^{s+1/2}_{\text{loc}}(\mathbb{R}^3)$ and $H^{-s}(\Gamma)$ as the dual space of $H^s(\Gamma)$ with respect to the L^2–scalar product. The corresponding norm is denoted by $\|\cdot\|_s$.

We consider Fredholm integral equations in the variational formulation

$$\begin{array}{l} \text{for given } f \in H^{\tilde{s}} \text{ find } u \in H^{s_1}(\Gamma) \text{ such that} \\ \quad (v, \lambda_1 u)_0 + (v, K_1 u)_0 = (v, \lambda_2 f)_0 + (v, K_2 f)_0 , \quad \forall v \in H^{s_1}(\Gamma) \end{array} \tag{1}$$

holds where the scalar product $(\cdot,\cdot)_0 : L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R}$ is identified with its extension onto $H^{s_1} \times H^{-s_1}$ by means of Riesz' representation theorem. In (1), λ_1, λ_2 are analytic functions and the integral operators $K_i, i = 1, 2$, are given by

$$(K_i w)(x) = \text{p.f.} \int_\Gamma k_i(x,y) w(y)\, dy$$

where the integral has to be understood in the finite part sense (cf. [10]).

The kernel functions k_i and/or the derivatives of k_i behaves singularly only for $x = y$. For $i = 1, 2$, the operator $\lambda_i I + K_i$ is assumed to be a continuous mapping from H^{s_i} onto H^{-s_i}, and $\tilde{s}$ is defined by $\tilde{s} = 2s_2 - s_1$. The left-hand side of (1) defines the bilinear form $a : H^{s_1} \times H^{s_1} \to \mathbb{C}$

$$a(u,v) := (v, \lambda_1 u)_0 + (v, K_1 u)_0 , \tag{2}$$

while the right-hand defines the functional

$$F(v) := (v, \lambda_2 f)_0 + (v, K_2 f)_0 . \tag{3}$$

Remark 1 *We assume that the bilinear form (2) is elliptic, i.e.,*

$$\|a(u,v)\| \geq C_e \|u\|^2_{s_1} - C_c \|u\|^2_t \qquad \forall u \in H^{s_1}(\Gamma)$$

with suitable $t < s_1$.

Throughout this paper, we assume that (1) has a unique solution for every $f \in H^{-s_1}(\Gamma)$.

Example 2 *Consider the Laplace equation in the exterior of the bounded domain $\Omega \subset \mathbb{R}^3$*

$$\begin{array}{rcll} \Delta u &=& 0 & \textit{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \\ \dfrac{\partial u}{\partial n} &=& f & \textit{on} \quad \Gamma, \\ |u(x)| &=& \mathcal{O}(\|x\|^{-1}) & \textit{for} \quad \|x\| \to \infty \end{array}$$

where $n(x)$ denotes the normal in $x \in \Gamma$ pointing into Ω and $\frac{\partial}{\partial n}$ the normal derivative.

The bilinear form $a : L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{C}$, i.e. $s_1 = 0$, is defined by

$$a(u,v) := \frac{1}{2}(u,v)_0 - p.f. \int_{\Gamma\times\Gamma} v(x) \left(\frac{\partial}{\partial n_y} \frac{1}{\|x-y\|} \right) u(y)\, dy\, dx \tag{4}$$

and the functional $F : L^2(\Gamma) \to \mathbb{C}$ of the right-hand side by

$$F(v) := -\int_{\Gamma\times\Gamma} v(x) \frac{1}{\|x-y\|} f(y)\, dy\, dx. \tag{5}$$

The variational formulation is discretized by means of the Galerkin method replacing the Sobolev space $H^{s_1}(\Gamma)$ by a finite dimensional subspace.

Let $\widetilde{\Gamma}$ be an interpolating polyhedron of the surface Γ, and $\tilde{\tau} := \{\widetilde{K}_1, \ldots, \widetilde{K}_N\}$ denotes a grid of the surface $\widetilde{\Gamma}$ consisting of plane triangles having the property

$$\begin{aligned} \widetilde{\Gamma} &= \overline{\bigcup_{\widetilde{K}\in\tilde{\tau}} \widetilde{K}}, \\ \widetilde{K} \cap \widetilde{K'} &= \emptyset. \end{aligned} \tag{6}$$

The grid $\tilde{\tau}$ induces a grid on the true surface Γ by $\tau := \{\eta(\widetilde{K}) : \widetilde{K} \in \tilde{\tau}\}$, where η is a piecewise analytic, bi–Lipschitz mapping from $\tilde{\Gamma}$ to Γ independent of the grid $\tilde{\tau}$. For a more detailed construction of boundary element grids see [7].

Let V^N be a finite dimensional subspace of the Sobolev space $H^{s_1}(\Gamma)$. The Galerkin discretization of (1) is given by replacing $H^{s_1}(\Gamma)$ with V^N leading to the problem

$$\begin{aligned} &\text{find } u_G \in V^N \text{ such that} \\ &\quad a(u_G, v) = F(v), \qquad \forall v \in V^N. \end{aligned} \tag{7}$$

Let $\{\varphi_i\}_{1\le i\le N}$ denote the usual local nodal basis of V^N. Since any function $w \in V^N$ has a unique representation

$$w(x) = \sum_{i=1}^{N} w_i \varphi_i(x), \tag{8}$$

we can rewrite (7) as follows

$$\mathbf{K}\mathbf{u_G} = \mathbf{F}. \tag{9}$$

The system matrix $\mathbf{K} = (K_{ij})_{i,j=1}^N$ is given by $K_{ij} = a(\varphi_i, \varphi_j)$ and the right–hand side by $\mathbf{F} = (F_i)_{i=1}^N$ with $F_i = F(\varphi_i)$. The solution of (9) is linked to the solution u_G of (7) via (8).

For the computation of the matrix $\mathbf{K}$ and the right–hand side $\mathbf{F}$ one has to evaluate integrals over pairs of surface patches.

$$a(u,v) \;=\; \sum_{K\in\tau} a_K(u,v) + \sum_{K_x\times K_y\in\tau\times\tau} a_{K_x\times K_y}(u,v) \qquad \text{and} \tag{10}$$

$$F(v) \;=\; \sum_{K\in\tau} F_K(v) + \sum_{K_x\times K_y\in\tau\times\tau} F_{K_x\times K_y}(v) \tag{11}$$

with

$$a_K(u,v) := \int_K \lambda_1 uv\, dx, \quad a_{K_x\times K_y}(u,v) := \lim_{\epsilon\to 0} \int_{\substack{K_x\times K_y \\ \|x-y\|\ge\epsilon}} v(x) k_1(x,y) u(y)\, dy\, dx. \tag{12}$$

$F_K(v)$ or $F_{K_x\times K_y}(v)$ are defined analogously. For details concerning this splitting, we refer to [3] and [6]. Analytic integration is only possible for a few special cases. Hence, one has to develop numerical cubature techniques to approximate the coefficients K_{ij} and F_i in (9).

We conclude this section with a collection of the characterristic properties of the considered kernel functions. They are Gâteaux derivatives of the fundamental solution of the underlying boundary value problem. The following properties are satisfied for all common kernels corresponding to elliptic boundary value problems.

We assume that $k(x,y)$ can be written in the form

$$k(x,y)=\sum_{|\alpha|=t}^{t+a} s_\alpha(x,y)\eta_\alpha(\|x-y\|)\frac{(y-x)^\alpha}{\|y-x\|^{\sigma+t}} \tag{13}$$

with functions $s_\alpha:\Gamma\times\Gamma\to\mathbb{C}$ being analytic on pairs of smooth surface patches of Γ. $\alpha\in\mathbb{N}_0^3$ is a multi–index and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. The functions η_α are assumed to be analytic. The integer σ denotes the order of the singularity. Furthermore, the kernel has to be anti–symmetric, i.e.

$$k(x,y)+k(y,x)=\sum_{|\alpha|=t+1}^{t+a} \tilde{s}_\alpha(x,y)\tilde{\eta}_\alpha(\|x-y\|)\frac{(y-x)^\alpha}{\|y-x\|^{\sigma+t}}. \tag{14}$$

This condition implies Cauchy singularity of the kernel k if $\sigma\le 2$. For $\sigma>2$ the $k(x,y)$ is hyper–singular and we recommend regularization by partial integration (cf. [4],[9]).

2 Cubature techniques

In this section, we present the cubature techniques for approximating the integrals in $a_{K_x\times K_y}(\cdot,\cdot)$ and $F_{K_x\times K_y}(\cdot)$ (see (10,11)) . The integrands have singular, nearly singular, or regular behaviour which requires different transformations removing the singularity. Then the integrals corresponding to $a_K(\cdot,\cdot)$ and $F_K(\cdot)$ are analytic and can be treated by standard cubature methods. These transformations do not contain any approximation process.

We have to distinguish the following four cases:

1. "Identical Panels": $K_x=K_y$,
2. "Common Edges" : K_x and K_y share exactly one edge,
3. "Common Vertex" : K_x and K_y share exactly one vertex,
4. "Regular Case" : $\overline{K_x}\cap\overline{K_y}=\emptyset$.

Without loss of generality, we assume that every pair of panels satisfies one of the four above cases. Otherwise, one has to subdivide the panels such that one of the conditions holds for every pair of the subpanels.

Let T_d denote the d–dimensional unit simplex

$$T_d:=\{\hat{x}\in\mathbb{R}_+^d:\sum_{i=1}^{d}\hat{x}_i<1\}$$

and $R_d:=(0,1)^d$.

2.1 The case of identical panels

In this subsection, we investigate the approximation of the following integral

$$I_i:=\lim_{\epsilon\to 0}\int_{\substack{K\times K\\ \|x-y\|\ge\epsilon}} H(x,y)k(x,y)\,dy\,dx, \tag{15}$$

where $H(x,y)$ equals either $v(x)u(y)$ or $f(y)v(x)$ respectively to $a_{K\times K}$ or F_K.

We define the master element K^0 by

$$K^0 := \{(x,y) : 0 \le x \le 1, 0 \le y \le x\}$$

and the transformation κ by

$$\kappa_K = \eta \circ \kappa_{\widetilde{K}}$$

where η is the mapping $\eta : \widetilde{\Gamma} \to \Gamma$ defined in Section 1 and $\kappa_{\widetilde{K}}$ defines the affine linear transformation of K^0 onto $\widetilde{K}$. Then the integral (15) is given in local coordinates by

$$I_i := \lim_{\epsilon\to 0} \int_0^1 \int_0^{\hat{x}_1} \int_0^1 \int_0^{\hat{y}_1}{}_{\|\hat{x}-\hat{y}\|\ge\epsilon} H_{\text{loc}}(\hat{x},\hat{y}) k_{\text{loc}}(\hat{x},\hat{y})\, d\hat{y}\, d\hat{x}, \tag{16}$$

where $H_{\text{loc}} = g_K(\hat{x}) g_K(\hat{y}) H(\kappa_K(\hat{x}), \kappa_K(\hat{y}))$ and $g_K(\cdot)$ denotes the Jacobian of the transformation κ_K. The kernel function in local coordinates is given by $k_{\text{loc}} = k(\kappa_K(\hat{x}), \kappa_K(\hat{y}))$.

We introduce relative coordinates to fix the singularity at the origin as proposed in [8] and [2]. The following representation formula is obtained by an appropriate subdivision of the integration domain. These domains are transformed onto the 4–d unit simplex by a four-dimensional rotation. Then, generalized Duffy coordinates render the integrand analytic

$$I_i := \int_{R_1} \int_{T_3} \omega_1 \sum_{j=1}^{6} H_{\text{loc}}(\hat{x}^{(j)}, \hat{y}^{(j)}) k_{\text{loc}}(\hat{x}^{(j)}, \hat{y}^{(j)})\, d\omega\, d\eta. \tag{17}$$

As shown in [8],[12], [7] the integrand of (17) is analytic and the integral can be approximated by standard cubature techniques. It remains to define the transformation $(\eta,\omega) \to (\hat{x}^{(j)}, \hat{y}^{(j)})$ for $j = 1, \dots, 6$:

$$\begin{pmatrix} \hat{x}_1^{(1)} \\ \hat{x}_2^{(1)} \\ \hat{y}_1^{(1)} \\ \hat{y}_2^{(1)} \end{pmatrix} = \begin{pmatrix} \omega_1+\omega_2+\omega_3 \\ \omega_1+\omega_2 \\ \omega_1(1-\eta)+\omega_2+\omega_3 \\ \omega_2 \end{pmatrix}, \quad \begin{pmatrix} \hat{x}_1^{(2)} \\ \hat{x}_2^{(2)} \\ \hat{y}_1^{(2)} \\ \hat{y}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \omega_1(1-\eta)+\omega_2+\omega_3 \\ \omega_1(1-\eta)+\omega_2 \\ \omega_1+\omega_2+\omega_3 \\ \omega_2 \end{pmatrix},$$

$$\begin{pmatrix} \hat{x}_1^{(3)} \\ \hat{x}_2^{(3)} \\ \hat{y}_1^{(3)} \\ \hat{y}_2^{(3)} \end{pmatrix} = \begin{pmatrix} \omega_1+\omega_2+\omega_3 \\ \omega_1\eta+\omega_2 \\ \omega_2+\omega_3 \\ \omega_2 \end{pmatrix}, \quad \begin{pmatrix} \hat{x}_1^{(4)} \\ \hat{x}_2^{(4)} \\ \hat{y}_1^{(4)} \\ \hat{y}_2^{(4)} \end{pmatrix} = \begin{pmatrix} \omega_1(1-\eta)+\omega_2+\omega_3 \\ \omega_2 \\ \omega_1+\omega_2+\omega_3 \\ \omega_1+\omega_2 \end{pmatrix},$$

$$\begin{pmatrix} \hat{x}_1^{(5)} \\ \hat{x}_2^{(5)} \\ \hat{y}_1^{(5)} \\ \hat{y}_2^{(5)} \end{pmatrix} = \begin{pmatrix} \omega_1+\omega_2+\omega_3 \\ \omega_2 \\ \omega_1(1-\eta)+\omega_2+\omega_3 \\ \omega_1(1-\eta)+\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \hat{x}_1^{(6)} \\ \hat{x}_2^{(6)} \\ \hat{y}_1^{(6)} \\ \hat{y}_2^{(6)} \end{pmatrix} = \begin{pmatrix} \omega_2+\omega_3 \\ \omega_2 \\ \omega_1+\omega_2+\omega_3 \\ \omega_1\eta+\omega_2 \end{pmatrix}.$$

2.2 The case of a common edge

Now, we will discuss the cubature of integrals over panels K_1, K_2 which share exactly one edge. The master elements are given by $K_1^0 = K^0$ and $K_2^0 = \{\hat{y} \in \mathbb{R}^2 : (\hat{y}^1, -\hat{y}^2)^T \in K^0\}$. Let the mapping $\kappa : \overline{K_1^0 \cup K_2^0} \to \overline{K_1 \cup K_2}$ be analytic on K_1^0 and K_2^0 and bi–Lipschitz continuous on $\overline{K_1^0 \cup K_2^0}$. The integral over $K_1 \times K_2$ in local coordinates is given by

$$I_e = \int_0^1 \int_0^{\hat{x_1}} \int_0^1 \int_{-\hat{y_1}}^0 H_{\text{loc}}(\hat{x},\hat{y}) k_{\text{loc}}(\hat{x},\hat{y})\, d\hat{y}\, d\hat{x}, \tag{18}$$

where $H_{\text{loc}}(\cdot,\cdot)$ and $k_{\text{loc}}(\cdot,\cdot)$ are defined analogously as in the case of identical panels.

The following transformations remove the singularity of the integrand. They map $R_2 \times T_2$ onto subsets of $K_1^0 \cup K_2^0$ and are explicitly given by

$$\begin{pmatrix} \hat{x}_1^{(1)} \\ \hat{x}_2^{(1)} \\ \hat{y}_1^{(1)} \\ \hat{y}_2^{(1)} \end{pmatrix} = \begin{pmatrix} \omega_1+\omega_2 \\ \omega_1\eta_1\eta_2 \\ \omega_1(1-\eta_1)+\omega_2 \\ \omega_1(\eta_1-1) \end{pmatrix}, \qquad \begin{pmatrix} \hat{x}_1^{(2)} \\ \hat{x}_2^{(2)} \\ \hat{y}_1^{(2)} \\ \hat{y}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \omega_1+\omega_2 \\ \omega_1\eta_1 \\ \omega_1(1-\eta_2\eta_1)+\omega_2 \\ \omega_1(\eta_2\eta_1-1) \end{pmatrix},$$

$$\begin{pmatrix} \hat{x}_1^{(3)} \\ \hat{x}_2^{(3)} \\ \hat{y}_1^{(3)} \\ \hat{y}_2^{(3)} \end{pmatrix} = \begin{pmatrix} \omega_1+\omega_2 \\ \omega_1 \\ \omega_1\eta_1+\omega_2 \\ -\omega_1\eta_2\eta_1 \end{pmatrix}, \qquad \begin{pmatrix} \hat{x}_1^{(4)} \\ \hat{x}_2^{(4)} \\ \hat{y}_1^{(4)} \\ \hat{y}_2^{(4)} \end{pmatrix} = \begin{pmatrix} \omega_1(1-\eta_1+\eta_2\eta_1)+\omega_2 \\ \omega_1\eta_2\eta_1 \\ \omega_1+\omega_2 \\ -\omega_1 \end{pmatrix},$$

$$\begin{pmatrix} \hat{x}_1^{(5)} \\ \hat{x}_2^{(5)} \\ \hat{y}_1^{(5)} \\ \hat{y}_2^{(5)} \end{pmatrix} = \begin{pmatrix} \omega_1\eta_1\eta_2+\omega_2 \\ \omega_1\eta_1\eta_2 \\ \omega_1 \\ -\omega_1\eta_1 \end{pmatrix}, \qquad \begin{pmatrix} \hat{x}_1^{(6)} \\ \hat{x}_2^{(6)} \\ \hat{y}_1^{(6)} \\ \hat{y}_2^{(6)} \end{pmatrix} = \begin{pmatrix} \omega_1\eta_1+\omega_2 \\ \omega_1\eta_1 \\ \omega_1+\omega_2 \\ -\omega_1\eta_2\eta_1 \end{pmatrix}.$$

The determinant of the Jacobi matrix equals $\eta_1\omega_1^2$ and the integral I_e can be written as

$$I_e = \int_{R_2}\int_{T_2} \eta_1\omega_1^2 \sum_{j=1}^{6} H_{\text{loc}}(\hat{x}^{(j)},\hat{y}^{(j)}) k_{\text{loc}}(\hat{x}^{(j)},\hat{y}^{(j)})\, d\omega\, d\eta. \tag{19}$$

As in the case of identical panels the integrand is analytic and the integral can be approximated to any desired accuracy by standard cubature methods with properly chosen order.

2.3 The case of a common vertex

Finally, we have to discuss the case that the two surface triangles have exactly one common point. The associated master elements are defined by $K_1^0 := K^0$ and $K_2^0 := \{\hat{y} \in \mathbb{R}^2 : -\hat{y} \in K^0\}$. The mapping $\kappa : \overline{K_1^0 \cup K_2^0} \to \overline{K_1 \cup K_2}$ is assumed to be bi–Lipschitz continuous and analytic on each reference element. The integral in local coordinates can be written as

$$I_p = \int_0^1 \int_0^{\hat{x_1}} \int_{-1}^0 \int_{\hat{y_1}}^0 H_{\text{loc}}(\hat{x},\hat{y}) k_{\text{loc}}(\hat{x},\hat{y})\, d\hat{y}\, d\hat{x}, \tag{20}$$

where we adopt the notations of the previous subsections. We split the domain of integration and define the transformation by

$$\begin{pmatrix} \hat{x}_1^{(1)} \\ \hat{x}_2^{(1)} \\ \hat{y}_1^{(1)} \\ \hat{y}_2^{(1)} \end{pmatrix} = \begin{pmatrix} \omega \\ \eta_2\omega \\ -\eta_1\omega \\ -\eta_3\eta_1\omega \end{pmatrix}, \qquad \begin{pmatrix} \hat{x}_1^{(2)} \\ \hat{x}_2^{(2)} \\ \hat{y}_1^{(2)} \\ \hat{y}_2^{(2)} \end{pmatrix} = \begin{pmatrix} \eta_1\omega \\ \eta_3\eta_1\omega \\ -\omega \\ -\eta_2\omega \end{pmatrix}$$

with $(\eta_1, \eta_2, \eta_3, \omega) \in R_3 \times T_1$. The Jacobian is given by $\omega_1 \eta^3$ resulting in

$$I_p = \int_{R_3} \int_{T_1} \omega_1 \eta^3 \sum_{j=1}^{2} H_{\text{loc}}(\hat{x}^{(j)}, \hat{y}^{(j)}) k_{\text{loc}}(\hat{x}^{(j)}, \hat{y}^{(j)})\, d\omega\, d\eta. \tag{21}$$

As in the other two cases, one can show that the integrand is analytic and (21) can be approximated to any required accuracy with standard methods of properly chosen order (see [8],[12], [7]).

In the regular case $\overline{K_x} \cap \overline{K_y} = \emptyset$ the integrand is analytic and can be approximated by standard cubature methods to any desired accuracy.

3 Cubature orders

In this section, we will discuss the methods for approximate calculation of the multiple integrals of the previous section. The cubature orders or the number of cubature points have to be chosen such that the asymptotic convergence of the Galerkin method is preserved.

Remark 3 *Assume that the polynomial degree of the finite dimensional subspace V^N of H^{s_1} satisfies $p \geq s_1$. We assume that the solution $u \in H^{s_1}$ of (1) is sufficiently smooth. Then, for small h the Galerkin discretization (7) has a unique solution $u_G \in V^N$, and it holds*

$$\|u - u_G\|_{s_1 - t} \leq C h^{p+1+t-s_1}$$

for $t \in [0, p+1-s_1]$ (cf. [13]).

For the integrals over R_d we use d–dimensional tensor product versions of Gauß cubature. The integrals over the simplices T_d are pulled back onto R_d by using simplex coordinates (cf. [11]) and then evaluated by tensor Gauß formulae. For the regular integrals the product domain $K_x \times K_y$ is pulled back onto $R_2 \times R_2$ by tensor product versions of simplex coordinates with respect to x and y. Summarizing, we conclude that all integrals are of the form

$$\int_{R_d \times R_{4-d}} g(\eta, \omega)\, d\eta\, d\omega$$

with $d = 1$ in the case of identical panels, $d = 2$ for both the case of a common edge and the regular case, and $d = 3$ for the case of a common vertex.

Let $Q^n_{R_d}$ denote the tensor Gauß cubature for approximating an integral over R_d with n Gauß points in each direction. Let $Q^{\mathbf{n}}_{R_{d_1} \times R_{d_2}}$ denote the compound formula of $Q^{n_1}_{R_{d_1}}$ and $Q^{n_2}_{R_{d_2}}$ to approximate an inetgral over $R_{d_1} \times R_{d_2}$ where $\mathbf{n} = (n_1, n_2)$.

Remark 4 *The integrals obtained in the singular cases shown in Section 2 have to be approximated by $Q^{\mathbf{n}}_{R_d \times R_{4-d}}$. The minimal numbers of Gauß points to preserve the asymptotic convergence of the Galerkin method are given by*

$$n_1 = \left\lceil \frac{p - s_1}{2}\ |\log h| \right\rceil, \quad n_2 = \lceil p - s_1 \rceil. \tag{22}$$

where $\lceil x \rceil$ denotes the smallest integer larger than or equal to x.

Remark 5 *Let σ be the order of the singularity of the kernel function k (see (13)). For the regular case we have to use the tensor Gauß formula $Q^n_{R_4}$ for the approximation of the integrals. The minimal number n of Gauß–points in each direction is given by*

$$n = \left\lceil \frac{1}{2}\left(2p + t - s_1 + (\sigma + p + t - s_1)D\right) \right\rceil \tag{23}$$

with $D := D(h,\delta) := \max\left(0, \frac{\log(\delta)}{\log(h/\delta)}\right)$ where $\delta := dist(K_x, K_y)$ is the distance of the panels K_x and K_y (cf. [5]).

4 Numerical results

As a test problem we consider Example 2 of Section 1 with $\Omega = K_1(0)$ denoting the unit ball. The right–hand side is given by

$$\frac{\partial u}{\partial n}(x) = \frac{< x, x - x_0 >}{\|x - x_0\|^3} \tag{24}$$

with $x_0 = (0.25, 0.25, 0.25)$. Then the exact solution is given by

$$u(x) = \frac{1}{\|x - x_0\|}. \tag{25}$$

The integrals in (4) and (5) are computed by applying the methods of Section 2 and using appropriate cubature rules as explained in Section 3.

The formulae (22–23) are used to determine the number of cubature points. For the considered problem we have $s_1 = 0$. We employ piecewise linear trial functions, i.e. $p = 1$. We have computed the problem on a hierarchy of grids approximating the unit ball. Table 1 shows the number of unknowns and panels for each grid.

Table 1:

grid	unknowns	panels
0	6	8
1	18	32
2	66	128
3	258	512
4	1026	2048
5	4098	8192

Let $\tilde{u}_{G,\ell}$ the discrete Galerkin solution computed on the ℓ–th grid. Then, the relative error on grid $\ell, \ell = 0, \ldots, 5$, is given by

$$\|e_\ell\| = \frac{\|u - \tilde{u}_{G,\ell}\|}{\|u\|} \tag{26}$$

with $\|\cdot\|$ denoting the L_2–norm, therefore t in (22–23) is zero. The contraction rate ρ_ℓ is defined by

$$\rho_\ell = \frac{\|e_{\ell-1}\|}{\|e_\ell\|}. \tag{27}$$

We refer to Remark 4 and Remark 5 to determine the minimal number of Gauß–points to preserve the convergence rate of the Galerkin method. The minimal numbers for the singular cases are given by

$$\mathbf{n} = (n_1, n_2) \quad \text{with} \quad n_1 = \left\lceil \frac{1}{2} |\log(h)| \right\rceil, \quad n_2 = 1, \tag{28}$$

and for the regular case by

$$n = \lceil 1 + 2D \rceil . \tag{29}$$

The numerical experiments were performed on a SUN Ultra station. The results are shown in Table 2–6.

Table 2:

level	$\|e\|$	ρ_ℓ	(n_1, n_2)	$\tilde{n}$	time
0	2.564418e-01	–	(1,1)	1	0.01s
1	1.450404e-01	1.79	(1,1)	1	0.08s
2	3.920217e-02	3.70	(1,1)	1	2s
3	4.523466e-03	8.67	(2,1)	1	69s
4	1.037374e-03	4.36	(2,1)	1	1167s
5	2.529199e-04	4.10	(2,1)	1	17484s

Table 3:

level	$\|e\|$	rate	(n_1, n_2)	$\tilde{n}$	time
0	2.697420e-01	–	(3,1)	1	0.57s
1	9.661654e-02	2.79	(3,1)	1	3s
2	2.071959e-02	4.67	(3,1)	1	13s
3	4.968625e-03	4.16	(4,1)	1	217s
4	1.240574e-03	4.01	(4,1)	1	1774s
5	3.165298e-04	3.91	(4,1)	1	20129s

Table 4:

level	$\|e\|$	ρ_ℓ	(n_1, n_2)	$\tilde{n}$	time
0	2.564418e-01	–	(1,1)	1	0.01s
1	1.450404e-01	1.79	(1,1)	1	0.08s
2	3.920217e-02	3.70	(1,1)	1	2s
3	1.147717e-02	3.42	(1,1)	1	62s
4	4.128453e-03	2.78	(1,1)	1	1142s
5	1.759973e-03	2.35	(1,1)	1	17346s

The fourth column of all tables contains the number of cubature points in the near field given by (28). In the fifth column the *minimal* number $\tilde{n}$ of cubature points occuring in the farfield integration given by (29) can be found. The last column contains the CPU–time used for generating the grid, assembling the system matrix and the right–hand side, and solving the equation system.

Formula (29) was used for determining the Gauß points for the farfield in the experiments shown in Table 2–4. Table 2 shows that (28) and (29) are sufficient. We can see that the contraction rate tends to four, which we have expected by the convergence rate of the Galerkin method. Comparing Table 2 and 3 it is obvious, that using higher order cubature formulae in the singular cases does not improve the accuracy but increases the computing time. On the other hand one cannot use less than n_1 cubature points in the near field, otherwise, the convergence deteriorates for the higher levels $\ell = 3, \ldots, 5$ as shown in Table 4.

Table 5:

level	$\|e\|$	ρ_ℓ	(n_1, n_2)	$\tilde{n}$	time
0	2.564418e-01	–	(1,1)	1	0.01s
1	1.450404e-01	1.79	(1,1)	1	0.08s
2	3.883874e-02	3.73	(1,1)	1	0.68s
3	4.959329e-03	7.83	(2,1)	1	17s
4	1.479656e-03	3.35	(2,1)	1	169s
5	6.232632e-04	2.37	(2,1)	1	2340s

Next, we have examined the cubature in the regular case. The experiments have validated that the number of cubature points for the farfield integration given by (29) are sufficient to get the expected contraction rate as shown in Table 2. To show that the logarithmic increase of the number of cubature points towards the diagonal is necessary, we have fixed the farfield rule for *all* integrals as the midpoint rule. The results in Table 5 show that the contraction rate decreases on higher grid levels. On the other hand, if one uses cubature rules of higher orders than determined by (29) this has no effect to ρ_ℓ as shown in Table 6 where we used $n + 1$ cubature points with n given by (29).

Table 6:

level	$\|e\|$	ρ_ℓ	(n_1, n_2)	$\tilde{n}$	time
0	2.542644e-01	–	(1,1)	2	0.02s
1	1.425245e-01	1.78	(1,1)	2	0.37s
2	3.881511e-02	3.67	(1,1)	2	13s
3	4.239961e-03	9.15	(2,1)	2	297s
4	1.220608e-03	3.47	(2,1)	2	11359s
5	2.220575e-04	5.49	(2,1)	2	76195s

Our numerical experiments have confirmed the formulae (28) and (29) are sufficient and necessary to preserve the asymptotic convergence rate of the Galerkin method.

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