Finite element methods for the Stokes problem on complicated domains

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Abstract. It is a standard assumption in the error analysis of finite element methods that the underlying finite element mesh has to resolve the physical domain of the modeled process. In case of complicated domains appearing in many applications such as ground water flows this requirement sometimes becomes a bottleneck. The resolution condition links the computational complexity a priorily to the number (and size) of geometric details. Therefore even the coarsest available discretization can lead to a huge number of unknowns. In this paper, we will relax the resolution condition and introduce coarse (optimal order) approximation spaces for Stokes problems on complex domains. The described method will be efficient in the sense that the number of unknowns is only linked to the properties of the solution and *not* to the problem data. The presentation picks up the concept of composite finite elements for the Stokes problem presented in a previous paper of the authors. Here, the a priori error and stability analysis of the proposed mixed method is generalized to quite general, i.e. slip and leak boundary conditions that are of great importance in practical applications.

Keywords. finite elements, mixed methods, Stokes equation, general boundary conditions, no-slip, slip, leak, non-conforming finite elements, space coarsening, complicated domain, microstructures

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1 Problem setting

We consider the stationary Stokes equations

$$\left.\begin{array}{rcl} -\Delta\mathbf{u} + \nabla p &=& \mathbf{f} \\ \operatorname{div}\mathbf{u} &=& 0 \end{array}\right\} \text{ in } \Omega \subseteq \mathbb{R}^d, \tag{1.1}$$

describing the motion of a viscous incompressible fluid in a bounded Lipschitz domain Ω under the general mixed boundary conditions proposed by Navier [25]

$$\left.\begin{array}{rcl} \lambda_\nu \mathbf{u}_\nu + (1-\lambda_\nu)\left(\mathbf{T}(\mathbf{u},p)\cdot\nu\right)_\nu &=& \mathbf{0} \\ \lambda_\tau \mathbf{u}_\tau + (1-\lambda_\tau)\left(\mathbf{T}(\mathbf{u},p)\cdot\nu\right)_\tau &=& \mathbf{0} \end{array}\right\} \text{ on } \partial\Omega. \tag{1.2}$$

Thereby we use the following notation

Ω		bounded Lipschitz domain in $\mathbb{R}^d$,	(1.3)
d	$\in \{2,3\}$	dimension,	
$\mathbf{u}$	$: \Omega \to \mathbb{R}^d$	velocity field,	
p	$: \Omega \to \mathbb{R}$	pressure distribution,	
$\mathbf{f}$	$: \Omega \to \mathbb{R}^d$	given force density,	
ν		outer normal of the domain Ω,	
$\mathbf{v}_\nu$	$= (\mathbf{v})_\nu := \langle \mathbf{v}, \nu\rangle\, \nu$	normal component of $\mathbf{v} \in \mathbb{R}^d$,	
$\mathbf{v}_\tau$	$= (\mathbf{v})_\tau := \mathbf{v} - \mathbf{v}_\nu$	tangential component of $\mathbf{v} \in \mathbb{R}^d$,	
$\lambda_\nu, \lambda_\tau$	$: \partial\Omega \to [0,1]$	coefficient functions,	
$\mathbf{Du}$	$:= \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$	symmetric gradient,	
$\mathbf{I}$		$d \times d$ identity matrix,	
$\mathbf{T}(\mathbf{u},p)$	$:= 2\mathbf{Du} - p\mathbf{I}$	stress tensor.	

Both, the equations (1.1) and the boundary conditions (1.2), can be generalized by replacing the zeros on the right hand sides by some given functions.

In this paper, we are especially interested in the limit cases of the boundary conditions, i.e., Dirichlet ($\lambda_\nu = \lambda_\tau = 1$), Neumann ($\lambda_\nu = \lambda_\tau = 0$), slip ($\lambda_\nu = 1, \lambda_\tau = 0$) and leak ($\lambda_\nu = 0, \lambda_\tau = 1$) boundary conditions. In particular, we assume the boundary $\Gamma := \partial\Omega$ to consist of four relatively closed disjoint parts

$$\Gamma = \Gamma_D \cup \Gamma_s \cup \Gamma_l \cup \Gamma_N, \text{ where } \Gamma_i \cap \Gamma_j = \emptyset \ \forall i,j \in \{D,s,l,N\} : i \neq j. \tag{1.4}$$

The leak and slip parts Γ_s and Γ_l are supposed to be of class C^1. We define the coefficient functions from (1.2) in the following way:

$$\lambda_\nu(\mathbf{x}) := \begin{cases} 1, & \mathbf{x} \in \Gamma_D \cup \Gamma_s \\ 0, & \mathbf{x} \in \Gamma_l \cup \Gamma_N \end{cases}, \quad \lambda_\tau(\mathbf{x}) := \begin{cases} 1, & \mathbf{x} \in \Gamma_D \cup \Gamma_l \\ 0, & \mathbf{x} \in \Gamma_s \cup \Gamma_N \end{cases}. \tag{1.5}$$

This choice leads to the following set of boundary conditions

$$\begin{array}{rclrcll} \mathbf{u} &=& \mathbf{0}, & & & & \text{on } \Gamma_D, \\ \langle \mathbf{u}, \nu \rangle &=& 0, & (\mathbf{Du}\cdot\nu)_\tau &=& \mathbf{0}, & \text{on } \Gamma_s, \\ \mathbf{u} - \langle \mathbf{u}, \nu\rangle\, \nu &=& \mathbf{0}, & 2\langle \mathbf{Du}\cdot\nu, \nu\rangle &=& p, & \text{on } \Gamma_l, \\ & & & 2\mathbf{Du}\cdot\nu &=& p\nu, & \text{on } \Gamma_N. \end{array} \tag{1.6}$$

While the mathematical literature on Dirichlet and Neumann boundary conditions is vast, leak and slip boundary conditions have been studied less extensively. However, they are of great practical interest. For theoretical studies of these boundary conditions we refer to [38; 13; 34] and for some applications to [19; 20].

The Sobolev space that contains those velocity fields which fulfill the essential parts (conditions on the left in (1.6)) of the boundary conditions is denoted by

$$\mathbf{H}^1_{\text{ess}} := \left\{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \mathbf{u}|_{\Gamma_D} = \mathbf{0},\ \mathbf{u}_\nu|_{\Gamma_s} = 0,\ \mathbf{u}_\tau|_{\Gamma_l} = \mathbf{0} \text{ in the sense of traces}\right\}. \tag{1.7}$$

The (mixed) weak formulation of problem (1.1) together with the boundary conditions (1.6) reads: *Find* $(\mathbf{u}, p) \in \mathbf{H}^1_{\text{ess}} \times \mathrm{L}^2$ *such that*

$$\begin{aligned} \mathfrak{a}(\mathbf{u},\mathbf{v}) + \mathfrak{b}(\mathbf{v},p) &= \langle \mathbf{f},\mathbf{v}\rangle_{\mathrm{L}^2(\Omega)}, & \forall \mathbf{v} \in \mathbf{H}^1_{\text{ess}}, \\ \mathfrak{b}(\mathbf{u},q) &= 0, & \forall q \in \mathrm{L}^2(\Omega). \end{aligned} \tag{1.8}$$

The bilinear forms $\mathfrak{a} : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ and $\mathfrak{b} : \mathbf{H}^1(\Omega) \times \mathrm{L}^2(\Omega) \to \mathbb{R}$ are defined by

$$\mathfrak{a}(\mathbf{u},\mathbf{v}) := 2\int_\Omega \mathbf{D}\mathbf{u} : \mathbf{D}\mathbf{v}, \quad \mathfrak{b}(\mathbf{v},q) := -\int_\Omega q \operatorname{div}\mathbf{v}. \tag{1.9}$$

In general, problem (1.8) is not uniquely solvable. The bilinear form $\mathfrak{a}$ has a nontrivial kernel given by the set of rigid body motions

$$\mathscr{R} := \{\mathbf{A}\cdot + \mathbf{b} \mid \mathbf{A} \in \mathbb{R}^{d\times d} \text{ skew symmetric, } \mathbf{b} \in \mathbb{R}^d\}. \tag{1.10}$$

Moreover, every $\mathbf{v} \in \mathscr{R}$ is divergence-free, i.e. the pairs (v,q), $q \in \mathbb{R}$, is a solution of the homogeneous Stokes problem. Thus, a solution of (1.8) can only be unique up to elements of $\mathbf{H}^1_{\text{ess}} \cap \mathscr{R}$.

Remark 1. *To assure unique solvability of problem* (1.8) *we assume the essential boundary to have a positive measure, i.e.*

$$|\Gamma_D \cup \Gamma_s \cup \Gamma_l| > 0. \tag{1.11}$$

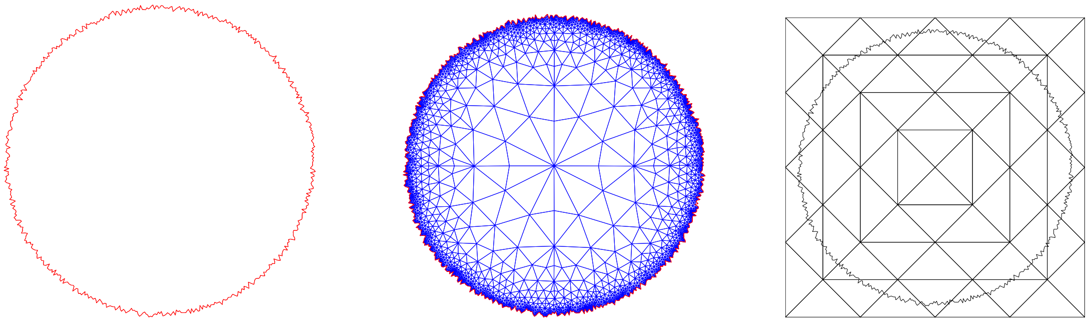
If $|\Gamma_D| = |\Gamma_l| = 0$, *we further have to exclude domains* Ω *having rotational symmetries (cf. [42; 43]). Uniqueness in the pressure variable of the solution can only be achieved up to constants if no additional constraint is given through the boundary condition, since the pressure component appears only as a gradient in the equations* (1.1). *That is why we assume boundary parts containing pressure constraints to have a positive measure as well:*

$$|\Gamma_l \cup \Gamma_N| > 0. \tag{1.12}$$

As an alternative to the assumptions (1.11) *and* (1.12) *weak formulations with respect to suitable quotient spaces or additional constraint formulations could be considered.*

Under the assumptions (1.11) and (1.12) there exists a unique solution $(\mathbf{u}, p) \in \mathbf{H}^1_{\text{ess}} \times \mathrm{L}^2(\Omega)$ for all right hand sides $\mathbf{f}$ in the dual space $\mathbf{H}^1_{\text{ess}}{}'$ of $\mathbf{H}^1_{\text{ess}}$. The unique solvability and regularity of the continuous problem (1.8) has been discussed in detail for the different boundary conditions for instance in [40], [38], [34] and [29]. The theory therein bases mainly on Korn's second inequality and its variants (cf. [26], [22], [11], [29]).

The classical finite element discretization approach is to replace the continuous spaces $\mathbf{H}^1_{\text{ess}}$ and $\mathrm{L}^2(\Omega)$ in the weak formulation (1.8) by suitable finite dimensional subspaces $\mathbf{X}_{\mathscr{T}}$ and $\mathrm{M}_{\mathscr{T}}$. That means that the essential boundary conditions are incorporated strongly in the velocity part of the approximation space. This is the standard procedure for the treatment of the Dirichlet boundary condition. An analysis for slip boundary conditions can be found in [42; 4; 21]. Typically, $\mathbf{X}_{\mathscr{T}}$ and $\mathrm{M}_{\mathscr{T}}$ contain continuous



(a) Domain with a rough boundary. ball.

(b) Resolving shape regular mesh.

(c) Coarse overlapping mesh.

Figure 1: A fitted and an unfitted mesh for a complicated domain.

functions that are piecewise polynomial with respect to some triangulation $\mathscr{T}$ of the domain Ω, as for instance the Mini element (cf. [8]) or the modified Taylor-Hood element (cf. [6]). The latter first order methods fulfill the classical error estimate

$$\|\mathbf{u}-\mathbf{u}_{\mathscr{T}}\|_{\mathbf{H}^1_{\mathrm{ess}}} + \|p-p_{\mathscr{T}}\|_{\mathrm{L}^2(\Omega)} \lesssim \inf_{\mathbf{v}\in\mathbf{X}_{\mathscr{T}}} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^1_{\mathrm{ess}}} + \inf_{q_{\mathscr{T}}\in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}}} \|p-q_{\mathscr{T}}\|_{\mathrm{L}^2(\Omega)}, \tag{1.13}$$

where the hidden constant depends only on the bilinear forms $\mathfrak{a}$ and $\mathfrak{b}$, i.e., on their continuity, the coercivity of $\mathfrak{a}$ and the infsup property of $\mathfrak{b}$. However, for the following reasons, the conformity condition $\mathbf{X}_{\mathscr{T}} \times \mathrm{M}_{\mathscr{T}} \subseteq \mathbf{H}^1_{\mathrm{ess}} \times \mathrm{L}^2(\Omega)$ which requires to resolve the details of the boundary exactly, can be too restrictive:

1. The triangulation $\mathscr{T}$ of the physical domain Ω needs to be "almost" exact, which can only be true for polygonal domains.

2. The approximation of general curved domains by simplicial meshes causes additional problems in the numerical treatment of leak or slip conditions, since the outer normal is not sufficiently well approximated by the (piecewise constant) outer normal of the polygonal mesh.

3. Due to 2. problems with leak and slip boundary are in general instable with respect to boundary perturbations. This fact has been investigated by Verfürth [42]. Therefore, polygonal approximations to general domains and approximations to the outer normal have to be chosen carefully.

4. Due to 1. to 3., the mesh density of suitable (shape regular) triangulations is determined by the domain geometry (cf. Figure 1b) and *not* by the local approximation properties of the finite element space. For domains containing a huge number of geometric details such as holes or rough boundaries, the number of vertices in a suitable triangulation, and therefore the dimension of a suitable approximation space, will be at least proportional to the number of details.

5. In practice, one is often interested in an only moderate accuracy, that should be achieved at a moderate effort. In addition, a mathematical model and its discretization is only an approximation of a real world process meaning that in general there are modeling and discretization errors anyway. Therefore it is possible to relax the boundary condition without increasing the overall error significantly.

In case of a piecewise smooth boundary isoparametric elements are often employed for its approximation. If the domain is not smooth then other approaches have to be chosen. One alternative is to impose the essential boundary conditions weakly as a side condition in a saddle point formulation [2; 42; 41]. Another approach is to incorporate boundary conditions via penalization (cf. [27; 1]). This is commonly used in Discontinuous Galerkin Methods for the Stokes problem (see for instance [12]), and also the basis of Fictitious Domain Methods [15] and Immersed Boundary Methods [28]. All these methods might be used with (overlapping) computational grids that are not fitted to the physical domain as depicted in Figure 1c (cf. [18; 5]). However, mesh compatibility conditions have to be imposed and boundary integrals that enter the variational formulations need to be evaluated which is problematic on very complicated domains, especially in three spatial dimensions.

In this paper, we will generalize the concept of composite finite elements [17; 16; 32; 36] to problems on complicated domains with Dirichlet, Neumann, slip, and leak boundary conditions. The concept of composite finite elements is as follows:

1. They are a generalization of classical finite element spaces which allow that boundary conditions on rough boundaries are resolved not necessarily by a very fine mesh and a huge number of degrees of freedom but allow the adaptation of the shape of the ansatz functions to the characteristic behavior of the solution via slave nodes.

2. To control this enrichment process in a problem-adapted way, a posteriori error estimators should be employed which allow to decide whether new degrees of freedoms are locally needed to reduce the error or whether it is enough to adapt the shape of the ansatz functions locally by using slave nodes..

3. A local a priori analysis allows to set up a (quasi-) optimal enrichment strategy based on the indications of the a posteriori error estimator.

In this paper we will concentrate on 1. and the derivation of an local a priori analysis. In a forthcoming paper this will be combined with an appropriate error estimator (see also [31] for the application in linear elasticity). By now, composite finite elements have been used successfully for Stokes problems with Dirichlet and slip boundary conditions [29; 30] where the *composite mini element* has been introduced. Here, we will generalize the theory to the Stokes problem with mixed Dirichlet, slip, leak and Neumann boundary conditions.

The paper is organized as follows. In Section 2 we introduce the composite mini element formulation of the problem. Section 3 will contain the main a priori error bound and its detailed proof. Finally, in Section 4 we will present some numerical experiments.

In this paper, various notations and conventions will be used. In order to improve readability, we have collected below the most relevant ones along with a link to their first appearance.

Notations

B_T	largest ball inscribed in the simplex T., 8
$C_1^{\mathcal{T}}, C_2^{\mathcal{T}}, C_3^{\mathcal{T}}$	mesh related constants, 8
$C_4^{\mathcal{T}}$	constant related to the local relative boundary length, 19
$C_{\mathscr{E}^N}, C_{\mathscr{E}^{\text{ess}}}$	constants from Lemma 4, 20
C_{ext}	constants related to the modified Stein extension operator, 16
C_{int}	constant from interpolation error estimates, 15
$C_{\text{np}}, C_{\text{size}}, C_{\text{dist}}$	constants related to the neighborhood property, 15

C_1^ν	Lipschitz constant of the outer normal, 18
C_2^ν	Constant related to the outer domain normal, 18
$\mathscr{E}^N, \mathscr{E}^D, \mathscr{E}^{\mathrm{ess}}$	extension operators, 9
$(\cdot)^\Gamma$	boundary projection, 9
$\mathbf{H}^1_{\mathrm{ess}}$	continuous velocity space, 3
$\mathscr{I}_{\mathscr{T}}, \mathscr{I}_T$	nodal interpolation operator with respect to the vertices of the triangulation $\mathscr{T}$ or the simplex T, 15
$\lambda_\nu, \lambda_\tau$	coefficient functions, 2
$\lesssim$	$a \lesssim b \Leftrightarrow \exists C > 0 : a \leq Cb$, 4
$(\cdot)_\nu$	normal component, 2
ν	outer normal of the domain Ω, 2
$\\|\cdot\\|_{k,p,\Omega}$, $\\|\cdot\\|_{k,p,\Omega}$	Sobolev norms, 14
$\|\cdot\|_{k,p,\Omega}$, $\|\cdot\|_{k,p,\Omega}$	Sobolev semi norms, 14
Ω	bounded Lipschitz domain, 8
Ω^{dof}	union of all elements of $\mathscr{T}^{\mathrm{dof}}$, 8
ω_T	neighborhood od the simplex T defined in Lemma 1, 16
$\Omega_{\mathscr{T}}$	union of all elements of $\mathscr{T}$, 8
$\mathrm{R}(T)$	ratio that measures the refinement of $T \in \mathscr{T}$ in $\mathscr{T}_{\mathrm{ess}}$, 19
$\rho_{\mathscr{T}}, \rho_{\mathscr{T}_{\mathrm{ess}}}$	shape regularity constants, 8
ρ_T	shape regularity constant of the simplex T, 15
$\mathscr{R}$	set of rigid body motions, 3
$S^{\mathrm{cfe}}_{\mathscr{T}}$	standard conforming $\mathbb{P}_1$ finite element space, 9
$S^{\mathrm{cfe}}_{\mathscr{T}}$, $S^{\mathrm{cfe}}_{\mathscr{T},D}$, $\mathbf{S}^{\mathrm{cfe}}_{\mathscr{T},\mathrm{ess}}$	composite $\mathbb{P}_1$ finite element spaces, 9, 10
$T_{(\cdot)}$	projection onto $\mathscr{T}^{\mathrm{dof}}$, 9
$(\cdot)_\tau$	tangential component, 2
Θ, Θ^{dof}, $\Theta^{\mathrm{dof}}_{\mathrm{ess}}, \ldots$	vertex sets, 9
$\mathscr{T}$	coarse overlapping mesh, 8
$\mathscr{T}^{\mathrm{dof}} = \mathscr{T}^{\mathrm{dof}}_{\mathrm{ess}}$	inner part of $\mathscr{T}$ and $\mathscr{T}$ resp., 8
$\mathscr{T}_{\mathrm{ess}}$	submesh of $\mathscr{T}$, 8
$\mathscr{T}^{\mathrm{slave}}$, $\mathscr{T}^{\mathrm{slave}}_{\mathrm{ess}}$	boundary parts of $\mathscr{T}$ and $\mathscr{T}_{\mathrm{ess}}$, 8

2 Composite mini element formulation

The choice of a suitable mixed finite element space for the problem (1.8) follows the concept of composite finite elements introduced in [17], [32] for Poisson problems and in [29] and [30] for Stokes problems. Instead of using conventional resolving triangulations we will define the composite mini element space with respect to an overlapping (and possibly structured) conforming triangulation $\mathscr{T}$ (in the sense of Ciarlet [9]). $\mathscr{T}$ contains closed simplices. Typically, $\mathscr{T}$ is a quasi uniform triangulation which does not contain extra boundary resolution. This technique allows the definition of coarse spaces even for very complicated geometries (see also Figure 1b). We mark the subset $\mathscr{T}^{\text{dof}}$ of the triangulation containing all triangles that are properly contained in Ω. Next, the finite element shape function will be defined with respect to $\mathscr{T}^{\text{dof}}$ and extended (smeared) to the remaining (outer) part $\mathscr{T}^{\text{slave}} := \mathscr{T} \setminus \mathscr{T}^{\text{dof}}$ in such a way that the essential parts of the boundary conditions are fulfilled in an approximative way. To be more precise, let $\mathscr{T} = \mathscr{T}^{\text{dof}} \cup \mathscr{T}^{\text{slave}}$ fulfill the subsequent conditions:

Overlap	$\Omega \subseteq \Omega_{\mathscr{T}} := \operatorname{int}\left(\bigcup_{T\in\mathscr{T}} T\right)$,
Shape regularity	$\exists \rho_{\mathscr{T}} > 0 : \operatorname{diam}(B_T) \geq \rho_{\mathscr{T}} \operatorname{diam}(T),\ \forall T \in \mathscr{T}$,
Admissible splitting	$\begin{cases} \emptyset \neq \mathscr{T}^{\text{dof}} \subseteq \mathscr{T} \text{ and } \mathscr{T}^{\text{slave}} = \mathscr{T} \setminus \mathscr{T}^{\text{dof}} \\ \Omega^{\text{dof}} := \operatorname{int}\left(\bigcup_{T\in\mathscr{T}^{\text{dof}}} T\right) \subseteq \Omega, \\ \exists C_1^{\mathscr{T}} : \operatorname{dist}(t, \partial\Omega) \leq C_1^{\mathscr{T}} \operatorname{diam}(t),\ \forall t \in \mathscr{T}^{\text{slave}}, \\ \exists C_2^{\mathscr{T}} : \operatorname{dist}(t, \Omega^{\text{dof}}) \leq C_2^{\mathscr{T}} \operatorname{diam}(t),\ \forall t \in \mathscr{T}^{\text{slave}}. \end{cases}$

(2.1)

In order to resolve the boundary part where essential boundary conditions are imposed we will employ a submesh $\mathscr{T}_{\text{ess}}$ which arises from $\mathscr{T}$ by standard finite element refinement patterns (cf. [32]). $\mathscr{T}_{\text{ess}}$ is refined toward the essential parts of the boundary in such a way that the subsequent assumptions hold:

Submesh property	$\mathscr{T}_{\text{ess}}^{\text{dof}} := \mathscr{T}^{\text{dof}} \subseteq \mathscr{T}_{\text{ess}}$,
Shape regularity	$\exists \rho_{\mathscr{T}_{\text{ess}}} > 0 : \operatorname{diam}(B_t) \geq \rho_{\mathscr{T}_{\text{ess}}} \operatorname{diam}(t),\ \forall t \in \mathscr{T}_{\text{ess}}$,
Admissibility	$\exists C_3^{\mathscr{T}} : \operatorname{dist}(t, \partial\Omega) \leq C_3^{\mathscr{T}} \operatorname{diam}(t),\ \forall t \in \mathscr{T}_{\text{ess}}^{\text{slave}} := \mathscr{T}_{\text{ess}} \setminus \mathscr{T}_{\text{ess}}^{\text{dof}}$.

(2.2)

In Figure 2 a typical choice of an admissible triangulation $\mathscr{T}$ and $\mathscr{T}_{\text{ess}}$ is visualized, some remarks are in order.

Remark 2. 1. *In order to resolve the boundary in such a way that optimal error estimates are preserved, the local mesh-width of $\mathscr{T}_{\text{ess}}$ in a neighborhood of $\Gamma \setminus \Gamma_{\text{N}}$ (i.e. close to essential boundary conditions) has to be of order $h^{\max\left(\frac{3}{2}-r,1\right)}$ (cf. Theorem 1), where h denotes the maximal mesh width of the initial mesh $\mathscr{T}$ and $r \in [0,1]$ is a parameter reflecting the regularity of the solution.*

2. *The resolution condition from 1. does not lead to an increase of degrees of freedom since it only restricts the choice of the submesh $\mathscr{T}_{\text{ess}}$ which only contains slave nodes.*

3. *The submesh $\mathcal{T}_{\text{ess}}$ will appear in practical computations only during the assembly of the system matrix and needs to be computed only locally.*

4. *The constants $C_1^{\mathcal{T}}$, $C_2^{\mathcal{T}}$, $C_3^{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ will be crucial in our analysis of the method, while the conditions $\Omega^{\text{dof}} \subseteq \Omega$ and $\mathcal{T}^{\text{dof}} \subseteq \mathcal{T}_{\text{ess}}$ can be relaxed.*

5. *The theory can be generalized to the case where $\mathcal{T}_{\text{ess}}$ contains hanging nodes as depicted in Figure 2b.*

We summarize further notations and definitions in connection with the meshes:

$$
\begin{array}{rl}
\text{Set of vertices of } \mathcal{T} & \Theta, \\
\text{Set of vertices of a simplex } T & V(T), \\
\text{Vertices of } \mathcal{T}^{\text{dof}} = \mathcal{T}_{\text{ess}}^{\text{dof}} & \Theta^{\text{dof}}, \\
\text{Slave nodes} & \Theta^{\text{slave}} := (\Theta \cup \Theta_{\text{ess}}) \setminus \Theta^{\text{dof}}, \\
\text{Maximal mesh-width in } \mathcal{T} & h_{\mathcal{T}} := \max_{T \in \mathcal{T}} \operatorname{diam}(T), \\
\text{Boundary projection}^1 & \begin{array}{rcl} (\cdot)^{\Gamma} : \Theta^{\text{slave}} & \to & \Gamma, \\ \mathbf{x} & \mapsto & \mathbf{x}^{\Gamma} \in \operatorname{arginf}_{\mathbf{y} \in \partial\Omega} \operatorname{dist}(\mathbf{x}, \mathbf{y}), \end{array} \\
\begin{array}{l}\text{Projection to the} \\ \text{closest inner triangle}\end{array} & \begin{array}{rcl} T_{(\cdot)} : \Theta^{\text{slave}} & \to & \mathcal{T}^{\text{dof}}, \\ \mathbf{x} & \mapsto & T_{\mathbf{x}} \in \operatorname{argmin}_{T \in \mathcal{T}^{\text{dof}}} \operatorname{dist}(\mathbf{x}, T). \end{array}
\end{array}
\tag{2.3}
$$

Our space definition is based on continuous piecewise affine functions and vector fields with respect to a triangulation $\mathcal{T}$:

$$
S_{\mathcal{T}} := \left\{ v \in C^0(\Omega_{\mathcal{T}}) \mid \forall T \in \mathcal{T} : v|_T \in \mathbb{P}_1 \right\}, \quad \mathbf{S}_{\mathcal{T}} := (S_{\mathcal{T}})^d. \tag{2.4}
$$

The composite finite element space (cf. [17; 16])

$$
S_{\mathcal{T}}^{\text{cfe}} := \mathcal{E}^N \left(S_{\mathcal{T}^{\text{dof}}} \right) \tag{2.5}
$$

is defined as the image of $S_{\mathcal{T}^{\text{dof}}}$ under a simple linear extension operator which is characterized by specifying its values at the nodal points explicitly by

$$
\mathcal{E}^N : S_{\mathcal{T}^{\text{dof}}} \to S_{\mathcal{T}} \subseteq S_{\mathcal{T}_{\text{ess}}}, \quad \left(\mathcal{E}^N q \right)(\mathbf{x}) := \begin{cases} q(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{dof}}, \\ q_{T_{\mathbf{x}}}(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{slave}}. \end{cases} \tag{2.6}
$$

This space is suitable for the use with Neumann boundary conditions as we will see later. In case of Dirichlet boundary conditions the space

$$
S_{\mathcal{T},D}^{\text{cfe}} := \mathcal{E}^D \left(S_{\mathcal{T}^{\text{dof}}} \right) \tag{2.7}
$$

is an appropriate choice (cf. [32; 29; 30]), where $\mathcal{E}^D : S_{\mathcal{T}^{\text{dof}}} \to S_{\mathcal{T}_{\text{ess}}}$ is defined by

$$
\left(\mathcal{E}^D q \right)(\mathbf{x}) := \begin{cases} q(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{dof}}, \\ q_{T_{\mathbf{x}}}(\mathbf{x}) - q_{T_{\mathbf{x}}}(\mathbf{x}^{\Gamma}) = \left\langle \nabla \mathbf{u}_{T_{\mathbf{x}}}, \mathbf{x} - \mathbf{x}^{\Gamma} \right\rangle, & \mathbf{x} \in \Theta_{\text{ess}}^{\text{slave}}. \end{cases} \tag{2.8}
$$

[1] The minimizer might not be unique and we fix one of them in this case.

In contrast to $\mathscr{E}^N$, the operator $\mathscr{E}^D$ is defined with respect to the refined mesh $\mathscr{T}^{\text{ess}}$. Shape functions in $S^{\text{cfe}}_{\mathscr{T},D}$ are not necessarily piecewise affine with respect to $\mathscr{T}$ *but composed of piecewise affine finite elements on the submesh* $\mathscr{T}_{\text{ess}}$. This composite construction allows to approximate the essential zero boundary condition in a very flexible way. For the definition of the approximation spaces that fulfill the essential parts the boundary conditions we will interpolate the vector valued versions $\mathscr{E}^{\text{N}}$ and $\mathscr{E}^{\text{D}}$ of (2.6) and (2.8) point-wise in the slave nodes with respect to the coefficients λ_ν and λ_ν to define $\mathscr{E}^{\text{ess}} : \mathbf{S}_{\mathscr{T}^{\text{dof}}} \to \mathbf{S}_{\mathscr{T}_{\text{ess}}}$ by

$$\left(\mathscr{E}^{\text{ess}}\mathbf{u}\right)(\mathbf{x}) := \begin{cases} \mathbf{u}(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{dof}}, \\ \quad \lambda_\nu(\mathbf{x}^\Gamma)(\mathscr{E}^{\text{D}}\mathbf{u}(\mathbf{x}))_{\nu(\mathbf{x}^\Gamma)} + (1-\lambda_\nu(\mathbf{x}^\Gamma))(\mathscr{E}^{\text{N}}\mathbf{u}(\mathbf{x}))_{\nu(\mathbf{x}^\Gamma)} & \\ \quad +\lambda_\tau(\mathbf{x}^\Gamma)(\mathscr{E}^{\text{D}}\mathbf{u}(\mathbf{x}))_{\tau(\mathbf{x}^\Gamma)} + (1-\lambda_\tau(\mathbf{x}^\Gamma))(\mathscr{E}^{\text{N}}\mathbf{u}(\mathbf{x}))_{\tau(\mathbf{x}^\Gamma)}, & \mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}}. \end{cases} \tag{2.9}$$

In the special case of slip boundary conditions for the unit disc the extrapolation procedure is shown for an example in Figure 3.

The operator $\mathscr{E}^{\text{ess}}$ can be rewritten explicitly (cf. (1.5)) by

$$\left(\mathscr{E}^{\text{ess}}\mathbf{u}\right)(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{dof}}, \\ \mathscr{E}^{\text{D}}\mathbf{u}(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}}, \mathbf{x}^\Gamma \in \Gamma_D, \\ (\mathscr{E}^{\text{D}}\mathbf{u}(\mathbf{x}))_{\nu(\mathbf{x}^\Gamma)} + (\mathscr{E}^{\text{N}}\mathbf{u}(\mathbf{x}))_{\tau(\mathbf{x}^\Gamma)}, & \mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}}, \mathbf{x}^\Gamma \in \Gamma_s, \\ (\mathscr{E}^{\text{N}}\mathbf{u}(\mathbf{x}))_{\nu(\mathbf{x}^\Gamma)} + (\mathscr{E}^{\text{D}}\mathbf{u}(\mathbf{x}))_{\tau(\mathbf{x}^\Gamma)}, & \mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}}, \mathbf{x}^\Gamma \in \Gamma_l, \\ \mathscr{E}^{\text{N}}\mathbf{u}(\mathbf{x}), & \mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}}, \mathbf{x}^\Gamma \in \Gamma_N. \end{cases} \tag{2.10}$$

We assume that

$$\{\mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}} \mid \mathbf{x}^\Gamma \in \Gamma_{\text{D}} \cup \Gamma_{\text{s}} \cup \Gamma_{\text{l}}\} \neq \emptyset \quad \text{and} \quad \{\mathbf{x} \in \Theta^{\text{slave}}_{\text{ess}} \mid \mathbf{x}^\Gamma \in \Gamma_{\text{N}} \cup \Gamma_{\text{l}}\} \neq \emptyset, \tag{2.11}$$

which can be seen as a discrete analogue to the conditions (1.11) and (1.12) of Remark 1.[2] $\mathbf{S}^{\text{cfe}}_{\mathscr{T},\text{ess}} := \mathscr{E}^{\text{ess}}\left(\mathbf{S}_{\mathscr{T}^{\text{dof}}}\right)$ will form the piecewise affine part of the composite mini element velocity space. In order to stabilize the method we will use simplex bubble functions (but only) on $\mathscr{T}^{\text{dof}}$

$$B_{\mathscr{T}^{\text{dof}}} := \operatorname{span}\left\{\psi_T : T \in \mathscr{T}^{\text{dof}}\right\}, \quad \psi_T := (d+1)^{d+1} \prod_{\mathbf{y} \in V(T)} \mathbf{b}_{\mathbf{y}}, \tag{2.12}$$

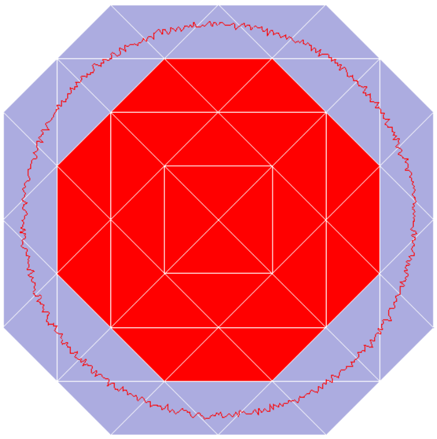
where $\mathbf{b}_{\mathbf{y}}$, $\mathbf{y} \in V(T)$, denote the barycentric coordinates of T. The composite mini element space is defined by

$$\mathbf{X}^{\text{cme}}_{\mathscr{T}} \times \mathrm{M}^{\text{cme}}_{\mathscr{T}} := (\mathbf{S}^{\text{cfe}}_{\mathscr{T},\text{ess}} \oplus \mathbf{B}_{\mathscr{T}^{\text{dof}}}) \times S^{\text{cfe}}_{\mathscr{T}}. \tag{2.13}$$

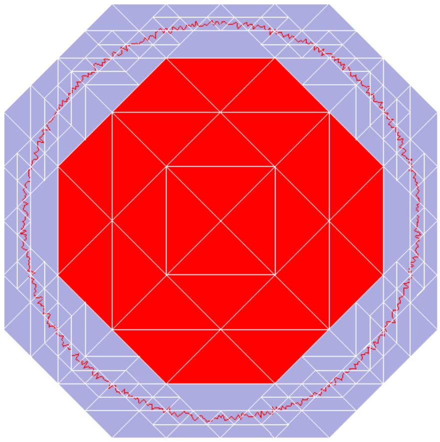
Due to (2.11) no quotient spaces have to be considered in (2.13). Note that, in general, the composite mini element is nonconforming because the Dirichlet boundary condition is satisfied only in an approximate way. This nonconformity can be controlled in an a priori or, respectively, in an a posteriori way by the local mesh size in $\mathscr{T}^{\text{slave}}_{\text{ess}}$. Note that there is no nonconformity arising from the pressure part of the space. A pair $(\mathbf{u}, p) \in \mathbf{X}^{\text{cme}}_{\mathscr{T}} \times \mathrm{M}^{\text{cme}}_{\mathscr{T}}$ defines the composite mini element approximation if it fulfills the discrete variational system:

$$\begin{aligned} \mathfrak{a}(\mathbf{u},\mathbf{v}) + \mathfrak{b}(\mathbf{v},p) &= \langle \mathbf{f}, \mathbf{v} \rangle_{L^2(\Omega)}, \quad && \forall \mathbf{v} \in \mathbf{X}^{\text{cme}}_{\mathscr{T}}, \\ \mathbf{b}(\mathbf{u},q) &= 0, && \forall q \in \mathrm{M}^{\text{cme}}_{\mathscr{T}}. \end{aligned} \tag{2.14}$$

[2] In the case of pure slip/Neumann boundary conditions we additionally have to make the technical assumption: If a rigid body motion $r \in \mathscr{R}$ fulfills $\left\langle r(\mathbf{x}^\Gamma), \nu(\mathbf{x}^\Gamma)\right\rangle = 0, \quad \forall \mathbf{x}^\Gamma \in \mathbf{V} := \{\mathbf{x}_i^\Gamma \mid \mathbf{x}_i \in \Theta^{\text{slave}}_{\text{ess}}, \mathbf{x}_i^\Gamma \in \Gamma_{\text{s}}\}$, then $r = 0$.

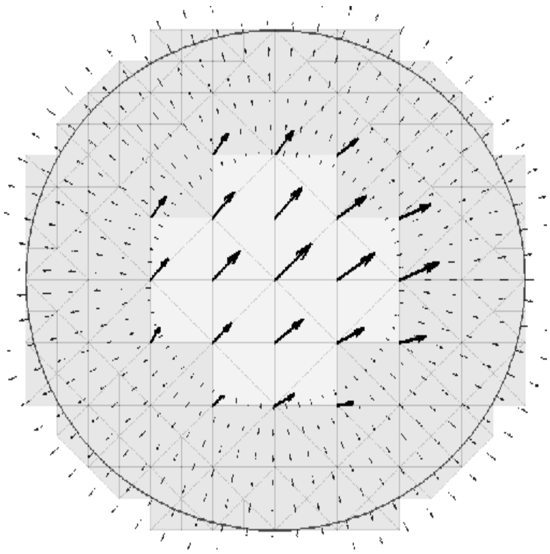


(a) Marked coarse overlapping mesh $\mathscr{T} = \mathscr{T}^{\text{dof}} \cup \mathscr{T}^{\text{slave}}$.

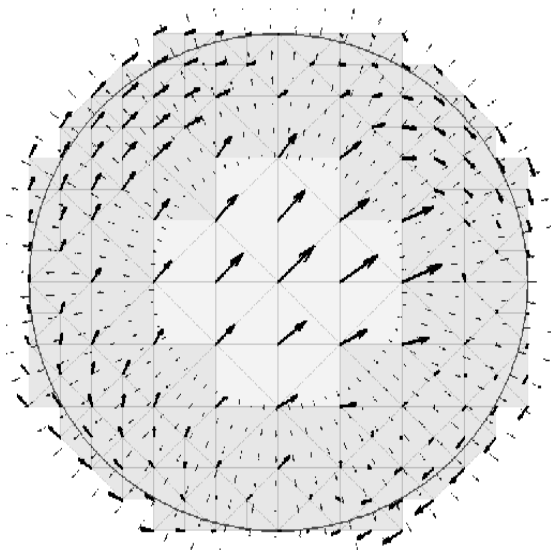


(b) Refined mesh $\mathscr{T}_{\text{ess}}$.

Figure 2: Admissible composite mini element triangulation of the domain from Figure 1b.



(a) Vector field $\mathbf{u}$ defined in the degrees of freedom.



(b) Extension of $\mathbf{u}$ to the slave vertices. The extended vector field is almost tangential to the boundary of the circle.

Figure 3: Extension of a vector field $\mathbf{u} \in \mathbf{S}_{\mathscr{T}^{\text{dof}}}$ in case of slip boundary conditions imposed on the unit circle. The small arrows represent the extended outer normal field of the domain. (The inner zone $\mathscr{T}^{\text{dof}}$ containing the degrees of freedom is kept small for visualization purposes.)

3 Error analysis

The main result of this paper concerning the unique solvability of the discrete problems (2.14) and optimal order a priori error is stated in the subsequent theorem.

Theorem 1. *The discrete problem* (2.14) *has always a unique solution* $(\mathbf{u}, p) \in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}} \times M_{\mathscr{T}}^{\mathrm{cme}}$. *Furthermore if* $(\mathbf{u}^*, p^*) \in \left(\mathbf{H}^1_{\mathrm{ess}} \cap \mathbf{H}^{1+r}(\Omega)\right) \times \left(L^2(\Omega) \cap H^r(\Omega)\right)$, $r \in [0,1]$, *is the solution of* (1.8) *then the following a priori error estimate holds:*

$$\|\mathbf{u}^* - \mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p^* - p\|_{L^2(\Omega)} \leq Ch^r \left(\|\mathbf{u}^*\|_{H^{1+r}(\Omega)} + \|p^*\|_{H^r(\Omega)}\right),$$

where the constant $C = C(\Omega, r)$ *does not depend on the mesh width parameter* $h := \max_{T \in \mathscr{T}} \operatorname{diam}(T)$.
The underlying submesh $\mathscr{T}_{\mathrm{ess}}$ *can be chosen equal to* $\mathscr{T}$ *in a neighborhood of* Γ_{N}. *In a neighborhood of* $\Gamma \setminus \Gamma_{\mathrm{N}}$ *(i.e. close to essential boundary conditions) the local mesh-width of* $\mathscr{T}_{\mathrm{ess}}$ *has to be of order* $h^{\max\left(\frac{3}{2} - r, 1\right)}$ *to ensure the above error estimate.*

The proof of Theorem 1 deserves some theoretical preparations and is left to the subsequent sections.

Remark 3. *We will add some remarks related to Theorem 1:*

1. *The unique solvability does not depend on the choice of the admissible refinement* $\mathscr{T}_{\mathrm{ess}}$*, i.e. it holds even in the case* $\mathscr{T} = \mathscr{T}_{\mathrm{ess}}$.
2. *Note, that resolution condition on the submesh does not depend on the domain geometry but only on the mesh-width parameter h of coarse overlapping mesh* $\mathscr{T}$.

3.1 Proof of Theorem 1

Theorem 1 is based on the general theory of (nonconforming) mixed finite element approximation as presented for example in [8]. The discrete problem (2.14) is uniquely solvable if the bounded bilinear form $\mathfrak{a}$ is coercive with respect to the velocity part of our finite element space $\mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}$, i.e.

$$\exists c_{\mathfrak{a}} > 0 : \ \mathfrak{a}(\mathbf{u}, \mathbf{u}) \geq c_{\mathfrak{a}} \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}, \quad \forall \mathbf{u} \in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}, \tag{3.1}$$

and the bounded bilinear form $\mathfrak{b}$ fulfills an inf-sup condition[3], i.e.

$$\exists c_{\mathfrak{b}} > 0 \, \forall p \in \mathrm{M}_{\mathscr{T}}^{\mathrm{cme}} \, \exists \mathbf{u} \in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}} : \ \mathfrak{b}(\mathbf{u}, p) \geq c_{\mathfrak{b}} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|p\|_{L^2(\Omega)}. \tag{3.2}$$

Both properties cannot be inherited from the continuous level where such inequalities hold (cf. [11], [26], [14]). We will prove (3.1) and (3.2) in Section 3.2.3. Once these conditions are fulfilled the error of the composite mini element approximation can be estimated by (cf. [8]):

$$\|\mathbf{u} - \mathbf{u}^{\mathrm{cme}}\|_{\forall p \in \mathrm{M}_{\mathscr{T}}^{\mathrm{cme}}} + \|p - p^{\mathrm{cme}}\|_{\mathrm{L}^2(\Omega)} \lesssim \inf_{\mathbf{v} \in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \inf_{q \in \mathrm{M}_{\mathscr{T}}^{\mathrm{cme}}} \|p - q\|_{\mathrm{L}^2(\Omega)} + \mathscr{K}, \tag{3.3}$$

[3](3.2) is known as Babuška-Brezzi-, Ladyshenskaja-Babuška-Brezzi- or LBB-condition.

where

$$\mathscr{K} := \sup_{0\neq \mathbf{v}\in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}} \frac{|\mathfrak{a}(\mathbf{u},\mathbf{v}) + \mathfrak{b}(p,\mathbf{v}) - \langle \mathbf{f},\mathbf{v}\rangle_{\mathrm{L}^2(\Omega)}|}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}} \tag{3.4}$$

reflects the nonconformity in the approximation space. In Section 3.2.1 we will investigate the first two terms of the error bound (3.3) and prove the approximability properties of our space under the assumption $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$

$$\inf_{\mathbf{v}\in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \lesssim h|\mathbf{u}|_{\mathbf{H}^2(\Omega)} \quad \text{and} \quad \inf_{q\in \mathrm{M}_{\mathscr{T}}^{\mathrm{cme}}} \|p-q\|_{\mathrm{L}^2(\Omega)} \lesssim h|p|_{H^1(\Omega)}.$$

This is indeed the same asymptotic error as for the classical stabilized $\mathbb{P}_1 \times \mathbb{P}_1$-elements. It remains to estimate $\mathscr{K}$. If the solution is sufficiently smooth, i.e. $(\mathbf{u}, p) \in \mathbf{H}^{\frac{3}{2}}(\Omega) \times H^{\frac{1}{2}}(\Omega)$, $\mathscr{K}$ can be estimated as follows:

$$\mathscr{K} \lesssim \sup_{0\neq \mathbf{v}\in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}} \frac{\|\lambda_\nu \langle \mathbf{v},\nu\rangle + \lambda_\tau \langle \mathbf{v},\tau\rangle\|_{\mathrm{L}^2(\Gamma\setminus\Gamma_N)} \|\mathbf{T}(\mathbf{u},p)\nu\|_{\mathrm{L}^2(\Gamma\setminus\Gamma_N)}}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}}.$$

We will show in Section 3.2.2 that $\mathscr{K}$ can further be bounded in terms of the mesh-width parameter:

$$\mathscr{K} \lesssim h(\|\mathbf{u}\|_{H^{\frac{3}{2}}(\Omega)} + \|p\|_{H^{\frac{1}{2}}(\Omega)}), \tag{3.5}$$

which finishes the proof of Theorem 1 for the case $r = 1$. Finally the interpolation theory of Sobolev spaces (see for instance [7, Theorem 12.3.3]) allows to relax the smoothness assumptions, since the error can always be bounded trivially by

$$\|\mathbf{u}-\mathbf{u}^{\mathrm{cme}}\|_{\mathbf{H}^1(\Omega)} + \|p-p^{\mathrm{cme}}\|_{\mathrm{L}^2(\Omega)} \lesssim \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{\mathrm{L}^2(\Omega)},$$

which leads to the (optimal) fractional convergence rate. This finishes the proof of Theorem 1. The following section is devoted to proofs of the referred statements.

Note that classical stabilized $\mathbb{P}_1 \times \mathbb{P}_1$-elements require a finite element mesh that resolves all boundary details in order to fulfill and error bound as (3.5).

3.2 Proof of the partial statements

We will now prove the assertions from the proof sketch of the previous section. Thereby we will use the following short notation for the norms in the Sobolev spaces $W_p^k(\Omega)$ containing functions with weak derivatives up to order k in $L^p(\Omega)$:

$$\|\cdot\|_{k,p,\Omega} := \|\cdot\|_{W_p^k(\Omega)}, \quad |\cdot|_{k,p,\Omega} := |\cdot|_{W_p^k(\Omega)}.$$

For $H(\Omega) = W_2^k(\Omega)$ we will write

$$\|\cdot\|_{k,\Omega} := \|\cdot\|_{H^k(\Omega)}, \quad |\cdot|_{k,\Omega} := |\cdot|_{H^k(\Omega)}.$$

3.2.1 Approximability

In this paragraph, we will show that solutions of the weak Stokes problem (1.8), i.e. elements of $\mathbf{H}^1_{\mathrm{ess}} \times \mathrm{L}^2$ can be approximated by composite mini element functions up to an error that decreases linearly in the maximal mesh-width h. Usually, a piecewise affine (quasi) interpolant $\mathscr{I}_{\mathscr{T}}$ with respect to the mesh $\mathscr{T}$ is used to prove this property. However, this is not possible in our situation because the vertices in

$\mathscr{T}^{\text{slave}}$ do not correspond to degrees of freedom, i.e. the interpolants are not contained in our space, in general. But we will prove that the extension operators of Section 2 are accurate enough to preserve the approximability properties with respect to the whole mesh.

Let us first recall some basic tools that we will use in the subsequent analysis:

Standard interpolation. It is well known (cf. [9, Theorem 16.1]) that, for an arbitrary simplex $T \subseteq \mathbb{R}^d$, $d = 2,3$, with shape regularity constant ρ_T, there exists a constant $C_{\text{int}} = C_{\text{int}}(m,p,d)$ such that

$$|u - \mathscr{I}_T u|_{m,p,T} \leq \frac{C_{\text{int}}}{\rho_T^m} \operatorname{diam}(T)^{\left(2-\frac{d}{2}+\frac{d}{p}-m\right)} |u|_{2,T}, \qquad \forall u \in \mathrm{H}^2(T), \tag{3.6}$$

where $m \in \{0,1\}$ and $1 \leq p \leq \infty$, provided $\mathrm{W}_p^m(\Omega) \subseteq \mathrm{H}^2(\Omega)$ [4]. $\mathscr{I}_T u \in \mathbb{P}_1(\mathbb{R}^d)$ denotes the linear interpolant of u in the vertices of T.

Inverse estimate. For $m \in \{0,1\}$ and $p \in \mathbb{N} \cup \{\infty\}$ it holds that

$$|q|_{m,p,T} \leq \left(\frac{2}{\rho_T}\right)^m h_T^{\left(\frac{d}{p}-m\right)} \|q\|_{0,\infty,T} \quad \forall q \in \mathbb{P}_1(\mathbb{R}^d) \tag{3.7}$$

and

$$|q|_{1,T} \leq \left(\frac{2}{\rho_T}\right)^m h_T^{\left(\frac{d}{2}-1\right)} \max_{\mathbf{x},\mathbf{y} \in V(T)} |q(\mathbf{x}) - q(\mathbf{y})| \quad \forall q \in \mathbb{P}_1(\mathbb{R}^d). \tag{3.8}$$

Neighborhood property. Let T be an arbitrary simplex with shape regularity constant ρ_T, t be an arbitrary simplex with regularity constant ρ_t. Let the ratio of the diameters of t and T be denoted by C_{size} and the distance between T and t relative to the size of T by C_{dist}, i.e.

$$C_{\text{size}} := \frac{\operatorname{diam}(t)}{\operatorname{diam}(T)} \quad \text{and} \quad C_{\text{dist}} := \frac{\operatorname{dist}(t,T)}{\operatorname{diam}(T)}.$$

Furthermore let $u \in \mathrm{H}^2(\operatorname{conv}(T \cup t))$ and let $\mathscr{I}_T u \in \mathbb{P}_1(\mathbb{R}^d)$ denote the affine interpolation of u at the vertices of T. Then, for $m \in \{0,1\}$ and $1 \leq p \leq \infty$, provided $\mathrm{W}_p^m(\Omega) \subseteq \mathrm{H}^2(\Omega)$, there exists a constant $C_{\text{np}} = C_{\text{np}}(C_{\text{int}}, d, C_{\text{size}}, C_{\text{dist}}, \rho_t, \rho_T) > 0$ such that

$$|u - \mathscr{I}_T u|_{m,p,t} \leq C_{\text{np}} \operatorname{diam}(T)^{\left(2-\frac{d}{2}\right)} \operatorname{diam}(t)^{\left(\frac{d}{p}-m\right)} |u|_{2,\operatorname{conv}(T\cup t)}. \tag{3.9}$$

The proof of (3.9) is given in [30, Lemma 1].

Bounded Extensions. Since in general $\Omega \subseteq \Omega_{\mathscr{T}}$, it will be useful to extend u to the larger domain Ω_T. It is known that, if Ω is bounded and Lipschitz, there exists a continuous, linear extension operator $\mathfrak{E} : \mathrm{H}^k(\Omega) \to \mathrm{H}^k(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, such that

$$\forall u \in \mathrm{H}^k(\Omega): \quad \mathfrak{E}u|_\Omega = u \ \text{ and } \ \|\mathfrak{E}u\|_{\mathrm{H}^k(\mathbb{R}^d)} \leq C_{\text{ext}} \|u\|_{\mathrm{H}^k(\Omega)} \tag{3.10}$$

with a constant C_{ext} depending only on k and Ω (cf. [39]). It is worth noting that for domains containing a large number of holes and a possibly rough outer boundary, there exists an extension operator with moderately small norm C_{ext} under mild assumptions on the geometry. For all details including the characterization of the class of domain geometries, we refer to [35]. In the following we always identify u with its minimal extension $\mathfrak{E}u$ without mentioning this explicitly. For $T \in \mathscr{T}^{\text{dof}}$ the approximation results are obvious corollaries of the classical interpolation estimate (3.6).

[4] The condition $\mathrm{W}_p^m(\Omega) \subseteq \mathrm{H}^2(\Omega)$ restricts the choices of m and p depending on the dimension d. The combinations of m and p that will be useful later ($(m,p) \in \{(0,2),(0,\infty),(1,2)\}$) are allowed in two as well as in three dimensions.

As a first step towards the approximation results we will show that an arbitrary $H^2(\Omega)$-function u, can be approximated sufficiently well by $\mathscr{E}^N\left(\mathscr{I}_{\mathscr{T}^{\text{dof}}}u\right)$, i.e. by the extension of the piecewise affine interpolation with respect to $\mathscr{T}^{\text{dof}}$. We will give local and global H^1-estimate.

Lemma 1. *Let $m \in \{0,1\}$.*
There is a constant $C = C(C_{\text{int}}, \rho_{\mathscr{T}}, C_1^{\mathscr{T}}, C_2^{\mathscr{T}}, d) > 0$ which does not depend on h such that

$$\|u - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{m,T} \leq C \operatorname{diam}(T)^{(2-m)} |u|_{2,\omega_T}, \quad \forall u \in H^2(\Omega_{\mathscr{T}}),\ \forall T \in \mathscr{T},$$

where $\omega_T = T$ for all $T \in \mathscr{T}^{\text{dof}}$ and $\omega_T = \operatorname{conv}\left(T \cup \left(\bigcup_{\mathbf{x}\in V(T)} T_x\right)\right)$ for slave simplices $T \in \mathscr{T}^{\text{slave}}$. Furthermore, the global estimate

$$\|u - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{m,\Omega} \leq C h^{(2-m)} |u|_{2,\Omega}, \quad \forall u \in H^2(\Omega_{\mathscr{T}}).$$

holds, where C depends only on the constant of the local estimate, $\rho_{\mathscr{T}}$ and C_{ext}.

Proof. For every $T \in \mathscr{T}^{\text{dof}}$ the local estimate is simply given by (3.6). For $T \in \mathscr{T}^{\text{slave}}$ we estimate the error as follows

$$\begin{aligned}\|u - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{m,T} &\leq \|u - \mathscr{I}_T u\|_{m,T} + \|\mathscr{I}_T u - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{m,T} \\ &\overset{(3.6),(3.7)}{\lesssim} \operatorname{diam}(T)|u|_{2,T} + \operatorname{diam}(T)^{\left(\frac{d}{2}-m\right)} \|\mathscr{I}_T u - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{\infty,T}.\end{aligned}$$

With the help of (3.9) we further get

$$\begin{aligned}\|\mathscr{I}_T u - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{\infty,T} &= \max_{\mathbf{x}\in V(T)} |\mathscr{I}_T u(\mathbf{x}) - \mathscr{E}^N \mathscr{I}_{\mathscr{T}^{\text{dof}}} u(\mathbf{x})| \\ &\overset{(2.6)}{=} \max_{\mathbf{x}\in V(T)} |\mathscr{I}_T u(\mathbf{x}) - \mathscr{I}_{T_{\mathbf{x}}} u(\mathbf{x})| \\ &\overset{(3.9),(2.1)}{\lesssim} \operatorname{diam}(T)^{2-\frac{d}{2}} |u|_{2,\omega_T}\end{aligned}$$

and therefore the local estimate follows. The global estimate follows immediately by summation over all $T \in \mathscr{T}$ since

$$\sum_{T\in\mathscr{T}^{\text{dof}}} |u|_{2,\omega_T}^2 \overset{(2.1),(3.10)}{\leq} C(\rho_{\mathscr{T}}, C_{\text{ext}}) |u|_{2,\Omega}^2.$$

Finally (3.10) allows to restrict the H^2-norm of u in the error bound to the physical domain Ω. ☐

Lemma 1 can be generalized easily to functions $u \in H^1$ by replacing the nodal interpolation operator by some bounded quasi interpolation operator $\Pi_{\mathscr{T}} : H^1(\Omega_{\mathscr{T}}) \to S_{\mathscr{T}}$ as introduced by Scott and Zhang (see e.g. [37]) or Clément (see [10] and [44],[45]). Instead of (3.6) we can use the error estimates from [37, Theorem 4.1 and Corollary 4.1)] to derive the approximation result of the pressure part of the composite mini element space.

Theorem 2 (Approximation property of $M_{\mathscr{T}}^{\text{cme}}$)**.** *Let $m \in \{0,1\}$. For all $p \in H^1(\Omega)$ there exists a $p^{\text{cme}} \in M_{\mathscr{T}}^{\text{cme}}$ such that*

$$\|p - p^{\text{cme}}\|_{m,\Omega} \leq C h^{1-m} \|p\|_{1,\Omega},$$

where the constant $C = C(C_{\text{qint}}, \rho_{\mathscr{T}}, C_1^{\mathscr{T}}, C_2^{\mathscr{T}}, d, C_{\text{ext}})$ does neither depend on h nor p.

Proof. The proof follows the line of the previous proof with $p^{\text{cme}} := \mathscr{E}^N \Pi_{\mathscr{T}^{\text{dof}}} p$. For technical details due to the use of quasi interpolation operators we refer to Theorem 4.8 in [29]. ☐

To prove a similar estimate in the presence of essential boundary conditions Theorem 2 cannot be simply applied component-wise. Its proof and especially the proof of Lemma 1 is based on the admissibility condition in (2.1), which assumes the distance between slave triangles and the degrees of freedom to be comparable to the diameter of the slaves. For essential boundary conditions, the constant $C_2^{\mathscr{T}}$ deteriorate with respect to the refined mesh $\mathscr{T}_{\text{ess}}$. Therefore a more precise analysis is required in this case. On the other hand the essential boundary conditions provide additional information about the functions to be approximated. This has been worked out in detail in [30; 29] for H_0^1-functions and the operator $\mathscr{E}^D$. We recall the result in the following Lemma.

Lemma 2. *Let $m \in \{0,1\}$.*
There is a constant $C = C(C_{\text{int}}, \rho_{\mathscr{T}}, C_1^{\mathscr{T}}, C_2^{\mathscr{T}}, d, C_{\text{ext}}) > 0$ which does not depend on h such that

$$\|u - \mathscr{E}^D \mathscr{I}_{\mathscr{T}^{\text{dof}}} u\|_{m,\Omega} \leq C h^{(2-m)} |u|_{2,\Omega}, \quad \forall u \in H^2(\Omega_{\mathscr{T}}).$$

holds, where C depends only on the constant of the local estimate, $\rho_{\mathscr{T}}$ and C_{ext}.

With the help of Lemma 1 and Lemma 2 we are now able to state the approximation property of the velocity space.

Theorem 3 (Approximation property of $\mathbf{X}_{\mathscr{T}}^{\text{cme}}$)**.** *Let $m \in \{0,1\}$.*
For all $\mathbf{u} \in \mathbf{H}^1_{\text{ess}}(\Omega) \cap \mathbf{H}^2(\Omega)$ there is a $\mathbf{u}^{\text{cme}} \in \mathbf{X}_{\mathscr{T}}^{\text{cme}}$ such that

$$\|\mathbf{u} - \mathbf{u}^{\text{cme}}\|_{m,\Omega} \leq C h^{(2-m)} |\mathbf{u}|_{2,\Omega}, \quad \forall \mathbf{u} \in \mathbf{H}^1_{\text{ess}}(\Omega) \cap \mathbf{H}^2(\Omega_{\mathscr{T}}),$$

where the constant $C = C(C_{\text{int}}, \rho_{\mathscr{T}}, \rho_{\mathscr{T}_{\text{ess}}}, C_1^{\mathscr{T}}, C_2^{\mathscr{T}}, C_1^{\nu}, d) > 0$ does neither depend on h nor on $\mathbf{u}$.

Proof. Let $\mathbf{u} \in \mathbf{H}^1_{\text{ess}}(\Omega) \cap \mathbf{H}^2(\Omega)$ and $\mathbf{u}^{\text{cme}} := \mathscr{E}^{\text{ess}} \mathscr{I}_{\mathscr{T}^{\text{dof}}} \mathbf{u}$. We denote the error by $\mathbf{e}^{\text{cme}} := \mathbf{u} - \mathbf{u}^{\text{cme}}$. It can be expressed in terms of $\mathbf{e}^{\text{D}} := \mathbf{u} - \mathscr{E}^{\text{D}} \mathscr{I}_{\mathscr{T}^{\text{dof}}} \mathbf{u}$ and $\mathbf{e}^{\text{N}} := \mathbf{u} - \mathscr{E}^{\text{N}} \mathscr{I}_{\mathscr{T}^{\text{dof}}} \mathbf{u}$ in every slave vertex $\mathbf{x} \in \Theta_{\text{ess}}$ in the following way

$$\begin{aligned} \mathbf{e}^{\text{cme}}(\mathbf{x}) &\overset{(2.9)}{=} \lambda_\nu \left(\mathbf{e}^{\text{D}}(\mathbf{x})\right)_{\nu(\mathbf{x}^\Gamma)} + (1-\lambda_\nu) \left(\mathbf{e}^{\text{N}}(\mathbf{x})\right)_{\nu(\mathbf{x}^\Gamma)} \\ &\quad + \lambda_\tau \left(\mathbf{e}^{\text{D}}(\mathbf{x})\right)_{\tau(\mathbf{x}^\Gamma)} + (1-\lambda_\tau) \left(\mathbf{e}^{\text{N}}(\mathbf{x})\right)_{\tau(\mathbf{x}^\Gamma)}. \end{aligned} \tag{3.11}$$

Recall that λ_ν and λ_τ are piecewise constant coefficient functions defined in (1.5). In the cases $\lambda_\nu = \lambda_\tau$ ($|\Gamma_l| = |\Gamma_s| = 0$) the error $\mathbf{e}^{\text{cme}}$ coincides with either $\mathbf{e}^{\text{D}}$ ($\lambda_\nu = \lambda_\tau = 1$) or $\mathbf{e}^{\text{N}}$ ($\lambda_\nu = \lambda_\tau = 0$) and can be estimated by either using Lemma 2 or Lemma 1.
Let us therefore concentrate on one of the remaining cases $\lambda_\nu = 1$ and $\lambda_\tau = 0$ [5]. First, we will investigate the local errors $|\mathbf{e}^{\text{cme}}|_{1,t}$ with respect to the slave simplices $t \in \mathscr{T}_{\text{ess}}^{\text{slave}}$:

$$\begin{aligned} |\mathbf{e}^{\text{cme}}|_{m,t} &\overset{(3.6),(3.7)}{\lesssim} \operatorname{diam}(t)^{\frac{d}{2}-m} \max_{\mathbf{x},\mathbf{y} \in V(t)} |\left(\mathbf{e}^{\text{cme}}(\mathbf{x})\right) - \left(\mathbf{e}^{\text{cme}}(\mathbf{y})\right)| + \operatorname{diam}(t) |\mathbf{u}|_{2,t} \\ &\overset{(3.11)}{\leq} \operatorname{diam}(t)^{\frac{d}{2}-m} \left(\mathscr{M}_{t,\nu}^{\text{D}} + \mathscr{M}_{t,\tau}^{\text{N}}\right) + \operatorname{diam}(t) |\mathbf{u}|_{2,t} \end{aligned} \tag{3.12}$$

with

$$\begin{aligned} \mathscr{M}_{t,\nu}^{\text{D}} &:= \max_{\mathbf{x},\mathbf{y} \in V(t)} \left| \left(\mathbf{e}^{\text{D}}(\mathbf{x})\right)_{\nu(\mathbf{x}^\Gamma)} - \left(\mathbf{e}^{\text{D}}(\mathbf{y})\right)_{\nu(\mathbf{y}^\Gamma)} \right|, \\ \mathscr{M}_{t,\tau}^{\text{N}} &:= \max_{\mathbf{x},\mathbf{y} \in V(t)} \left| \left(\mathbf{e}^{\text{N}}(\mathbf{x})\right)_{\tau(\mathbf{x}^\Gamma)} - \left(\mathbf{e}^{\text{N}}(\mathbf{y})\right)_{\tau(\mathbf{y}^\Gamma)} \right|. \end{aligned}$$

[5] The opposite case can be proved equivalently.

Noting that $\mathbf{e}^{\mathrm{N}}(\mathbf{x}) = \mathscr{I}_t\mathbf{u}(\mathbf{x}) - \mathscr{E}^N\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}(\mathbf{x})$ for all vertices $\mathbf{x} \in V(t)$ the maximum $\mathscr{M}^{\mathrm{N}}_{t,\nu}$ can be estimated as follows:

$$\begin{aligned}\mathscr{M}^{\mathrm{N}}_{t,\tau} &\le \max_{\mathbf{x},\mathbf{y}\in V(t)} \left|\mathbf{e}^{\mathrm{N}}(\mathbf{x}) - \mathbf{e}^{\mathrm{N}}(\mathbf{y})\right| + \left|\mathbf{e}^{\mathrm{N}}(\mathbf{y})\right| \left|\nu(\mathbf{x}^\Gamma) - \nu(\mathbf{y}^\Gamma)\right| \\ &\overset{\Gamma_s\in C^1(p.2)}{\lesssim} \operatorname{diam}(t)\,|\mathscr{I}_t\mathbf{u} - \mathscr{E}^N\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}|_{1,\infty,t} + \underbrace{\|\nu\|_{C^1(\Gamma^s)}}_{=:C_1^\nu} \operatorname{diam}(t)\,|\mathscr{I}_t\mathbf{u} - \mathscr{E}^N\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}|_{0,\infty,t} \\ &\overset{(3.7)}{\lesssim} \operatorname{diam}(t)^{(1-\frac{d}{2})}\|\mathscr{I}_t\mathbf{u} - \mathscr{E}^N\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}\|_{1,t} \\ &\overset{(3.6)}{\lesssim} \operatorname{diam}(t)^{(1-\frac{d}{2})}\|\mathbf{e}^{\mathrm{N}}\|_{1,t} + \operatorname{diam}(t)^{(2-\frac{d}{2})}|\mathbf{u}|_{2,t}.\end{aligned} \tag{3.13}$$

To estimate $\mathscr{M}^{\mathrm{D}}_{t,\nu}$ we choose a vector field $\mathbf{u}_\nu \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\Omega)$ such that

$$\mathbf{u}_\nu(\mathbf{x}) = (\mathbf{u}_\nu(\mathbf{x}))_{\nu(\mathbf{x}^\Gamma)} = (\mathbf{u}(\mathbf{x}))_{\nu(\mathbf{x}^\Gamma)}, \forall \mathbf{x} \in \Theta^{\mathrm{slave}} \quad \text{and} \quad \|\mathbf{u}_\nu\|_{2,\Omega} \le C_2^\nu \|\mathbf{u}\|_{2,\Omega}. \tag{3.14}$$

The vector field $\mathbf{u}_\nu$ could be defined by interpolating $\mathbf{u}(\cdot)_{\nu(\cdot^\Gamma)}$ in the slave nodes using a C^1-interpolation operator as defined for instance in (cf. [7, Theorem 4.4.20]). The auxiliary function $\mathbf{u}_\nu$ contains only the (extended) normal component of $\mathbf{u}$. The constant C_2^ν can be bounded in terms of the C^2-norm of ν independent from $\mathbf{u}$. Therefore Γ^s is implicitly assumed to be of class C^2. This smoothness assumption on the slip boundary could be circumvented by following the proof of Theorem 4.7 in [29], which makes only use of the constant C_1^ν. However, the use of the auxiliary function $\mathbf{u}_\nu$ simplifies the presentation and avoids many technical difficulties. Note that

$$\left(\mathbf{e}^{\mathrm{D}}(\mathbf{x})\right)_{\nu(\mathbf{x}^\Gamma)} = \left(\mathbf{e}^{\mathrm{D}}_\nu(\mathbf{x})\right)_{\nu(\mathbf{x}^\Gamma)} := \left(\mathbf{u}_\nu - \mathscr{E}^{\mathrm{D}}\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}_\nu\right)(\mathbf{x}), \forall \mathbf{x} \in \Theta^{\mathrm{slave}}, \tag{3.15}$$

which leads to

$$\begin{aligned}\mathscr{M}^{\mathrm{D}}_{t,\nu} &= \max_{\mathbf{x},\mathbf{y}\in V(t)} \left|\left(\mathbf{e}^{\mathrm{D}}_\nu(\mathbf{x})\right)_{\nu(\mathbf{x}^\Gamma)} - \left(\mathbf{e}^{\mathrm{D}}_\nu(\mathbf{y})\right)_{\nu(\mathbf{y}^\Gamma)}\right| \\ &\le \max_{\mathbf{x},\mathbf{y}\in V(t)} \left|\mathbf{e}^{\mathrm{D}}_\nu(\mathbf{x}) - \mathbf{e}^{\mathrm{D}}_\nu(\mathbf{y})\right| + \left|\mathbf{e}^{\mathrm{D}}_\nu(\mathbf{y})\right| \left|\nu(\mathbf{x}^\Gamma) - \nu(\mathbf{y}^\Gamma)\right| \\ &\overset{\Gamma_s\in C^1(p.2)}{\lesssim} \operatorname{diam}(t)\,|\mathscr{I}_t\mathbf{u}_\nu - \mathscr{E}^D\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}_\nu|_{1,\infty,t} + C_1^\nu \operatorname{diam}(t)\,|\mathscr{I}_t\mathbf{u}_\nu - \mathscr{E}^D\mathscr{I}_{\mathscr{T}^{\mathrm{dof}}}\mathbf{u}_\nu|_{0,\infty,t} \\ &\overset{(3.7),(3.6),(3.14)}{\lesssim} \operatorname{diam}(t)^{(1-\frac{d}{2})}\|\mathbf{e}^{\mathrm{D}}_\nu\|_{1,t} + \operatorname{diam}(t)^{(2-\frac{d}{2})}|\mathbf{u}|_{2,t}.\end{aligned} \tag{3.16}$$

Now summing up all the local errors gives the following global bound

$$\begin{aligned}\|\mathbf{e}^{\mathrm{cme}}\|^2_{m,\Omega} &= \sum_{T\in\mathscr{T}} \|\mathbf{e}^{\mathrm{cme}}\|^2_{m,T} = \sum_{T\in\mathscr{T}} \sum_{t\in\mathscr{T}_{\mathrm{ess}}:t\subseteq T} \|\mathbf{e}^{\mathrm{cme}}\|^2_{m,t} \\ &\overset{(3.12),(3.13),(3.16)}{\lesssim} \sum_{T\in\mathscr{T}} \sum_{t\in\mathscr{T}_{\mathrm{ess}}:t\subseteq T} \operatorname{diam}(t)^{2(1-m)}\left(\|\mathbf{e}^{\mathrm{D}}_\nu\|^2_{1,t} + \|\mathbf{e}^{\mathrm{N}}\|^2_{1,t}\right) + \operatorname{diam}(t)^{2(2-m)}|\mathbf{u}|^2_{2,t} \\ &\overset{(3.10)}{\le} h^{2(1-m)}\left(\|\mathbf{e}^{\mathrm{D}}_\nu\|^2_{1,\Omega} + \|\mathbf{e}^{\mathrm{N}}\|^2_{1,\Omega}\right) + h^{2(2-m)}|\mathbf{u}|^2_{2,\Omega} \\ &\overset{\text{Lem.2,Lem.1,(3.14)}}{\lesssim} h^{2(2-m)}|\mathbf{u}|^2_{2,\Omega}.\end{aligned}$$

□

3.2.2 Nonconformity

We have seen at the beginning of Section 3.1 that for essential boundary conditions, the composite mini element space is nonconforming in the sense that these boundary conditions are fulfilled only in an approximative way. We will now see that this nonconformity can be controlled by the local mesh refinement in the slave part $\mathscr{T}^{\text{slave}}$ of the mesh $\mathscr{T}$, more precisely, by the ratio

$$\mathrm{R}(T) := \max_{t\in\mathscr{T}_{\text{ess}}:T\supseteq t,t\cap(\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{s}}\cup\Gamma_{\mathrm{l}})\neq\emptyset} \frac{\operatorname{diam}(t)}{\operatorname{diam}(T)}, \quad T\in\mathscr{T}^{\text{dof}}, \tag{3.17}$$

which can be assigned to every extrapolation simplex. Before we state the result we introduce the constant

$$C_4^{\mathscr{T}} := \max_{T\in\mathscr{T}} \frac{|\Gamma_{\mathrm{s}}\cap T|}{\operatorname{diam}(T)^{(d-1)}},$$

which is assumed to be independent of the mesh width parameter.

Lemma 3 (Nonconformity)**.** *There is a constant $C>0$ depending on $\rho_{\mathscr{T}_{\text{ess}}}, C_3^{\mathscr{T}}, C_4^{\mathscr{T}} > 0$ and the curvature of $\Gamma_{\mathrm{s}}\cup\Gamma_{\mathrm{l}}$ such that*

$$\|\lambda_\nu\langle\mathbf{u},\nu\rangle+\lambda_\tau\langle\mathbf{u},\tau\rangle\|_{L^2(\Gamma\setminus\Gamma_N)} \le C\big(\max_{T\in\mathscr{T}^{\text{dof}}}\mathrm{R}(T)\big)h^{\frac{1}{2}}|\mathbf{u}|_{1,\Omega} \quad \forall\mathbf{u}\in\mathbf{X}_{\mathscr{T}}^{\text{cme}}.$$

Proof. We will only prove the case $\lambda_\nu=1,\lambda_\tau=0$. The opposite case $\lambda_\nu=0,\lambda_\tau=1$ can be treated analogously and $\lambda_\nu=1,\lambda_\tau=1$ follows by combination of the first two cases. Let $t\in\mathscr{T}^{\text{slave}}$ satisfy $t\cap\partial\Omega\neq\emptyset$. In the proof of Lemma 4.11 in [29] it was shown that

$$\|\langle\mathbf{u},\nu\rangle\|_{0,\infty,\Gamma_{\mathrm{s}}\cap t} \le \|\mathscr{E}^{\mathrm{D}}\mathbf{u}\|_{0,\infty,t} + C\operatorname{diam}(t)\|\mathscr{E}^{\mathrm{N}}\mathbf{u}\|_{0,\infty,t}, \tag{3.18}$$

where the constant C depends only on the maximal curvature of $\Gamma_{\mathrm{s}}\cap t$. Therefore we can prove the following local $\mathbf{L}^\infty$-estimate:

$$\begin{aligned}\|\langle\mathbf{u},\nu\rangle\|_{0,\infty,\Gamma_{\mathrm{s}}\cap t} &\overset{(2.8),(3.18),(2.2)}{\lesssim} \operatorname{diam}(t)\|\nabla\mathbf{u}\|_{1,\infty,\omega_t}\\ &\overset{(2.2),(3.7)}{\lesssim} \frac{\operatorname{diam}(t)}{\operatorname{diam}(T)^{\frac{d}{2}}}\|\nabla\mathbf{u}\|_{1,\omega_T},\end{aligned} \tag{3.19}$$

where $T\in\mathscr{T}^{\text{dof}}$ is chosen in such a way that $T_{\mathbf{x}}=T$ for some $\mathbf{x}\in V(t)$. A simple summation gives the final result:

$$\begin{aligned}\|\langle\mathbf{u},\nu\rangle\|_{0,\Gamma_{\mathrm{s}}}^2 &\lesssim \sum_{T\in\mathscr{T}:T\cap\Gamma_{\mathrm{s}}\neq\emptyset}\ \sum_{t\in\mathscr{T}_{\text{ess}}:T\supseteq t,t\cap\Gamma_{\mathrm{s}}\neq\emptyset} |\Gamma_{\mathrm{s}}\cap t|\,\|\langle\mathbf{u},\nu\rangle\|_{0,\infty,\Gamma_{\mathrm{s}}\cap t}^2\\ &\overset{(3.19),(3.7)}{\lesssim} \sum_{T\in\mathscr{T}} |\Gamma_{\mathrm{s}}\cap T|\,\mathrm{R}(T)^2\operatorname{diam}(T)^{2-d}|\mathbf{u}|_{1,\omega_T}^2\\ &\overset{(2.1)}{\lesssim} \underbrace{\max_{T\in\mathscr{T}}\frac{|\Gamma_{\mathrm{s}}\cap T|}{\operatorname{diam}(T)^{(d-1)}}}_{=C_4^{\mathscr{T}}}\left(\max_{T\in\mathscr{T}^{\text{dof}}}\mathrm{R}(T)^2\right)h\|\mathbf{u}\|_{1,\Omega}^2.\end{aligned}$$

□

3.2.3 Discrete stability and coercivity

In this section, we will investigate the unique solvability of the discrete composite mini element systems.

Lemma 4. *The extension operators* $\mathscr{E}^N$ *and* $\mathscr{E}^{\text{ess}}$ *defined in* (2.6) *and* (2.9) *are uniformly bounded, i.e. there are constants* $C_{\mathscr{E}^N}$ *and* $C_{\mathscr{E}^{\text{ess}}}$ *which only on* $\rho_{\mathscr{T}}$, *d and* $C_2^{\mathscr{T}}$ *and not on the local mesh-size such that*

$$\|\mathscr{E}^N u\|_{m,\Omega} \leq C_{\mathscr{E}^N} \|u\|_{m,\Omega^{\text{dof}}} \quad \textit{and} \quad \|\mathscr{E}^{\text{ess}} \mathbf{u}\|_{m,\Omega} \leq C_{\mathscr{E}^{\text{ess}}} \|\mathbf{u}\|_{m,\Omega^{\text{dof}}}$$

for all $u \in S_{\mathscr{T}^{\text{dof}}}$, $\mathbf{u} \in \mathbf{S}_{\mathscr{T}^{\text{dof}}}$ *and* $m \in \{0,1\}$.

Proof. For $u \in S_{\mathscr{T}^{\text{dof}}}$ and $m \in \{0,1\}$ there holds

$$\|\mathscr{E}^N u\|_{m,\Omega}^2 \leq \|u\|_{m,\Omega^{\text{dof}}}^2 + \sum_{T \in \mathscr{T}^{\text{slave}}} \|u\|_{m,T}^2 \overset{(3.7)}{\leq} \|\mathscr{E}^N u\|_{m,\Omega^{\text{dof}}}^2 + \sum_{T \in \mathscr{T}^{\text{slave}}} \operatorname{diam}(T)^d \|\mathscr{E}^N u\|_{m,\infty,T}^2. \tag{3.20}$$

Let $T \in \mathscr{T}^{\text{slave}}$. Since $\mathscr{E}^N u|_T$ takes its maximum in a vertex $\mathbf{x} \in V(T)$, there holds

$$\|\mathscr{E}^N u\|_{m,\infty,T} \leq \|u_{T_\mathbf{x}}\|_{m,\infty,T} \overset{(2.1),(3.7)}{\lesssim} \left(1 + \frac{\operatorname{diam}(T_\mathbf{x})}{\operatorname{diam}(T)}\right) \|u_{T_\mathbf{x}}\|_{m,\infty,T_\mathbf{x}} \overset{(2.1),(3.7)}{\lesssim} |T_\mathbf{x}|^{-\frac{1}{2}} \|u\|_{m,T_\mathbf{x}},$$

where $u_{T_\mathbf{x}}$ denotes the extension of $u|_{T_\mathbf{x}}$ (by itself) to $\mathbb{R}^d$. We plug this into (3.20) which finishes the proof for $\mathscr{E}^N$, since the resulting overlap can be bounded in terms of the shape regularity constant $\rho_{\mathscr{T}}$.

Next, we prove the boundedness of $\mathscr{E}^D$ defined in (2.8). For $u \in S_{\mathscr{T}^{\text{dof}}}$ and $m \in \{0,1\}$ there holds

$$\begin{aligned}
\|\mathscr{E}^D u\|_{m,\Omega}^2 &\leq \|u\|_{m,\Omega^{\text{dof}}}^2 + \sum_{t \in \mathscr{T}_{\text{ess}}^{\text{slave}}} \|\mathscr{E}^D u\|_{m,t}^2 \\
&\overset{(3.7)}{\leq} \|u\|_{m,\Omega^{\text{dof}}}^2 + \sum_{t \in \mathscr{T}_{\text{ess}}^{\text{slave}}} \operatorname{diam}(t)^{(d-2m)} \|\mathscr{E}^D u\|_{0,\infty,t}^2 \\
&\overset{(2.9),(3.7)}{\lesssim} \|u\|_{m,\Omega^{\text{dof}}}^2 + \sum_{T \in \mathscr{T}^{\text{slave}}} \sum_{t \in \mathscr{T}_{\text{ess}}^{\text{slave}} : t \subseteq T} \frac{\operatorname{diam}(t)^{(d-2m+2)}}{\operatorname{diam}(T)} \|u\|_{1,\omega_T \cap \Omega^{\text{dof}}}^2 \\
&\overset{d-2m+2 \geq d,(2.1)}{\lesssim} \|u\|_{1,\Omega^{\text{dof}}}^2.
\end{aligned} \tag{3.21}$$

The boundedness of $\mathscr{E}^{\text{ess}}$ follows from the results for $\mathscr{E}^N$ and $\mathscr{E}^D$ in a straight forward way. □

Theorem 4 (Stability)**.** $\mathbf{X}_{\mathscr{T}}^{\text{cme}} \times M_{\mathscr{T}}^{\text{cme}}$ *is a stable pairing, i.e. there is a constant* β^{cme} *which does not depend on the mesh size h and the choice of the submesh* $\mathscr{T}_{\text{ess}}$ *such that*

$$\inf_{p \in M_{\mathscr{T}}^{\text{cme}}} \sup_{\mathbf{0} \neq \mathbf{u} \in \mathbf{X}_{\mathscr{T}}^{\text{cme}}} \frac{\mathfrak{b}(\mathbf{u}, p)}{\|\mathbf{u}\|_{1,\Omega} \|p\|_{0,\Omega}} \geq \beta^{\text{cme}}. \tag{3.22}$$

Proof. To keep things clear we will restrict the proof to the case of a quasi uniform triangulation $\mathscr{T}$, i.e. $h \approx \operatorname{diam}(T)$ for all $T \in \mathscr{T}$. The general case can be proved by using standard localization techniques.

Note that the pressure part of the composite mini element space $\mathrm{M}_{\mathscr{T}}^{\text{cme}}$ can be decomposed in the following way

$$\mathrm{M}_{\mathscr{T}}^{\text{cme}} = \left(\mathrm{M}_{\mathscr{T}}^{\text{cme}} \cap \mathrm{L}_0^2(\Omega)\right) \oplus \mathbb{R},$$

where $\mathrm{L}^2_0(\Omega) := \{v \in \mathrm{L}^2(\Omega) \mid \int_\Omega v = 0\}$. Due to (2.11), it is easy to construct a vector field $\mathbf{u} \in \mathbf{X}^{\mathrm{cme}}_{\mathscr{T}}$ such that

$$\int_\Omega c \operatorname{div} \mathbf{u} = c \int_\Omega \operatorname{div} \mathbf{u} = c \int_{\Gamma_\mathrm{N} \cup \Gamma_\mathrm{l}} \langle \mathbf{u}, \nu \rangle \geq \beta_1^{\mathrm{cme}} \|c\|_{0,\Omega} \|\mathbf{u}\|_{1,\Omega}$$

for all constant pressures $c \in \mathbb{R}$. The constant $\beta_1^{\mathrm{cme}} > 0$ will depend on the relative length of $\Gamma_\mathrm{N} \cup \Gamma_\mathrm{l}$.

It is left to bound

$$\inf_{p \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega)} \sup_{\mathbf{0} \neq \mathbf{u} \in \mathbf{X}^{\mathrm{cme}}_{\mathscr{T}}} \frac{\mathfrak{b}(\mathbf{u}, p)}{\|\mathbf{u}\|_{1,\Omega} \|p\|_{0,\Omega}}$$

uniformly from below. Recall that the velocity part $\mathbf{X}^{\mathrm{cme}}_{\mathscr{T}}$ of the composite mini element space is the sum a piecewise affine part $\mathscr{E}^{\mathrm{ess}}(\mathbf{S}_{\mathscr{T}^{\mathrm{dof}}})$ and a stabilization part $\mathbf{B}_{\mathscr{T}^{\mathrm{dof}}}$ containing simplex bubble functions with respect to the elements with degrees of freedom (cf. (2.13)). We define a mapping $\mathscr{P}_\mathbf{B} : \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}^{\mathrm{dof}}} \cap \mathrm{L}^2_0(\Omega) \to \mathbf{B}_{\mathscr{T}^{\mathrm{dof}}}$ by

$$\mathscr{P}_\mathbf{B}(p)(\mathbf{x}) := \sum_{T \in \mathscr{T}^{\mathrm{dof}}} (\nabla p|_T) \psi_T(\mathbf{x}),$$

where ψ_T denotes the normalized simplex bubble on T defined in (2.12). We can bound $\mathscr{P}_\mathbf{B}$ by

$$\begin{aligned} \|\mathscr{P}_\mathbf{B}(p)\|^2_{1,\Omega} = \|\mathscr{P}_\mathbf{B}(p)\|^2_{1,\Omega^{\mathrm{dof}}} &= \sum_{T \in \mathscr{T}^{\mathrm{dof}}} \|\nabla p\|^2_{0,\infty,T} \|\psi_T\|^2_{1,T} \\ &\overset{(3.7),\text{Poinc. inequ.}}{\leq} (C_1)^2 h_T^{-2} \|p\|^2_{0,\Omega^{\mathrm{dof}}}, \quad \forall p \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega). \end{aligned} \tag{3.23}$$

Since

$$C_2 |T| \leq \int_T \psi_T, \tag{3.24}$$

we can estimate

$$\begin{aligned} |\mathfrak{b}(\mathscr{P}_\mathbf{B}(p), p)| &= \left| \sum_{T \in \mathscr{T}^{\mathrm{dof}}} \left\langle \nabla p|_T, \int_T \mathscr{P}_\mathbf{B}(p) \right\rangle \right| = \sum_{T \in \mathscr{T}^{\mathrm{dof}}} \|\nabla p|_T\|^2_{0,\infty,T} \int_T \psi_T \\ &\overset{(3.24)}{\geq} C_2 \|\nabla p\|^2_{0,\Omega^{\mathrm{dof}}} \overset{\text{Lem.4,Poinc. ineq.}}{\geq} \frac{C_2}{C_{\mathscr{E}^N}} \|p\|^2_{0,\Omega} > 0 \end{aligned} \tag{3.25}$$

for all $p \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega)$. Although (3.25) does not imply stability of the mixed space $\mathbf{B}_{\mathscr{T}^{\mathrm{dof}}} \times (\mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega))$, it guarantees that the problem

$$\begin{aligned} \mathfrak{a}(\mathbf{u}, \mathbf{v}) + \mathfrak{b}(\mathbf{v}, p) &= 0, & \forall \mathbf{v} \in \mathbf{B}_{\mathscr{T}^{\mathrm{dof}}}, \\ \mathbf{b}(\mathbf{u}, q) &= g(q), & \forall q \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega), \end{aligned} \tag{3.26}$$

has a unique solution $(\mathbf{u}_g, p_g)$ for all $g \in (\mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega))'$. Furthermore, we get

$$\|\mathbf{u}_g\|^2_{1,\Omega} \overset{\mathbf{B}_{\mathscr{T}^{\mathrm{dof}}} \subseteq \mathbf{H}^1_0(\Omega),\text{Korn ineq.},(3.26)}{\leq} \frac{1}{\alpha_\mathrm{K}} \|g\|_{(\mathrm{M}^{\mathrm{cme}}_{\mathscr{T}})'} \|p_g\|_{0,\Omega}, \tag{3.27}$$

$$\|p_g\|^2_{0,\Omega} \overset{(3.25)}{\leq} \frac{C_{\mathscr{E}^N}}{C_2} |\mathfrak{b}(\mathscr{P}_\mathbf{B}(p_g), p_g)| \overset{(3.26),(3.23)}{\leq} \frac{C_1 C_{\mathscr{E}^N}}{C_2} h^{-1} \|\mathbf{u}_g\|_{1,\Omega} \|p_g\|_{0,\Omega} \tag{3.28}$$

and therefore

$$\|\mathbf{u}_g\|_{1,\Omega} \overset{(3.27),(3.28)}{\leq} \underbrace{\frac{C_1 C_{\mathscr{E}^N}}{C_2 \alpha_\mathrm{K}}}_{=:C_3} h^{-1} \|g\|_{(\mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega))'}, \tag{3.29}$$

It is well known (cf. [14, Lemma 3.2]) that the inf-sup condition holds on the continuous level, i.e.

$$\forall p \in \mathrm{L}^2_0(\Omega) \exists \mathbf{u}_p \in \mathbf{H}^1_{\mathrm{ess}}(\Omega): \; \mathfrak{b}(\mathbf{u}_p, p) > \beta \|\mathbf{u}_p\|_{1,\Omega} \|p\|_{0,\Omega}. \tag{3.30}$$

Let $p \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega)$ and $\mathbf{u}_p \in \mathbf{H}^1_{\mathrm{ess}}(\Omega)$ denote the associated velocity field according to (3.30). By $\mathbf{u}_p^{\mathrm{cme}}$ we denote the projection of $\mathbf{u}_p$ onto the piecewise affine part $\mathscr{E}^{\mathrm{ess}}(\mathbf{S}_{\mathscr{T}^{\mathrm{dof}}})$ of $\mathbf{X}^{\mathrm{cme}}_{\mathscr{T}}$. We deduce from Theorem 2 by simple interpolation arguments (cf. [7, Theorem 12.3.3]) that

$$\|\mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}}\|_{0,\Omega} \leq C_4 h \|\mathbf{u}_p\|_{1,\Omega}. \tag{3.31}$$

Based on $\mathbf{u}_g$ we choose the functional $g \in (\mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega))'$ from (3.26) by

$$g(q) := \mathfrak{b}(\mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}}, q). \tag{3.32}$$

There is a unique $\mathbf{u}_g \in \mathbf{B}_{\mathscr{T}}$ such that $\mathfrak{b}(\mathbf{u}_g, q) = g(q)$ for all $q \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega)$ and

$$\begin{aligned}
\|\mathbf{u}_g\|_{1,\Omega} &\overset{(3.29)}{\leq} C_3 h^{-1} \|g\|_{(\mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega))'} \\
&\leq C_3 h^{-1} \sup_{q \in \mathrm{M}^{\mathrm{cme}}_{\mathscr{T}} \cap \mathrm{L}^2_0(\Omega)} \frac{\left|\int_\Omega \left\langle \nabla q, \mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}} \right\rangle\right| + \left|\int_{\partial\Omega} q \left\langle \mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}}, \nu \right\rangle\right|}{\|q\|_{0,\Omega}} \\
&\overset{p \in \mathrm{H}^1, \text{Poinc. ineq.}}{\leq} C_P C_3 h^{-1} (\|\mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}}\|_{0,\Omega} + \|\mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}}\|_{-\frac{1}{2},\partial\Omega}) \\
&\overset{p \in \mathrm{H}^1, \text{trace th.}}{\leq} C_P C_3 C_{\mathrm{tr}} h^{-1} \|\mathbf{u}_p - \mathbf{u}_p^{\mathrm{cme}}\|_{0,\Omega} \overset{(3.31)}{\leq} \underbrace{C_P C_4 C_3 C_{\mathrm{tr}}}_{=:C} \|\mathbf{u}_p\|_{1,\Omega}.
\end{aligned} \tag{3.33}$$

Finally, we estimate

$$\begin{aligned}
\mathfrak{b}(\mathbf{u}_p^{\mathrm{cme}} + \mathbf{u}_g, p) &\overset{(3.27)}{=} \mathfrak{b}(\mathbf{u}_p^{\mathrm{cme}}, p) + g(p) \overset{(3.31)}{=} \mathfrak{b}(\mathbf{u}_p, p) \\
&\overset{(3.25)}{\geq} \beta \|\mathbf{u}_p\|_{1,\Omega} \|p\|_{0,\Omega} \overset{(3.33)}{\geq} \frac{\beta}{1+C} \|\mathbf{u}_p^{\mathrm{cme}} + \mathbf{u}_g\|_{1,\Omega} \|p\|_{0,\Omega}
\end{aligned}$$

□

Next, we have to investigate the coercivity of the bilinear form $\mathfrak{a}$ with respect to the discrete space $\mathbf{X}^{\mathrm{cme}}_{\mathscr{T}}$. Due to the assumption (1.11) coercivity of $\mathfrak{a}$ is fulfilled with respect to the discrete spaces

$$\exists \alpha > 0: \; \mathfrak{a}(\mathbf{u}, \mathbf{u}) \geq \alpha \|\mathbf{u}\|_{1,\Omega}, \quad \forall \mathbf{u} \in \mathbf{H}^1_{\mathrm{ess}}. \tag{3.34}$$

We refer to [26; 22; 11; 29] for a proof of (3.34). Since $\mathbf{X}^{\mathrm{cme}}_{\mathscr{T}} \not\subseteq \mathbf{H}^1_{\mathrm{ess}}$ this result needs to be extended to a certain neighborhood of $\mathbf{H}^1_{\mathrm{ess}}$. This neighborhood will be controlled in terms of the $\mathbf{L}^2$-norm of the trace.

Lemma 5 (Equivalent norms in $\mathbf{H}^1_{\mathrm{ess}}$)**.** *For all* $\mathbf{u} \in \mathbf{H}^1_{\mathrm{ess}}$ *there holds*

$$\|\mathbf{u}\|^2_{1,\Omega} \lesssim \mathfrak{a}(\mathbf{u}, \mathbf{u}) + \|\lambda_\nu \mathbf{u}_\nu + \lambda_\tau \mathbf{u}_\tau\|^2_{0,\partial\Omega},$$

where the hidden constant does not depend on $\mathbf{u}$.

Lemma 5 is a straight forward generalization of Lemma 4.12 in [29], where the cases of Dirichlet and slip boundary conditions are discussed. The case of leak boundary conditions can be proved in an analog way. The pure Neumann was excluded by Remark 1. Lemma 5 implies that $\mathfrak{a}$ is coercive on the composite space $\mathbf{X}^{\mathrm{cme}}_{\mathscr{T}}$ if the violation of the essential boundary conditions is not too large.

Finally, we will discuss coercivity of the bilinear form $\mathfrak{a}(\cdot,\cdot)$ with respect to the discrete space.

Theorem 5 (Discrete coercivity)**.** *There is a constant* α^{cme} *that does not depend on h such that* $\mathfrak{a}(\mathbf{u},\mathbf{u}) \geq \alpha^{\mathrm{cme}}\|\mathbf{u}\|_{1,\Omega}^2$ *for all* $\mathbf{u} \in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}$.

Proof. In Lemma 3 we have seen that the nonconformity in the velocity space can be controlled by the ratios $\mathrm{R}(T)$. As a consequence, $\mathfrak{a}$ is coercive on the composite space $\mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}$, if the submesh is fine enough. However, here we want to avoid constraints on $\mathrm{R}(T)$ to ensure well-posedness of the discrete problem 2.14 independently from the choice of $\mathscr{T}_{\mathrm{ess}}$.

$\mathscr{R}(T)$ (cf. (3.17)) is bounded by a constant C independent of the mesh size h and the right hand side in Lemma 3 is always bounded by $C\sqrt{h}$. In view of Lemma 5, there is an h_0 such that the bilinear form is coercive for all triangulations $\mathscr{T}$ with mesh size $h \leq h_0$. The case $h > h_0$ is discussed in what follows. The bilinear $\mathfrak{a}$ has a nontrivial kernel given by the finite dimensional set of rigid body motions $\mathscr{R}$ (cf. (1.10)) and it is therefore coercive on $\mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}$ if and only if $\mathbf{X}_{\mathscr{T}}^{\mathrm{cme}} \cap \mathscr{R} = \{\mathbf{0}\}$. The latter has already been proved for Dirichlet boundary conditions in [30]. The case of leak boundary conditions follows by similar arguments. The generalization to the case $|\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{l}}| > 0$ is straight forward. In the case of pure slip/Neumann boundary conditions, i.e. $\Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{l}} = \emptyset$, let $\mathbf{u} \in \mathbf{X}_{\mathscr{T}}^{\mathrm{cme}}$, $\mathbf{A} \in \mathbb{R}^{d\times d}$ be skew symmetric and $\mathbf{b} \in \mathbb{R}^d$ such that $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$. Then, by definition (2.9), we get

$$\left\langle \mathbf{A}\mathbf{x}^{\Gamma} - \mathbf{b}, \nu(\mathbf{x}^{\Gamma}) \right\rangle = 0, \quad \forall \mathbf{x}^{\Gamma} \in \mathbf{V} := \{\mathbf{x}_i^{\Gamma} \mid \mathbf{x}_i \in \Theta_{\mathrm{ess}}^{\mathrm{slave}}, \mathbf{x}_i^{\Gamma} \in \Gamma_{\mathrm{s}}\}. \tag{3.35}$$

Under the assumptions 2.11 the latter implies that $\mathbf{A} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$. □

Theorem 4 and 5 imply the unique solvability of the discrete problem (2.14). Note that this result does not depend on the choice of submesh $\mathscr{T}_{\mathrm{ess}}$ and remains true for $\mathscr{T}_{\mathrm{ess}} = \mathscr{T}$. The dependence on the mesh $\mathscr{T}$ is only minimal.

Therefore all assertions from the proof sketch in Section 3.1 have been verified.

4 Numerical Experiments

In this section we will report on the results of some numerical experiments. Extensive numerical parameter studies which systematically investigate the performance of the composite mini element with respect to the roughness of the domain boundary have been published in [29; 30]. They clearly show that the composite mini element is a very robust generalization of the standard mini element to very coarse, non resolving meshes. [31; 32; 24]).

Here, we have studied the convergence behavior of the composite mini element with respect to the approximation error in case of a small hole that is not resolved by the computational grid. We will start with the following parametrized class of model problems. For $0 < \frac{r}{2} < 1$ we define $\mathscr{M}(r)$ by

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f}_r,\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_r := \mathscr{B}_1(\mathbf{0}) \setminus \mathscr{B}_{\frac{r}{2}}(\mathbf{0}),\ 0 < \frac{r}{2} < 1, \tag{4.1}$$

$$\langle \mathbf{u}, \nu \rangle = 0,\ (\mathbf{Du} \cdot \nu)_\tau = \mathbf{0} \quad \text{on } \Gamma_s := \partial\mathscr{B}_{\frac{r}{2}}(\mathbf{0}), \tag{4.2}$$

$$2\mathbf{Du} \cdot \nu = p\nu \quad \text{on } \Gamma_N := \partial\mathscr{B}_1(\mathbf{0}), \tag{4.3}$$

where $\mathbf{f}_r := -\Delta(\mathbf{u}_r)$ is the Laplacian of $\mathbf{u}_r$ given by

$$\mathbf{u}_r(\mathbf{x}) := (r - \|x\|)(1 - \|x\|).$$

Obviously the pair $(\mathbf{u}_r, p) \in \left(\mathbf{H}^1_{\text{ess}} \cap \mathbf{H}^2(\Omega)\right) \times \left(\mathrm{L}^2(\Omega) \cap \mathrm{H}^1(\Omega)\right)$ is a solution of the model problem $\mathscr{M}(r)$ for all constant pressures $p \in \mathbb{R}$ and all radii $0 < \frac{r}{2} < 1$. The solution flow is visualized for $r = 0.5$ in Figure 4. In a numerical computation the non-uniqueness in the pressure variable can be fixed by adding a constraint like $\int_\Omega p = 0$ to the system of equations. We will use uniform overlapping triangulations (cf. Figure 4b) arising from the initial triangulation

$$\mathscr{T} = \{\operatorname{conv}\{(-1,-1),(1,-1),(1,1)\}, \operatorname{conv}\{(1,1),(-1,1),(-1,-1)\}\}$$

by uniform refinements. Note that none of the meshes will resolve the domain Ω_r, i.e. neither the outer boundary is resolved nor the hole. Especially for small values of r the hole will be much smaller than the mesh width of the finest triangulation. In Figure 5a the convergence history of the composite mini element method is depicted for different hole sizes. Obviously, the optimal order of convergence is present right from the coarsest levels. This is not surprising, since the solution remains very smooth in a neighborhood of the hole when r tends to 0. In general, we expect the method to give similar reasonable approximations on complicated domains if the solution is smooth enough to be well represented by the coarse composite space. Since we only adapt the basis to the boundary conditions and no additional degrees of freedom are placed locally at the boundary, the method is not able to capture local behavior of the solution within the slave part of the mesh. This can be seen in the second example where the solution depends crucially on the size of the hole. For $n \in \mathbb{N}$, $n \geq 2$, we define $\mathscr{M}(n)$ by

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f}_n,\ \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega := \mathscr{B}_1(\mathbf{0}) \setminus \mathscr{B}_r(\mathbf{0}),\ 0 < r := \frac{2}{1+n} < 1, \tag{4.4}$$

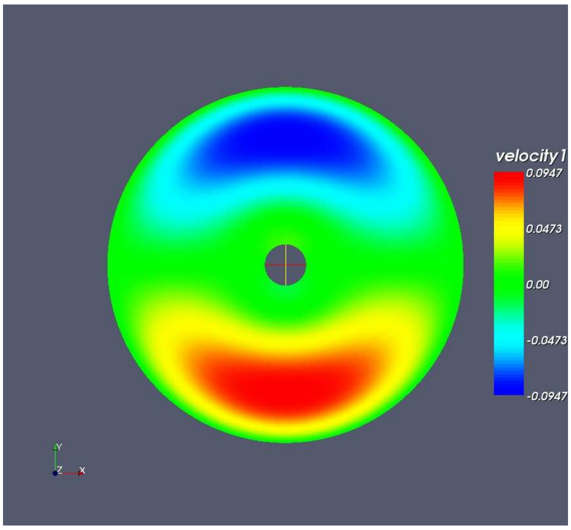
$$\langle \mathbf{u}, \nu \rangle = 0,\ (\mathbf{Du} \cdot \nu)_\tau = \mathbf{0} \quad \text{on } \Gamma_s := \partial\mathscr{B}_r(\mathbf{0}), \tag{4.5}$$

$$2\mathbf{Du} \cdot \nu = p\nu \quad \text{on } \Gamma_N := \partial\mathscr{B}_1(\mathbf{0}), \tag{4.6}$$

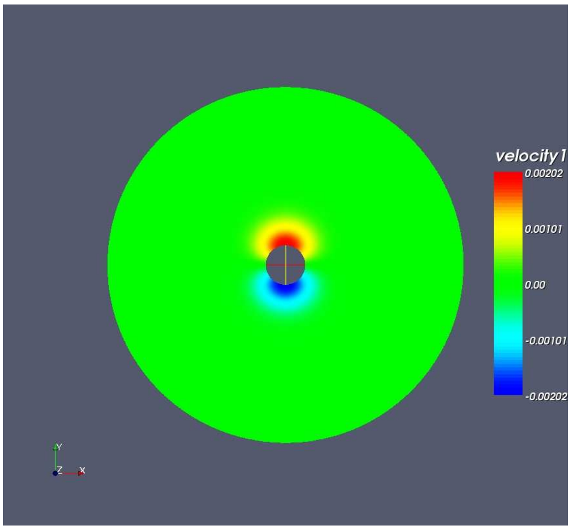
where $\mathbf{f}_n := -\Delta(\mathbf{u}_n)$ is the Laplacian of $\mathbf{u}_n$ given by

$$\mathbf{u}_n(\mathbf{x}) := \|x\|(1-\|x\|)^n.$$

Obviously the pair $(\mathbf{u}_n, p) \in \left(\mathbf{H}^1_{\mathrm{ess}} \cap \mathbf{H}^2(\Omega)\right) \times \left(\mathrm{L}^2(\Omega) \cap \mathrm{H}^1(\Omega)\right)$ is the unique solution of the model problem $\mathscr{M}(n)$ for all constant pressures $p \in \mathbb{R}$. A solution flow is visualized in Figure 4. Using the same meshes as before, results in the convergence history of the composite mini element method as depicted in Figure 5b. As expected, the local bump of the solution is not captured by the composite mini element approximation until the global mesh size is small enough. Therefore we observe a suboptimal convergence depending on the size of the hole. However, for all investigated radii optimal convergence order starts long before the hole is resolved by the mesh. These examples show that small holes might influence the singular behavior of the solution in some case while in other cases the solution is harmless and degrees of freedom are not necessary from the view point of approximability. As explained in the introduction composite finite elements, conceptually, allow to enrich the finite element space in an optimal way. Future research will be directed to control the optimal enrichment by a posteriori error indicators.

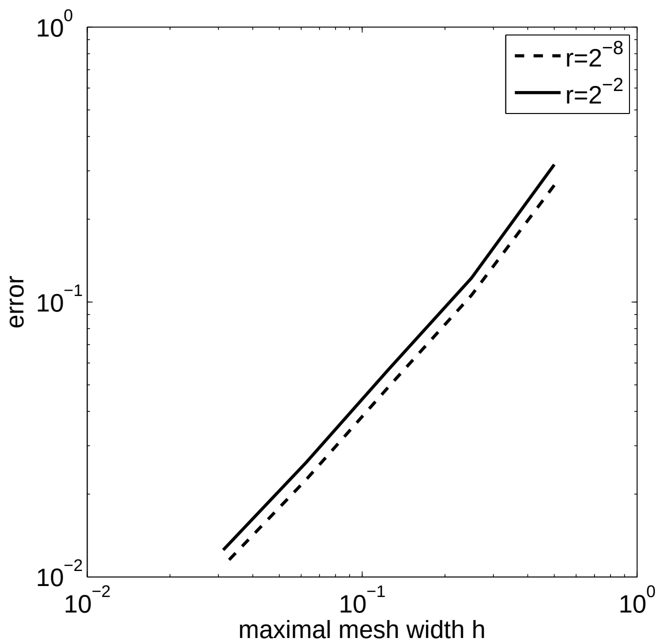


(a) Solution flow (first component) of $\mathscr{M}_1(\frac{1}{4})$ ($r = \frac{1}{8}$).

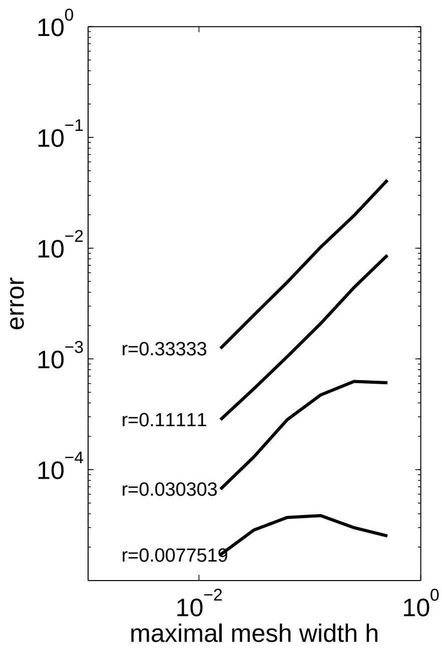


(b) Solution flow (first component) of $\mathscr{M}_2(15)$ ($r = \frac{1}{8}$).

Figure 4: Model problems: Stokes flows on the unit disc with a circular hole, Dirichlet boundary condition on the outer boundary, slip boundary condition on the inner circle.



(a) $\mathbf{H}^1$-error (velocity) of CME applied to $\mathscr{M}_1(2r)$ vs. max. mesh width.



(b) $\mathbf{H}^1$-error (velocity) of CME applied to $\mathscr{M}_1(n)(r = \frac{2}{1+n})$ vs. max. mesh width.

Figure 5: Convergence history of CME applied to the model problems.

5 Conclusion

We have described a mixed finite element method for the Stokes problem that does not require the underlying finite element mesh to resolve the physical domain. Overlapping, and possibly structured, meshes are used instead. Therefore arbitrary coarse approximation spaces can be defined even if the domain is very complicated. In contrast to other coarsening strategies (cf. [3; 23; 46]), the asymptotic error estimates are preserved on the coarse meshes. Furthermore, the application of the method is not restricted to the standard Dirichlet and Neumann boundary conditions. Boundary conditions of leak and slip type can be treated as well. Additionally, our error analysis requires only minimal smoothness of the domain.

Compared to homogenization approaches we did not make any periodicity assumptions. Furthermore, the definition of the basis functions is fully explicit, no local problems have to be solved. Therefore the complexity of the method will be proportional to the number of degrees of freedom which can be chosen almost independent from the geometry. The only difficulty lies in the integration over the intersections of elements and the domain. However, from a practical point of view, integration in space is much simpler then integration over the complicated boundary or solving a whole sub-problem on a fine scale mesh.

In cases where the domain contains rough boundaries or holes the method still allows to derive reasonable approximations at moderate effort. Although, as in standard finite element methods, details of the solution that are smaller than the mesh width cannot be captured, the composite approximation will always be a reasonable and cheap initial guess in an adaptive process of an adaptive enrichment of the finite element space driven via an a-posteriori error estimation and mesh refinement (cf. [32; 33]). As a consequence, geometric details will only be resolved where it is required by the solution.

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