Frequency Explicit Regularity Estimates for the Electric Field Integral Operator

L. Banjai* S. Sauter†

Abstract

In this paper, we derive regularity estimates for the electric field integral operator which arises when formulating the time-harmonic Maxwell problem as a boundary integral equation. More precisely, we show that the regularity constants can be bounded polynomially in terms of the frequency, where the degree of the polynomial depends on the regularity order and is given explicitly. The paper concludes with an application of these results to the electric field integral equation with distributional right-hand side.

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1 Introduction

Computation of the propagation of electromagnetic waves scattered by a bounded obstacle is important in applications, e.g., the design and analysis of antennas, but also as a model problem for numerical methods developed to compute wave propagation in infinite domains. An important class of such methods are the time-domain boundary integral equations.

The formulation of time-dependent electromagnetic scattering problems as time-domain boundary integral equations (TDBIEs) goes back to the 1960s (cf. [14]), while the efficient and accurate numerical solution is an active field of research. Important discretization techniques include Galerkin methods based on space-time variational formulations (cf. [5, 31, 3, 1, 16, 29]) and methods based on bandlimited interpolation and extrapolation (cf. [34]).

An alternative approach to solve TDBIEs numerically is to employ convolution quadrature – a method for discretizing convolutions introduced in the 1980s (cf. [20, 21]). Convolution quadrature based on linear multistep methods has been applied to numerous problems (cf. [22, 8, 30, 33, 12]); fast numerical

*Department of Mathematics, Heriot-Watt University, Edinburgh, EH14 4AS,UK, (l.banjai@hw.ac.uk); most of the work has been done while this author has been working at the Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany.

†(stas@math.uzh.ch), Institut für Mathematik, Universität Zürich, Winterthurerstr 190, CH-8057 Zürich, Switzerland

implementations were developed in [18, 17, 19]. For a review of convolution quadrature and its applications we refer to [23, 9]. The advantages of this discretization scheme for TDBIEs include its excellent stability properties and the fact that only the Laplace transform of the time-domain fundamental solution is required and thus distributional kernels are avoided. An important assumption for the stability of convolution quadrature is the A-stability of the underlying time-discretization method. Since A-stable linear multistep methods cannot exceed a convergence order of 2, convolution quadratures based on Runge-Kutta methods have recently been considered and analyzed in order to obtain high order and low dissipation schemes (cf. [24, 6, 7]). For the TDBIE of the Maxwell Equations on a three-dimensional scatterer, this has been developed in [4].

A key ingredient in the convergence theory in [4] is a regularity assumption for the electric field boundary integral equation of the time-harmonic Maxwell equations which is explicit with respect to complex frequencies. This problem is interesting in its own right and is in the spirit of the coefficient-explicit regularity results for *scalar* second order elliptic boundary value problems (cf. [25]).

To be more precise we consider the propagation of time-dependent electromagnetic fields in a homogeneous medium arising from the scattering of incoming waves at a perfectly conducting obstacle. The formulation of this problem as an integral equation results in the time-domain electric field integral equation (EFIE). For the analysis of *convolution quadrature* for the convolution in time, the EFIE operator $\mathcal{V}(s)$ in the Laplace domain has to be investigated for complex frequencies in the half-plane

$$I_{\sigma_0} := \{s \in \mathbb{C} \mid \operatorname{Re} s \geq \sigma_0\} \quad \text{for some } \sigma_0 > 0, \tag{1}$$

more precisely, estimates of the norm of the inverse operator $\mathcal{V}^{-1}(s)$ which are explicit in $s \in I_{\sigma_0}$ are needed. Such estimates are known, see [4], in the energy norm, however estimates in more regular spaces are to the best of our knowledge not available in the literature.

Although the regularity theory for systems of elliptic equations is well established – see, e.g., the monograph [13], the derivation of frequency-explicit regularity estimates for systems of boundary integral equations is far from trivial. It requires the use of high-order Maxwell-harmonic extension operators along with indexed Sobolev norms on the boundary and on the domain. We will show that the regularity constant grows polynomially with higher order regularity in terms of the complex frequency. As an application of our theory we will derive frequency-explicit regularity estimates for the dual problem which arises when the *Aubin-Nitsche trick* (cf., e.g., [28, Section 4.2.5]) is applied to estimate the error of field point evaluations (cf. [4]).

The paper is structured as follows. In Section 2 we will introduce the relevant Sobolev spaces for Euclidean domains and Lipschitz manifolds and recall the definitions of the tangential trace operators. In Section 3 we will define the Laplace domain *electric field integral operator* on the boundary of the domain. Since trace liftings will play an essential role for the regularity estimates we introduce corresponding interior and exterior Maxwell problems in a form that allows to apply classical regularity estimates to this system. In Section 4 we will derive the regularity results for the electric field integral operators where the dependence of the regularity constant on the complex frequency is explicit. Finally, in Section 5 we apply our theory to estimate the potential of the electric field in field points of the exterior domain explicitly in terms of the frequency.

2 Setting

Let Ω_- be an open bounded set in $\mathbb{R}^3$ with Lipschitz boundary Γ, unitary outer normal $\mathbf{n}$, and complement $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega_-}$. In this paper we will consider the propagation of time-dependent electromagnetic fields near a perfectly conducting body. We consider three-dimensional exterior scattering problems in a homogeneous, isotropic medium with constant, positive electric permittivity ε and constant, positive magnetic permeability μ. Furthermore we assume that there are no external sources.

First, we have to introduce some notation. The inner product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$ is denoted by $\langle \mathbf{a}, \mathbf{b} \rangle$; as a convention, the complex conjugation is always applied to the first argument in scalar products. The usual vector product is denoted by $\mathbf{a} \times \mathbf{b}$. We will use the notation Ω to indicate that a statement is true for both Ω_- and Ω_+. The standard square-integrable Lebesgue and Sobolev spaces are denoted by $L^2(\Omega)$ (with norm $\|\cdot\|_\Omega$) and $H^s(\Omega)$ (with norm $\|\cdot\|_{H^s(\Omega)}$) (cf. [2]). We will use bold letters $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^s(\Omega)$ for the corresponding spaces of vector-valued functions. Let

$$\mathbf{H}(\mathrm{curl}, \Omega) := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \mathrm{curl}\, \mathbf{v} \in \mathbf{L}^2(\Omega) \right\},$$
$$\mathbf{H}(\mathrm{div}, \Omega) := \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \mathrm{div}\, \mathbf{v} \in \mathbf{L}^2(\Omega) \right\},$$

equipped with their graph norms

$$\|\mathbf{w}\|_{\mathrm{curl},\Omega} = \left(\|\mathbf{w}\|_\Omega^2 + \|\mathrm{curl}\, \mathbf{w}\|_\Omega^2 \right)^{1/2} \qquad \forall \mathbf{w} \in \mathbf{H}(\mathrm{curl}, \Omega),$$
$$\|\mathbf{v}\|_{\mathrm{div},\Omega} = \left(\|\mathbf{v}\|_\Omega^2 + \|\mathrm{div}\, \mathbf{v}\|_\Omega^2 \right)^{1/2} \qquad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega).$$

We emphasize that our goal in this paper is to develop regularity estimates which are explicit with respect to the complex frequency s – the dependence on ε and μ will mostly be hidden in the constants. However, some of the estimates become sharper if one uses the following indexed norm

$$\|\mathbf{v}\|_{\mathrm{curl},\Omega,s} := \sqrt{\|\sqrt{\mu\varepsilon} s \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathrm{curl}\, \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2}, \quad s \in I_{\sigma_0}.$$

Let $\mathcal{D}\left(\overline{\Omega}\right) = \left\{ \phi|_\Omega \mid \phi \in C^\infty_{\mathrm{comp}}\left(\mathbb{R}^3\right) \right\}$. The closure of $\mathcal{D}^3\left(\overline{\Omega}\right)$ with respect to the $\|\cdot\|_{\mathrm{curl},\Omega}$-norm is

$$\mathbf{H}_0(\mathrm{curl}, \Omega) := \overline{\mathcal{D}\left(\overline{\Omega}\right)}^{\|\cdot\|_{\mathrm{curl},\Omega}}.$$

We will further require square integrable tangential fields

$$\mathbf{L}_t^2(\Gamma) := \{\mathbf{v} \in \mathbf{L}^2(\Gamma) | \langle \mathbf{n}, \mathbf{v} \rangle = 0 \text{ on } \Gamma\}$$

and the following trace operators Π_τ and γ_τ mapping from $\mathcal{D}(\overline{\Omega})$ to $\mathbf{L}_t^2(\Gamma)$

$$\Pi_\tau : \mathbf{u} \mapsto \mathbf{n} \times (\mathbf{u} \times \mathbf{n})\,|_\Gamma \quad \text{and} \quad \gamma_\tau : \mathbf{u} \mapsto \mathbf{u}|_\Gamma \times \mathbf{n}.$$

Following [10], we define the Hilbert spaces

$$V := \mathbf{H}^{1/2}(\Gamma), \qquad V_\gamma := \gamma_\tau(\mathbf{H}^1(\Omega)), \qquad V_\Pi := \Pi_\tau(\mathbf{H}^1(\Omega))$$

with norms that assure the continuity of the trace operators

$$\|\boldsymbol{\lambda}\|_{V_\gamma} = \inf_{\mathbf{u} \in V} \{\|\mathbf{u}\|_V \mid \gamma_\tau \mathbf{u} = \boldsymbol{\lambda}\}$$

and

$$\|\boldsymbol{\lambda}\|_{V_\Pi} = \inf_{\mathbf{u} \in V} \{\|\mathbf{u}\|_V \mid \Pi_\tau \mathbf{u} = \boldsymbol{\lambda}\}.$$

Further, we denote by V_Π' and V_γ' the respective dual spaces with $\mathbf{L}_t^2(\Gamma)$ as the pivot space and their natural norms. We denote the *surface divergence* by div_Γ and the *surface curl* by curl_Γ (cf., e.g., [27], [10]). We are now ready, see [10], to introduce the following Hilbert spaces on Γ:

$$\begin{aligned} \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) &:= \{\mathbf{v} \in V_\gamma' \mid \mathrm{div}_\Gamma\, \mathbf{v} \in H^{-1/2}(\Gamma)\}, \\ \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) &:= \{\mathbf{v} \in V_\Pi' \mid \mathrm{curl}_\Gamma\, \mathbf{v} \in H^{-1/2}(\Gamma)\} \end{aligned}$$

with norms defined as

$$\begin{aligned} \|\mathbf{v}\|_{-1/2,\mathrm{curl}_\Gamma} &:= \left\{ \|\mathbf{v}\|_{V_\gamma'}^2 + \|\mathrm{curl}_\Gamma\, \mathbf{v}\|_{H^{-1/2}(\Gamma)}^2 \right\}^{1/2}, \\ \|\mathbf{v}\|_{-1/2,\mathrm{div}_\Gamma} &:= \left\{ \|\mathbf{v}\|_{V_\Pi'}^2 + \|\mathrm{div}_\Gamma\, \mathbf{v}\|_{H^{-1/2}(\Gamma)}^2 \right\}^{1/2}. \end{aligned} \tag{2}$$

The following theorem shows that $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ are the correct spaces for these densities.

Theorem 1 *Let $\Omega \in \{\Omega_-, \Omega_+\}$. The trace mappings*

$$\begin{aligned} \Pi_\tau &: \mathbf{H}(\mathrm{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\ \gamma_\tau &: \mathbf{H}(\mathrm{curl}, \Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \end{aligned}$$

are continuous and surjective. Moreover, there exist continuous liftings for these trace operators in $\mathbf{H}(curl, \Omega)$.

For a proof we refer to [11, 27].

3 Electric Field Integral Equations

We have now collected all the ingredients to define the Laplace domain *electric field integral operator* on the boundary $\mathcal{V}(s):\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)\to\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)$ for $s\in I_{\sigma_0}$ by

$$\begin{aligned}(\mathcal{V}(s)\boldsymbol{\varphi})(\mathbf{y}):=&-\mu\Pi_\tau\int_\Gamma s^2\,K(s,\mathbf{x}-\mathbf{y})\,\boldsymbol{\varphi}(\mathbf{x})\,d\Gamma_{\mathbf{x}}\\&+\frac{1}{\varepsilon}\nabla_\Gamma\int_\Gamma K(s,\mathbf{x}-\mathbf{y})\,\operatorname{div}_\Gamma\boldsymbol{\varphi}(\mathbf{x})\,d\Gamma_{\mathbf{x}}\qquad\mathbf{y}\in\Gamma.\end{aligned}\tag{3}$$

Further denote by $\mathcal{S}(s):\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)\to\mathbf{H}(\operatorname{curl},\Omega)$ the operator

$$\begin{aligned}\left(\mathcal{S}(s)\hat{\mathbf{j}}\right)(\mathbf{y}):=&-\mu\int_\Gamma s\,K(s,\mathbf{x}-\mathbf{y})\hat{\mathbf{j}}(\mathbf{x})\,d\Gamma_{\mathbf{x}}\\&+\frac{1}{\varepsilon}\nabla\int_\Gamma\frac{1}{s}K(s,\mathbf{x}-\mathbf{y})\,\operatorname{div}_\Gamma\hat{\mathbf{j}}(\mathbf{x})\,d\Gamma_{\mathbf{x}},\qquad\mathbf{y}\in\Omega.\end{aligned}\tag{4}$$

Note that $\mathcal{V}(s)$ is the tangential trace of the domain operator $\mathcal{S}(s)$ scaled by s:

$$\mathcal{V}(s)=s\Pi_\tau\mathcal{S}(s).$$

Our goal is to analyze the mapping properties of the inverse operator $\mathcal{V}^{-1}(s)$. For $\boldsymbol{\psi}\in\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)$, we define

$$\boldsymbol{\varphi}:=\mathcal{V}^{-1}(s)\boldsymbol{\psi}.\tag{5}$$

We relate this equation to the following exterior and interior, time-harmonic Maxwell problem. Let $\Omega\in\{\Omega_-,\Omega_+\}$. Find $(\mathbf{E}_\Omega,\mathbf{H}_\Omega)\in\mathbf{H}(\operatorname{curl},\Omega)\times\mathbf{H}(\operatorname{curl},\Omega)$ such that

$$\begin{aligned}-s\varepsilon\mathbf{E}_\Omega+\operatorname{curl}\ \mathbf{H}_\Omega&=\mathbf{0}&&\text{in }\Omega,\\s\mu\mathbf{H}_\Omega+\operatorname{curl}\ \mathbf{E}_\Omega&=\mathbf{0}&&\text{in }\Omega,\\\gamma_\tau\mathbf{E}_\Omega&=\tfrac{1}{s}\boldsymbol{\psi}\times\mathbf{n}&&\text{on }\Gamma.\end{aligned}\tag{6}$$

This problem admits a unique solution for all $\operatorname{Re}s>0$ as proved, e.g., in [31] and [32, Lemma 3.3]:

$$\|\mathbf{E}_\Omega\|_{\operatorname{curl},\Omega,s}\leq C\frac{1}{\sigma_0}\left(\frac{1}{\sigma_0}+\sqrt{\mu\varepsilon}\right)|s|\,\|\boldsymbol{\psi}\|_{-1/2,\operatorname{curl}_\Gamma}.\tag{7}$$

Next we will investigate the regularity of $\mathbf{E}_\Omega$ for smoother data $\boldsymbol{\psi}$. Let the boundary of Ω be in the class $C^{k,1}$ for some $k\in\mathbb{N}_0$ and let $\boldsymbol{\psi}\in\mathbf{H}^{k+1/2}(\Gamma)$. Apply the vector product $\mathbf{n}\times(\cdot)$ to the last equation in (6) and observe that $\mathbf{n}\times\gamma_\tau\mathbf{E}=\Pi_\tau\mathbf{E}$ and, since $\boldsymbol{\psi}$ is a tangential field, $\mathbf{n}\times(\boldsymbol{\psi}\times\mathbf{n})=\boldsymbol{\psi}$. Thus, under these assumptions on $\boldsymbol{\psi}$ we deduce that the boundary condition

$$\Pi_\tau\mathbf{E}_\Omega=\frac{1}{s}\boldsymbol{\psi}$$

is equivalent to the boundary condition in (6). After eliminating $\mathbf{H}_\Omega$ we get the system

$$\begin{aligned} s^2\varepsilon\mu\mathbf{E}_\Omega + \operatorname{curl}\operatorname{curl}\mathbf{E}_\Omega &= \mathbf{0} && \text{in } \Omega, \\ \Pi_\tau\mathbf{E}_\Omega &= \tfrac{1}{s}\boldsymbol{\psi} && \text{on } \Gamma. \end{aligned} \tag{8}$$

For any $\boldsymbol{\varphi}$,

$$\mathbf{E}_\Omega = \mathcal{S}(s)\boldsymbol{\varphi}$$

satisfies the first equation in (8) and the choice (5) ensures that also the boundary condition is satisfied.

In the next step, we will employ an extension operator to transform (8) to a problem with homogeneous boundary conditions. We recall (see [35, Satz 8.8]) that for domains with compact $C^{k,1}$ boundary, $k \in \mathbb{N}_0$, there exists, a linear and continuous lifting operator $Z : H^{k+1/2}(\Gamma) \to H^{k+1}(\Omega)$ such that

$$(Zw)|_\Gamma = w \qquad \forall w \in H^{k+1/2}(\Gamma)$$

and

$$\|Zw\|_{H^{k+1}(\Omega)} \leq C_{\mathrm{ext}} \|w\|_{H^{k+1/2}(\Gamma)} \qquad \forall w \in H^{k+1/2}(\Gamma), \tag{9}$$

where the positive constant C_{ext} depends on Γ and k. We define $\mathbf{Z} : \mathbf{H}^{k+1/2}(\Gamma) \to \mathbf{H}^{k+1}(\Omega)$ by $\mathbf{Zw} = (Zw_1, Zw_2, Zw_3)^\intercal$ and observe that $\boldsymbol{\psi}$ in (8) satisfies

$$\Pi_\tau\mathbf{Z}\boldsymbol{\psi} = \Pi_\tau\mathbf{Z}\Pi_\tau\mathbf{E}_\Omega = \Pi_\tau^2\mathbf{E}_\Omega = \Pi_\tau\mathbf{E}_\Omega = \boldsymbol{\psi}.$$

By using the ansatz

$$\mathbf{E}_\Omega = \mathbf{E}_\Omega^0 + \frac{1}{s}\mathbf{Z}\boldsymbol{\psi} \tag{10}$$

we obtain that $\mathbf{E}_\Omega^0$ satisfies

$$\begin{aligned} s^2\varepsilon\mu\mathbf{E}_\Omega^0 + \operatorname{curl}\operatorname{curl}\mathbf{E}_\Omega^0 &= -s\varepsilon\mu\mathbf{Z}\boldsymbol{\psi} - \tfrac{1}{s}\operatorname{curl}\operatorname{curl}\mathbf{Z}\boldsymbol{\psi} && \text{in } \Omega, \\ \Pi_\tau\mathbf{E}_\Omega^0 &= \mathbf{0} && \text{on } \Gamma. \end{aligned} \tag{11}$$

Moving

$$\mathbf{J} := -s^2\varepsilon\mu\mathbf{E}_\Omega^0$$

to the right-hand side and applying the divergence operator to the first equation of (11) results in

$$\begin{aligned} \operatorname{curl}\operatorname{curl}\mathbf{E}_\Omega^0 &= \mathbf{J} - s\varepsilon\mu\mathbf{Z}\boldsymbol{\psi} - \tfrac{1}{s}\operatorname{curl}\operatorname{curl}\mathbf{Z}\boldsymbol{\psi} && \text{in } \Omega, \\ \operatorname{div}\mathbf{E}_\Omega^0 &= -\tfrac{1}{s}\operatorname{div}\mathbf{Z}\boldsymbol{\psi} && \text{in } \Omega, \\ \Pi_\tau\mathbf{E}_\Omega^0 &= \mathbf{0} && \text{on } \Gamma. \end{aligned} \tag{12}$$

Recall that the existence, uniqueness, and well-posedness (cf. (7)) of the solution $\mathbf{E}_\Omega$ of (6) is already stated which also implies the existence of $\mathbf{E}_\Omega^0 = \mathbf{E}_\Omega - \frac{1}{s}\mathbf{Z}\boldsymbol{\psi}$. We use (12) only for proving regularity estimates for $\mathbf{E}_\Omega$.

Let

$$\mathbf{X} := \{\mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega) \mid \Pi_\tau\mathbf{v} = \mathbf{0}\}.$$

A regularized variational formulation of (12) is given by: Find $\mathbf{E}_\Omega^0 \in \mathbf{X}$ such that

$$a_\Omega^{\mathrm{reg}}\left(\mathbf{E}_\Omega^0, \mathbf{F}\right) = \mathcal{R}\left(\mathbf{F}\right) \quad \forall \mathbf{F} \in \mathbf{X}, \tag{13}$$

with

$$a_\Omega^{\mathrm{reg}}\left(\mathbf{E}_\Omega^0, \mathbf{F}\right) := \left(\operatorname{curl} \mathbf{E}_\Omega^0, \operatorname{curl} \mathbf{F}\right)_\Omega + \left(\operatorname{div} \mathbf{E}_\Omega^0, \operatorname{div} \mathbf{F}\right)_\Omega \quad \text{and} \quad \mathcal{R}\left(\mathbf{F}\right) := \left(\mathbf{Q}, \mathbf{F}\right)_\Omega$$

and

$$\mathbf{Q} := -s^2 \varepsilon \mu \mathbf{E}_\Omega^0 - \left(s\varepsilon\mu - \frac{1}{s}\boldsymbol{\Delta}\right)\mathbf{Z}\boldsymbol{\psi} = -s^2\varepsilon\mu\mathbf{E}_\Omega + \frac{1}{s}\boldsymbol{\Delta}\mathbf{Z}\boldsymbol{\psi}. \tag{14}$$

Here $(\cdot,\cdot)_\Omega$ denotes the standard $\mathbf{L}^2(\Omega)$-scalar product if the arguments are vector fields and the standard $L^2(\Omega)$-scalar product if the arguments are scalar functions. That (13) is a correct variational formulation for (12) follows from the calculation

$$\begin{aligned}\left(\operatorname{div} \mathbf{E}_\Omega^0, \operatorname{div} \mathbf{F}\right)_\Omega &= -\left(\operatorname{grad}\operatorname{div} \mathbf{E}_\Omega^0, \mathbf{F}\right)_\Omega \\ &= \left(\tfrac{1}{s}\operatorname{grad}\operatorname{div} \mathbf{Z}\boldsymbol{\psi}, \mathbf{F}\right)_\Omega \\ &= \left(\tfrac{1}{s}\boldsymbol{\Delta}\mathbf{Z}\boldsymbol{\psi} + \tfrac{1}{s}\operatorname{curl}\operatorname{curl}\mathbf{Z}\boldsymbol{\psi}, \mathbf{F}\right)_\Omega.\end{aligned}$$

Theorem 2 *We have*

$$a_\Omega^{\mathrm{reg}}\left(\mathbf{E}, \mathbf{E}\right) \geq c \left\|\mathbf{E}\right\|_{\mathrm{curl},\Omega,}^2 \qquad \forall \mathbf{E} \in \mathbf{X}$$

and

$$\left|a_\Omega^{\mathrm{reg}}\left(\mathbf{E}, \mathbf{F}\right)\right| \leq \left\|\mathbf{E}\right\|_{\mathrm{curl},\Omega}\left\|\mathbf{F}\right\|_{\mathrm{curl},\Omega} \qquad \forall \mathbf{E}, \mathbf{F} \in \mathbf{X}.$$

Proof. From [15, Chap 1, Corollary 2.10] we conclude that for any subset $\Omega \subset \mathbb{R}^3$

$$\mathbf{H}_0\left(\mathrm{curl}, \Omega\right) \cap \mathbf{H}\left(\mathrm{div}, \Omega\right) \subset \mathbf{H}_{\mathrm{loc}}^1\left(\Omega\right).$$

Hence, the operators γ_τ and Π_τ are mappings from $\mathbf{H}\left(\mathrm{curl}, \Omega\right) \cap \mathbf{H}\left(\mathrm{div}, \Omega\right)$ into $\mathbf{H}^{1/2}\left(\Gamma\right)$. For any $\mathbf{E} \in \mathbf{H}\left(\mathrm{curl}, \Omega\right) \cap \mathbf{H}\left(\mathrm{div}, \Omega\right)$ the following implications hold

$$\begin{aligned}&\gamma_\tau \mathbf{E} = 0 \implies \Pi_\tau \mathbf{E} = \mathbf{n} \times \gamma_\tau \mathbf{E} = \mathbf{0} \\ &\Pi_\tau \mathbf{E} = 0 \implies \gamma_\tau \mathbf{E} = \mathbf{E} \times \mathbf{n} - \left\langle \mathbf{n}, \mathbf{E}\right\rangle\left(\mathbf{n} \times \mathbf{n}\right) = \left(\mathbf{E} - \left\langle \mathbf{n}, \mathbf{E}\right\rangle \mathbf{n}\right) \times \mathbf{n} = \left(\Pi_\tau \mathbf{E}\right) \times \mathbf{n} = \mathbf{0}.\end{aligned} \tag{15}$$

As in [13, Notation 3.4.3] we introduce for $s > 1/2$ the space

$$\mathbf{H}_N^s\left(\Omega\right) := \left\{\mathbf{E} \in \mathbf{H}^s\left(\Omega\right) \mid \Pi_\tau \mathbf{E} = \mathbf{0}\right\}. \tag{16}$$

From this the first equality in

$$\mathbf{X} = \left\{\mathbf{v} \in \mathbf{H}\left(\mathrm{curl}, \Omega\right) \cap \mathbf{H}\left(\mathrm{div}, \Omega\right) \mid \gamma_\tau \mathbf{v} = \mathbf{0}\right\} \overset{[13,\ \mathrm{p.}\ 158]}{=} \mathbf{H}_N^1\left(\Omega\right) \tag{17}$$

follows. From [13, p. 158] we conclude that

$$a_\Omega^{\mathrm{reg}}\left(\mathbf{E}, \mathbf{E}\right) \geq c_{\mathrm{ell}} \left\|\mathbf{E}\right\|_{\mathbf{H}^1(\Omega)}^2 \qquad \forall \mathbf{E} \in \mathbf{H}_N^1\left(\Omega\right) \overset{(17)}{=} \mathbf{X} \tag{18}$$

holds. ■

4 Frequency Explicit Regularity Estimates

To derive frequency explicit regularity results we first define a trace lifting which is adjusted to a frequency dependent norm on the boundary for $r \in \mathbb{N}_{\geq 1}$ which is defined by

$$\|\psi\|_{r-1/2,s,\Gamma}^2 := |s| \|\psi\|_{\mathbf{L}^2(\Gamma)}^2 + \sum_{\ell=1}^{r} |s|^{2-2\ell} |\psi|_{\mathbf{H}^{\ell-1/2}(\Gamma)}^2 .$$

Since Γ is compact there exists some $R > 0$ such that the open ball B_R about the origin satisfies $\Gamma \subset B_R$. We define an extension operator $Z_R : H^{1/2}(\Gamma) \to H^1(\Omega \cap B_R)$ as the solution of: For given $\varphi \in H^{1/2}(\Gamma)$ find $u_\varphi \in H^1(\Omega \cap B_R)$ with $u_\varphi|_\Gamma = \varphi$ and $u_\varphi|_{\Omega\cap\partial B_R} = 0$ such that

$$-\Delta u_\varphi + |s|^2 u_\varphi = 0 \qquad \text{in } \Omega. \tag{19}$$

We set $Z_R\varphi := u_\varphi$. The following lemma follows from [26, Lemma 4.22] via a bootstrapping argument. Let

$$r_+ := \max\{2, r\}. \tag{20}$$

Lemma 3 *Let Γ be of class C^{r_+} for some $r \geq 1$. If $\varphi \in H^{r-1/2}(\Gamma)$, then $Z_R\varphi \in H^r(\Omega)$ and the following regularity estimate holds*

$$\|Z_R\varphi\|_{H^r(\Omega\cap B_R)} \leq C_r |s|^{r-1} \|\varphi\|_{r-1/2,s,\Gamma} \qquad \forall s \in I_{\sigma_0}, \tag{21}$$

where C_r depends on r and on $\sigma_0 > 0$ (cf (1)).

For $r \geq 2$, it holds

$$\|\Delta Z_R\varphi\|_{H^{r-2}(\Omega\cap B_R)} \leq C_r |s|^{r-1} \begin{cases} \|\varphi\|_{1/2,s,\Gamma} & r = 2, \\ \|\varphi\|_{r-5/2,s,\Gamma} & r \geq 3. \end{cases} \tag{22}$$

Proof. For $r = 1$, the estimate (21) follows from (9) via

$$\|Z_R\varphi\|_{H^1(\Omega\cap B_R)} \leq C_{\text{ext}} \|\varphi\|_{H^{1/2}(\Gamma)} \leq C_1 \|\varphi\|_{1/2,s,\Gamma}$$

with $C_1 := C_{\text{ext}}\left(1 + \sigma_0^{-1}\right)$. For the following we assume $r \geq 2$.

Elliptic regularity theory applied to the inhomogeneous Dirichlet problem (19) implies the following statement. Let Γ be of class C^r for some $r \geq 2$. If $\varphi \in H^{r-1/2}(\Gamma)$ then $Z_R\varphi \in H^r(\Omega)$ and

$$\|Z_R\varphi\|_{H^r(\Omega\cap B_R)} \leq C_r \left(|s|^2 \|Z_R\varphi\|_{H^{r-2}(\Omega\cap B_R)} + \|\varphi\|_{H^{r-1/2}(\Gamma)}\right). \tag{23}$$

We set

$$w_r := \|Z_R\varphi\|_{H^r(\Omega\cap B_R)} \quad \text{and} \quad v_r := C_r \|\varphi\|_{H^{r-1/2}(\Gamma)}$$

and obtain from (23) the recursion

$$w_r \leq C_r |s|^2 w_{r-2} + v_r.$$

This recursion can be resolved and, for any $1 \leq m \leq \lfloor r/2 \rfloor$, it holds

$$w_r \leq \tilde{C}_{r,m} |s|^{2m} w_{r-2m} + \sum_{\ell=0}^{m-1} \tilde{C}_{r,\ell} |s|^{2\ell} v_{r-2\ell} \quad \text{with} \quad \tilde{C}_{r,q} := \prod_{\ell=0}^{q-1} C_{r-2\ell}. \tag{24}$$

We consider the cases r even/odd separately to estimate the first term in the right-hand side in (24).

- r is even. We choose $m = r/2$ and use

$$|s|^r w_0 = |s|^r \|Z_R\varphi\|_{L^2(\Omega\cap B_R)} \overset{[26,\ (4.34)]}{\leq} C|s|^{r-1} \|\varphi\|_{1/2,s,\Gamma}.$$

- r is odd. We choose $m = \frac{r-1}{2}$ and use

$$|s|^{r-1} w_1 = |s|^{r-1} \|Z_R\varphi\|_{H^1(\Omega\cap B_R)} \leq C|s|^{r-1} \|\varphi\|_{1/2,s,\Gamma},$$

where the constant C in addition depends on σ_0.

Hence,

$$w_r \leq C\tilde{C}_{r,\lfloor r/2\rfloor} |s|^{r-1} \|\varphi\|_{1/2,s,\Gamma} + \sum_{\ell=0}^{\lfloor r/2\rfloor -1} \tilde{C}_{r,\ell} |s|^{2\ell} v_{r-2\ell}$$

so that

$$\|Z_R\varphi\|_{H^r(\Omega\cap B_R)} \leq C_r |s|^{r-1} \|\varphi\|_{r-1/2,s,\Gamma},$$

where C_r depend on r and on σ_0.

For $r \geq 3$, estimate (22) is a simple consequence of the first statement: We have

$$\|\Delta Z_R\varphi\|_{H^{r-2}(\Omega\cap B_R)} = |s|^2 \|Z_R\varphi\|_{H^{r-2}(\Omega\cap B_R)} \leq C_{r-2} |s|^{r-1} \|\varphi\|_{r-5/2,s,\Gamma},$$

while, for $r = 2$, the estimate follows from

$$\|\Delta Z_R\varphi\|_{L^2(\Omega\cap B_R)} = |s|^2 \|Z_R\varphi\|_{L^2(\Omega\cap B_R)} \overset{[26,\ (4.34)]}{\leq} C|s| \|\varphi\|_{1/2,s,\Gamma}.$$

■

The frequency-explicit higher order regularity estimate is based on Theorem 3.4.5 in [13] which we recall for convenience.

Theorem 4 *Let the boundary of Ω be of class C^k for some $k \geq 2$ and assume that $\mathbf{Q} \in \mathbf{H}^{k-2}(\Omega)$. Then $\mathbf{E}_\Omega^0$ belongs to $\mathbf{H}^k(\Omega)$ and satisfies*

$$\|\mathbf{E}_\Omega^0\|_{\mathbf{H}^t(\Omega)} \leq C_{\text{er}} \left(\|\mathbf{Q}\|_{\mathbf{H}^{t-2}(\Omega)} + \|\mathbf{E}_\Omega^0\|_{\mathbf{H}^1(\Omega)} \right) \qquad \forall t \in (3/2, 2]. \tag{25}$$

We come now to the frequency-explicit regularity estimate of the electric field.

Theorem 5 *Let the boundary of Ω be of class C^{k+} for some $k \geq 1$ and assume that $\boldsymbol{\psi} \in \mathbf{H}^{k-1/2}(\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{curl}, \Gamma)$. Then, the solution of (6) is in $\mathbf{H}^k(\Omega)$ and satisfies the estimate*

$$\|\mathbf{E}_\Omega\|_{\mathbf{H}^k(\Omega)} \leq C\left(|s|^{k-2}\|\boldsymbol{\psi}\|_{k-1/2,s,\Gamma} + |s|^{k-1}\|\boldsymbol{\psi}\|_{1/2,s,\Gamma} + |s|^{k+1}\|\boldsymbol{\psi}\|_{-1/2,\mathrm{curl}_\Gamma}\right),$$

where C depends on σ_0, ε, μ, and k.

Proof. For $r \in \mathbb{N}_{\geq 1}$, let

$$w_r := \|\mathbf{E}_\Omega\|_{\mathbf{H}^r(\Omega)} \quad \text{and} \quad v_r := C_r \|\boldsymbol{\psi}\|_{r-1/2,s,\Gamma},$$

where C_r is the constant in[1] (cf. Lemma 3)

$$\|\mathbf{Z}_\mathbf{R}\boldsymbol{\psi}\|_{\mathbf{H}^r(\Omega)} \leq C_r |s|^{r-1}\|\boldsymbol{\psi}\|_{r-1/2,s,\Gamma}.$$

We first prove the case $k = 1$. The coercivity estimate (18) implies for $\mathbf{Q} \in \mathbf{L}^2(\Omega)$ the estimate

$$\left\|\mathbf{E}_\Omega^0\right\|_{\mathbf{H}^1(\Omega)} \leq c_{\mathrm{ell}}^{-1}\|\mathbf{Q}\|_{\mathbf{L}^2(\Omega)}.$$

From (10) we deduce

$$\|\mathbf{E}_\Omega\|_{\mathbf{H}^1(\Omega)} \leq c_{\mathrm{ell}}^{-1}\|\mathbf{Q}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{|s|}\|\mathbf{Z}_\mathbf{R}\psi\|_{\mathbf{H}^1(\Omega)} \leq c_{\mathrm{ell}}^{-1}\|\mathbf{Q}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{|s|}v_1.$$

For $r \geq 0$, the norm of $\mathbf{Q}$ can be estimated by using (14) and (22):

$$\|\mathbf{Q}\|_{\mathbf{H}^r(\Omega)} \leq |s|^2\varepsilon\mu\|\mathbf{E}_\Omega\|_{\mathbf{H}^r(\Omega)} + \frac{1}{|s|}\|\boldsymbol{\Delta}\mathbf{Z}_\mathbf{R}\boldsymbol{\psi}\|_{\mathbf{H}^r(\Omega)} \leq |s|^2\varepsilon\mu\|\mathbf{E}_\Omega\|_{\mathbf{H}^r(\Omega)} + |s|^r \times \begin{cases} v_1 & r = 0, \\ v_r & r \geq 1. \end{cases}$$

Hence ,

$$\begin{aligned} \|\mathbf{E}_\Omega\|_{\mathbf{H}^1(\Omega)} &\leq c_{\mathrm{ell}}^{-1}\left(|s|\|\mathbf{E}_\Omega\|_{\mathrm{curl},\Omega,s} + v_1\right) + \frac{1}{|s|}v_1 \\ &\overset{(7)}{\leq} C\left(|s|^2\|\boldsymbol{\psi}\|_{-1/2,\mathrm{curl}_\Gamma} + v_1\right), \end{aligned} \tag{26}$$

where C depends on ε and σ_0.

Next, we consider the case $k \geq 2$. Note that

$$\begin{aligned} \|\mathbf{E}_\Omega\|_{\mathbf{H}^k(\Omega)} &\leq \left\|\mathbf{E}_\Omega^0\right\|_{\mathbf{H}^k(\Omega)} + \frac{1}{|s|}\|\mathbf{Z}_\mathbf{R}\boldsymbol{\psi}\|_{\mathbf{H}^k(\Omega)} \\ &\overset{\text{Thm. 4}}{\leq} C_1\left(|s|^2\varepsilon\mu\|\mathbf{E}_\Omega\|_{\mathbf{H}^{k-2}(\Omega)} + |s|^{k-2}v_k + \left\|\mathbf{E}_\Omega^0\right\|_{\mathbf{H}^1(\Omega)}\right) \quad \text{with} \quad C_1 = C_{\mathrm{er}} + 1. \end{aligned}$$

Thus,

$$w_k \leq C_1\left(|s|^2\varepsilon\mu w_{k-2} + |s|^{k-2}v_k + \kappa_0\right) \quad \text{with} \quad \kappa_0 := \left\|\mathbf{E}_\Omega^0\right\|_{\mathbf{H}^1(\Omega)}.$$

[1] $\mathbf{Z}_R\psi$ is the defined as the application of the operator Z_R as in Lemma 3 to each component of ψ.

This recursion can resolved and we obtain for $1 \leq m \leq \lfloor k/2 \rfloor$

$$w_k \leq \left(C_1 |s|^2 \varepsilon\mu \right)^m w_{k-2m} + C_1 |s|^{k-2} \sum_{\ell=0}^{m-1} (C_1 \varepsilon\mu)^\ell v_{k-2\ell} + C_1 \kappa_0 \sum_{\ell=0}^{m-1} \left(C_1 |s|^2 \varepsilon\mu \right)^\ell . \tag{27}$$

To estimate the first term, we distinguish again between even an odd k.

1. Let k be even. Choose $m = k/2$ to obtain

$$(C_1\varepsilon\mu)^{k/2} |s|^k w_0 \leq (C_1\varepsilon\mu)^{k/2} |s|^k \|\mathbf{E}_\Omega\|_{\mathbf{L}^2(\Omega)} \overset{(7)}{\leq} (C_1\varepsilon\mu)^{k/2} |s|^k \|\boldsymbol{\psi}\|_{-1/2,\operatorname{curl}_\Gamma} .$$

2. Let k be odd. We choose $m = \frac{k-1}{2}$ and get

$$\begin{aligned} |s|^{k-1} w_1 &\leq |s|^{k-1} \|\mathbf{E}_\Omega\|_{\mathbf{H}^1(\Omega)} \\ &\overset{(26)}{\leq} C |s|^{k-1} \left(|s|^2 \|\boldsymbol{\psi}\|_{-1/2,\operatorname{curl}_\Gamma} + \|\boldsymbol{\psi}\|_{1/2,s,\Gamma} \right). \end{aligned}$$

Since the bound for odd k grows faster in $|s|$ we use this one for estimating the first term in (27). We finally obtain

$$w_k \leq C_k \left(|s|^{k-2} \|\boldsymbol{\psi}\|_{k-1/2,s,\Gamma} + |s|^{k-1} \|\boldsymbol{\psi}\|_{1/2,s,\Gamma} + |s|^{k+1} \|\boldsymbol{\psi}\|_{-1/2,\operatorname{curl}_\Gamma} \right).$$

■

Corollary 6 *From the second equation in (6), the regularity estimates for $\mathbf{E}_\Omega$ carry over to regularity estimates for $\mathbf{H}_\Omega$. Let the boundary of Ω be of class C^{1+k_+} for some $k \geq 1$ and assume that $\boldsymbol{\psi} \in \mathbf{H}^{k+1/2}(\Gamma) \cap \mathbf{H}^{-1/2}(\operatorname{curl}, \Gamma)$. Then, the solution $\mathbf{H}_\Omega$ of (6) is in $\mathbf{H}^k(\Omega)$ and satisfies the estimate*

$$\|\mathbf{H}_\Omega\|_{\mathbf{H}^k(\Omega)} \leq C \left(|s|^{k-2} \|\boldsymbol{\psi}\|_{k+1/2,s,\Gamma} + |s|^{k-1} \|\boldsymbol{\psi}\|_{1/2,s,\Gamma} + |s|^{k+1} \|\boldsymbol{\psi}\|_{-1/2,\operatorname{curl}_\Gamma} \right). \tag{28}$$

Let

$$\begin{aligned} \mathbf{H}^k(\operatorname{curl}, \Omega) &:= \left\{ \mathbf{v} \in \mathbf{H}^k(\Omega) \mid \operatorname{curl} \mathbf{v} \in \mathbf{H}^k(\Omega) \right\}, \\ \mathbf{H}^{k-1/2}(\operatorname{div}_\Gamma, \Gamma) &:= \left\{ \boldsymbol{\psi} \in \mathbf{H}^{k-1/2}(\Gamma) \mid \operatorname{div}_\Gamma \boldsymbol{\psi} \in \mathbf{H}^{k-1/2}(\Gamma) \right\} \end{aligned}$$

with norms

$$\begin{aligned} \|\mathbf{v}\|_{k,\operatorname{curl}} &:= \sqrt{\|\mathbf{v}\|^2_{\mathbf{H}^k(\Omega)} + \|\operatorname{curl} \mathbf{v}\|^2_{\mathbf{H}^k(\Omega)}}, \\ \|\boldsymbol{\psi}\|_{k-1/2,\operatorname{div}_\Gamma} &:= \sqrt{\|\boldsymbol{\psi}\|^2_{\mathbf{H}^{k-1/2}(\Gamma)} + \|\operatorname{div}_\Gamma \boldsymbol{\psi}\|^2_{\mathbf{H}^{k-1/2}(\Gamma)}}. \end{aligned}$$

The next corollary states that we are able to obtain a slightly better estimate for $\|\mathbf{H}_\Omega\|_{k,\operatorname{curl}}$ than just simply bounding it by $\|\mathbf{H}_\Omega\|_{\mathbf{H}^{k+1}(\Omega)}$.

Corollary 7 *Under the same conditions as in Corollary 6 the following holds*

$$\|\mathbf{H}_\Omega\|_{k,\mathrm{curl}} \leq C\Big(|s|^{k-1}\|\boldsymbol{\psi}\|_{k-1/2,s,\Gamma} + |s|^{k-2}\|\boldsymbol{\psi}\|_{k+1/2,s,\Gamma} + |s|^{k}\|\boldsymbol{\psi}\|_{1/2,s,\Gamma} + |s|^{k+2}\|\boldsymbol{\psi}\|_{-1/2,\mathrm{curl}_\Gamma}\Big).$$

Proof. From the first equation in (6) we get

$$\|\mathbf{H}_\Omega\|_{k,\mathrm{curl}} = \sqrt{\|\mathbf{H}_\Omega\|^2_{\mathbf{H}^k(\Omega)} + |s\varepsilon|^2\|\mathbf{E}_\Omega\|^2_{\mathbf{H}^k(\Omega)}}.$$

The combination of Theorem 5 with Corollary 6 leads to the assertion. ■

From the regularity estimate for $\mathbf{H}_\Omega$ it is easy to derive the mapping property of the operator $\mathcal{V}^{-1}(s)$ by using the jump relation

$$\varphi = \gamma_\tau^{\Omega_-}\mathbf{H}_{\Omega_-} - \gamma_\tau^{\Omega_+}\mathbf{H}_{\Omega_+}. \tag{29}$$

Theorem 8 *Let the boundary of Ω be of class C^{1+k_+} for some $k \geq 1$ and assume that $\boldsymbol{\psi} \in \mathbf{H}^{k+1/2}(\Gamma) \cap \mathbf{H}^{-1/2}(\mathrm{curl},\Gamma)$. Then the function $\boldsymbol{\phi} = \mathcal{V}^{-1}(s)\boldsymbol{\psi}$ is in $\mathbf{H}^{k-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and satisfies the estimate*

$$\|\boldsymbol{\phi}\|_{k-1/2,\mathrm{div}_\Gamma} \leq C\Big(|s|^{k-1}\|\boldsymbol{\psi}\|_{k-1/2,s,\Gamma} + |s|^{k-2}\|\boldsymbol{\psi}\|_{k+1/2,s,\Gamma} + |s|^{k}\|\boldsymbol{\psi}\|_{1/2,s,\Gamma} + |s|^{k+2}\|\boldsymbol{\psi}\|_{-1/2,\mathrm{curl}_\Gamma}\Big).$$

Proof. From [11, Prop. 10], we conclude that $\gamma_\tau : \mathbf{H}^k(\mathrm{curl},\Omega) \to \mathbf{H}^{k-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is continuous. Then, the estimate follows by combining the result of Corollary 7 with the jump relation (29). ■

5 An Application: The Electric Field Integral Equation with a Functional as the Right-Hand Side

Let $\boldsymbol{\psi} \in \mathbf{H}^{-1/2}(\mathrm{curl},\Gamma)$ and $\boldsymbol{\zeta} = \mathcal{V}^{-1}(s)\boldsymbol{\psi}$. Then, the corresponding electromagnetic potential in the Laplace domain at a field point $\mathbf{y} \in \Omega^+$ is given by

$$\begin{aligned}\ell_i(\boldsymbol{\zeta}) := \delta_{\mathbf{y}}\mathcal{S}_i(s)(\boldsymbol{\zeta}) &= -\mu\int_\Gamma sK(s,\mathbf{x}-\mathbf{y})\,\boldsymbol{\zeta}_i(\mathbf{x})\,d\Gamma_{\mathbf{x}} \\ &\quad + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\int_\Gamma \frac{1}{s}K(s,\mathbf{x}-\mathbf{y})\,\mathrm{div}_\Gamma\,\boldsymbol{\zeta}(\mathbf{x})\,d\Gamma_{\mathbf{x}} \\ &= \int_\Gamma \Big\langle \overline{\mathbf{K}_i^{\mathrm{tot}}(s,\mathbf{x}-\mathbf{y})}, \boldsymbol{\zeta}(\mathbf{x})\Big\rangle\, d\Gamma_{\mathbf{x}},\end{aligned}$$

where

$$\mathbf{K}_i^{\mathrm{tot}}(s,\mathbf{x}-\mathbf{y}) = -\mu sK(s,\mathbf{x}-\mathbf{y})\,\mathbf{e}_i - \frac{1}{s\varepsilon}\frac{\partial}{\partial y_i}\nabla_{\mathbf{x}}K(s,\mathbf{x}-\mathbf{y})$$

and $\mathbf{e}_i \in \mathbb{R}^3$ is the i-th canonical unit vector.

Theorem 9 *Let the boundary of Ω be of class C^{1+k_+} for some $k \geq 1$ and assume that $\boldsymbol{\psi} \in \mathbf{H}^{k+1/2}(\Gamma) \cap \mathbf{H}^{-1/2}(\operatorname{curl}, \Gamma)$. Consider the electric field integral equation with the field point evaluation in $\mathbf{y} \in \Omega_+$ as the right-hand side functional:*

$$\text{find } \mathbf{w}_i \in \mathbf{H}^{-1/2}(\operatorname{div}, \Gamma) \quad \text{s.t.} \quad (\mathbf{w}_i, \mathcal{V}(s)\boldsymbol{\zeta})_\Gamma = \ell_i(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathbf{H}^{-1/2}(\operatorname{div}, \Gamma). \tag{30}$$

Then, the solution $\mathbf{w}_i$ is in $\mathbf{H}^{k-1/2}(\operatorname{div}, \Gamma)$ and satisfies the estimate

$$\|\mathbf{w}_i\|_{k-1/2,\operatorname{div}_\Gamma} \leq C \, \mathrm{e}^{-\operatorname{dist}(\mathbf{y},\Gamma)\operatorname{Re} s} |s|^{k+2}.$$

Proof. Since $\boldsymbol{\zeta}$ is a tangential field, $\Pi_\tau \boldsymbol{\zeta} = \boldsymbol{\zeta}$ holds and we conclude that

$$\begin{aligned}
\int_\Gamma \left\langle \overline{\mathbf{K}_i^{\mathrm{tot}}(s, \mathbf{x}-\mathbf{y})}, \boldsymbol{\zeta}(\mathbf{x}) \right\rangle d\Gamma_\mathbf{x} &= \int_\Gamma \left\langle \overline{\mathbf{K}_i^{\mathrm{tot}}(s, \mathbf{x}-\mathbf{y})}, \Pi_\tau \boldsymbol{\zeta}(\mathbf{x}) \right\rangle d\Gamma_\mathbf{x} \\
&= \int_\Gamma \left\langle \overline{\Pi_\tau \mathbf{K}_i^{\mathrm{tot}}(s, \mathbf{x}-\mathbf{y})}, \boldsymbol{\zeta}(\mathbf{x}) \right\rangle d\Gamma_\mathbf{x} \\
&= \left(\overline{\Pi_\tau \mathbf{K}_i^{\mathrm{tot}}(s, \cdot - \mathbf{y})}, \boldsymbol{\zeta} \right)_\Gamma,
\end{aligned}$$

where $(\cdot,\cdot)_\Gamma$ denotes the continuous extension of the $\mathbf{L}_t^2(\Gamma)$-scalar product to a sesqui-linear duality pairing (again the complex conjugation in $(\cdot,\cdot)_\Gamma$ is on the first argument)

$$(\cdot,\cdot)_\Gamma : \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \times \mathbf{H}^{-1/2}(\operatorname{curl}, \Gamma) \to \mathbb{C}.$$

Hence,

$$\ell_i(\boldsymbol{\zeta}) = \left(\overline{\Pi_\tau \mathbf{K}_i^{\mathrm{tot}}(s, \cdot - \mathbf{y})}, \boldsymbol{\zeta} \right)_\Gamma. \tag{31}$$

By taking into account (31) the solution of (30) is given by

$$\mathbf{w}_i = \overline{\mathcal{V}^{-1}(s)\boldsymbol{\psi}_i} \quad \text{with} \quad \boldsymbol{\psi}_i := \Pi_\tau \mathbf{K}_i^{\mathrm{tot}}(s, \cdot - \mathbf{y}).$$

In order to obtain the final result, it suffices to bound higher order norms of $\boldsymbol{\psi}_i$. Recall

$$K(s, \mathbf{x}-\mathbf{y}) = \frac{\mathrm{e}^{-s|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

In the following, let $d := \operatorname{dist}(\mathbf{y}, \Gamma) > 0$. First note that the standard trace inequality results in

$$\begin{aligned}
\|\boldsymbol{\psi}_i\|_{k-1/2,\Gamma} &\leq C \left\| \mu s K(s, \cdot - \mathbf{y})\mathbf{e}_i + \frac{1}{\varepsilon s} \partial_{y_i} \nabla_\mathbf{x} K(s, \cdot - \mathbf{y}) \right\|_{k,\Omega_-} \\
&\leq C(d) \, \mathrm{e}^{-d \operatorname{Re} s} |s|^{k+1}
\end{aligned}$$

so that

$$\begin{aligned}
\|\boldsymbol{\psi}_i\|_{k-1/2,s,\Gamma} &= \sqrt{|s| \, \|\psi_i\|^2_{\mathbf{L}^2(\Gamma)} + \sum_{\ell=1}^{r} |s|^{2-2\ell} \, |\psi_i|^2_{\mathbf{H}^{\ell-1/2}(\Gamma)}} \\
&\leq \mathrm{e}^{-d \operatorname{Re} s} \sqrt{C_1 |s|^4} \leq C \, \mathrm{e}^{-d \operatorname{Re} s} |s|^2.
\end{aligned}$$

Theorem 1 gives us

$$\|\boldsymbol{\psi}_i\|_{-1/2,\mathrm{curl}_\Gamma} \leq C \left\|\mathbf{K}_i^{\mathrm{tot}}\left(s, \cdot - \mathbf{y}\right)\right\|_{\mathbf{H}(\mathrm{curl},\Omega)} \leq C\,\mathrm{e}^{-d\,\mathrm{Re}\,s}\,|s|^2 .$$

The combination with Theorem 8 then leads to the assertion. ■

Remark 10 *The above result is, e.g., needed in an Aubin-Nitsche duality argument used in [4, Theorem 4.10 (b)].*

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