Classical BEM with linear complexity

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Abstract

Alternative representations of boundary integral operators corresponding to elliptic boundary value problems are developed as a starting point for numerical approximations as, e.g., Galerkin boundary elements including numerical quadrature and panel-clustering. These representations have the advantage that the integrands of the integral operators have a reduced singular behaviour allowing to choose the order of the numerical approximations much lower than for the classical formulations.

Low order discretizations for the single layer integral equations as well as for the classical double layer potential and the hypersingular integral equation are considered. We will present fully discrete Galerkin boundary element methods where the storage amount and the CPU-time grow only linearly (without any logarithmic terms) with respect to the number of unknowns.

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1   Introduction

The integral equation method is an elegant tool to transform homogeneous linear boundary value problems with constant coefficients into boundary integral equations (BIE) on the boundary of the domain (see, e.g., [12], [17]). The boundary element method is a flexible discretisation technique for solving these equations numerically. In the last twenty years, efficient algorithms have been developed for overcoming the major bottlenecks of this method: Quadrature methods for evaluating singular and nearly singular surface integrals have been established (cf. [15], [7], [24], [29]) and

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sparse representations for the non-local integral operators have been developed ([14], [22], [23], [29]). By employing such fast algorithms, the computational complexity and the storage amount for solving boundary integral equations were reduced from $\mathcal{O}(n^2)$ to $\mathcal{O}(n \log^\kappa n)$, where n denotes the number of degrees of freedom of the discretisation and $\kappa \sim 6$. Asymptotically, with increasing n, the term $\log^\kappa n$ becomes negligible compared to n while for practical problem sizes the factor $\log^\kappa n$ has a significant effect on the run-time behaviour of the algorithms.

In [23], the model problem of the classical double layer potential was considered on a smooth surface discretised with piecewise constant boundary elements. It was shown that for the nearfield integrals where the integrand is singular and nearly singular the approximation by zero yields a consistent and stable discretisation and the "expansion orders" in the panel-clustering representation of the integral operator can be chosen such that the complexity of the resulting algorithm and the storage amount is $\mathcal{O}(n)$ with the same constant in the $\mathcal{O}(\cdot)$-estimate as before.

This approach is extended in this paper to more general boundary integral equations by re-formulating the boundary integral equation such that the kernel function has a reduced singular behaviour. This concept can be regarded as an extension of the concept of partial integration introduced in [19].

We emphasize that the main goal of this paper is to present a panel-clustering type algorithm which solves numerically the classical boundary integral equations in $\mathbb{R}^3$ on piecewise smooth surfaces in $\mathcal{O}(n)$ operations. The key ingredients are

- partial integration which reduces the singular behaviour of the integral kernels,

- a modified panel-clustering method which employs a globally Lipschitz continuous approximation of the kernel function,

- a variable expansion order for the panel clustering (small on small surface regions and large on large regions)

- the error analysis which determines the expansion orders for the panel-clustering.

The paper is at some point a bit technical since the detailed analysis and subtle combination of all these ingredients are essential to achieve the prescribed goal.

Some consequences are listed below:

1. Most nearfield integrals in the alternative representation can be replaced by 0! No complicated evaluation of singular surface integrals are necessary any more.

2. The changes from the classical panel-clustering algorithm to the new one are only moderate. Various shift and summation operations simply have to be cut earlier as in the classical versions.

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The main results are the alternative representations (3.3), (3.8), and (3.10) of the integral operators, the panel-clustering approximations of the arising bilinear forms (4.33), the choice of the expansion orders (4.32), the algorithmic description of the method at the end of Section 5 and the justification of the choices of the expansion orders by the error analysis in Section 6.

2   Galerkin discretisation of integral operators

Throughout this paper, $\Omega \subset \mathbb{R}^3$ denotes a bounded Lipschitz domain with boundary Γ and normal vector field n (oriented to the exterior of Ω). We define the Sobolev space $H^s(\Gamma)$, $s \geq 0$, in the usual way (see, e.g., [17]). Note that the range of s for which $H^s(\Gamma)$ is defined may be limited, depending on the global smoothness of the surface Γ. For $s < 0$, the spaces $H^s(\Gamma)$ are the dual of $H^{-s}(\Gamma)$.

We will consider the general integral equation

$$(\lambda I + \mathcal{K}) u(x) := \lambda u(x) + \int_\Gamma k(x, y) u(y) \, ds_y = f(x), \qquad x \in \Gamma, \qquad (2.1)$$

for some given scalar $\lambda \in \mathbb{R}$ kernel function k and sufficiently smooth right-hand side f. The corresponding weak form is

Find $u \in H$ such that $a(u, v) := ((\lambda I + \mathcal{K})u, v) = (f, v)$ for all $v \in H$.   (2.2)

Here H equals $H^\mu(\Gamma)$ or is a closed subspace of $H^\mu(\Gamma)$ for some $\mu \in \{-1/2, 0, 1/2\}$. (The bracket $(\cdot, \cdot)$ denotes the continuous extension of the $L^2(\Gamma)$ scalar product to the $H^{-\mu}(\Gamma) \times H^\mu(\Gamma)$ duality pairing.) Typical examples are: the classical single layer, double layer and hypersingular operators for the Laplacian where $a(\cdot, \cdot) = \lambda(\cdot, \cdot) + \hat{a}(\cdot, \cdot)$ with:

Single layer potential:
$$\lambda = 0, \; H = H^{-1/2}(\Gamma), \; \hat{a}(u, v) := \int_{\Gamma \times \Gamma} \frac{v(x) u(y)}{4\pi |x - y|} ds_x \, ds_y, \qquad (2.3a)$$

Double layer potential:
$$\lambda = \pm\tfrac{1}{2}, \; H = L^2(\Gamma), \; \hat{a}(u, v) := \int_{\Gamma \times \Gamma} v(x) u(y) \frac{\partial}{\partial n_y} \frac{1}{4\pi |x - y|} ds_y ds_x, \qquad (2.3b)$$

Hypersingular operator:
$$\lambda = 0, \; H = H^{1/2}(\Gamma)/\mathbb{R}, \; \hat{a}(u, v) := \int_\Gamma v(x) \frac{\partial}{\partial n_x} \int_\Gamma u(y) \frac{\partial}{\partial n_y} \frac{1}{4\pi |x - y|} ds_y ds_x. \qquad (2.3c)$$

In the standard, conforming Galerkin method we select a subspace $\mathcal{S} \subset H$ and approximate (2.2) by seeking $U \in \mathcal{S}$, such that

$$a(U, V) = (f, V) \qquad \text{for all } V \in \mathcal{S}. \qquad (2.4)$$

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In the context of the boundary element method, these subspaces are finite element spaces lifted to the surface Γ.

Definition 2.1 *(a) The* master element *$\hat{t} \subseteq \mathbb{R}^2$ is the open triangle with vertices $(0,0)^\intercal$, $(0,1)^\intercal$ and $(1,1)^\intercal$.*

(b) A set $\mathcal{T} = \{t_1, t_2, \ldots, t_n\}$ consisting of open and disjoint (possibly curved) triangles in $\mathbb{R}^3$ such that there is a C^k-diffeomorphism $\Psi_t : \hat{t} \to \overline{t}$ for each $t \in \mathcal{T}$ is a surface triangulation *of Γ if it satisfies*

$$\Gamma = \bigcup_{t \in \mathcal{T}} \overline{t}.$$

(c) The triangulation is compatible *if the intersection $\overline{t} \cap \overline{t'} =: e$ of non-identical triangles $t, t' \in \mathcal{T}$ is either empty, a common vertex, or a common edge and, in the case that e is an edge, there exists an affine mapping $\gamma : \hat{\overline{t}} \to \hat{\overline{t}}$ such that $\Psi_t^{-1}\big|_e = (\Psi_{t'} \circ \gamma)^{-1}\big|_e$.*

The stepsize of a grid is given by

$$h = \max_{t \in \mathcal{T}} \operatorname{diam}(t) \tag{2.5}$$

For $k \in \{0, 1\}$ and $p \in \mathbb{N}_0$, the finite element space $\mathcal{S}(k, p, \mathcal{T})$ is defined by

$$\mathcal{S}(k, p, \mathcal{T}) := \{u \in H^k(\Gamma) \mid \forall t \in \mathcal{T} : u \circ \Psi_t \in \mathbb{P}_p\}.$$

Remark 2.2 *If $k = 1$, we assume throughout the paper that the triangulation is compatible.*

We will consider boundary elements of lowest order for the discretisation of the integral operators, i.e.,

- $\mathcal{S} = \mathcal{S}(0, 0, \mathcal{T})$ for the single layer operator and

- $\mathcal{S} = \mathcal{S}(1, 1, \mathcal{T})/\mathbb{R}$ for the hypersingular operator.

- The boundary element space for the double layer potential operator is a subset of $\mathcal{S}(0, 0, \mathcal{T})$ which contains all functions vanishing in a certain neighbourhood of the corners and edges of the surface Γ. For the detailed definition of the *finite section method* we refer to [6].

Let $(b_i)_{i=1}^n$ be the local basis of $\mathcal{S}(k, p, \mathcal{T})$. Hence, every function $u \in \mathcal{S}$ is characterised uniquely by the coefficient vector $\mathbf{u} = (u_i)_{i=1}^n \in \mathbb{R}^n$ with respect to this basis

$$u = \sum_{i=1}^n u_i b_i. \tag{2.6}$$

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If $u \in \mathcal{S}$ and $\mathbf{u} \in \mathbb{R}^n$ appear in the same context we will assume throughout the paper that (2.6) holds. Plugging this representation into equation (2.4) and testing by the basis functions, we can rewrite the variational equation as a system of linear equations

$$\mathbf{Ku} = \mathbf{F} \tag{2.7}$$

with

$$\mathbf{K}_{ij} = a(b_i, b_j) \quad \text{and} \quad \mathbf{F} = F(b_j) \tag{2.8}$$

for $i, j \in \{1, \ldots, n\}$.

The basis representation of the Galerkin method transforms the continuous problem into a system of linear equations. The direct numerical realisation of this approach suffers from two bottlenecks, (a) the computation of the matrix entries requires the evaluation of singular, nearly singular and regular surface integrals over pairs of panels, (b) the system matrix is not sparse but fully populated and the computational and storage costs are at least of order n^2.

In recent years, numerical methods have been developed to overcome these drawbacks of the boundary element method. Regularising transformations for evaluating singular and nearly singular surface integrals (see [26], [15], [7]), the panel-clustering method (see [14], [13],) and multipole methods (see [22], [10]) have been introduced. The use of these methods reduces the storage and computational complexity to $\mathcal{O}\left(n \log^{\kappa} n\right)$, where $\kappa = 4$ for some special kernel functions in (2.1) and $\kappa \sim 6, 7$ for general kernels.

However, the logarithmic terms and the large constants in the $\mathcal{O}\left(\cdot\right)$-estimates are clearly visible for practical problem sizes and significantly spoil the "linear" scaling of the complexity.

In this paper, we will present alternative representations of boundary integral equations which allow the use of quadrature techniques and panel-clustering methods with reduced approximation order while preserving the overall convergence order of the discretisation. The storage and computational complexity of the approach is $\mathcal{O}\left(n\right)$ without any logarithmic factors. The remaining part of the paper is organised as follows.

In Section 3, we will introduce alternative representations of classical boundary integral operators. These representations allow the use of quadrature and panel-clustering methods of low approximation order.

The panel clustering method is based on an approximation of the integral kernels. In Section 4, we will introduce a new panel-clustering method to approximate the integral kernels arising via the alternative representations of integral operators. Its algorithmic formulation is presented in Section 5.

The error analysis is developed in Section 6 and justifies the choices of the approximation orders in the description of the algorithm.

3 Alternative Representation of Classical Boundary Integral Operators

In this section, we will derive alternative representation of classical boundary integral equations, where the kernels are replaced by tangential derivatives of some *generator functions* which have reduced singular behaviour. In our applications this generator function is one of

$$G_1(x,y) := \frac{1}{4\pi \|x - y\|}, \qquad G_2(x,y) := \frac{\|x - y\|}{4\pi}. \tag{3.1}$$

(Note that G_1 is the fundamental solution of the Laplace operator in three dimensions.)

The advantage compared to the classical formulations is as follows. If we replace the generator function G on $\Gamma \times \Gamma$ by some globally Lipschitz continuous approximation $\widehat{G} \in C^{0,1}(\Gamma \times \Gamma)$ we may obtain an approximation $\widehat{k}$ of the kernel function k by applying the tangential derivative, say D, to $\widehat{G}$. Typically, the approximation $G - \widehat{G}$ is of higher order compared to the approximation of $k - \widehat{k} = D\left(G - \widehat{G}\right)$. For the error analysis, we apply partial integration pulling the derivative D to the test and trial function and may take advantage of the possible regularity of the exact solution. The continuity of $G - \widehat{G}$ avoids the appearance of line integrals due to partial integration. Thus, in the error estimates the difference $G - \widehat{G}$ instead of $k - \widehat{k}$ enters allowing to reduce the order of approximation.

3.1 Classical double layer potential

It is well known that

$$\Upsilon_\Gamma : \Gamma \to \mathbb{R}, \quad x \mapsto -\int_\Gamma \frac{\partial}{\partial n_y} G_1(x,y)\, ds_y = -\frac{1}{4\pi} \int_\Gamma \frac{\langle n_y, x - y \rangle}{\|x - y\|^3}\, ds_y \tag{3.2}$$

defines a function in $L^2(\Gamma)$ which has constant value $1/2$ almost everywhere on Γ. Hence, the constant function $1/2$ on Γ coincides in the L^2-sense with Υ_Γ.

This means that we can rewrite the bilinear form a_D of the double layer potential (cf. (2.3b)) as

$$a_D(u,v) = \left(\lambda - \frac{1}{2}\right)(u,v) + \int_{\Gamma \times \Gamma} v(x)(u(y) - u(x))\frac{\partial G_1(x,y)}{\partial n_y}\, ds_y\, ds_x \tag{3.3}$$

for all $u, v \in L^2(\Gamma)$. The difference to the standard form in (2.3b) is that the integrand of the second term in (3.3) has a reduced singular behaviour if the function u has some regularity, e.g., $u \in H^1(\Gamma)$.

3.2 Hypersingular operator

Partial integration in the form of Stokes' theorem can be applied to equation (2.3c) for Lipschitz surfaces (cf. [19]) to obtain for the bilinear form a_H of the hypersingular operator

$$a_H(u,v) = \int_{\Gamma \times \Gamma} \langle \overrightarrow{\mathrm{curl}}_\Gamma v(x), \overrightarrow{\mathrm{curl}}_\Gamma u(y) \rangle G_1(x,y)\, ds_y\, ds_x. \tag{3.4}$$

Here, the *surface curl* $\overrightarrow{\mathrm{curl}}_\Gamma$ is a tangential differential operator. For functions $u \in H^{1/2}(\Gamma)$ having an extension $u^\star \in H^1(\mathcal{U})$ into a three-dimensional neighbourhood $\mathcal{U}$ of Γ, the surface gradient ∇_Γ and the surface curl are defined by

$$\nabla_\Gamma u := (\nabla u^\star)|_\Gamma \quad \text{and} \quad \overrightarrow{\mathrm{curl}}_\Gamma u := -n \times \nabla_\Gamma u \qquad \text{on } \Gamma. \tag{3.5}$$

For a differentiable vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, we introduce the scalar counterpart of the surface curl by

$$\mathrm{curl}_\Gamma F := \langle n, \mathrm{curl}\, F \rangle \qquad \text{on } \Gamma,$$

where curl denotes the curl operator in the Euclidean space $\mathbb{R}^3$. The composition of the surface and tangential curl operator leads to the Laplace-Beltrami operator

$$\Delta_\Gamma u = -\mathrm{curl}_\Gamma \overrightarrow{\mathrm{curl}}_\Gamma u = \left(\Delta u^* - \frac{\partial^2}{\partial n^2} u^* \right)\Big|_\Gamma.$$

Simple tensor analysis yields

$$\frac{1}{\|y-x\|} = \Delta_{\Gamma,y}\|y-x\| + \langle n_y, y-x \rangle \frac{\partial}{\partial n_y} \frac{1}{\|y-x\|} \tag{3.6}$$

for all $y \in \Gamma$ and $x \in \mathbb{R}^3 \setminus \{y\}$. The second derivatives in $\Delta_{\Gamma,y}$ with respect to y would complicate the panel-clustering method and require higher global smoothness of the approximation. Thus, we express the Laplace-Beltrami operator (3.6) by a composition of first order surface derivatives with respect to x and y.

Lemma 3.1 *Let $\nabla_\Gamma^t := \nabla_\Gamma - n\partial/\partial n$ denote the tangential gradient and let $\Delta_{\Gamma,xy} := -\langle \nabla_{\Gamma,x}, \nabla_{\Gamma,y}^t \rangle$. Then*

$$\Delta_{\Gamma,y} G_2(x,y) = \Delta_{\Gamma,xy} G_2(x,y) \qquad \forall y \in \Gamma,\ \forall x \in \mathbb{R}^3 \setminus \{y\},$$

where G_2 is as in (3.1).

   Proof. The proof follows from

$$\Delta_{\Gamma,y}\|x-y\| = (\mathrm{curl}_{\Gamma,y}(n_z \times \nabla_y \|x-y\|))|_{z=y} = -(\mathrm{curl}_{\Gamma,y}(n_z \times \nabla_x \|x-y\|))|_{z=y}$$

$$= \left\langle \nabla_x, n_y \frac{\partial}{\partial n_y} - \nabla_y \right\rangle \|x-y\| = -\langle \nabla_x, \nabla_{\Gamma,y}^t \rangle \|x-y\|.$$

∎

Remark 3.2 *For a differentiable function $F : U_\Gamma \to \mathbb{R}$ which is defined in a tubular neighbourhood U_Γ of Γ we introduce (cf. (3.5)) the* mixed surface curl *by*

$$\overrightarrow{\mathrm{curl}}_{\Gamma,xy} := -n_y \times \nabla_{\Gamma,x}$$

and obtain

$$\Delta_{\Gamma,xy} = \mathrm{curl}_{\Gamma,y}\,\overrightarrow{\mathrm{curl}}_{\Gamma,xy}. \tag{3.7}$$

Replacing the kernel in (3.4) by (3.6), we obtain the representation

$$a_H(u,v) \tag{3.8}$$
$$= \int_\Gamma \int_\Gamma \langle \overrightarrow{\mathrm{curl}}_\Gamma u(y), \overrightarrow{\mathrm{curl}}_\Gamma v(x) \rangle \left(\Delta_{\Gamma,xy} G_2(x,y) + \langle n_y, y - x \rangle \frac{\partial G_1}{\partial n_y}(x,y) \right) ds_y\,ds_x.$$

3.3 Single layer potential

In order to work out the principal idea for the regularization of the single layer potential (2.3a), we assume that the Lipschitz boundary Γ is the surface of a polyhedron, i.e., there exist disjoint open plane faces Γ_i, $i \in \{1,\dots,q\}$, such that

$$\Gamma = \bigcup_{i=1}^q \overline{\Gamma_i}.$$

The oriented distance of x from the extended infinite plane $\Gamma_i^\star$ through Γ_i is given by

$$d_i(x) := \langle n_i, x - c_i \rangle$$

where n_i is the constant outer normal vector on Γ_i and $c_i \in \Gamma_i$. A simple consequence is that

$$\langle n_y, x - y \rangle = d_i(x) \tag{3.9}$$

holds for all $y \in \Gamma_i$.

Due to (3.6), the bilinear form a_S for the single layer operator (2.3a) has the representation

$$a_S(u,v) = \int_\Gamma \int_\Gamma v(x) G_1(x,y) u(y)\,ds_y\,ds_x$$
$$= \frac{1}{4\pi} \int_\Gamma \int_\Gamma v(x) u(y) \left(\Delta_{\Gamma,xy}\|y - x\| + \langle n_y, y - x \rangle \frac{\partial}{\partial n_y} \frac{1}{\|y - x\|} \right) ds_y\,ds_x$$
$$= \int_\Gamma \int_\Gamma v(x) u(y) \Delta_{\Gamma,xy} G_2(x,y)\,ds_y\,ds_x$$
$$+ \int_\Gamma \int_\Gamma v(x)(u(y) - u(x)) \langle n_y, y - x \rangle \frac{\partial}{\partial n_y} G_1(x,y)\,ds_y\,ds_x$$
$$+ \int_\Gamma v(x) u(x) \sum_{i=1}^q \int_{\Gamma_i} \langle n_y, y - x \rangle \frac{\partial}{\partial n_y} G_1(x,y)\,ds_y\,ds_x.$$

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By using (3.9), we can simplify the last term in order to get

$$a_S(u,v) = \int_\Gamma \int_\Gamma v(x)u(y)\Delta_{\Gamma,xy}G_2(x,y)\, ds_y\, ds_x \tag{3.10}$$

$$+ \int_\Gamma \int_\Gamma v(x)(u(y)-u(x))\langle n_y, y-x\rangle \frac{\partial}{\partial n_y}G_1(x,y)\, ds_y\, ds_x$$

$$+ (\rho u, v)\,,$$

where $\rho(x) := \sum_{i=1}^q d_i(x)\Upsilon_i(x)$ is a weight function and the spherical angle $\Upsilon_i(x)$ is given by

$$\Upsilon_i(x) := -\int_{\Gamma_i} \frac{\partial}{\partial n_y}G_1(x,y)\, ds_y.$$

Note that, for polyhedral surfaces, Υ_i can be computed analytically (see, e.g., [8]).

4 Kernel approximation

A matrix-vector multiplication appears as a basic arithmetic operation in every step of an iterative solution method for solving the linear system in (2.7). The result of a multiplication of the system matrix corresponding to a bilinear form $a(\cdot,\cdot)$ with a vector $\mathbf{u}$ is $(a(b_i,u))_{i=1}^n$, where b_i denotes the basis of the boundary element space and u is the boundary element function corresponding to the coefficient vector $\mathbf{u}$.

The representation of the Galerkin discretisation with respect to the nodal basis of the boundary element space leads to a full matrix and the computational and storage costs of assembling the matrix and of a matrix-vector multiplication are of order n^2. The panel-clustering method allows to represent the Galerkin discretisation with $O(n\log^\kappa n)$ quantities in a non-matrix form (our new approach results in $\kappa = 0$). This representation allows the efficient evaluation of a matrix-vector multiplication and, hence, iterative solvers can be employed for solving the linear systems.

All kernels in the bilinear forms a_S, a_D, a_H (cf. (3.10), (3.3), (3.8)) are Gâteaux derivatives of scalar generator functions defined in $\mathbb{R}^3$. Our concept for the kernel approximation is to approximate the generator functions and then to approximate the kernel by the Gâteaux derivative of the expansion.

4.1 Generalized cluster tree

The standard panel clustering method is based on the local approximation of the kernel function by a degenerate kernel on non-overlapping (six-dimensional) subsets of the domain $\Gamma \times \Gamma$. Since we need a globally continuous approximation (cf. Section 3), we would have to ensure continuity along the one- to five-dimensional intersections of the respective subsets, which would lead to a complicated algorithm.

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Instead, we construct approximations on overlapping subsets of $\Gamma \times \Gamma$ and blend them by using a partition of unity $\left(w^{(\sigma,s)}\chi^{(\sigma,s)}\right)_{(\sigma,s)\in P}$ consisting of non-negative, globally continuous cut-off functions $\chi^{(\sigma,s)} \in C^{0,1}(\Gamma \times \Gamma)$ and positive weights $\left(w^{(\sigma,s)}\right)_{(\sigma,s)\in P} \subset \mathbb{R}$.

Thus, any function $f : \Gamma \times \Gamma \to \mathbb{R}$ has the representation

$$f(x,y) = \sum_{(\sigma,s)\in P} w^{(\sigma,s)}\chi^{(\sigma,s)}(x,y) f(x,y).$$

The construction of the functions $\chi^{(\sigma,s)}$ will result in a tensor structure, i.e., $\chi^{(\sigma,s)} = \chi^\sigma \otimes \chi^s$ with functions $\chi^\sigma, \chi^s \in C^{0,1}(\Gamma)$ which have local support. In our application, the function f will be the generator function $G \in \{G_1, G_2\}$. If, for $\mathbf{b} = (\sigma, s) \in P$, the supports of χ^σ and χ^s are well separated, the kernel function can be approximated by a degenerate expansion.

Definition 4.1 (Hierarchical partition of unity) *A function system*

$$(\chi^\sigma)_{\sigma\in T} \quad with \quad \chi^\sigma : \Gamma \to \mathbb{R} \qquad \forall \sigma \in T \tag{4.1}$$

along with a disjoint partition $T = (T_\ell)_{\ell=0}^L$ of the index set T in (4.1) is a hierarchical partition of unity, *if,*

1. *for all $\ell \in \{0, \ldots, L\}$, the family $(\chi^\sigma)_{\sigma\in T_\ell}$ is a partition of unity for Γ*

2. *for each $\ell \in \{0, \ldots, L-1\}$ and each $\sigma \in T_\ell$, there are a set $\mathrm{sons}(\sigma) \subset T_{\ell+1}$ and positive coefficients $(\gamma^{\sigma,\sigma'})_{\sigma'\in\mathrm{sons}(\sigma)}$ satisfying*

$$\chi^\sigma = \sum_{\sigma'\in\mathrm{sons}(\sigma)} \gamma^{\sigma,\sigma'}\chi^{\sigma'}. \tag{4.2}$$

The level of a cluster $\sigma \in T_\ell$ is denoted by $\mathrm{level}(\sigma) := \ell$. The elements in T are called clusters. In contrast to the classical panel-clustering method the set T is not a tree but defines a simple hierarchy of clusters via the relation $\sigma \to \mathrm{sons}(\sigma)$.

Remark 4.2 *For the classical panel clustering method, the clusters are subsets of the set of freedoms for the boundary element discretisation and the hierarchy of clusters is defined via set inclusion. The equation (4.2) is the analogue of this condition*

Example 4.3 *Let $\Gamma = (0,1)$ and, for $\ell = 0, 1, \ldots, L$, define the index sets $T_\ell := \{(\mu, \ell) : 0 \le \mu \le n_\ell\}$ where $n_\ell = 2^\ell$.*
For $0 \le \ell \le L$, let $(x_{\mu,\ell})_{\mu=0}^{n_\ell} := (\mu/n_\ell)_{\mu=0}^{n_\ell}$ denote the set of equidistant grid points and let $(\chi^\sigma)_{\sigma\in T_\ell}$ be the continuous, piecewise linear Lagrange basis corresponding to the nodal points $(x_\sigma)_{\sigma\in T_\ell}$.
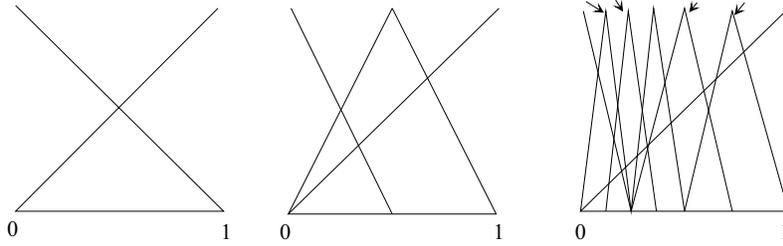
Figure 1: Cut-off functions arising in the multiscale representations 1, 2, 4 of f as in (4.3).

The function system $(\chi^\sigma)_{\sigma \in T_\ell}$ forms a $C^{0,1}(\Gamma)$-partition of unity for Γ and satisfies, for $\sigma = (\mu, \ell) \in T_\ell$ with $0 \le \ell < L$, the relation

$$\chi^\sigma = \sum_{\sigma' \in \mathrm{sons}(\sigma)} \gamma^{\sigma, \sigma'} \chi^{\sigma'}$$

with $\mathrm{sons}(\sigma) = \{(2\mu - 1, \ell + 1), (2\mu, \ell + 1), (2\mu + 1, \ell + 1)\} \cap T_{\ell+1}$ and

$$\gamma^{\sigma, \sigma'} := \left\{ \begin{array}{ll} 1 & x_\sigma = x_{\sigma'}, \\ 1/2 & \text{otherwise.} \end{array} \right\} \qquad \forall \sigma' \in \mathrm{sons}(\sigma).$$

There exist numerous (multiscale) decompositions of a function $f : \Gamma \to \mathbb{R}$ corresponding to different subsets $P \subset T$; some examples are shown below and illustrated in Figure 1

$$\begin{aligned} f &= \chi^{(0,0)} f + \chi^{(1,0)} f = \chi^{(0,1)} f + \tfrac{1}{2}\chi^{(1,1)} f + \chi^{(1,0)} f \\ &= \chi^{(0,2)} f + \tfrac{3}{4}\chi^{(1,2)} f + \tfrac{1}{2}\chi^{(2,2)} f + \tfrac{1}{4}\chi^{(3,2)} f + \chi^{(1,0)} f \\ &= \chi^{(0,2)} f + \tfrac{3}{8}\chi^{(1,3)} f + \tfrac{3}{4}\chi^{(2,3)} f + \tfrac{3}{8}\chi^{(3,3)} f + \tfrac{1}{2}\chi^{(2,2)} f + \tfrac{1}{4}\chi^{(3,2)} f + \chi^{(1,0)} f. \end{aligned} \qquad (4.3)$$

Remark 4.4 *The generalisation of the approach of Example 4.3 for the construction of hierarchical partitions of unity to sequences $(\mathcal{G}_\ell)_{\ell=0}^L$ of nested surface triangulations is straightforward.*

Note that the definition of a hierarchy of partitions of unity does not necessarily require a sequence of nested meshes. One might employ techniques developed in the fields of composite finite elements (cf. [16]), agglomeration methods (cf. [1], [28]), the partition of unity method (PUM) (cf. [18]) or meshless methods (cf. [11]) for this purpose.

For the efficiency of the algorithm, it is essential that there exists a constant $C_{\mathrm{sons}} \in \mathbb{N}$ such that

$$|\mathrm{sons}(\sigma)| \le C_{\mathrm{sons}} \qquad (4.4)$$

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holds for all $\sigma \in T$.

We will use tensor-product interpolation on axis-parallel boxes $Q^\sigma \subset \mathbb{R}^3$ satisfying the condition

$$\operatorname{supp} \chi^\sigma \subseteq Q^\sigma \tag{4.5}$$

for the construction of the degenerate kernel approximation. In order to find a stable interpolation scheme, we require that the partition of unity consists of functions in $W^{1,\infty}(\Gamma)$ and that there are constants $C_{\text{stab}}, C_{\text{supp}}, C_\sharp \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}
&\|\chi^\sigma\|_{W^{1,\infty}(\Gamma)} \le C_{\text{stab}} 2^{\operatorname{level}(\sigma)}, && \operatorname{diam} Q^\sigma \le C_{\text{supp}} 2^{-\operatorname{level}(\sigma)}, \\
&\max_{\tau \in \mathcal{T}} \operatorname{card} \{\sigma \in T_L : \tau \subset \operatorname{supp} \chi^\sigma\} \le C_\sharp,
\end{aligned} \tag{4.6}$$

Furthermore, we assume that there is an extension χ_+^σ of χ^σ into a tubular neighbourhood of Γ such that the normal derivative $\partial \chi_+^\sigma / \partial n$ is in $L^\infty(\Gamma)$ and

$$\partial \chi_+^\sigma / \partial n = 0 \qquad \text{almost everywhere on } \Gamma. \tag{4.7}$$

In the following, we skip the index "+" in χ_+^σ and identify χ^σ with its extension.

4.2 Generalized partitions

The kernel function in (2.1) which we want to approximate is defined on $\Gamma \times \Gamma$, while T corresponds to partitions of unity for Γ. We will use a tensor-product approach in order to construct a suitable partition of unity for $\Gamma \times \Gamma$: We set $T_\ell^2 := T_\ell \times T_\ell$ and $T^2 := \bigcup_{\ell=0}^L T_\ell^2$. The elements in T^2 are called *blocks*. Note that, for each $\mathbf{b} = (\sigma, s) \in T^2$, the clusters σ, s belong to the same level and we define $\operatorname{level}(\mathbf{b}) := \operatorname{level}(\sigma) = \operatorname{level}(s)$. Let

$$\chi^{\mathbf{b}} := \chi^\sigma \otimes \chi^s \quad \text{and} \quad \operatorname{sons}(\mathbf{b}) := \begin{cases} \operatorname{sons}(\sigma) \times \operatorname{sons}(s) & \text{if } \operatorname{level}(\mathbf{b}) < L, \\ \emptyset & \text{if } \operatorname{level}(\mathbf{b}) = L, \end{cases}$$

for all $\mathbf{b} = (\sigma, s) \in T^2$.

In order to approximate the kernel function, we need a covering of the domain $\Gamma \times \Gamma$. It will turn out in the analysis of the interpolation error that the relative distance of a pair of boxes has to be controlled by some parameter.

Definition 4.5 (Admissibility) *A pair $(\sigma, s) \in T \times T$ is called η-admissible for $\eta \in \mathbb{R}_{>0}$, if*

$$\operatorname{diam}(Q^\sigma \times Q^s) \le \eta \operatorname{dist}(Q^\sigma, Q^s) \tag{4.8}$$

holds.

In order to derive an efficient approximation of the integral kernels, we will specify a minimal index set $P \subset T^2$ along with corresponding weights $(w^{\mathbf{b}})_{\mathbf{b} \in P}$ in $\mathbb{R}$ such that $(w^{\mathbf{b}} \chi^{\mathbf{b}})_{\mathbf{b} \in P}$ is a partition of unity for $\Gamma \times \Gamma$ and each $\mathbf{b} \in P$ is either η-admissible or a leaf with $\operatorname{level}(\mathbf{b}) = L$.

Definition 4.6 (Admissible partition of unity) *A family $(w^{\mathbf{b}})_{\mathbf{b}\in P} \subset \mathbb{R}_{\geq 0}$ along with an index set $P \subset T^2$ is an* admissible weight *corresponding to $\eta \in \mathbb{R}_{>0}$, if*

$$\sum_{\mathbf{b}\in P} w^{\mathbf{b}} \chi^{\mathbf{b}} \equiv 1 \tag{4.9}$$

holds and if each block $\mathbf{b} = (\sigma, s) \in P$ with $\text{level}(\mathbf{b}) < L$ is η-admissible. If $(w^{\mathbf{b}})_{\mathbf{b}\in P}$ is an admissible weight, then $(w^{\mathbf{b}}\chi^{\mathbf{b}})_{\mathbf{b}\in P}$ is an admissible partition of unity *for $\Gamma \times \Gamma$.*

The weights are the tensor versions of the weighting factors which appear, e.g., in the decompositions (4.3). The procedure **subdivide** computes a minimal admissible weight. It is called by

$$w := 0; \quad P_{\text{far}} := \emptyset; \quad P_{\text{near}} := \emptyset; \quad \textbf{for } \mathbf{b} \in T_0 \times T_0 \textbf{ do subdivide}(\mathbf{b}, P_{\text{near}}, P_{\text{far}}, 1, w);$$

and defined by

> **procedure subdivide**$(\mathbf{b}, P_{\text{near}}, P_{\text{far}}, c, w)$;
> **begin**
>   **if b** is admissible **then begin**
> $w^{\mathbf{b}} := w^{\mathbf{b}} + c; \quad P_{\text{far}} := P_{\text{far}} \cup \{\mathbf{b}\};$
>   **end else if** $\text{level}(\mathbf{b}) = L$ **then begin**
> $w^{\mathbf{b}} := w^{\mathbf{b}} + c; \quad P_{\text{near}} := P_{\text{near}} \cup \{\mathbf{b}\};$
>   **end else for** $\mathbf{b}' \in \text{sons}(\mathbf{b})$ **do subdivide**$(\mathbf{b}', P_{\text{near}}, P_{\text{far}}, \gamma^{\sigma,\sigma'}\gamma^{s,s'}c, w)$;
> **end;**

Here, we used the conventions $\mathbf{b} = (\sigma, s)$, $\mathbf{b}' = (\sigma', s')$ and the definition of the coefficients $\gamma^{\sigma,\sigma'}$ as in (4.2).

The index set $P \subset T^2$ is the union of P_{far} and P_{near}. Note that all blocks in P_{far} are η-admissible and all blocks P_{near} belong to the finest level: $\text{level}(\mathbf{b}) = L$. The supports $\text{supp}\,\chi^{\mathbf{b}} \subset \Gamma \times \Gamma$ corresponding to $\mathbf{b} = (\sigma, s) \in P$ form a covering of $\Gamma \times \Gamma$.

4.3 Construction of the kernel approximation

Let us first consider the standard polynomial approximation of the kernel. The idea is to approximate the generator functions (cf. (3.1)),

$$G_1(x, y) = \frac{1}{4\pi \|x - y\|} \quad \text{and} \quad G_2(x, y) = \frac{\|x - y\|}{4\pi} \tag{4.10}$$

on domains $Q^{\sigma} \times Q^{s}$ corresponding to admissible blocks $(\sigma, s) \in P_{\text{far}}$.

For a continuous function $u \in C^0\left(\overline{Q^\sigma}\right)$, we denote the tensorised m-th order Chebyshev interpolation by

$$\mathcal{I}^\sigma[u] := \sum_{|\nu|_\infty \leq m^\sigma} u\left(x_\nu^\sigma\right) \mathcal{L}_\nu^\sigma, \tag{4.11}$$

where, for $\nu \in \mathbb{N}_0$, $|\nu|_\infty := \max\{\nu_i : 1 \leq i \leq 3\}$ denotes the maximum norm, $x_\nu^\sigma = \left(x_{\nu_i}^{J_i}\right)_{i=1}^3$ are the Chebyshev interpolation nodes scaled to the intervals $J_1 \times J_2 \times J_3 := Q^\sigma$ and $\mathcal{L}_\nu^\sigma(x) := \mathcal{L}_{\nu_1}^{J_1}(x_1) \mathcal{L}_{\nu_2}^{J_2}(x_2) \mathcal{L}_{\nu_3}^{J_3}(x_3)$ are the corresponding tensorised Lagrange polynomials. The choice of the approximation orders m^σ will be fixed later in (4.32).

Let $G \in \{G_1, G_2\}$ be a generator function (cf. (4.10)). For each $\mathbf{b} = (\sigma, s) \in P$, we define approximations $\tilde{G}^{\mathbf{b}}$ of $G^{\mathbf{b}} := \chi^{\mathbf{b}} G$ by

$$\tilde{G}^{\mathbf{b}} := \chi^{\mathbf{b}} \begin{cases} \mathcal{I}^{\mathbf{b}} G & \text{if } \mathbf{b} \text{ is admissible} \\ G & \text{otherwise} \end{cases} \quad \text{with} \quad \mathcal{I}^{\mathbf{b}} := \mathcal{I}^\sigma \otimes \mathcal{I}^s$$

and combine these functions by means of the hierarchical partition of unity:

$$\tilde{G} := \sum_{\mathbf{b} \in P} w^{\mathbf{b}} \tilde{G}^{\mathbf{b}}. \tag{4.12}$$

Thus, for $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$, we have defined blockwise degenerate resp. separable approximations

$$\tilde{G}^{\mathbf{b}}(x, y) = \chi^{\mathbf{b}} \sum_{|(\nu,\mu)|_\infty \leq m^{\mathbf{b}}} G(x_\nu^\sigma, x_\mu^s) \mathcal{L}_\nu^\sigma(x) \mathcal{L}_\mu^s(y) \quad \text{with} \quad |(\nu, \mu)|_\infty := \max\{|\nu|_\infty, |\mu|_\infty\}. \tag{4.13}$$

Our construction of blocks $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$ implies that the clusters σ, s belong to the same cluster level. The choice of the expansion orders (cf. (4.32)) will be constant on each level, i.e., $m^\sigma = m^s$ for all $\sigma, s \in T_\ell$. This motivates the definition $m^{\mathbf{b}} := m^\sigma$ in (4.13), while generalisation might be advantageous in special situations (cf. [2, Section 1.6.2]).

Example 4.7 (Approximation properties) *Let $G \in \{G_1, G_2\}$ be a generator function (cf. (4.10)). If we have a globally uniform local error bound, i.e., an $\varepsilon \in \mathbb{R}_{>0}$ satisfying $\|G - \mathcal{I}^{\mathbf{b}} G\|_{\infty, Q^\sigma \times Q^s} \leq \varepsilon$ for all $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$, we find*

$$\|G - \tilde{G}\|_\infty = \left\| \sum_{\mathbf{b} \in P} w^{\mathbf{b}} \chi^{\mathbf{b}} (G - \mathcal{I}^{\mathbf{b}} G) \right\|_\infty \leq \sum_{\mathbf{b} \in P_{\text{far}}} w^{\mathbf{b}} \chi^{\mathbf{b}} \|G - \mathcal{I}^{\mathbf{b}} G\|_{\infty, \text{supp}\,\chi^{\mathbf{b}}}$$

$$\leq \sum_{\mathbf{b} = (\sigma,s) \in P_{\text{far}}} w^{\mathbf{b}} \chi^{\mathbf{b}} \|G - \mathcal{I}^{\mathbf{b}} G\|_{\infty, Q^\sigma \times Q^s} \leq \sum_{\mathbf{b} \in P_{\text{far}}} w^{\mathbf{b}} \chi^{\mathbf{b}} \varepsilon = \varepsilon,$$

so good local approximations can be combined to form a good global approximation.

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4.4 Application to kernel functions

Now we will apply the approximation scheme described in the preceding subsections to the bilinear forms given in Section 3.

The relevant kernel functions are related to the generator functions by

$$k_1(x,y) := \frac{\partial}{\partial n_y} G_1(x,y), \quad k_2(x,y) := \langle n_y, y - x \rangle \frac{\partial}{\partial n_y} G_1(x,y) \qquad (4.14)$$

$$k_3(x,y) := \Delta_{\Gamma,xy} G_2(x,y), \qquad (4.15)$$

and (3.3), (3.8) and (3.10) take the form

$$a_D(u,v) = \left(\lambda - \frac{1}{2}\right)(u,v) + \int_\Gamma \int_\Gamma v(x)(u(y) - u(x)) k_1(x,y)\, ds_y\, ds_x, \qquad (4.16)$$

$$a_H(u,v) = \int_\Gamma \int_\Gamma \langle \overrightarrow{\mathrm{curl}}_\Gamma u(y), \overrightarrow{\mathrm{curl}}_\Gamma v(x) \rangle (k_2(x,y) + k_3(x,y))\, ds_y\, ds_x, \qquad (4.17)$$

$$a_S(u,v) = \int_\Gamma \int_\Gamma v(x) u(y) k_3(x,y) + v(x)(u(y) - u(x)) k_2(x,y)\, ds_y\, ds_x + (\rho u, v). \qquad (4.18)$$

Approximations of the kernel functions k_1, k_2, k_3 will be constructed by applying the surface derivatives $\partial/\partial n_y$, $\Delta_{\Gamma,xy}$, and the multiplication by $\langle n_y, x - y \rangle$ to the approximations of the generator functions.

We will not discuss the numerical treatment of $\left(\lambda - \frac{1}{2}\right)(u,v)$ and $(\rho u, v)$ since their representations with respect to the local boundary element basis result in sparse matrices where the computation of the entries can be performed by simple quadrature methods. Apart from these local operators the bilinear forms a_D, a_H, and a_S can be written as sums of terms of the form

$$a(u,v) = \int_{\Gamma \times \Gamma} v_{op}(x)\, u_{op}(y)\, G_{op}(x,y)\, ds_y ds_x, \qquad (4.19)$$

where, for $u, v \in S$, we put $u_{op} = D_I u$, $v_{op} = D_{II} u$, $G_{op} = (D_{III} \otimes D_{IV}) G$, where the operators

$$D_I, D_{II}, D_{III}, D_{IV} \in \left\{ I, \frac{\partial}{\partial n_y}, \langle n_y, y - x \rangle \frac{\partial}{\partial n_y}, \overrightarrow{\mathrm{curl}}_i, -\nabla_{\Gamma,x}, \nabla_{\Gamma,y}^t \right\} \qquad (4.20)$$

depend on the underlying problem (cf. Lemma 3.1, (4.14), (4.17)).

Remark 4.8 *Note that for all operators the algorithmic realisation is based on the same panel-clustering algorithm which can be realised in an abstract way independently of the concrete kernel function. This fact reduces the complexity of an implementation substantially.*

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Remark 4.9 *Note that the term $\int_{\Gamma\times\Gamma} v(x)\,u(x)\,k_1(x,y)\,ds_y ds_x$ in (4.16) is not of the form (4.19). However, by substituting $u_{op}(y) \leftarrow 1$ and $v_{op}(x) \leftarrow u(x)\,v(x)$ in (4.19), this term can be treated in a completely analogous way.*

Notation 4.10 *In Section 4.3, we have introduced the approximation $\tilde{G}$ of the generator function G. In view of the definition of G_{op}, we introduce $\tilde{G}_{op}$ as a shorthand for $(D_{III} \otimes D_{IV})\tilde{G}$. Similarly, $G_{op}^{\mathbf{b}}$ is short for $(D_{III} \otimes D_{IV})(\chi^{\mathbf{b}}G)$ and $\tilde{G}_{op}^{\mathbf{b}} := (D_{III} \otimes D_{IV})\tilde{G}^{\mathbf{b}}$.*

4.4.1 Approximation of a

We replace G by $\tilde{G}$ in (4.19) to obtain

$$
\tilde{a}(u,v) := \sum_{\mathbf{b}\in P} w^{\mathbf{b}} \int_\Gamma \int_\Gamma v_{op}(x) u_{op}(y) \tilde{G}_{op}^{\mathbf{b}}(x,y)\, ds_y\, ds_x
$$

$$
= \sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{near}}} w^{\mathbf{b}} \int_\Gamma \int_\Gamma v_{op}(x) u_{op}(y) G_{op}^{\mathbf{b}}(x,y)\, ds_y\, ds_x \tag{4.21}
$$

$$
+ \sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{far}}} w^{\mathbf{b}} \sum_{|(\nu,\mu)|\le m^{\mathbf{b}}} G(x_\nu^\sigma, x_\mu^s) \left(\int_\Gamma v_{op} D_{III}\left(\chi^\sigma \mathcal{L}_\nu^\sigma\right)\, ds \right) \tag{4.22}
$$

$$
\times \left(\int_\Gamma u_{op} D_{IV}\left(\chi^s \mathcal{L}_\mu^s\right)\, ds \right).
$$

Let $(b_i)_{i=1}^n$ denote the basis of the boundary element space S. To obtain a more compact representation we introduce

1. for $1 \le i, j \le n$, the nearfield matrix

$$
\mathbf{N}_{i,j} := \sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{near}}} w^{\mathbf{b}} \int_\Gamma \int_\Gamma (b_i)_{op}(x)\,(b_j)_{op}(y) G_{op}^{\mathbf{b}}(x,y)\, ds_y\, ds_x, \tag{4.23}
$$

2. for $\sigma \in T$ and $|\nu|_\infty \le m^\sigma$, the farfield coefficients

$$
L_\nu^\sigma(v) := \int_\Gamma v_{op} D_{III}\left(\chi^\sigma \mathcal{L}_\nu^\sigma\right)\, ds, \tag{4.24a}
$$

$$
R_\nu^\sigma(u) := \int_\Gamma u_{op} D_{IV}\left(\chi^\sigma \mathcal{L}_\nu^\sigma\right)\, ds, \tag{4.24b}
$$

3. for $\mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}$ and $|(\nu,\mu)|_\infty \le m^{\mathbf{b}}$ (cf. (4.13)), the expansion coefficients

$$
\kappa_{\nu,\mu}^{\mathbf{b}} := w^{\mathbf{b}} G(x_\nu^\sigma, x_\mu^s). \tag{4.25}
$$

Then, $\tilde{a}(u,v)$ can be expressed by (cf. (2.6))

$$\tilde{a}(u,v) = \mathbf{v}^{\mathsf{T}} \mathbf{N} \mathbf{u} + \sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{far}}} \sum_{|(\nu,\mu)|_\infty \leq m^{\mathbf{b}}} L_\nu^\sigma(v) \, \kappa_{\nu,\mu}^{\mathbf{b}} R_\mu^s(u). \qquad (4.26)$$

Remark 4.11 *As a consequence of the local supports of the boundary element basis functions b_i and the cutoff functions χ^σ (cf. (4.6)), the nearfield matrix is sparse. The integral over $\Gamma \times \Gamma$ in (4.23) can be replaced by an integral over*

$$\sum_{\mathbf{b}=(k,\ell)\in P_{\mathrm{near}}} \int_{\mathrm{supp}\, b_i \cap \mathrm{supp}\, \chi^k} \int_{\mathrm{supp}\, b_j \cap \mathrm{supp}\, \chi^\ell} \cdots$$

and the number of blocks $\mathbf{b} \in P_{\mathrm{near}}$ per index pair (i,j), where the integration domain has positive measure, is $O(1)$.

The farfield coefficients $L_\nu^\sigma(v)$, $R_\nu^\sigma(u)$ can be evaluated recursively by using the hierarchical structure of T. The details and the algorithmic realisation of a matrix-vector multiplication will be discussed in Section 5.

4.4.2   Variable Expansion Order

For the efficiency of the panel clustering method, it is essential that the farfield coefficients can be evaluated by a recurrence relation. The function systems which we have introduced in the previous sections can be written in the form

$$D_{III}\left(\chi^\sigma \mathcal{L}_\nu^\sigma\right), \qquad D_{IV}\left(\chi^\sigma \mathcal{L}_\nu^\sigma\right) \qquad \forall \sigma \in T, |\nu|_\infty \leq m^\sigma. \qquad (4.27)$$

where $D_{III}, D_{IV} \in \left\{I, \partial/\partial n, \langle n_y, y-x\rangle \frac{\partial}{\partial n_y}, \nabla_\Gamma, \nabla_\Gamma^t\right\}$ depend on the underlying operator (cf. (4.20)).

For the efficiency of the algorithm it is essential to evaluate the farfield coefficients L_ν^σ, R_μ^s via a recursion which is based on a **refinement relation** of the expansion system

$$\mathcal{L}_\nu^\sigma = \sum_{|\nu'|_\infty \leq m^{\sigma'}} \lambda_{\nu,\nu'}^{\sigma,\sigma'} \mathcal{L}_{\nu'}^{\sigma'} \qquad \forall\, |\nu|_\infty \leq m^\sigma, \forall \sigma \in T \text{ with } \mathrm{level}(\sigma) < L, \forall \sigma' \in \mathrm{sons}(\sigma)$$

$$(4.28)$$

with suitable shift coefficients $\lambda_{\nu,\nu'}^{\sigma,\sigma'}$. The functions $\mathcal{L}_\nu^\sigma$ are polynomials and, hence, relation (4.28) implies that the expansion orders are constant $m^\sigma = m^{\sigma'}$ (since a polynomial cannot be represented by a polynomial of lower degree).

On the other hand, a panel-clustering method which has linear complexity requires *low* expansion orders on small blocks and *higher* expansion orders on larger blocks. This leads to a variable distribution of the expansion orders and the condition $m^{\sigma'} < m^\sigma$ for $\sigma' \in \mathrm{sons}(\sigma)$. This is a conflict to (4.28) if we choose polynomials as the expansion system.

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To overcome this problem, we will replace the Lagrange polynomials by suitable approximations and employ the refinement relation directly for the definition of the expansion system.

Definition 4.12 *Let* $(\mathcal{L}_\nu^\sigma)_{\substack{\sigma \in T \\ |\nu|_\infty \leq m^\sigma}}$ *denote the Lagrange polynomials as in (4.11). For any given set of order distribution* $(m^\sigma)_{\sigma \in T}$ *which satisfies*

$$m^{\sigma'} \leq m^\sigma \qquad \text{for all } \sigma \in T \text{ with } \operatorname{level}(\sigma) < L \text{ and for all } \sigma' \in \operatorname{sons}(\sigma),$$

the approximated *Lagrange polynomials* $(\tilde{\mathcal{L}}_\nu^\sigma)_{\substack{\sigma \in T \\ |\nu|_\infty \leq m^\sigma}}$ *are given, for all* $\sigma \in T$ *and* ν *with* $|\nu|_\infty \leq m^\sigma$, *by*

$$\tilde{\mathcal{L}}_\nu^\sigma := \begin{cases} \mathcal{L}_\nu^\sigma & \text{if } \operatorname{level}(\sigma) = L, \\ \displaystyle\sum_{\sigma' \in \operatorname{sons}\sigma} \sum_{|\nu'|_\infty \leq m^{\sigma'}} \left(\gamma^{\sigma,\sigma'} \lambda_{\nu,\nu'}^{\sigma,\sigma'} \right) \tilde{\mathcal{L}}_{\nu'}^{\sigma'} & \text{otherwise} \end{cases} \qquad (4.29)$$

with $\lambda_{\nu,\nu'}^{\sigma,\sigma'} = \mathcal{L}_\nu^\sigma \left(x_{\nu'}^{\sigma'} \right)$ *and* $\gamma^{\sigma,\sigma'}$ *as in (4.2).*

Remark 4.13 *The construction of an approximated function system which is based on a refinement relation can be applied to more general function systems than the Lagrange polynomials as well. For instance, the construction of function systems which are based on a refinement relation is very common in the context of wavelets.*

Remark 4.14 *If the expansion order is the same on all clusters, i.e.,* $m^\sigma = m$ *for all* $\sigma \in T$, *then, the approximated functions system coincides with the original one in (4.27).*

Replacing the expansion systems in (4.27) by the approximated systems $\left(D\tilde{\mathcal{L}}_\nu^\sigma \right)_{\substack{\sigma \in T \\ |\nu|_\infty \leq m^\sigma}}$, $D \in \{D_{III}, D_{IV}\}$, leads to approximated farfield coefficients $\tilde{L}_\nu^\sigma$, $\tilde{R}_\nu^\sigma$ in (4.24).

Lemma 4.15 *Let*

$$\tilde{L}_\nu^\sigma (v) := \int_\Gamma v_{op} D_{III} \tilde{\mathcal{L}}_\nu^\sigma \, ds, \qquad (4.30)$$

$$\tilde{R}_\mu^s (u) := \int_\Gamma u_{op} D_{IV} \tilde{\mathcal{L}}_\nu^\sigma \, ds.$$

Then the recursion for the farfield coefficients

$$\tilde{L}_\nu^\sigma (v) = \sum_{\sigma' \in \operatorname{sons}(\sigma)} \sum_{|\nu'|_\infty \leq m^{\sigma'}} \gamma^{\sigma,\sigma'} \lambda_{\nu,\nu'}^{\sigma,\sigma'} \tilde{L}_{\nu'}^{\sigma'} (v) \qquad (4.31)$$

holds for all $\sigma \in T$ *with* $\operatorname{level}(\sigma) < L$ *and* ν *with* $|\nu|_\infty \leq m^\sigma$. *The coefficients* $\tilde{R}_\mu^s$ *satisfy the same recurrence.*

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Finally, we will specify the expansion order m^σ. Recall the definition of the axis-parallel cubes $Q^\sigma = J_1^\sigma \times J_2^\sigma \times J_3^\sigma$ as in (4.5). The following assumption reduces technicalities.

Assumption 4.16 *The length of subsequent intervals is strictly decreasing. There exist constants $0 < \underline{q} \leq \overline{q} < 1$ such that*

$$0 < \underline{q} \leq \left| J_j^{\sigma'} \right| / \left| J_j^\sigma \right| \leq \overline{q} < 1 \qquad \forall \sigma \in T \backslash T_L \quad \forall \sigma' \in \mathrm{sons}\,(\sigma) \quad \forall 1 \leq j \leq 3.$$

The expansion order on a cluster $\sigma \in T$ with $\ell := \mathrm{level}\,\sigma$ is given by

$$m^\sigma := m_\ell := \beta + (L - \ell)\,\alpha \qquad \forall 1 \leq j \leq 3 \tag{4.32}$$

for some integers $\beta \geq 0$ and $\alpha \geq 1$.

Remark 4.17 *If Assumption 4.16 does not hold and clusters do not shrink on (many) subsequent levels we recommend to use an anisotropic order distribution as proposed in [2, Section 1.6.2].*

4.4.3 Panel-Clustering with variable expansion order

Let $\alpha, \beta \in \mathbb{N}_0$ and $0 < \overline{q} \leq 1$ be fixed. Recall the definition of the expansion orders m^σ as in (4.32) and, as in (4.13), we set, for $\mathbf{b} \in P_{\mathrm{far}}$ with $\ell := \mathrm{level}\,(\mathbf{b})$, $m^{\mathbf{b}} := m_\ell$. Let the farfield coefficients $\tilde{L}_\nu^\sigma$, $\tilde{R}_\nu^\sigma$ be defined by (4.30). Then, the panel clustering approximation with variable order of the bilinear form $a\,(\cdot,\cdot)$ is given by

$$\tilde{a}(u, v) = \mathbf{v}^{\mathsf{T}} \mathbf{N} \mathbf{u} + \sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{far}}} \sum_{|(\nu,\mu)|_\infty \leq m^{\mathbf{b}}} \tilde{L}_\nu^\sigma\,(v)\,\kappa_{\nu,\mu}^{\mathbf{b}}\,\tilde{R}_\mu^s\,(u) \tag{4.33}$$

5 Algorithm

In this section, we will explain the efficient algorithmic realisation of a matrix-vector multiplication, i.e., the evaluation of $(\tilde{a}\,(b_i, u))_{i=1}^n$. The farfield coefficients $\tilde{L}_\nu^\sigma$, $\tilde{R}_\mu^s$ are of the abstract form (4.30), and the expansion coefficients $\kappa_{\nu,\mu}^{\mathbf{b}}$ are the evaluation of the generator function at the interpolation nodes:

$$\kappa_{\nu,\mu}^{\mathbf{b}} := w^{\mathbf{b}} G\,\left(x_\nu^\sigma, x_\mu^s\right) \qquad \forall \mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}, \quad \forall (\nu, \mu) \in \mathbb{N}_0^3 \times \mathbb{N}_0^3 : |(\nu, \mu)|_\infty \leq m^{\mathbf{b}}, \tag{5.1}$$

where the choice of $G \in \{G_1, G_2\}$ depends on the underlying operator.

5.1   Numerical Quadrature

The computation of the nearfield matrix $\mathbf{N}$ involves the evaluation of singular and nearly singular surface integrals.

The nearfield integrals are approximated by the quadrature technique introduced in [26], [15], [7]. We distinguish between the *proper* singular case, where the kernel function is singular in the domain of integration and the *nearly* singular case, where the kernel function is analytic but its derivatives blow up as the distance of the panels tends to zero.

For the proper singular integrals, regularising coordinate transforms (relative coordinates) are introduced rendering the integrands analytic (cf. [26], [15], [7], [25]). Then, properly scaled tensor Gauß quadrature rules are employed for the numerical approximation.

The nearly singular integrals are directly treated by properly scaled tensor Gauß rules where the order is chosen according to the error estimates. We do not recapitulate the quadrature formulae but refer to [7], [25] for a compact reference.

The error analysis in Section 6 will show that these quadrature techniques require only $O(1)$ kernel evaluations per matrix entry (independent of the mesh width) to obtain an approximation $\tilde{\mathbf{N}}$ of the nearfield matrix such that the corresponding perturbed Galerkin solution converges with the same rate as the exact Galerkin solution.

5.2   Setup Phase of the Panel Clustering Algorithm

In this section, we will develop an efficient algorithm to evaluate the second term in (4.33), namely the farfield part

$$\tilde{a}_{\mathrm{far}}(u,v) := \sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{far}}} \sum_{|(\nu,\mu)|_\infty \le m^{\mathbf{b}}} \kappa^{\mathbf{b}}_{\nu,\mu} \tilde{L}^{\sigma}_{\nu}(v)\tilde{R}^{s}_{\mu}(u). \tag{5.2}$$

The farfield coefficients can be represented by the *farfield coefficient matrices* $\mathbf{L}^{\sigma} = \left(\mathbf{L}^{\sigma}_{i,\nu}\right)_{\substack{1\le i\le n \\ |\nu|_\infty \le m^{\sigma}}}$, $\mathbf{R}^{\sigma} = \left(\mathbf{R}^{\sigma}_{i,\nu}\right)_{\substack{1\le i\le n \\ |\nu|_\infty \le m^{\sigma}}}$ which are defined for all clusters $\sigma \in T$ and all $1 \le i \le n$ by

$$\mathbf{L}^{\sigma}_{i,\nu} := \int_{\Gamma}(b_i)_{op}\, D_{III}\tilde{\mathcal{L}}^{\sigma}_{\nu}\ ds, \tag{5.3}$$

$$\mathbf{R}^{\sigma}_{i,\nu} := \int_{\Gamma}(b_i)_{op}\, D_{IV}\tilde{\mathcal{L}}^{\sigma}_{\nu}\ ds. \tag{5.4}$$

The expansion coefficients are the entries of the *interaction matrices* $\mathbf{S}^{\mathbf{b}} = \left(\kappa^{\mathbf{b}}_{\nu,\mu}\right)_{|(\nu,\mu)|_\infty \le m^{\mathbf{b}}}$ with the expansion coefficients as in (5.2). Using these matrices, the bilinear form $\tilde{a}_{\mathrm{far}}(\cdot,\cdot)$ in (5.2) equals

$$\mathbf{v}^{\intercal}\left(\sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{far}}}\mathbf{L}^{\sigma}\mathbf{S}^{\mathbf{b}}\left(\mathbf{R}^{s}\right)^{\intercal}\right)\mathbf{u} := \sum_{i,j=1}^{n}\mathbf{v}_i\left(\sum_{\mathbf{b}=(\sigma,s)\in P_{\mathrm{far}}}\sum_{|(\nu,\mu)|_\infty \le m^{\mathbf{b}}}\mathbf{S}^{\mathbf{b}}_{\nu,\mu}\mathbf{L}^{\sigma}_{i,\nu}\mathbf{R}^{s}_{j,\mu}\right)\mathbf{u}_j.$$

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In the next step, we will use the recurrence relation (4.31) to avoid the storage of the farfield coefficient matrices $\mathbf{L}^\sigma$, $\mathbf{R}^\sigma$. Instead we define, for all $\sigma \in T$ with level $(\sigma) < L$ and $\sigma' \in \text{sons}(\sigma)$ the *shift matrix* $\mathbf{B}^{\sigma',\sigma} = \left(\mathbf{B}^{\sigma',\sigma}_{\nu',\nu}\right)_{\substack{|\nu|_\infty < m^\sigma \\ |\nu'|_\leq \leq m^{\sigma'}}}$ by

$$\mathbf{B}^{\sigma',\sigma}_{\nu',\nu} := \gamma^{\sigma,\sigma'} \lambda^{\sigma,\sigma'}_{\nu,\nu'} \qquad \text{with} \qquad \gamma^{\sigma,\sigma'} \text{ as in (4.2) and } \lambda^{\sigma,\sigma'}_{\nu,\nu'} \text{ as in (4.28).}$$

The relation (4.31) implies

$$\mathbf{R}^\sigma = \sum_{\sigma' \in \text{sons}(\sigma)} \mathbf{R}^{\sigma'}\mathbf{B}^{\sigma',\sigma} := \left(\sum_{\sigma' \in \text{sons}(\sigma)} \sum_{|\nu'|_\infty \leq m^{\sigma'}} \mathbf{R}^{\sigma'}_{i,\nu'}\mathbf{B}^{\sigma',\sigma}_{\nu',\nu}\right)_{\substack{1 \leq i \leq n \\ |\nu|_\infty \leq m^\sigma}}$$

and the same relation holds for the farfield coefficient matrices $\mathbf{L}^\sigma$.

In summary, the fast evaluation of the bilinear form $\tilde{a}_{\text{far}}(\cdot,\cdot)$ in (5.2) requires a setup phase which consists of the following steps.

1. Compute and store the interaction matrices $\mathbf{S^b}$ for each farfield block $\mathbf{b} \in P_{\text{far}}$.

2. For all $\sigma \in T$ with level $(\sigma) < L$ and all $\sigma' \in \text{sons}(\sigma)$ compute and store the shift matrices $\mathbf{B}^{\sigma',\sigma}$.

3. For all $\sigma \in T$ with level $(\sigma) = L$ compute and store the basis farfield coefficient matrix

$$\mathbf{L}^\sigma_{i,\nu} := \begin{cases} \displaystyle\int_\Gamma (b_i)_{op} D_{III}\tilde{\mathcal{L}}^\sigma_\nu \, ds & \forall i \in \{1, 2, \ldots, n\} : |\text{supp } b_i \cap \text{supp } \chi^\sigma| > 0, \\ \\ 0 & \text{otherwise,} \end{cases}$$

in a sparse format. Proceed with the basis farfield coefficient matrix $\mathbf{R}^\sigma_{i,\nu}$ in the same manner.

5.3   Fast Matrix-Vector Multiplication

Let $\mathbf{K}_{\text{far}}$ denote the representation of the bilinear form $\tilde{a}_{\text{far}}(\cdot,\cdot)$ with respect to the basis $(b_i)_{i=1}^n$ of the boundary element space. Recall the convention (2.6) relating a coefficient vector $\mathbf{u} = (u_i)_{i=1}^n \in \mathbb{R}^n$ with the corresponding boundary element function u.

In this section, we will present an algorithm for the fast evaluation of $\mathbf{K}_{\text{far}}\mathbf{u} = (\tilde{a}_{\text{far}}(b_i, u))_{i=1}^n$. For $\sigma \in T$, let

$$P_{\text{far}}(\sigma) := \{s \in T \ : \ (\sigma, s) \in P_{\text{far}}\}. \tag{5.5}$$

For a given vector $\mathbf{u} \in \mathbb{R}^n$, we define

$$\mathbf{r}^s := (\mathbf{r}^s_\nu)_{|\nu|_\infty \leq m^s} := (\mathbf{R}^s)^\mathsf{T}\,\mathbf{u}, \quad \text{and} \quad \mathbf{s}^\sigma := \sum_{s \in P_{\mathrm{far}}(\sigma)} \mathbf{S}^{\sigma,s}\mathbf{r}^s$$

and find

$$\mathbf{y} := \mathbf{K}_{\mathrm{far}}\mathbf{u} = \sum_{\mathbf{b}=(\sigma,s) \in P_{\mathrm{far}}} \mathbf{L}^\sigma \mathbf{S}^{\mathbf{b}}\,(\mathbf{R}^s)^\mathsf{T}\,\mathbf{u} = \sum_{\mathbf{b}=(\sigma,s) \in P_{\mathrm{far}}} \mathbf{L}^\sigma \mathbf{S}^{\mathbf{b}}\mathbf{r}^s = \sum_{\sigma \in T} \mathbf{L}^\sigma \mathbf{s}^\sigma. \qquad (5.6)$$

For $\sigma \in T$ with $\mathrm{level}\,(\sigma) < L$, the computation of $\mathbf{r}^\sigma$ can be written in the form

$$\mathbf{r}^\sigma = (\mathbf{R}^\sigma)^\mathsf{T}\,\mathbf{u} = \sum_{\sigma' \in \mathrm{sons}(\sigma)} \left(\mathbf{B}^{\sigma',\sigma}\right)^\mathsf{T} \left(\mathbf{R}^{\sigma'}\right)^\mathsf{T}\mathbf{u} = \sum_{\sigma' \in \mathrm{sons}(\sigma)} \left(\mathbf{B}^{\sigma',\sigma}\right)^\mathsf{T}\mathbf{r}^{\sigma'},$$

so we can compute $\mathbf{r} = (\mathbf{r}^\sigma)_{\sigma \in T}$ by the recursive procedure **forward** which starts from the coarsest cluster level $T_0 \subset T$. The procedure is called by

> **for** $\sigma_0 \in T_0$ **do** **forward**$(\mathbf{u}, \sigma_0, \mathbf{r})$;

and defined by

> **procedure forward**$(\mathbf{u}, \sigma, \mathbf{r})$;
> **begin**
>   **if** $\mathrm{level}\,(\sigma) = L$ **then** $\mathbf{r}^\sigma := \mathbf{R}^\sigma\mathbf{u}$
>   **else begin**
> $\mathbf{r}^\sigma := 0$;
>   **for** $\sigma' \in \mathrm{sons}\,(\sigma)$ **begin**
>   **forward**$(\mathbf{u}, \sigma', \mathbf{r})$; $\mathbf{r}^\sigma := \mathbf{r}^\sigma + \left(\mathbf{B}^{\sigma',\sigma}\right)^\mathsf{T}\mathbf{r}^{\sigma'}$
>   **end**
>   **end**
> **end;**

By the similar procedure **backward** we can use the matrices $\mathbf{B}^{\sigma',\sigma}$ to compute $\mathbf{y}$ from the coefficients $\mathbf{s}^\sigma$. The procedure is called by

> **for** $\sigma_0 \in T_0$ **do** **backward**$(\mathbf{y}, \sigma_0, \mathbf{s})$;

and defined by

> **procedure backward**$(\mathbf{y}, \sigma, \mathbf{s})$;
> **begin**
>   **if** $\mathrm{level}\,(\sigma) = L$ **then**
> $\mathbf{y} := \mathbf{y} + \mathbf{L}^\sigma\mathbf{s}^\sigma$
>   **else for** $\sigma' \in \mathrm{sons}\,(\sigma)$ **do begin**
> $\mathbf{s}^{\sigma'} := \mathbf{s}^{\sigma'} + \mathbf{B}^{\sigma',\sigma}\mathbf{s}^\sigma$; **backward**$(\mathbf{y}, \sigma', \mathbf{s})$;
>   **end**
> **end;**

We can combine the forward and backward transformation in order to devise a fast algorithm for the matrix-vector multiplication. Recall the definition of $P_{\text{far}}(\sigma)$ as in (5.5).

```
procedure mvm(u, y);
begin
    for   σ₀ ∈ T₀   do   forward(u, σ₀, r);
    for   σ ∈ T   do   begin
        sσ := 0; for  s ∈ P_far(σ) do sσ := sσ + S^(σ,s)r^s
    end
    for   σ₀ ∈ T₀   do   backward(y, σ₀, s);
end;
```

6   Error analysis

In this section, we develop the error analysis for the variable order panel-clustering method applied to the alternative representations of boundary integral operators. As a result we obtain the distribution of the expansion order on different cluster levels and the quadrature orders for the nearfield part.

We restrict here to low order boundary element discretisations with quasi-uniform and shape regular meshes (cf. Definition 6.15 and 6.16). For higher order boundary elements, the use of graded meshes becomes important which are neither quasi-uniform nor shape regular. The panel-clustering method for those types of meshes will be discussed in [9].

In the error estimates, C denotes a generic constant which is independent of the levels ℓ and L and may vary at each appearance.

6.1   Interpolation error estimate

Let $G \in \{G_1, G_2\}$. G is *asymptotically smooth*, i.e., we have

$$|\partial_x^\nu \partial_y^\mu G(x,y)| \le C_{\text{asymp}}(\nu + \mu)! c_0^{|\nu+\mu|} \|x - y\|^{-g-|\nu+\mu|} \tag{6.1}$$

for constants $C_{\text{asymp}}, c_0 \in \mathbb{R}_{>0}$. This estimate follows from Cauchy's integral formula (see, e.g., ([14, Appendix B]), [23, Lemma 4.24]). The order of singularity g equals 1 for $G = G_1$ and is -1 for $G = G_2$.

For each $m \in \mathbb{N}$, let $\Lambda_m \in \mathbb{R}_{\ge 1}$ be the smallest constant such that

$$\|\mathcal{I}^m u\|_{\infty,[-1,1]} \le \Lambda_m \|u\|_{\infty,[-1,1]} \tag{6.2}$$

for all $u \in C([-1,1])$, i.e., Λ_m is the Lebesgue constant for the m-order interpolation operator $\mathcal{I}^m$ on $[-1,1]$. For our analysis, it is not essential that $\mathcal{I}^m$ is the Chebyshev interpolation operator as long as there are constants $C_\lambda, \lambda \in \mathbb{R}_{>0}$ satisfying

$$\Lambda_m \le C_\lambda (m+1)^\lambda \tag{6.3}$$

for all $m \in \mathbb{N}$, i.e., that the interpolation scheme is stable (in the case of Chebyshev interpolation, we have $C_\lambda = \lambda = 1$, cf. [21]).

In [2, Corollary 3.8], the fundamental estimate

$$\|u - \mathcal{I}^k_{J'}u\|_{\infty,J'} \leq C(1 + \Lambda_k)(4p)^{p-k-1}\left(\frac{|J'|}{|J|}\right)^{k+1}\|u\|_{\infty,J} \tag{6.4}$$

has been proven for polynomials $u \in \mathcal{P}^p$ and intervals $J' \subseteq J$. Since our kernel approximations are constructed as derivatives of variable-order interpolants of the generating functions G_1 and G_2, we need to introduce derivatives into the above estimate.

Lemma 6.1 *Let $J' \subseteq J$ be intervals. For $p, k, l \in \mathbb{N}_0$ with $l \leq k \leq p$ and for all $u \in \mathcal{P}^p$, we have*

$$\|u^{(l)} - (\mathcal{I}^k_{J'}u)^{(l)}\|_{\infty,J'} \leq \frac{C}{l!}(1 + \Lambda_k)p^{2l}(4p)^{p-k-1}\left(\frac{|J'|}{|J|}\right)^{k-l+1}\|u^{(l)}\|_{\infty,J}.$$

Proof. Let $z \in J$ be the midpoint of J and let $\bar{u}$ be the l-th order Taylor expansion of u in the point z. Then we have

$$\|u - \bar{u}\|_{\infty,J} \leq \frac{1}{l!}\left(\frac{|J|}{2}\right)^l\|u^{(l)}\|_{\infty,J}. \tag{6.5}$$

Markov's inequality (cf., e.g., [5, Thm. 4.1.4])

$$\|u^{(l)}\|_{\infty,J'} \leq \left(\frac{2}{|J'|}\right)^l \begin{bmatrix} p \\ l \end{bmatrix}^2 \|u\|_{\infty,J'} \quad \text{with} \quad \begin{bmatrix} p \\ l \end{bmatrix} = p!/(p-l)! \tag{6.6}$$

allows to apply verbatim the proof of Corollary 3.8 in [2] to prove the assertion. $\blacksquare$

By similar techniques, we find

$$\|(\mathcal{I}^k_J w)^{(l)}\|_{\infty,J} \leq \Lambda_k k^{2l}\|w^{(l)}\|_{\infty,J},$$

for all $k, l \in \mathbb{N}$ with $l \leq k$ and all $w \in C^l([-1,1])$.

Using (6.3), these estimates take the form

$$\|u^{(l)} - (\mathcal{I}^k_{J'}u)^{(l)}\|_{\infty,J'} \leq \frac{C'}{l!}(p+1)^{\lambda+2l}(4p)^{p-k-1}\left(\frac{|J'|}{|J|}\right)^{k-l+1}\|u^{(l)}\|_{\infty,J}, \tag{6.7a}$$

$$\|(\mathcal{I}^k_{J'}u)^{(l)}\|_{\infty,J'} \leq \left\{1 + \frac{C'}{l!}(p+1)^{\lambda+2l}(4p)^{p-k-1}\left(\frac{|J'|}{|J|}\right)^{k-l+1}\right\}\|u^{(l)}\|_{\infty,J}, \tag{6.7b}$$

$$\|(\mathcal{I}^k_J w)^{(l)}\|_{\infty,J} \leq C_\lambda(k+1)^{\lambda+2l}\|w^{(l)}\|_{\infty,J} \tag{6.7c}$$

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for $u \in \mathcal{P}^p$, $w \in C^l([-1,1])$ and a constant $C' \in \mathbb{R}_{>0}$. This means that the only difference between the original estimates for the interpolants and our estimates for their derivatives are the increased stability constants and the decreased approximation order.

Next, we will investigate the approximation property of polynomials for analytic functions.

Lemma 6.2 *Let J be a closed interval and $u \in C^\infty(J)$ such that, for some C_u, $\gamma_u > 0$, there holds*
$$\|u^{(n)}\|_{\infty,J} \le C_u \gamma_u^n n! \qquad \forall n \in \mathbb{N}.$$
Then
$$\inf_{p \in \mathcal{P}^k} \|u - p\|_{\infty,J} \le CC_u \left(1 + \frac{1}{c_0}\right)^{-k-1} \qquad with \quad c_0 := \gamma_u |J|,$$
where C only depends on c_0.

Proof. The proof is a direct consequence of [2, Lemma 3.13]. ∎

Definition 6.3 (Cluster sequence) *Let $\sigma \in T$, $\ell := \text{level}(\sigma)$. A sequence $\overrightarrow{\sigma} = (\sigma_i)_{i=\ell}^L \subset T$ satisfying $\sigma_\ell = \sigma$ and $\sigma_{i+1} \in \text{sons}(\sigma_i)$ for all $i \in \{\ell, \dots, L-1\}$ is a* cluster sequence *corresponding to σ. The set of all such cluster sequences is denoted by $\overrightarrow{\mathfrak{C}}^\sigma$.*

For each cluster sequence $\overrightarrow{\sigma} = (\sigma_i)_{i=\ell}^L \in \overrightarrow{\mathfrak{C}}^\sigma$ and $\ell \le k \le L$, the composite interpolation operator *is given by*
$$\mathcal{I}_k^{\overrightarrow{\sigma}} := \mathcal{I}^{\sigma_L} \circ \mathcal{I}^{\sigma_{L-1}} \cdots \circ \mathcal{I}^{\sigma_k}.$$

If $k = \ell$, we skip the index k and write $\mathcal{I}^{\overrightarrow{\sigma}}$ instead of $\mathcal{I}_\ell^{\overrightarrow{\sigma}}$. The relation of the iterated interpolation operator on cluster sequences and the approximated function systems (cf. (4.29)) is given by

$$D\tilde{\mathcal{L}}_\nu^\sigma = \sum_{\overrightarrow{\sigma} \in \overrightarrow{\mathfrak{C}}^\sigma} \gamma^{\overrightarrow{\sigma}} D\left(\chi^{\sigma_L} \mathcal{I}^{\overrightarrow{\sigma}}(\mathcal{L}_\nu^\sigma)\right) \tag{6.8}$$

where, for $\overrightarrow{\sigma} = (\sigma_i)_{i=\ell}^L$, we set $\gamma^{\overrightarrow{\sigma}} := \prod_{i=\ell}^{L-1} \gamma^{\sigma_i, \sigma_{i+1}}$. In (6.8), σ_L is the last component of the sequence $\overrightarrow{\sigma}$, χ^{σ_L} is the corresponding cutoff function. $\mathcal{L}_\nu^\sigma$ are the Lagrange polynomials as in (4.11) and $\tilde{\mathcal{L}}_\nu^\sigma$ denote the approximated ones (cf. (4.29)).

Next, we will express the variable order approximation of the kernel functions by using the iterated interpolation operator and start with the approximation of the generator function. We have for the farfield approximation

$$G_{\text{far}} := \sum_{\mathbf{b} \in P_{\text{far}}} w^{\mathbf{b}} \underbrace{\sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} G}_{=G^{\mathbf{b}}} \approx \widehat{G}_{\text{far}} = \sum_{\mathbf{b} \in P_{\text{far}}} w^{\mathbf{b}} \underbrace{\sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} \mathcal{I}^{\overrightarrow{\mathbf{b}}} G}_{=:\widehat{G}^{\mathbf{b}}}, \tag{6.9}$$

where, for $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$, we employ the notation $\overrightarrow{\mathfrak{C}}^{\mathbf{b}} := \overrightarrow{\mathfrak{C}}^{\sigma} \times \overrightarrow{\mathfrak{C}}^{s}$ and, for $\overrightarrow{\sigma} = (\sigma_i)_{i=\ell}^{L}$, $\overrightarrow{s} = (s_i)_{i=\ell}^{L}$ and a corresponding *block sequence* $\overrightarrow{\mathbf{b}} = (\overrightarrow{\sigma}, \overrightarrow{s}) \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}$, we put

$$\chi^{\overrightarrow{\mathbf{b}}} := \chi^{\sigma_L} \otimes \chi^{s_L}, \qquad \gamma^{\overrightarrow{\mathbf{b}}} := \gamma^{\overrightarrow{\sigma}} \gamma^{\overrightarrow{s}}, \quad \text{and} \quad \mathcal{I}_k^{\overrightarrow{\mathbf{b}}} := \mathcal{I}_k^{\overrightarrow{\sigma}} \otimes \mathcal{I}_k^{\overrightarrow{s}}.$$

Again, for $k = \ell$, we skip the index k in $\mathcal{I}_k^{\overrightarrow{\mathbf{b}}}$.

Remark 6.4 *The function system* $\left(w^{\mathbf{b}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} \right)_{\substack{\mathbf{b} \in P \\ \overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}}}$ *forms a partition of unity of* $\Gamma \times \Gamma$ *and it holds*

$$\chi^{\mathbf{b}} = \sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}}. \tag{6.10}$$

Farfield approximations to the kernel function in the farfield G_{far} (cf. (4.19)) are derived by applying the operator $D_{III} \otimes D_{IV}$ to $\widehat{G}_{\text{far}}$. Recalling (6.9), we get

$$\left(\widehat{G}_{\text{far}} \right)_{op} := \sum_{\mathbf{b} \in P_{\text{far}}} w^{\mathbf{b}} \sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \left(\gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right)_{op} (x, y). \tag{6.11}$$

The farfield part of the exact kernel is defined by replacing the term $\left(\chi^{\overrightarrow{\mathbf{b}}} \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right)_{op}$ in (6.11) by $\left(\chi^{\overrightarrow{\mathbf{b}}} G \right)_{op}$.

Let us consider an admissible block $(\sigma, s) \in P_{\text{far}}$. Let $\overrightarrow{\sigma} = (\sigma_i)_{i=\ell}^{L}$ and $\overrightarrow{s} = (s_i)_{i=\ell}^{L}$ be cluster sequences corresponding to σ and s. Let the parameter $\beta \in \mathbb{N}_0$ (cf. (4.32)) satisfy $\beta > 1$. Using our results on differentiated interpolants (cf. (6.7), we can generalize [2, Theorem 4.3] in order to find a polynomial C and constants $\overline{\alpha} > 0$ and $0 \leq c < 1$ satisfying

$$\|\partial_x^{\nu} \partial_y^{\mu} (G - \mathcal{I}^{\overrightarrow{\mathbf{b}}} G)\|_{\infty, \text{supp} \chi^{\overrightarrow{\mathbf{b}}}} \leq C \operatorname{dist}(Q^{\sigma}, Q^{s})^{-g - |\nu + \mu|} c^{\beta + \min\{(\beta\overline{\alpha}, \alpha\}(L - \ell)}, \tag{6.12}$$

for all multi-indices $\nu, \mu \in \mathbb{N}_0^3$ with $|\nu|, |\mu| \in \{0, 1\}$.

Under reasonable assumptions the right-hand side in (6.12) can be expressed in terms of the levels ℓ, L. The first one concerns the support of the cut-off functions. Let $\Gamma_{\sigma} := \operatorname{supp} \chi^{\sigma}$, $\sigma \in T$.

Assumption 6.5 *There exist constants $C < \infty$ and $1 < C < \infty$ so that, for all $0 \leq \ell \leq L$ and any $\sigma \in T_{\ell}$,*

$$\operatorname{diam} \Gamma_{\sigma} \leq \operatorname{diam} Q^{\sigma} \leq Ch2^{L - \ell},$$
$$C^{-1} 2^{-\ell} \leq \operatorname{diam} \Gamma_{\sigma} \leq \operatorname{diam} Q^{\sigma} \leq C 2^{-\ell}.$$

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The admissibility condition controls the maximal diameters of admissible pairs of clusters. Assumption 6.6 implies that the diameters of admissible pairs of clusters are *comparable* with their distance.

Assumption 6.6 *There exists a constant $0 < C \leq 1$ such that, for all admissible blocks $(\sigma, s) \in P_{\text{far}}$, the estimate*

$$\max\{\text{diam}(Q^\sigma), \text{diam}(Q^s)\} \geq C\eta \, \text{dist}\,(Q^\sigma, Q^s)$$

holds.

Proposition 6.7 *Let $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$ with $\ell := \text{level}\,\mathbf{b}$. Assumptions 6.5 and 6.6 imply*

$$C^{-1}\frac{2^{-\ell}}{\eta} \leq \text{dist}\,(Q^\sigma, Q^s) \leq C\frac{2^{-\ell}}{\eta}.$$

Proof. Let $\mathbf{b}$ be as in the proposition. Then, Assumptions 6.5 and 6.6 imply

$$\text{dist}\,(Q^\sigma, Q^s) \leq \frac{1}{C\eta}\max\{\text{diam}(Q^\sigma), \text{diam}(Q^s)\} \leq C\frac{2^{-\ell}}{\eta}.$$

For the lower bound, we use the admissibility condition (4.8) and Assumption 6.5 to obtain

$$\text{dist}(Q^\sigma, Q^s) \geq \eta^{-1}\,\text{diam}\,(Q^\sigma \times Q^s) \geq \eta^{-1}\,\text{diam}\,Q^\sigma \geq C^{-1}\frac{2^{-\ell}}{\eta}.$$

■

Lemma 6.8 *Let (4.6) and Assumptions 6.5, 6.6 be valid. Let $g + |\mu + \nu| \geq 0$. Then, the right-hand side in (6.12) can be estimated from above and we obtain the estimate*

$$\|\partial_x^\nu \partial_y^\mu (G - \mathcal{I}^{\vec{\mathbf{b}}}G)\|_{\infty, \text{supp}\,\chi^{\vec{\mathbf{b}}}} \leq C2^{\ell(g+|\mu+\nu|)}c^{m_\ell}, \tag{6.13}$$

where $C < \infty$ and $0 < c < 1$ are independent of the level ℓ and L.

Next, we will derive an estimate of $\partial_x^\nu \partial_y^\mu\left(G_{\text{far}} - \widehat{G}_{\text{far}}\right)$ on $\text{supp}\,\chi^{\mathbf{b}}$ for some $\mathbf{b} \in P_{\text{far}}$. By using the splitting (6.11) we obtain

$$G_{\text{far}} - \widehat{G}_{\text{far}} = \sum_{\mathbf{b} \in P_{\text{far}}} w^{\mathbf{b}} \sum_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\,\mathbf{b}} \gamma^{\vec{\mathbf{b}}}\chi^{\vec{\mathbf{b}}}\left(G - \mathcal{I}^{\vec{\mathbf{b}}}G\right).$$

Since $\left(w^{\mathbf{b}}\gamma^{\vec{\mathbf{b}}}\chi^{\vec{\mathbf{b}}}\right)_{\substack{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\,\mathbf{b} \\ \mathbf{b} \in P}}$ is a partition of unity, the local estimate (6.12) carries over to

$$\left\|G_{\text{far}} - \widehat{G}_{\text{far}}\right\|_{\infty, \text{supp}\,\chi^{\mathbf{b}}} \leq \max_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\,\mathbf{b}} \|G - \mathcal{I}^{\vec{\mathbf{b}}}G\|_{\infty, \text{supp}\,\chi^{\vec{\mathbf{b}}}} \overset{(6.13)}{\leq} C2^{\ell g}c^{m_\ell} \tag{6.14}$$

(cf. Example 4.7).

Next, we will estimate first derivatives of the approximation, more precisely, for $\mathbf{b} \in P_{\text{far}}$ the difference

$$\partial \left(G^{\mathbf{b}}_{\text{far}} - \widehat{G}^{\mathbf{b}}_{\text{far}} \right) = \partial \sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} \left(G - \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right), \qquad (6.15)$$

where ∂ is a first order derivative. Leibniz' product rule yields

$$\partial \left(G^{\mathbf{b}}_{\text{far}} - \widehat{G}^{\mathbf{b}}_{\text{far}} \right) = \sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \left(\partial \chi^{\overrightarrow{\mathbf{b}}} \right) \left(G - \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right) + \sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} \partial \left(G - \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right).$$

$$(6.16)$$

The second sum can be treated as before and, again, the local estimates carry over to the global ones:

$$\left\| \sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi^{\overrightarrow{\mathbf{b}}} \partial \left(G - \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right) \right\|_{\infty, \text{supp} \chi^{\mathbf{b}}} \leq \max_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \| \partial \left(G - \mathcal{I}^{\overrightarrow{\mathbf{b}}} G \right) \|_{\infty, \text{supp} \chi^{\overrightarrow{\mathbf{b}}}} \overset{(6.13)}{\leq} C 2^{(g+1)\ell} c^{m_\ell}.$$

$$(6.17)$$

Hence, we focus in the following to the first sum in (6.16). The difficulty here is that, in a straightforward estimate, the term $\left(\partial \chi^{\overrightarrow{\mathbf{b}}} \right)$ blows up like 2^L (cf. (4.6)) while the approximation property for a block $\mathbf{b}$ with $\ell = $ level $\mathbf{b}$ decreases as $c^{L-\ell}$ which does not compensate the 2^L-term in an adequate way. The following proposition overcomes this problem by a more subtle estimate of the difference $G - \mathcal{I}^{\mathbf{b}} G$.

Proposition 6.9 *Let* $G \in \{G_1, G_2\}$*. Let* $\mathbf{b} \in P_{\text{far}}$ *with* $\ell := $ *level* $\mathbf{b}$ *and* $\overrightarrow{\mathbf{b}} = (\mathbf{b}_j)_{j=\ell}^{L} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}$ *with* $\mathbf{b}_j = (\sigma_j, s_j)$*. Then, there exist constants* $\delta, C > 0$ *independent of* $\ell, k,$ *and* L *such that*

$$\left\| G - \mathcal{I}^{\mathbf{b}_k} G \right\|_{\infty, Q^{\sigma_k} \times Q^{s_k}} \leq C \left(\eta 2^\ell \right)^{g+1} 2^{-k} \left(1 + \frac{\delta}{\eta 2^{\ell - k}} \right)^{1 - m_k} \qquad \forall \ell \leq k \leq L.$$

Proof. We start with the one-dimensional case. Let J be an interval and $\mathcal{I}^k_J$ the kth order interpolant on J. Then, for any $u \in C^1 (J)$, we have for all $x \in J$

$$\left| \left(u - \mathcal{I}^k_J u \right) (x) \right| = \left| \int_{x_0}^x \left(u - \mathcal{I}^k_J u \right)' \right| \leq |J| \left\| \left(u - \mathcal{I}^k_J u \right)' \right\|_{\infty, J},$$

where $x_0 \in J$ is some interpolation point for $\mathcal{I}^k_J$. Let $p_{k-1} \in \mathcal{P}^{k-1}$ be any polynomial of maximal degree $k - 1$ and $p_k \in \mathcal{P}^k$ an antiderivative satisfying $p'_k = p_{k-1}$. Then, by using the projection property of $\mathcal{I}^k$, we get

$$\left\| (u - \mathcal{I}_k u)' \right\|_{\infty, J} = \left\| (u - p_k + p_k - \mathcal{I}_k u)' \right\|_{\infty, J}$$

$$\leq \left\| u' - p_{k-1} \right\|_{\infty, J} + \left\| \left(\mathcal{I}^k (p_k - u) \right)' \right\|_{\infty, J}.$$

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The stability estimate (6.7c) leads to

$$\left\| \left(\mathcal{I}^k \left(p_k - u \right) \right)' \right\|_{\infty,J} \le C_\lambda (k+1)^{\lambda+2} \| u' - p_{k-1} \|_{\infty,J}.$$

Together, we have proved

$$\left\| u - \mathcal{I}_J^k u \right\|_{\infty,J} \le \left\{ 1 + C_\lambda (k+1)^{\lambda+2} \right\} |J| \inf_{v \in \mathcal{P}^{k-1}} \| u' - v \|_{\infty,J}.$$

For functions u, where u' satisfies the assumption of Lemma 6.2, we conclude from Lemma 6.2 that

$$\left\| u - \mathcal{I}_J^k u \right\|_{\infty,J} \le C C_{u'} \left\{ 1 + C_\lambda (k+1)^{\lambda+2} \right\} |J| \left(1 + \frac{1}{c_0'} \right)^{-k} \tag{6.18}$$

with $c_0' := \gamma_{u'} |J|$. In our application, c_0' will depend on the cluster level ℓ and we will assume (and prove below) uniform boundedness $c_0' \le \overline{c_0}$, where $\overline{c_0}$ is independent of ℓ and L. Under this assumption and for $k \ge 1$, we find a constant $\delta > 0$ depending only on C_λ, $\overline{c_0}$, and λ such that

$$\left\| u - \mathcal{I}_J^k u \right\|_{\infty,J} \le C C_{u'} \left(1 + \frac{1}{c_0'} \right)^{1-k} |J|.$$

By a classical tensor argument, we transfer this result to the approximation on any block $\mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}$ with $\ell := \mathrm{level}(\mathbf{b})$. Let $\overrightarrow{\mathbf{b}} = ((\sigma_j, s_j))_{j=\ell}^L \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}$. Fix $\ell \le k \le L$. From Proposition 6.7 we derive

$$\mathrm{diam}\,(Q^{\sigma_k} \times Q^{s_k}) \le \left(C \eta 2^k \right)^{-1}.$$

Let ∂ denote a first order partial derivative. For the generator functions $G \in \{G_1, G_2\}$ (cf. (3.1)), the estimate

$$|\partial G|_{W^{n,\infty}(Q^\sigma \times Q^s)} \le C_{G'} \gamma_{G'}^n n! \qquad \forall n \in \mathbb{N} \quad \text{with} \quad C_{G'} = C \left(\eta 2^\ell \right)^{g+1}, \quad \gamma_{G'} = C \eta 2^\ell.$$

follows from (6.1) in combination with Proposition 6.7. Note that the constant c_0' in the multidimensional case is given (and can be estimated) by

$$c_0' := \gamma_{G'} \, \mathrm{diam}\,(Q^{\sigma_k} \times Q^{s_k}) \le C \eta 2^{\ell-k}.$$

Hence, we can apply (6.18) componentwise and, by a standard tensor argument, we derive

$$\left\| G - \mathcal{I}^{\mathbf{b}_k} G \right\|_{\infty, Q^{\sigma_k} \times Q^{s_k}} \le C \left(\eta 2^\ell \right)^{g+1} \mathrm{diam}\,(Q^{\sigma_k} \times Q^{s_k}) \left(1 + \frac{\delta}{\eta 2^{\ell-k}} \right)^{1-m_k}$$

$$\le C \left(\eta 2^\ell \right)^{g+1} 2^{-k} \left(1 + \frac{\delta}{\eta 2^{\ell-k}} \right)^{1-m_k}.$$

∎

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Lemma 6.10 *There exists $\theta > 0$ independent of ℓ and L such that, for any $\mathbf{b} \in P_{\mathrm{far}}$ with $\ell :=$ level $\mathbf{b}$, there holds*

$$\left\| \partial \left(G_{\mathrm{far}}^{\mathbf{b}} - G_{\mathrm{far}}^{\mathbf{b}} \right) \right\|_{\infty, \mathrm{supp}\, \chi^{\mathbf{b}}} \leq C 2^{\ell(g+1)} \left(1 + \theta \right)^{-m_\ell} . \tag{6.19}$$

Proof. Let $\mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}$ with $\ell :=$ level $\mathbf{b}$. Then,

$$E^{\mathbf{b}} := \sum_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\, \mathbf{b}} \gamma^{\vec{\mathbf{b}}} \left(\partial \chi^{\vec{\mathbf{b}}} \right) \left(G - \mathcal{I}^{\vec{\mathbf{b}}} G \right) = \sum_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\, \mathbf{b}} \gamma^{\vec{\mathbf{b}}} \left(\partial \chi^{\vec{\mathbf{b}}} \right) \sum_{k=\ell}^{L} \mathcal{I}_{k+1}^{\vec{\mathbf{b}}} \left(G - \mathcal{I}^{\mathbf{b}_k} G \right) .$$

By using the inverse inequality (4.6) and Proposition 6.9 we obtain

$$\left\| E^{\mathbf{b}} \right\|_{\infty, Q^\sigma \times Q^s} \leq C C_{\mathrm{stab}} \sum_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\, \mathbf{b}} \gamma^{\vec{\mathbf{b}}} \chi_0^{\vec{\mathbf{b}}} 2^L \sum_{k=\ell}^{L} s_{k+1}^{\vec{\mathbf{b}}} \left(\eta 2^\ell \right)^{g+1} 2^{-k} \left(1 + \frac{\delta}{\eta 2^{\ell-k}} \right)^{1-m_k} ,$$

where, for $\vec{\mathbf{b}} = ((\sigma_j, s_j))_{j=\ell}^{L}$ and $\ell \leq k \leq L - 1$, the stability of the iterated interpolation is

$$s_k^{\vec{\mathbf{b}}} := \left\| \mathcal{I}_k^{\vec{\mathbf{b}}} \right\|_{L^\infty(Q^{\sigma_L} \times Q^{s_L}) \leftarrow L^\infty(Q^{\sigma_k} \times Q^{s_k})}$$

and $\chi_0^{\vec{\mathbf{b}}} : \Gamma \times \Gamma \to \mathbb{R}$ denotes the characteristic function of $\mathrm{supp}\, \chi^{\sigma_L} \times \mathrm{supp}\, \chi^{s_L}$. In [2, Theorem 3.11, Proof of Lemma 4.2] it is shown that

$$s_{k+1}^{\mathbf{b}} \leq C \Lambda_{m_k}^6$$

holds. Hence, by choosing $\beta \geq 1 + \alpha$ in (4.32), we obtain

$$\left\| E^{\mathbf{b}} \right\|_{\infty, Q^\sigma \times Q^s} \leq C C_{\mathrm{stab}} \left(\eta 2^\ell \right)^{g+1} \sum_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\, \mathbf{b}} \gamma^{\vec{\mathbf{b}}} \chi_0^{\vec{\mathbf{b}}} \sum_{k=\ell}^{L} \Lambda_{m_k}^6 \left(\frac{\left(1 + \frac{\delta}{\eta 2^{\ell-k}} \right)^\alpha}{2} \right)^{k-L-1}$$

$$\leq C M C_{\mathrm{stab}} \left(\eta 2^\ell \right)^{g+1} \sum_{\vec{\mathbf{b}} \in \vec{\mathfrak{c}}\, \mathbf{b}} \gamma^{\vec{\mathbf{b}}} \chi_0^{\vec{\mathbf{b}}} \sum_{k=\ell}^{L} \Lambda_{m_k}^6 \tag{6.20}$$

with

$$M := \max_{\ell \leq k \leq L} \left(\frac{\left(1 + \frac{\delta}{\eta 2^{\ell-k}} \right)^\alpha}{2} \right)^{k-L-1} .$$

The maximum is attained for $k = L$, yielding

$$M \leq \frac{2}{\left(1 + \frac{\delta}{\eta 2^{\ell-L}} \right)^\alpha} \leq C 2^{-m_\ell}$$

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with C depending only on δ, η, and α. The sum over the powers of the stability constants in (6.20) can be estimated by

$$\sum_{k=\ell}^{L} \Lambda_{m_k}^{6} \leq C_{\lambda}^{6} \left(m_\ell + 1\right)^{7}.$$

By fixing a constant $0 < \theta < 1$, we obtain

$$M \sum_{k=\ell}^{L} \Lambda_{m_k}^{6} \leq C \left(1 + \theta\right)^{-m_\ell}.$$

Next, we estimate the remaining term $\sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi_0^{\overrightarrow{\mathbf{b}}}$ in (6.20). Let $\mathfrak{S}^{\mathbf{b}} \in T_L \times T_L$ be the set of terminal blocks in $\overrightarrow{\mathfrak{C}}^{\mathbf{b}}$:

$$\mathfrak{S}^{\mathbf{b}} := \left\{ \mathbf{b}_L : \overrightarrow{\mathbf{b}} = (\mathbf{b}_j)_{j=\ell}^{L} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}} \right\}$$

and, for $\mathbf{b}' \in \mathfrak{S}^{\mathbf{b}}$, we define $\overleftarrow{\mathfrak{C}}^{\mathbf{b}, \mathbf{b}'} \subset \overrightarrow{\mathfrak{C}}^{\mathbf{b}}$ as

$$\overleftarrow{\mathfrak{C}}^{\mathbf{b}, \mathbf{b}'} := \left\{ \overrightarrow{\mathbf{b}} = (\mathbf{b}_j)_{j=\ell}^{L} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}} : \mathbf{b}_L = \mathbf{b}' \right\}.$$

Then, we can write

$$\sum_{\overrightarrow{\mathbf{b}} \in \overrightarrow{\mathfrak{C}}^{\mathbf{b}}} \gamma^{\overrightarrow{\mathbf{b}}} \chi_0^{\overrightarrow{\mathbf{b}}} = \sum_{\mathbf{b}' \in \mathfrak{S}^{\mathbf{b}}} \chi_0^{\mathbf{b}'} \sum_{\overrightarrow{\mathbf{b}} \in \overleftarrow{\mathfrak{C}}^{\mathbf{b}, \mathbf{b}'}} \gamma^{\overrightarrow{\mathbf{b}}} = \sum_{\mathbf{b}' \in \mathfrak{S}^{\mathbf{b}}} \chi_0^{\mathbf{b}'} w^{\mathbf{b}'} \tag{6.21}$$

with $w^{\mathbf{b}} \in [0, 1]$ as in (4.9). Since the supports of the basis functions have finite overlap (4.6), the right-hand side in (6.21) is bounded by constant which is independent of the levels ℓ and L.

Altogether, we have proved

$$\left\| E^{\mathbf{b}} \right\|_{\infty, Q^{\sigma} \times Q^{s}} \leq C \left(1 + \theta\right)^{-m_\ell} \left(\eta 2^{\ell}\right)^{g+1}.$$

The combination with (6.17) and Lemma 6.8 proves the assertion. ∎

The proof in the case of mixed derivatives $\partial_x \partial_y \left(G_{\text{far}}^{\mathbf{b}} - \widehat{G}_{\text{far}}^{\mathbf{b}}\right)$ can be derived in the same fashion as for the first order derivatives.

Lemma 6.11 *There exists $\theta > 0$ independent of ℓ and L such that, for any $\mathbf{b} \in P_{\text{far}}$ with $\ell := \text{level } \mathbf{b}$, there holds*

$$\left\| \partial_x \partial_y \left(G_{\text{far}}^{\mathbf{b}} - G_{\text{far}}^{\mathbf{b}}\right) \right\|_{\infty, \text{supp} \chi^{\mathbf{b}}} \leq C 2^{\ell(g+2)} \left(1 + \theta\right)^{-m_\ell}. \tag{6.22}$$

Remark 6.12 *Since* $\text{supp} \left(G_{\text{far}}^{\mathbf{b}} - \widehat{G}_{\text{far}}^{\mathbf{b}}\right) = \text{supp} \chi^{\mathbf{b}}$, *we may replace the norm* $\|\cdot\|_{\infty, \text{supp} \chi^{\mathbf{b}}}$ *in (6.14) and (6.19) by the norm* $\|\cdot\|_{\infty, \Gamma \times \Gamma}$.

6.2 Local analysis of the kernel approximation

We have presented alternative integral equation formulations for the single and double layer potential and the hypersingular equation. For the error analysis, we will impose certain assumptions on the smoothness of the surface.

Assumption 6.13 *For the*

$$\text{Single layer potential: } \Gamma \text{ is the surface of a Lipschitz polyhedron,} \tag{6.23a}$$

$$\text{Double layer potential: } \Gamma \text{ is the surface of a Lipschitz domain} \tag{6.23b}$$
$$\text{and is piecewise smooth,}$$

$$\text{Hypersingular equation: } \Gamma \text{ is globally smooth.} \tag{6.23c}$$

Under these assumptions, local error estimates can be derived for the approximation of the kernels k_1, k_2 and k_3 (cf. (4.14), (4.15)). For $j = 1, 2, 3$, let $k_j^{\mathbf{b}} := w^{\mathbf{b}} \chi^{\mathbf{b}} k_j$ and (cf. (6.9))

$$\widehat{k}_1^{\mathbf{b}}(x,y) := w^{\mathbf{b}} \tfrac{\partial}{\partial n_y} \widehat{G}_1^{\mathbf{b}}(x,y), \quad \widehat{k}_2^{\mathbf{b}}(x,y) := w^{\mathbf{b}} \langle n_y, y - x \rangle \tfrac{\partial}{\partial n_y} \widehat{G}_1^{\mathbf{b}}(x,y),$$
$$\widehat{k}_3^{\mathbf{b}}(x,y) := w^{\mathbf{b}} \Delta_{\Gamma,xy} \widehat{G}_2^{\mathbf{b}}(x,y).$$

The following Theorem is a direct consequence of (6.14), (6.19), and (6.22). Recall the definition of the operator $\overrightarrow{\mathrm{curl}}_{\Gamma,xy}$ as in (3.7).

Theorem 6.14 *Let (6.23b) be satisfied. There exist $\overline{\eta} > 0$ and positive constants $C_1 < \infty$ and $c_2 < 1$ such that, for all $0 < \eta < \overline{\eta}$ (cf. (4.8)), there holds,*

- *for all admissible blocks* $\mathbf{b} = (\sigma, s)$ *with* $\ell = \mathrm{level}\,(\mathbf{b})$ *and all* $(x, y) \in \Gamma \times \Gamma$

$$\left| k_2^{\mathbf{b}}(x,y) - \widehat{k}_2^{\mathbf{b}}(x,y) \right| + \left| \Delta_{\Gamma,xy} \left(G_2^{\mathbf{b}}(x,y) - \widehat{G}_2^{\mathbf{b}}(x,y) \right) \right| \leq \frac{C_1 c_2^{m_\ell}}{\mathrm{dist}\,(Q^\sigma, Q^s)},$$
$$\left| k_1^{\mathbf{b}}(x,y) - \widehat{k}_1^{\mathbf{b}}(x,y) \right| \leq \frac{C_1 c_2^{m_\ell}}{\mathrm{dist}^2\,(Q^\sigma, Q^s)}. \tag{6.24}$$

- *For all blocks* $\mathbf{b} \in P_{\mathrm{far}}$ *with* $\ell = \mathrm{level}\,(\mathbf{b})$ *and all* $(x, y) \in \Gamma \times \Gamma$, *we have*

$$\left| \overrightarrow{\mathrm{curl}}_{\Gamma,xy} \left(G_2^{\mathbf{b}}(x,y) - \widehat{G}_{\mathbf{2}}^{\mathbf{b}}(x,y) \right) \right| \leq C_1 c_2^{m_\ell}. \tag{6.25}$$

- *If, in addition, assumption (6.23c) holds, the estimate[1]*

$$\left| k_2^{\mathbf{b}}(x,y) - \widehat{k}_2^{\mathbf{b}}(x,y) \right| \leq C_1 c_2^{m_\ell} \tag{6.26}$$

is fulfilled for all admissible blocks $\mathbf{b}$ *with* $\ell = \mathrm{level}\,(\mathbf{b})$ *and all* $(x, y) \in \Gamma \times \Gamma$.

---

[1]Estimate (6.26) can be improved with a substantial increase of technicalities to an upper bound $C_1 c_2^{m_\ell} \mathrm{dist}\,(Q_\sigma, Q_s)$. However, it will turn out that (6.26) is sufficient for our applications.

In view of this lemma, we introduce the order of singularity g by

$$g := \begin{cases} 1 & \text{if } k = k_2 \text{ or } k = k_3, \\ 2 & \text{if } k = k_1, \\ 0 & \text{if } k = k_2 \text{ and } (6.23c) \text{ holds.} \end{cases} \tag{6.27}$$

6.3 Abstract assumption on the mesh, the cluster tree and the block covering

The following abstract assumptions concern the covering P. They are proved for shape regular and quasi-uniform meshes in [23].

Definition 6.15 *The uniformity of a mesh $\mathcal{G}$ is characterized by the smallest constant C_{uni} satisfying*

$$h \leq C_{\mathrm{uni}} h_\tau, \qquad \forall \tau \in \mathcal{G}$$

where h is as in (2.5) and

$$h_\tau := \operatorname{diam} \tau.$$

Definition 6.16 *The shape regularity of panels is characterized by the smallest constant C_q satisfying*

$$h_\tau^2 \leq C_q \left| \tau \right|, \qquad \forall \tau \in \mathcal{G}.$$

Remark 6.17 *Since $\mathcal{G}$ only contains finitely many panels, the constants C_{uni}, C_q are always bounded. However, it will turn out that the constants in the estimates below behaves critically with increasing values of C_q, C_{uni} and we assume here that C_q and C_{uni} are of moderate size.*

Assumption 6.18 *The constants α, β in (4.32) are chosen so that α in (4.32) satisfies $\alpha > 1$ and $2^{2-g} c_2^\alpha =: C_3 < 1/2$ holds with c_2 as in Theorem 6.14 and g as in (6.27).*

We need an assumption estimating, for $\sigma \in T_\ell$, the number of clusters s forming a block (σ, s) in P_{far}.

Assumption 6.19 *There exists a positive constant $C < \infty$ so that*

$$\max_{\sigma \in T} \sharp P_{\mathrm{far}} (\sigma) = C.$$

Note that the symmetry of the admissibility condition (cf. (4.8)) implies

$$\max_{\sigma \in T} \sharp \left\{ \tilde{\sigma} \in T : \sigma \in P_{\mathrm{far}} (\tilde{\sigma}) \right\} = C.$$

In order to estimate functions in the $H^1 (\Gamma)$-norm on blocks $(\sigma, s) \in P_{\mathrm{far}}$, we have to introduce neighbourhoods of $\sigma \cup s$ which are connected (cf. Figure 2). More precisely, we will approximate derivatives of functions on Γ by difference quotients with respect to the surface metric and express the geodetic distance (approximately) by triangle neighbourhoods.
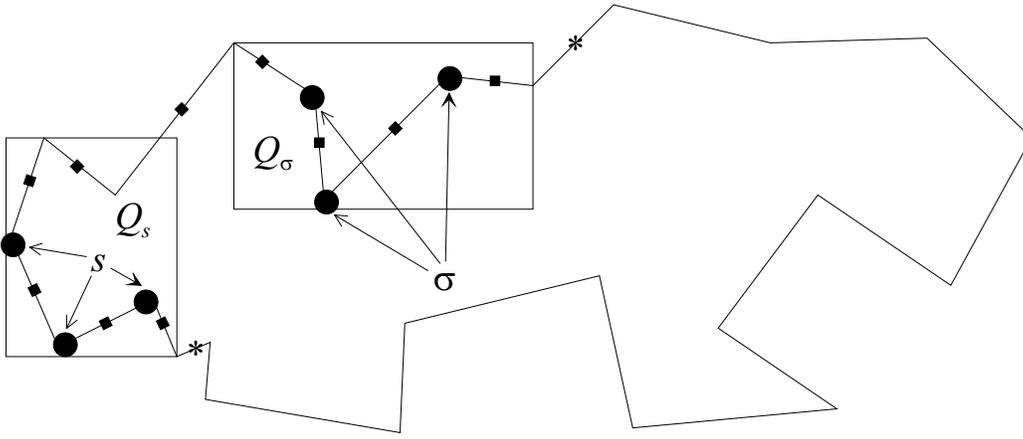
Figure 2: Illustration of the sets $\mathcal{G}(\mathbf{b})$ and $\mathcal{G}^\star(\mathbf{b})$ for a two-dimensional domain with polygonal boundary. The panels are the segments of the polygon. The panels in $\mathcal{G}(\mathbf{b})$ are marked by a black square while the *additional* panels in $\mathcal{G}^\star(\mathbf{b})$ are marked by an asterisk.

Notation 6.20 *A sequence $(\tau_i)_{i=0}^q$ of triangles in $\mathcal{G}$ is "edge-connected" if, for all $1 \le i \le q$, the triangles τ_{i-1} and τ_i share a common edge.*

For $\mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}$, let $\mathcal{G}(\mathbf{b}) \subset \mathcal{G}$ denote a smallest set having the properties

a. *all triangles $\tau \in \mathcal{G}$ with $|\tau \cap (Q^\sigma \cup Q^s)| > 0$ are contained in $\mathcal{G}(\mathbf{b})$,*

b. *for any two triangles $\tau, t \in \mathcal{G}(\mathbf{b})$, there exists an edge-connected sequence $(\tau_i)_{i=0}^q$ of triangles in $\mathcal{G}(\mathbf{b})$ with $\tau_0 = \tau$ and $\tau_q = t$.*

Notation 6.21 *The union of all elements in $\mathcal{G}(\mathbf{b})$ defines an edge-connected neighbourhood of $(Q^\sigma \cup Q^s) \cap \Gamma$*

$$\mathcal{U}(\mathbf{b}) := \bigcup_{\tau \in \mathcal{G}(\mathbf{b})} \overline{\tau}.$$

Furthermore, we need the extended sets $\mathcal{G}^\star(\mathbf{b})$ and $\mathcal{U}^\star(\mathbf{b})$ defined by

$$\mathcal{G}^\star(\mathbf{b}) := \{\tau \in \mathcal{G} : \overline{\tau} \cap \mathcal{U}(\mathbf{b}) \ne \emptyset\} \quad and \quad \mathcal{U}^\star(\mathbf{b}) := \bigcup_{\tau \in \mathcal{G}^\star(\mathbf{b})} \overline{\tau}.$$

Assumption 6.22 *For $\sigma \in T$, let*

$$\mathcal{U}^\star(\sigma) := \bigcup_{s \in P_{\mathrm{far}}(\sigma)} \mathcal{U}^\star(\sigma, s).$$

There is a positive constant $C < \infty$ such that for all $\sigma \in T_\ell$ the estimate

$$\sharp\{s \in T_\ell : \sigma \subset \mathcal{U}^\star(s)\} \le C$$

holds.

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Lemma 6.23 *Let Assumptions 6.5, 6.18, and 6.6 be satisfied. There exists a constant $C < \infty$ so that, for all $\mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}$ with $\ell := \mathrm{level}\,(\mathbf{b})$ there holds*

$$\left(\sqrt{|\Gamma_\sigma|\,|\Gamma_s|} + |\Gamma_s|\right) c_2^{m_\ell} \operatorname{dist}^{-g}(Q^\sigma, Q^s) \leq C c_2^\beta \eta^g h^{2-g} C_3^{L-\ell}, \qquad (6.28)$$

where g is as in (6.27) and β is as in (4.32).

Proof. Recall that $0 \leq c_2 < 1$ from Theorem 6.14. Let $\mathbf{b} = (\sigma, s) \in P_{\mathrm{far}}$. Without loss of generality we assume that

$$\operatorname{diam}\Gamma_\sigma = \max\{\operatorname{diam}\Gamma_\sigma, \operatorname{diam}\Gamma_s\}.$$

Hence,

$$\left(\sqrt{|\Gamma_\sigma|\,|\Gamma_s|} + |\Gamma_s|\right) \leq C\,(\operatorname{diam}\Gamma_\sigma)^2,$$

where C depends only on (the curvature of) the surface Γ. Using (4.8), Assumptions 6.5 and 6.6 we obtain

$$\left(\sqrt{|\Gamma_\sigma|\,|\Gamma_s|} + |\Gamma_s|\right) c_2^{m_\ell} \operatorname{dist}^{-g}(Q^\sigma, Q^s) \leq C\eta^g h^{2-g} 2^{(L-\ell)(2-g)} c_2^{m_\ell}$$

and, by employing Assumption 6.18,

$$\left(\sqrt{|\Gamma_\sigma|\,|\Gamma_s|} + |\Gamma_s|\right) c_2^{m_\ell} \operatorname{dist}^{-g}(Q^\sigma, Q^s) \leq C c_2^\beta \eta^g h^{2-g} \left(2^{2-g} c_2^\alpha\right)^{(L-\ell)} = C c_2^\beta \eta^g h^{2-g} C_3^{L-\ell}.$$

∎

6.4 Quasi-interpolants

For the error analysis of the panel-clustering approximation, we will need some results concerning the approximation of functions in $H^1(\Gamma)$. Let $P : H^1(\Gamma) \to \mathcal{S} := \mathcal{S}(0,0,\mathcal{G})$ denote the L^2-projection:

$$(Pu)\,|_\tau = \frac{1}{|\tau|} \int_\tau u \, ds \quad \text{for all } \tau \in \mathcal{G}. \qquad (6.29)$$

Lemma 6.24 *For $u \in H^1(\Gamma)$, let $U := Pu$ denote the L^2-projection as in (6.29). Then,*

$$\|u - U\|_{L^2(\Gamma)} \leq Ch\,\|u\|_{H^1(\Gamma)}. \qquad (6.30)$$

For all $\mathbf{b} \in P_{\mathrm{far}}$, there holds

$$\sup_{(x,y)\in\mathbf{b}} |U(y) - U(x)| \leq C\,|u|_{H^1(\mathcal{U}^\star(\mathbf{b}))}. \qquad (6.31)$$

Proof. Estimate (6.30) is a standard approximation result. We omit the proof and proceed directly with (6.31).

For $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$, let $\Gamma_{\mathbf{b}} := \Gamma_\sigma \times \Gamma_s = \operatorname{supp} \chi^\sigma \times \operatorname{supp} \chi^s$. Choose $\tau, t \in \mathcal{G}$ with $\tau, t \in \mathcal{G}(\mathbf{b})$ such that

$$\sup_{(x,y) \in \Gamma_{\mathbf{b}}} |U(y) - U(x)| = |U_\tau - U_t|,$$

where $U_K := U|_K$ for any $K \in \mathcal{G}$. Hence,

$$\sup_{(x,y) \in \Gamma_{\mathbf{b}}} |U(y) - U(x)| = \frac{1}{|\tau|} \int_\tau u \; ds - \frac{1}{|t|} \int_t u \; ds. \tag{6.32}$$

Let $P_{Cl} : H^1(\Gamma) \to \mathcal{S}(1, 1, \mathcal{G})$ denote the Clément interpolation operator (cf. [3]). For $u \in H^1(\Gamma)$ and $\widehat{U} = P_{Cl} u$, we obtain for the difference in the right-hand side of (6.32)

$$\frac{1}{|\tau|} \int_\tau u \; ds - \frac{1}{|t|} \int_t u \; ds = \frac{1}{|\tau|} \int_\tau \left(u - \widehat{U} \right) ds - \frac{1}{|t|} \int_t \left(u - \widehat{U} \right) ds$$

$$+ \frac{1}{|\tau|} \int_\tau \widehat{U} \; ds - \frac{1}{|t|} \int_t \widehat{U} \; ds. \tag{6.33}$$

The first difference in the right-hand side of (6.33) can be estimated by

$$\left| \frac{1}{|\tau|} \int_\tau \left(u - \widehat{U} \right) ds - \frac{1}{|t|} \int_t \left(u - \widehat{U} \right) ds \right| \leq C \sum_{K \in \{\tau, t\}} \left\| u - \widehat{U} \right\|_{L^2(K)} |K|^{-1/2} \tag{6.34}$$

$$\leq C |u|_{H^1(\mathcal{U}^\star(\mathbf{b}))}$$

and we proceed with the second difference in (6.33). Choose an edge-connected sequence $(\tau_i)_{i=0}^q$ with $\tau_0 = \tau$ and $\tau_q = t$ and a sequence of points $M_i \in \overline{\tau_i} \cap \overline{\tau_{i-1}}$. Then,

$$\frac{1}{|\tau|} \int_\tau \widehat{U} \; ds = \frac{1}{|\tau|} \int_\tau \left(\widehat{U}(M_1) + \left\langle \nabla \widehat{U}_\tau, y - M_1 \right\rangle \right) ds$$

$$= \widehat{U}(M_1) + \frac{1}{|\tau|} \int_\tau \left\langle \nabla \widehat{U}_\tau, y - M_1 \right\rangle ds$$

$$= \widehat{U}(M_q) + \sum_{i=1}^{q-1} \left\langle \nabla \widehat{U}_{\tau_i}, M_i - M_{i+1} \right\rangle + \frac{1}{|\tau|} \int_\tau \left\langle \nabla \widehat{U}_\tau, y - M_1 \right\rangle ds.$$

Consequently, by using well-established properties of the Clément interpolation we get for the second difference in the right-hand side of (6.33)

$$\left| \frac{1}{|\tau|} \int_\tau \widehat{U} \; ds - \frac{1}{|t|} \int_t \widehat{U} \; ds \right| \leq \sum_{i=0}^q \left\| \nabla \widehat{U}_{\tau_i} \right\| h_{\tau_i} \leq C \left| \widehat{U} \right|_{H^1(\mathcal{U}(\mathbf{b}))} \leq C |u|_{H^1(\mathcal{U}^\star(\mathbf{b}))}. \tag{6.35}$$

The combination of (6.32), (6.33), (6.34), and (6.35) finishes the proof. ∎

The next lemma concerns an inverse inequality for piecewise constant functions.

Lemma 6.25 *There exists a constant C depending only on the constants C_{uni} and C_q (cf. Definitions 6.15 and 6.16) such that*

$$\|U\|_{L^2(\Gamma)} \leq Ch^{-1/2}\|U\|_{H^{-1/2}(\Gamma)} \qquad \forall U \in \mathcal{S}(0,0,\mathcal{G}).$$

For a proof, we refer to [4, Theorem 4.7] and [9], where the assumptions on the mesh are even relaxed.

6.5 Single layer potential operator

We consider the discretisation of the single layer potential operator by piecewise constant boundary elements $\mathcal{S} := \mathcal{S}(0,0,\mathcal{G})$ on a quasi-uniform mesh $\mathcal{G}$. We assume that Γ is a polyhedral Lipschitz surface. If the solution u is in $H^1(\Gamma)$, the quasi-optimality of the Galerkin discretisation and the approximation property of $\mathcal{S}$ imply

$$\|u - U\|_{H^{-1/2}(\Gamma)} \leq Ch^{3/2}\|u\|_{H^1(\Gamma)}.$$

The effect of numerical integration in the nearfield part and the panel-clustering representation is studied in the framework of the first Strang Lemma.

Let $\tilde{a}_S : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ denote the bilinear form a_S as in (4.18) where the nearfield integrals (related to P_{near}) are replaced by numerical quadrature and the farfield integrals (related to P_{far}) by the panel-clustering approximation. The corresponding solution is denoted by $\tilde{U}$. The solution $\tilde{U}$ exists provided the *stability estimate*

$$|a_S(U,V) - \tilde{a}_S(U,V)| \leq C\|U\|_{H^{-1/2}(\Gamma)}\|V\|_{H^{-1/2}(\Gamma)} \tag{6.36}$$

holds for all $U, V \in \mathcal{S}$ with a sufficiently small constant $0 < C = O(1)$. The convergence is of optimal order provided the *consistency estimate*

$$\inf_{U \in \mathcal{S}}|a_S(U,V) - \tilde{a}_S(U,V)| \leq Ch^{3/2}\|u\|_{H^1(\Gamma)}\|V\|_{H^{-1/2}(\Gamma)} \tag{6.37}$$

is satisfied for all $V \in \mathcal{S}$ and all $u \in H^1(\Gamma)$. We discuss the effect of the panel-clustering approximation first. We assume that the bilinear form $(v, \rho u)$ in the definition of a_S (cf. 4.18) is treated without further numerical approximations.

Let $E_S := E_{P,S} + E_{Q,S} := a_S(U,V) - \tilde{a}_S(U,V)$ with (cf. (6.11))

$$E_{P,S} := E_{P,S}^I + E_{P,S}^{II} := \sum_{\mathbf{b} \in P_{\mathrm{far}}} \int_\Gamma \int_\Gamma \left\{ V(x)U(y)\left(k_3^{\mathbf{b}}(x,y) - \widehat{k}_3^{\mathbf{b}}(x,y)\right) \right.$$

$$\left. + V(x)(U(y) - U(x))\left(k_2^{\mathbf{b}}(x,y) - \widehat{k}_2^{\mathbf{b}}(x,y)\right) \right\} ds_y ds_x$$

and

$$E_{Q,S} := E_{Q,S}^I + E_{Q,S}^{II} := \sum_{\mathbf{b} \in P_{\mathrm{near}}} (I_{\Gamma \times \Gamma} - Q_{\Gamma \times \Gamma})\left((V \otimes U)k_3^{\mathbf{b}} + ((V \otimes U) - (VU) \otimes 1)k_2^{\mathbf{b}}\right),$$

where $I_{\Gamma \times \Gamma}$ denotes the integral over $\Gamma \times \Gamma$ and $Q_{\Gamma \times \Gamma}$ the quadrature approximation.

6.5.1 Panel-Clustering Approximation

We start with estimating the panel-clustering approximation error and begin with $E_{P,S}^{II}$. If ℓ and $\mathbf{b}$ appear in the same context, they are linked by $\ell = \text{level}(\mathbf{b})$.

We employ estimate (6.24) to derive

$$\left|E_{P,S}^{II}\right| \leq \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{far}}} C_1 c_2^m \, \text{dist}^{-1}\left(Q^\sigma, Q^s\right) \int_{\Gamma_\sigma \times \Gamma_s} \left|V(x)\right| \left|U(y) - U(x)\right| ds_x ds_y. \quad (6.38)$$

First, we establish the stability estimate in (6.36). The integral in (6.38) can be estimated by Hölder's inequality

$$\int_{\Gamma_\sigma \times \Gamma_s} \left|V(x)\right| \left|U(y) - U(x)\right| ds_x ds_y \leq \sqrt{|\Gamma_\sigma||\Gamma_s|} \, \|V\|_{L^2(\Gamma_\sigma)} \|U\|_{L^2(\Gamma_s)}$$

$$+ |\Gamma_s| \, \|V\|_{L^2(\Gamma_\sigma)} \|U\|_{L^2(\Gamma_\sigma)}$$

$$\leq \left(\sqrt{|\Gamma_\sigma||\Gamma_s|} + |\Gamma_s|\right) \|V\|_{L^2(\Gamma_\sigma)} \|U\|_{L^2(\Gamma_\sigma \cup \Gamma_s)}.$$

Estimate (6.28) leads to

$$\left|E_{P,S}^{II}\right| \leq C c_2^\beta \eta h \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{far}}} C_3^{L-\ell} \|V\|_{L^2(\Gamma_\sigma)} \|U\|_{L^2(\Gamma_\sigma \cup \Gamma_s)} \quad (6.39)$$

$$\leq C c_2^\beta \eta h \sum_{\ell=0}^{L} C_3^{L-\ell} \sum_{\sigma\in T_\ell} \|V\|_{L^2(\Gamma_\sigma)} \sum_{s\in P_{\text{far}}(\sigma)} \|U\|_{L^2(\Gamma_\sigma \cup \Gamma_s)}$$

$$\leq C c_2^\beta \eta h \sum_{\ell=0}^{L} C_3^{L-\ell} \sum_{\sigma\in T_\ell} \|V\|_{L^2(\Gamma_\sigma)} \|U\|_{L^2(\Gamma_\sigma \cup \mathcal{U}^\star(\sigma))}$$

$$\leq C c_2^\beta \eta h \sum_{\ell=0}^{L} C_3^{L-\ell} \|V\|_{L^2(\Gamma)} \sqrt{\sum_{\sigma\in T_\ell} \|U\|_{L^2(\Gamma_\sigma \cup \mathcal{U}^\star(\sigma))}^2}$$

$$\leq C c_2^\beta \eta h^{1/2} \|V\|_{H^{-1/2}(\Gamma)} \|U\|_{L^2(\Gamma)} \sum_{\ell=0}^{L} C_3^{L-\ell}$$

$$\leq \frac{C}{1-C_3} c_2^\beta \eta \|V\|_{H^{-1/2}(\Gamma)} \|U\|_{H^{-1/2}(\Gamma)}.$$

Hence, for sufficiently small $0 < \eta = O(1)$ or sufficiently large $0 < \beta = O(1)$ (cf. (4.32)), the stability requirements of the first Strang Lemma are satisfied for the term $E_{P,S}^{II}$.

For consistency, we proceed in (6.38) with $U = Pu$ (cf. (6.29)) and employ (6.28)

and Lemma 6.24 to obtain

$$
\left| E_{P,S}^{II} \right| \le C \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{far}}} \sqrt{|\Gamma_s|} \, \|V\|_{L^2(\sigma)} \, |u|_{H^1(\mathcal{U}^\star(\sigma))} \, \eta h C_3^{L-\ell}
$$

$$
\le C\eta h \sum_{\ell=0}^{L} C_3^{L-\ell} 2^{-\ell} \sum_{\sigma\in T_\ell} \|V\|_{L^2(\Gamma_\sigma)} \, |u|_{H^1(\mathcal{U}^\star(\sigma))} \, .
$$

By using Assumptions 6.22 and 6.5, we get

$$
\left| E_{P,S}^{II} \right| \le C\eta h^2 \sum_{\ell=0}^{L} C_3^{L-\ell} 2^{L-\ell} \sum_{\sigma\in T_\ell} \|V\|_{L^2(\Gamma_\sigma)} \, |u|_{H^1(\mathcal{U}^\star(\sigma))}
$$

$$
\le C\eta h^2 \sum_{\ell=0}^{L} (2C_3)^{L-\ell} \|V\|_{L^2(\Gamma)} \sqrt{\sum_{\sigma\in T_\ell} |u|_{H^1(\Gamma_\sigma)}^2 \sum_{s:\sigma\subset\mathcal{U}^\star(s)} 1}
$$

$$
\le \frac{C}{1-2C_3} \eta h^{3/2} \|V\|_{H^{-1/2}(\Gamma)} \, |u|_{H^1(\Gamma)} \tag{6.40}
$$

and this is the consistency estimate.

In the next step, we estimate the error $E_{P,S}^I$ and employ the representation

$$
E_{P,S}^I = \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{far}}} \int_{\Gamma_\sigma\times\Gamma_s} V(x)\,U(y)\,\left(\Delta_{xy}^\Gamma e_{\mathbf{b}}\right)(x,y)\,ds_y ds_x
$$

with

$$
e_{\mathbf{b}}(x,y) := \sum_{\vec{\mathbf{b}}\in\vec{\mathfrak{C}}^{\,\mathbf{b}}} \chi^{\vec{\mathbf{b}}} \left(G_2 - \mathcal{I}^{\vec{\mathbf{b}}} G_2\right).
$$

The stability estimate is completely analogous as for $E_{P,S}^{II}$ due to (6.24) and hence omitted. We proceed with the consistency.

Let $u \in H^1(\Gamma)$ and $U := Pu$ denote the L^2-projection of u onto $\mathcal{S}(0,0,\mathcal{G})$ (cf. (6.29)). Then,

$$
E_{P,S}^I = E_{P,S}^I(U,V) = E_{P,S}^I(U-u,V) + E_{P,S}^I(u,V). \tag{6.41}
$$

We begin with the estimate of the first term. Analogous estimates as for the proof of stability lead, in combination with (6.30), to

$$
\left| E_{P,S}^I(U-u,V) \right| \le Ch^{1/2} \|V\|_{H^{-1/2}(\Gamma)} \|u-U\|_{L^2(\Gamma)} \le Ch^{3/2} \|V\|_{H^{-1/2}(\Gamma)} \|u\|_{H^1(\Gamma)}
$$

We proceed with the estimate of the second term in (6.41). Since $u \in H^1(\Gamma)$, (each component of) the surface curl, $\overrightarrow{\text{curl}}_\Gamma u$, is in $L^2(\Gamma)$ and we may employ partial integration in Γ to the $\text{curl}_{\Gamma,y}$ operator in (3.7). Due to the smoothness and localness of the integrands the boundary term vanishes and we obtain the representation

$$
E_{P,S}^I(u,V) = \sum_{\mathbf{b}\in P_{\text{far}}} \int_{\Gamma\times\Gamma} V(x) \left\langle \overrightarrow{\text{curl}}_\Gamma u(y), \overrightarrow{\text{curl}}_{\Gamma,xy} e_{\mathbf{b}}(x,y) \right\rangle ds_y ds_x.
$$

The integral can be estimated, for $\mathbf{b} = (\sigma, s) \in P_{\text{far}}$, by

$$\int_{\Gamma_\sigma \times \Gamma_s} |V| \left| \overrightarrow{\text{curl}}_\Gamma u \right| ds_y ds_x \leq \int_{\Gamma_\sigma} |V| \, ds \int_{\Gamma_s} \left| \overrightarrow{\text{curl}}_\Gamma u \right| \, ds \leq \sqrt{|\Gamma_\sigma| |\Gamma_s|} \, \|V\|_{L^2(\Gamma_\sigma)} \, |u|_{H^1(\Gamma_s)} \, .$$

Combining these two estimates with Lemma 6.28 and arguing as in (6.39) results in

$$\left| E_{P,S}^I (u, V) \right| \leq C c_2^\beta h^2 \sum_{\mathbf{b} \in P_{\text{far}}} C_3^{L-\ell} \|V\|_{L^2(\Gamma_\sigma)} |u|_{H^1(\Gamma_s)}$$

$$\leq \frac{C}{1 - C_3} c_2^\beta h^{3/2} \|V\|_{H^{-1/2}(\Gamma)} |u|_{H^1(\Gamma)} \, .$$

6.5.2 Estimate of the Quadrature Error

We begin with studying the nearfield quadrature error $E_{Q,S}^{II}$. For the nearfield integrals we employ the coordinate transforms which are described in [7] to regularise the integrands and standard Gauß rules for their quadrature approximation with m point per space dimension. Since we are dealing in the Galerkin boundary element method with four-dimensional integrals the amount of work for approximating a matrix entry is proportionally to $O(m^4)$.

In [26], [15], [29], [27], error estimates have been developed for the quadrature approximation. It was shown that for a pair of panels $\tau \times t$ the quadrature error satisfies

$$\left| (I_{\tau \times t} - Q_{\tau \times t}^m)(W) \right| \leq C h^3 (1 + \delta)^{-2m} \|W\|_{L^\infty(\Gamma_\sigma \times \Gamma_s)}$$

holds for some $0 < \delta = O(1)$ and any piecewise polynomial W. By assuming this quadrature error estimate and the inverse inequalities

$$\|V\|_{L^\infty(\tau)} \leq C h^{-1} \|V\|_{L^2(\tau)}, \qquad \|U\|_{L^\infty(\tau \times t)} \leq C h^{-1} \|U\|_{L^2(\tau \times t)},$$

we derive, in a similar fashion as for the panel-clustering approximation, the stability estimate

$$\left| E_{Q,S}^{II} \right| \leq C h^3 (1 + \delta)^{-2m} \sum_{\mathbf{b} = (\sigma, s) \in P_{\text{near}}} \|V\|_{L^\infty(\Gamma_\sigma)} \|U \otimes 1\|_{L^\infty(\Gamma_\sigma \times \Gamma_s)}$$

$$\leq C h (1 + \delta)^{-2m} \sum_{\sigma \in T_L} \|V\|_{L^2(\Gamma_\sigma)} \|U\|_{L^2(\mathcal{U}^\star(\sigma))}$$

$$\leq C h (1 + \delta)^{-2m} \|V\|_{L^2(\Gamma)} \|U\|_{L^2(\Gamma)} \, .$$

In combination with the inverse inequality

$$\|V\|_{L^2(\Gamma)} \|U\|_{L^2(\Gamma)} \leq C h^{-1} \|V\|_{H^{-1/2}(\Gamma)} \|U\|_{H^{-1/2}(\Gamma)}$$

we obtain stability for sufficiently large $0 \leq m = O(1)$.

For the consistency estimates we put $U = Pu$ (cf. 6.29)) and employ Lemma 6.24 to argue as in (6.40) resulting in

$$
\begin{aligned}
\left|E_{Q,S}^{II}\right| &\leq Ch^3 \left(1+\delta\right)^{-2m} \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{near}}} \left(\|V\|_{L^\infty(\Gamma_\sigma)} \sup_{(x,y)\in\Gamma_\sigma\times\Gamma_s} |U(y) - U(x)| \right) \\
&\leq Ch^2 \left(1+\delta\right)^{-2m} \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{near}}} \left(\|V\|_{L^2(\Gamma_\sigma)} |u|_{H^1(\mathcal{U}^\star(\sigma))} \right) \\
&\leq Ch^2 \left(1+\delta\right)^{-2m} \sum_{\sigma\in T_L} \|V\|_{L^2(\Gamma_\sigma)} |u|_{H^1(\mathcal{U}^\star(\sigma))} \\
&\leq Ch^2 \left(1+\delta\right)^{-2m} \|V\|_{L^2(\Gamma)} |u|_{H^1(\Gamma)} \\
&\leq Ch^{3/2} \left(1+\delta\right)^{-2m} \|V\|_{H^{-1/2}(\Gamma)} |u|_{H^1(\Gamma)}
\end{aligned}
$$

and this is consistency for any $m \geq 0$.

The quadrature error for the bilinear form a_s^I can be analysed in a completely analogue way.

6.6   Hypersingular integral operator

The kernel functions arising in the hypersingular integral equation (4.17) are already analysed in the context of the single layer kernel. However, due to the smoothness assumption (6.23c) on the surface, we may employ (6.26) for the approximation quality of the kernel function $k^{(2)}$ to obtain by simply repeating all steps in the analysis of the single layer potential operator the required stability and consistency estimates. For all $U, V \in \mathcal{S}(1, 1, \mathcal{G})$, there holds

$$
|a_H(U,V) - \tilde{a}_H(U,V)| \leq C\rho_{m,\beta} \|U\|_{H^{1/2}(\Gamma)} \|V\|_{H^{1/2}(\Gamma)}
$$

with a function $\rho_{m,\beta}$ which depends on the local quadrature order m of the nearfield integrals and the minimal approximation order β of the panel clustering approximation and satisfies $\rho_{m,\beta} \to 0$ as $m, \beta \to \infty$. In other words, the constant $C\rho_{m,\beta}$ is sufficiently small provided $m, \beta = O(1)$ are chosen sufficiently large.

The consistency estimate is derived as for the single layer potential. For all $V \in \mathcal{S}(1, 1, \mathcal{G})$ and $u \in H^2(\Gamma)$, there holds

$$
|a_H(U,V) - \tilde{a}_H(U,V)| \leq Ch^{3/2} \|u\|_{H^2(\Gamma)} \|V\|_{H^{1/2}(\Gamma)} .
$$

Both estimates guarantee that the panel-clustering and quadrature approximation converges with optimal rate provided the undisturbed Galerkin method converges with optimal order $O\left(h^{3/2}\right)$.

6.7   Double layer potential operator

The panel-clustering error of the double-layer potential has the representation

$$E_{P,D} := \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{far}}} \int_{\Gamma_\sigma \times \Gamma_s} V(x)(U(y) - U(x)) e_{1,\mathbf{b}}(x,y)\, ds_y ds_x$$

with (cf. (6.11))

$$e_{1,\mathbf{b}}(x,y) := k_1^{\mathbf{b}}(x,y) - \widehat{k}_1^{\mathbf{b}}(x,y). \tag{6.42}$$

We employ estimate (6.24) to derive

$$|E_{P,D}| \leq \sum_{\mathbf{b}=(\sigma,s)\in P_{\text{far}}} C_1 c_2^{m_\ell} \operatorname{dist}^{-2}(Q^\sigma, Q^s) \int_{\Gamma_\sigma \times \Gamma_s} |V(x)|\,|U(y) - U(x)|\, ds_x ds_y.$$

The stability estimate (cf. 6.36)

$$|E_{P,D}| \leq \frac{C}{1-C_3} c_2^\beta \eta^2 \|V\|_{L^2(\Gamma)} \|U\|_{L^2(\Gamma)}$$

is derived in a completely analogous fashion as for the single layer potential. Hence, for sufficiently small $0 < \eta = O(1)$ or sufficiently large $0 < \beta = O(1)$, the stability requirements of the first Strang Lemma are satisfied for this term.

For consistency, we proceed again as for the single layer potential operator. Setting $U = Pu$ (cf. (6.29) leads to the consistency estimate

$$|E_{P,D}| \leq \frac{C}{1-2C_3} \eta h \|V\|_{L^2(\Gamma)} |u|_{H^1(\Gamma)}.$$

The quadrature error analysis is again simply a repetition of the arguments in Section 6.5.2 and shows that the choice of $m = O(1)$ to be sufficiently large leads to a consistent and stable perturbation.

7   Conclusions

In this paper, we have presented alternative representations of boundary integral operators corresponding to the Laplace problem in three dimensions. These representations give raise to low order quadrature formulae and panel-clustering approximations with variable order while preserving the optimal order of convergence. The key role for the error estimates of the single layer and hypersingular kernel plays a *globally* Lipschitz continuous approximation of a suitable antiderivative of the kernel function. This approximation is obtained by employing a hierarchy of partitions of unity as cut-off functions.

The complexity of the algorithms behaves as $O(n)$, where n denotes the number of unknowns *without* further logarithmic terms. The complexity estimates can be

carried out as in [23, Section 5] and will be presented, along with numerical results, in a forthcoming paper.

The panel clustering with variable order is based on "approximate" Lagrange polynomials and interpolation. The abstract assumptions on the local approximation error are proved under moderate assumptions on the tree structure and the mesh in [2]. For Taylor expansions and monomials (centred at cluster centred) these estimates are proved in [23, Section 5].

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