Efficient Numerical Solution of Neumann Problems on Complicated Domains

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July 15, 2005

Abstract

In this paper, we will consider elliptic PDEs with Neumann boundary conditions on complicated domains. The discretization is performed by composite finite elements.

The a-priori error analysis **typically** is based on the precise knowledge of the regularity of the solution. However, the constants in the regularity estimates, possibly, depend critically on the geometric details of the domain and the analysis of their quantitative influence is rather involved.

Here, we will consider a polygonal/-hedral Lipschitz domain Ω with, possibly, a huge number of geometric details ranging from size $O(\varepsilon)$ to $O(1)$. We assume that Ω is a perturbation of a simpler Lipschitz domain $\widehat{\Omega}$. We will prove error estimates where only the regularity of the PDE on $\widehat{\Omega}$ is needed along with some bounds of the norm of some extension operators which are explicit in appropriate geometric parameters.

Since Composite Finite Elements allow a multiscale discretization of problems on complicated domains, the arising linear system can be solved by a simple multigrid method. We will show that this multigrid method converges at optimal rate independent of the geometric structure of the problem.

1 Galerkin discretization of Neumann Problems on Complicated Domains by Composite Finite Elements

1.1 The Model Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with polygonal/ -hedral boundary Γ. We are interested in applications, where Ω has a rough boundary, i.e., the number of straight segments in the polygonal/ -hedral boundary, possibly, is huge. We assume that Ω can be regarded as a perturbation of a simpler polygonal/-hedral Lipschitz domain $\widehat{\Omega}$ in the sense that

$$\operatorname{area}\left(\Omega^{\text{diff}}\right) =: \varepsilon \quad \text{with} \quad \Omega^{\text{diff}} := \left(\widehat{\Omega} \backslash \Omega\right) \cup \left(\Omega \backslash \widehat{\Omega}\right). \tag{1}$$

is small (cf. Figure 1). Although the method does not require the existence of such a domain $\widehat{\Omega}$ we state the condition "area$\left(\Omega^{\text{diff}}\right)$ is small" already here, since the convergence analysis is strongly based on this fact.

For any bounded domain $D \subset \mathbb{R}^d$, we define the Sobolev space $H^s(D)$, $s \geq 0$, in the usual way (see, e.g., [12] or [17]). For $s \in \mathbb{R}$, we set

$$\mathcal{H}^s(D) := \begin{cases} H^s(D) & s \geq 0, \\ \left(H^{-s}(D)\right)' & s < 0. \end{cases}$$

If D equals Ω, we write $\mathcal{H}^s$ short for $\mathcal{H}^s(\Omega)$. The L^2-scalar product is denoted by $(\cdot,\cdot)_{L^2(D)}$ and identified with its continuous extension to the dual pairing $\langle\cdot,\cdot\rangle_{\mathcal{H}^s(D)\times\mathcal{H}^{-s}(D)}$.

Let the right-hand side $f \in L^2(\Omega)$ be given. Consider the problem: Find $u \in \mathcal{H}^1$ such that

$$a(u,v) := \int_\Omega \langle \nabla u, \nabla v\rangle + uv = \int_\Omega fv \qquad \forall v \in \mathcal{H}^1. \tag{2}$$

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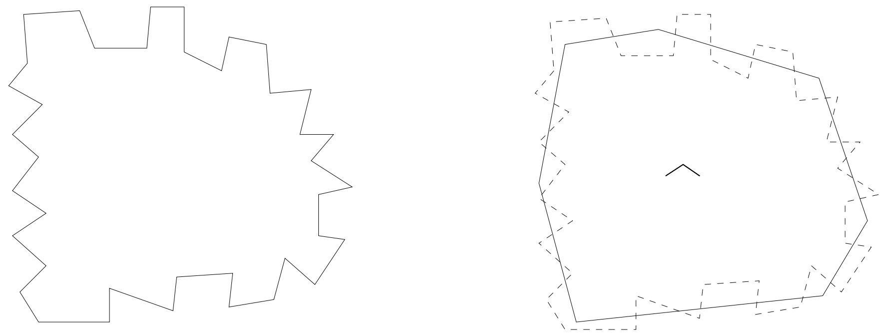


Figure 1: Physical domain Ω and simpler domain $\widehat{\Omega}$.

1.2 Composite Finite Element Discretization

Let $\mathcal{G} := \{\tau_i : 1 \leq i \leq N\}$ be a shape regular triangulation in the sense of Ciarlet of an overlapping domain $\Omega^\star$. The maximal mesh width is denoted by $h_\mathcal{G}$. In the case that $\varepsilon = O(h_\mathcal{G})$, the geometric structure can be resolved by a finite element mesh in a standard way. In this light, we assume that the measure for the size of the non-resolved geometric details satisfy

$$\varepsilon \leq C_{\mathrm{res}} h_\mathcal{G}^{1+\nu} \tag{3}$$

for some $\nu > 0$.

Let $S_\mathcal{G}^\star$ denote the standard finite element space on $\Omega^\star$

$$S_\mathcal{G}^\star := \left\{ u \in C^0(\Omega^\star) \mid \forall \tau \in \mathcal{G} : u|_\tau \in \mathbb{P}_1 \right\}.$$

The composite finite element space on the domain Ω is defined as the restriction

$$S_\mathcal{G} := S_\mathcal{G}^\star\big|_\Omega := \left\{ u|_\Omega : u \in S_\mathcal{G}^\star \right\}.$$

The Galerkin discretization of (2) via composite finite elements is given by: Find $u_\mathcal{G} \in S_\mathcal{G}$ such that

$$a(u_\mathcal{G}, v) = \int_\Omega f v \qquad \forall v \in S_\mathcal{G}. \tag{4}$$

As a basis for the space $S_\mathcal{G}$ we choose the restrictions $(\varphi_z|_\Omega)_{z \in \Theta_\mathcal{G}}$ of the standard nodal basis φ_z for the space $S_\mathcal{G}^\star$. Here and in the sequel, $\Theta_\mathcal{G}$ denotes the set of mesh points in $\mathcal{G}$. The basis representation of (4) leads to the linear system

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \tag{5}$$

where the system matrix $\mathbf{A} \in \mathbb{R}^{\Theta_\mathcal{G} \times \Theta_\mathcal{G}}$ and the right-hand side $\mathbf{f} \in \mathbb{R}^{\Theta_\mathcal{G}}$ is given by

$$\mathbf{A}_{x,y} = a\left(\varphi_y|_\Omega, \varphi_x|_\Omega\right) \quad \text{and} \quad \mathbf{f}_x := \int_\Omega f\, \varphi_x|_\Omega .$$

The solution of (5) is linked to the solution of (4) via

$$u_\mathcal{G} = \sum_{z \in \Theta_\mathcal{G}} \mathbf{u}_z \varphi_z.$$

The efficient assembling of $\mathbf{A}$ and $\mathbf{f}$ is explained in [13] and we do not discuss this aspect here.

1.3 Multigrid Methods for Neumann Problems on Complicated Domains

The basis representation of the Galerkin method leads to a system of linear equation of the form

$$\mathbf{A}\mathbf{u} = \mathbf{f}. \tag{6}$$

Typically, the dimension of $\mathbf{A}$ is huge and iterative solvers should be employed for its solution. Multigrid methods are among the fastest iterative solvers and we will formulate and analyze a multigrid method for the Neumann problem on complicated domains. For a detailed description of multigrid methods we refer to [11].

The efficiency of multigrid methods is based on a multi-scale discretization of the boundary value problem. It is a combination of an iterative solver (called smoother) on each discretization level and a recursive coarse grid correction. Formally, we introduce a parameter $\ell \in \mathbb{N}$ with $0 \leq \ell \leq L$ describing the discretization level. We start with the *given* fine grid equations (6) and rename them as

$$\mathbf{A}_L \mathbf{u}_L = \mathbf{f}_L,$$

where the number of levels L is not known a priori. Analogously, we rename the finite element space $S_{\mathcal{G}}$ as S_L and its basis as $\varphi_{L,x}$, $x \in \Theta_L$, where Θ_L is the set of all mesh points in $\mathcal{G}_L$.

1.4 Multigrid Algorithm

Let $(\mathcal{G}_\ell)_{\ell=0}^L$ be a sequence of finite element meshes which arise by applying recursively a standard refinement strategies to an initial mesh $\mathcal{G}_0$.

Notation: The domain covered by a finite element mesh $\mathcal{G}_\ell$ is denoted by Ω_ℓ. The set of mesh points in $\mathcal{G}_\ell$ is denoted by Θ_ℓ.

The precise requirements on the mesh sequence $(\mathcal{G}_\ell)_{\ell=0}^L$ are

1. Overlap property:

   (a)
   $$\Omega_0 \supseteq \Omega_1 \supseteq \ldots \supseteq \Omega_L \supseteq \Omega,$$

   (b) For all $0 \leq \ell \leq L$ and $\tau \in \mathcal{G}_\ell$:
   $$\operatorname{area}(\tau \cap \Omega) > 0$$

2. Nestedness: For all $0 \leq \ell \leq L-1$ and $\tau \in \mathcal{G}_\ell$, there exists a "set of sons" $\operatorname{sons}(\tau) \subset \mathcal{G}_{\ell+1}$ such that
   $$\bigcup_{t \in \operatorname{sons}(\tau)} t \subset \tau.$$

Let $\varphi_{\ell,x}$, $x \in \Theta_\ell$, denote the standard continuous, piecewise linear Lagrange basis on $\mathcal{G}_\ell$. For any grid function $\mathbf{u} \in \mathbb{R}^{\Theta_\ell}$, we associate a finite element function on the *overlapping domain* Ω_ℓ by

$$P_\ell[\mathbf{u}](x) := \sum_{z \in \Theta_\ell} \mathbf{u}(z) \varphi_{\ell,z}(x) \,. \tag{7}$$

From $\Theta_{\ell+1} \subset \overline{\Omega_\ell}$ we conclude that the function $P_\ell[\mathbf{u}]$ can be evaluated at the grid points $\Theta_{\ell+1}$ of the finer mesh. In this light, the inter-grid prolongation $\mathbf{p}_{\ell+1,\ell} : \mathbb{R}^{\Theta_\ell} \to \mathbb{R}^{\Theta_{\ell+1}}$ is defined by

$$\mathbf{p}_{\ell+1,\ell}[\mathbf{u}](x) := P_\ell[\mathbf{u}](x) \,, \qquad x \in \Theta_{\ell+1} \,,$$

and the matrix representation is

$$\mathbf{p}_{\ell+1,\ell} \in \mathbb{R}^{\Theta_{\ell+1} \times \Theta_\ell} : \;\; \mathbf{p}_{\ell+1,\ell}(x,y) = P_\ell[\varphi_{\ell,y}](x)$$

for all $x \in \Theta_{\ell+1}$ and $y \in \Theta_\ell$. The restriction is the transposed of $\mathbf{p}_{\ell+1,\ell}$, i.e.,

$$\mathbf{r}_{\ell,\ell+1} \in \mathbb{R}^{\Theta_\ell \times \Theta_{\ell+1}} : \;\; \mathbf{r}_{\ell,\ell+1}(x,y) = \mathbf{p}_{\ell+1,\ell}(y,x) \,.$$

Coarse grid operators $\mathbf{A}_\ell$ are recursively defined, for $\ell < L$, via the Galerkin product

$$\mathbf{A}_\ell := \mathbf{r}_{\ell,\ell+1} \mathbf{A}_{\ell+1} \mathbf{p}_{\ell+1,\ell}. \tag{8}$$

In order to define the multi-grid algorithm we have to specify a (classical) iterative solver on each single grid. We restrict here to linear solvers of the form

$$\mathbf{u}_\ell^{(i+1)} := \mathbf{u}_\ell^{(i)} - \mathbf{N}_\ell \left(\mathbf{A}_\ell \mathbf{u}_\ell^{(i)} - \mathbf{f}_\ell \right) . \tag{9}$$

The application of ν iterations of the form (9) defines a mapping $\mathbf{S}_\ell^{(\nu)}\left(\mathbf{u}_\ell^{(i)}, \mathbf{f}_\ell\right) := \mathbf{u}_\ell^{(i+\nu)}$.

The multi-grid algorithm is a recursive procedure which requires as input parameters ν_1, $\nu_2 \in \mathbb{N}$ specifying the number of pre- and postsmoothing steps and a parameter $\gamma \in \{1,2\}$ controlling whether a V- or a W-cycle is employed (for details we refer to [11]). The multi-grid algorithm is called by

$$\mathbf{u}_L := \mathbf{0}; \qquad \mathbf{mg}\left(\mathbf{u}_L, \mathbf{f}_L, L\right);$$

and defined by

procedure $\mathbf{mg}(\mathbf{u}_\ell, \mathbf{f}_\ell, \ell)$;
begin
  if $(\ell = 0)$ **then** $\mathbf{u}_\ell := \mathbf{A}_\ell^{-1}\mathbf{f}_\ell$ **else begin**
    $\mathbf{u}_\ell := \mathbf{S}_\ell^{(\nu_1)}\left(\mathbf{u}_\ell, \mathbf{f}_\ell\right)$;
    $\mathbf{d}_\ell := \mathbf{A}_\ell \mathbf{u}_\ell - \mathbf{f}_\ell$;
    $\mathbf{d}_{\ell-1} := \mathbf{r}_{\ell-1,\ell}\mathbf{d}_\ell$;
    $\mathbf{v}_{\ell-1} := \mathbf{0}$;
    for $j := 1$ **to** γ **do** $\mathbf{mg}(\mathbf{v}_{\ell-1}, \mathbf{d}_{\ell-1}, \ell-1)$;
    $\mathbf{u}_\ell := \mathbf{u}_\ell - \mathbf{p}_{\ell,\ell-1}\mathbf{v}_{\ell-1}$;
    $\mathbf{u}_\ell := \mathbf{S}_\ell^{(\nu_2)}\left(\mathbf{u}_\ell, \mathbf{f}_\ell\right)$;
  end;
end;

2 Convergence Analysis for the Galerkin Discretization

In this section, we will derive convergence estimates for the Galerkin solutions. Emphasis is taken on the explicit tracking of constants on parameters describing the geometry of the domain. For the space dimension, we assume in the sequel $d \in \{1,2,3\}$.

2.1 Analysis of Perturbations in the Domain

Since the quantitative regularity of (2) might be very complicated, we compare the solution with a related problem on the simpler domain $\widehat{\Omega}$ (cf. (1)). In this light, we extend the data f to $\widehat{\Omega}$ by zero and denote the resulting function again by f.

Let $\widehat{u} \in \mathcal{H}^1\left(\widehat{\Omega}\right)$ denote the unique solution of

$$a_0\left(\widehat{u}, v\right) := \int_{\widehat{\Omega}} \langle \nabla \widehat{u}, \nabla v\rangle + \widehat{u}v = \int_{\widehat{\Omega}} fv \qquad \forall v \in \mathcal{H}^1\left(\widehat{\Omega}\right). \tag{10}$$

First, we will investigate the error $e := u - \left.\widehat{u}\right|_\Omega$ in the $\mathcal{H}^1$-norm.

As prerequisite we will discuss the dependence of the norm of extension operators for Sobolev spaces on geometric parameters describing the domain.

2.1.1 Extension Operators

In this section, we will define extension operators $\mathfrak{E} : H^k(\Omega) \to H^k\left(\widehat{\Omega}\right)$, for any $k \in \mathbb{N}$ so that the supremum

$$\sup_{v \in H^k(\Omega)\backslash\{0\}} \|\mathfrak{E}v\|_{H^k(\widehat{\Omega})} / \|v\|_{H^k(\Omega)} =: C_{\text{ext}}^k < \infty.$$

is moderately bounded for a large class of domains, which may contain a huge number of geometric details. We will employ the extension operator which was developed in [21], [22] as a refinement of some extension operators in [20], [18], [16]. The proofs of the Theorems in this section can be found in [21], [22].

The construction consists of several steps.

1. The rough boundary (details of size ε) of the domain is simplified by extending to locally cuboid neighborhoods Q_i of the original domain. We assume that, after a few iterations, the extended domain contains only details of size $O(1)$ (cf. Figure 2).

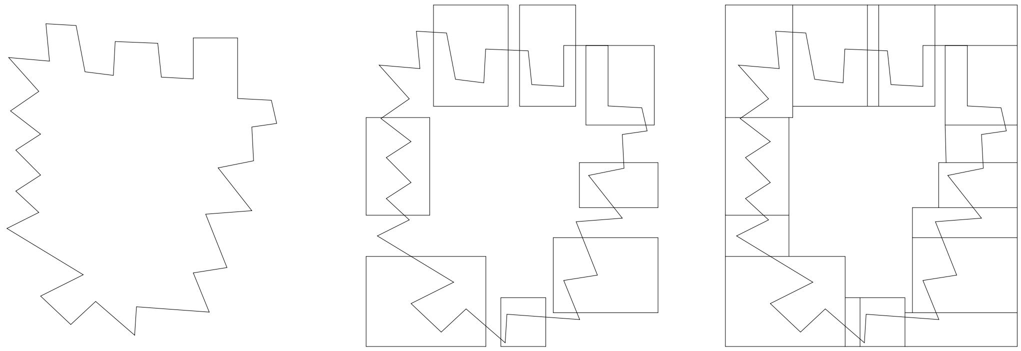


Figure 2: Complicated domain and locally cuboid neighborhoods. After two iterations the resulting extended domain is a rectangle.

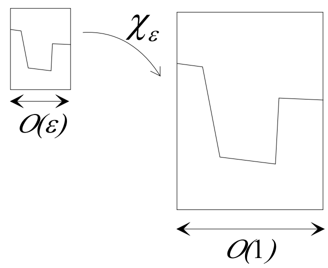


Figure 3: Scaling of local cuboid neighborhoods.

2. The extension operator is defined locally from the intersections $Q_i \cap \Omega$ to Q_i. The diameters of these cubes, should be of the same order as the size of the underlying details of the original domain (cf. Figure 3)

3. These $\varepsilon-$cubes Q_i along with their intersections $Q_i \cap \Omega$ are scaled to reference cubes $\widehat{Q_i}$ and scaled intersections $\widehat{Q_i \cap \Omega}$ of diameter $O(1)$.

We will prove that the norm of the extension operator mainly depend on the norm of the minimal extension operator for the reference cubes, i.e., $\mathfrak{E} : H^k\left(\widehat{Q_i \cap \Omega}\right) \to H^k\left(\widehat{Q_i}\right)$.

Definition 1 *For $\varepsilon > 0$, $N \in \mathbb{N}$ and $M > 0$, the domain $\Omega \subset \mathbb{R}^d$ is of class (ε, M, N) if there exists a family $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of subsets in $\mathbb{R}^d$ with*

1. *For all $x \in \partial\Omega$ there exists $i \in \mathbb{N}$ such that $B_\varepsilon(x) \subset U_i$,*

2. *For any $x \in \partial\Omega$, there holds*
$$\operatorname{card}\{U \in \mathcal{U} : x \in U\} \leq N.$$

3. *For any i, the intersection $U_i \cap \partial\Omega$ is locally the graph of a Lipschitz curve with Lipschitz constant smaller than or equal to M.*

Definition 2 *A bounded open set $D \subset \mathbb{R}^d$ has the* ***property*** *X (cf. Figure 4) if there is an axes-parallel cuboid $Q = \bigotimes_{i=1}^{d} (a_i, b_i) \subset \mathbb{R}^d$ such that set $\omega := \overline{Q} \backslash \overline{D}$ contains one full side of the cube Q, i.e., there is $1 \leq i_\star \leq d$ and $r \in \{a_{i_\star}, b_{i_\star}\}$ such that*

$$\bigotimes_{i=1}^{i_\star - 1} (a_i, b_i) \times \{r\} \times \bigotimes_{i=i_\star+1}^{d} (a_i, b_i) \subset \overline{\omega}.$$

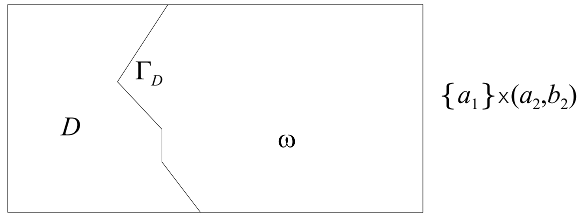


Figure 4: A bounded open set $D \subset \mathbb{R}^2$ with property X. The common boundary is the X-boundary Γ_D.

*The boundary $\Gamma_D := \overline{D} \cap \overline{\omega}$ is the **X**-boundary of D.*

*A pair (ω, U) of bounded domains with $\omega \subset U \subset \mathbb{R}^d$ is **admissible for extension** if*

1. *ω is a Lipschitz domain,*
2. *$\omega^c := U \backslash \overline{\omega}$ has property X,*
3. *$\Gamma_{\omega^c} \subset (\partial\omega) \cap (\partial\omega^c)$, where Γ_{ω^c} is the X-boundary of ω^c.*

Theorem 3 *Let $k \in \mathbb{N}$ and let $\omega \in \mathbb{R}^d$ be of class (ε, M, N). Let (ω, U) be admissible for extension. Then, there exists an extension operator $\mathfrak{E}_k : H^k(\omega) \to H^k(U)$ with bounded operator norm*

$$\|\mathfrak{E}_k f\|_{H^k(U)} \leq C_2 (1 + C_4) \|f\|_{H^k(U)} \quad \forall f \in H^k(\Omega)$$

which satisfies

$$|\mathfrak{E}_k f|_{H^k(U)} \leq C_3 (1 + C_4) |f|_{H^k(\omega)} \quad \forall f \in H^k(\Omega).$$

The constants C_2, C_3 only depends on ε, M, N and $\operatorname{diam} \omega^c$, where $\omega^c := U \backslash \overline{\omega}$. C_4 is the constant in the Poincaré inequality, i.e.

$$\forall f \in H^k(\omega^c): \quad \|f\|^2_{H^k(\omega^c)} \leq C_4 |f|^2_{H^k(\omega^c)} + C_4 \sum_{|\alpha|<k} \left| \int_\Omega D^\alpha f \right|^2.$$

The essential observation for the extension on complicated domains is that the constants in Theorem 3 remains unchanged if ω is scaled. In this light, let $\omega \subset \mathbb{R}^d$ be a subset with positive diameter $\operatorname{diam} \omega > 0$. We define the scaling operator $\chi_\omega : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\chi_\omega(x) := \frac{x}{\operatorname{diam} \omega} \quad \text{and} \quad \hat{\omega} := \{\chi_\omega(x) : x \in \omega\}.$$

Theorem 4 *Let $k \in \mathbb{N}$ and let $\omega \subset U \subset \mathbb{R}^d$. Assume that the normalized domain $\hat{\omega}$ is of class (ε, M, N) and that $\left(\hat{\omega}, \hat{U}\right)$ is admissible for extension. Let $\widehat{\mathfrak{E}}_k : H^k(\omega) \to H^k(U)$ be the extension operator as in Theorem 3. Then, the operator $\mathfrak{E}_k := H^k(\omega) \to H^k(U)$ defined by*

$$\mathfrak{E}_k f = \left(\widehat{\mathfrak{E}}_k \left(f \circ \chi_\omega^{-1}\right)\right) \circ \chi_\omega \tag{11}$$

is an extension operator with

$$\|\mathfrak{E}_k\|_{H^k(U) \leftarrow H^k(\omega)} \leq \widehat{C_2} \left(1 + \widehat{C_4}\right) \left(1 + \operatorname{diam}^k \omega\right),$$

where $\widehat{C_2}$, $\widehat{C_4}$ are the constants as in Theorem 3 for the operator $\widehat{\mathfrak{E}}_k$ on the normalized domains $\hat{\omega}, \hat{U}$.

Now, we will compose the global extension operator from Ω to an overlapping (simpler) domain $\Omega^\star$.

Definition 5 *Let Ω be a bounded domain. A finite family of disjoint axes parallel cubes $\mathcal{Q} = \{Q_i : 1 \leq i \leq q\}$ is **admissible for extension** for the domain Ω if, for all $1 \leq i \leq q$, the pairs $(Q_i \cap \Omega, Q_i)$ are admissible for extension. The extension operator $\mathfrak{E}_{k,i} : H^k(Q_i \cap \Omega) \to H^k(Q_i)$ is given as in (11) where ω is replaced by $Q_i \cap \Omega$ and U by Q_i. Let $\Omega_\mathcal{Q} := \Omega \cup \bigcup_{i=1}^q Q_i$ and let the extension operator $\mathfrak{E}_k : H^k(\Omega) \to H^k(\Omega_\mathcal{Q})$ be defined by*

$$(\mathfrak{E}_k f)(x) := \begin{cases} \mathfrak{E}_{k,i}\left(f|_{Q_i \cap \Omega}\right)(x) & x \in Q_i, \\ f(x) & \textit{otherwise.} \end{cases}$$

Theorem 6 *Let Ω be a bounded domain and let*

$$\mathcal{Q} = \{Q_i : 1 \leq i \leq q\} \tag{12}$$

be admissible for extension for Ω. The extension operator $\mathfrak{E}_k : H^k(\Omega) \to H^k(\Omega_Q)$ as in Definition 5 is bounded

$$\|\mathfrak{E}_k\|_{H^k(\Omega_{\mathcal{Q}}) \leftarrow H^k(\Omega)} \leq C_{\mathrm{ol}} \max_{1 \leq i \leq q} \|\mathfrak{E}_k\|_{H^k(Q_i) \leftarrow H^k(\Omega \cap Q_i)} =: C_{\mathcal{Q}},$$

where C_{ol} is the overlap constant

$$C_{\mathrm{ol}} := \sqrt{\sup_{x \in \Omega} \sharp \{i : x \in Q_i\}}. \tag{13}$$

Finally, we may allow finite iterations of the extension to families of cubes.

Definition 7 *Let Ω be a bounded domain and let $\overrightarrow{\mathcal{Q}} = (\mathcal{Q}_i : 1 \leq i \leq p)$ be a finite sequence of families of axes parallel cubes. Recursively, we put $\Omega_0 := \Omega$ and, for $1 \leq i \leq p$,*

$$\Omega_i = \Omega_{i-1} \cup \bigcup_{Q \in \mathcal{Q}_i} Q$$

If, for all $1 \leq i \leq p$, the family $\mathcal{Q}_i$ is admissible for extension for the domain Ω_{i-1}, we say that $\overrightarrow{\mathcal{Q}}$ is ***admissible for extension*** *from Ω to Ω_p. Let $\mathfrak{E}_{k,i} : H^k(\Omega_{i-1}) \to H^k(\Omega_i)$ be constructed as in Definition 5. Then $\mathfrak{E}_k : H^k(\Omega_0) \to H^k(\Omega_p)$ is the composition*

$$\mathfrak{E}_k = \mathfrak{E}_{k,p} \circ \mathfrak{E}_{k,p-1} \circ \ldots \circ \mathfrak{E}_{k,1}.$$

Theorem 8 *Let Ω be a bounded domain and let*

$$\overrightarrow{\mathcal{Q}} = \{\mathcal{Q}_i : 1 \leq i \leq p\} \tag{14}$$

be admissible for extension from Ω to Ω_p. The extension operator $\mathfrak{E}_k : H^k(\Omega) \to H^k(\Omega_p)$ as in Definition 5 is bounded by

$$\|\mathfrak{E}_k\|_{H^k(\Omega_{\mathcal{Q}}) \leftarrow H^k(\Omega)} \leq \prod_{i=1}^{p} C_{\mathcal{Q}_i}.$$

In summary, we have shown that the extension operator $\mathfrak{E}_k$ from a domain Ω to a domain Ω_p which is the iterated cuboid extension of Ω is *independent* of

1. the number q of geometric details (cf. (12)),
2. and of the size $\operatorname{diam} Q_i$ of the geometric details (cf. (12))

but depends on

1. the norm of the local extension operator $\widehat{\mathfrak{E}}_k : H^k\left(\widehat{Q_i \cap \Omega}\right) \to H^k\left(\widehat{Q_i}\right)$ on the *normalized* domains $\widehat{Q_i \cap \Omega}$ and $\widehat{Q_i}$
2. the overlap constant C_{ol} of the cuboids (cf. (13)),
3. the number p of iterations in the extension process (cf. (14)).

2.1.2 Bounds on the Perturbation Error

Let Ω be the given (Lipschitz) domain and $\widehat{\Omega}$ the simplified domain as explained in Section 1. To reduce technicalities, we assume that $\Omega \subset \widehat{\Omega}$.

Let $\mathfrak{E}_1 : H^1(\Omega) \to H^1\left(\widehat{\Omega}\right)$ be the minimal extension operator. We consider equation (10) and employ test functions of the form $\mathfrak{E}_1 v \in H^1\left(\widehat{\Omega}\right)$ for any $v \in H^1(\Omega)$. Subtracting this equation from (2) yields

$$a(e, v) - \int_{\widehat{\Omega} \backslash \Omega} \left(\langle \nabla \widehat{u}, \nabla \mathfrak{E}_1 v \rangle + \widehat{u} \mathfrak{E}_1 v\right) = 0$$

for all $v \in H^1(\Omega)$. Choosing $v = e$, we get

$$\|e\|_{H^1(\Omega)}^2 = -(\widehat{u}, \mathfrak{E}_1 e)_{H^1(\widehat{\Omega}\backslash\Omega)}.$$

Cauchy-Schwarz inequalities and the boundedness of $\mathfrak{E}_1$ lead to

$$\|e\|_{H^1(\Omega)}^2 \le C_{\text{ext}}^1 \|\widehat{u}\|_{H^1(\widehat{\Omega}\backslash\Omega)} \|e\|_{H^1(\Omega)}. \tag{15}$$

Hence, the perturbation error $\|e\|_{H^1(\Omega)}$ can be estimated by the norm of the solution $\widehat{u}$ in the small strip $\widehat{\Omega}\backslash\Omega$. This norm will be estimated in Lemma 10 under the weak assumption that

$$\widehat{\Omega}\backslash\Omega \subset S_\varepsilon,$$

where S_ε is a strip of width $O(\varepsilon)$ along the boundary of $\widehat{\Omega}$, i.e.,

$$S_\varepsilon := \{x \in \widehat{\Omega} : \operatorname{dist}(x, \partial\widehat{\Omega}) < c_w \varepsilon\}, \tag{16}$$

for some $c_w > 0$. Furthermore, we assume some minimal regularity for the homogenous Neumann problem.

Assumption 9 (H^λ-regularity) *There exists $\lambda \in \left]\frac{3}{2}, 2\right]$ and $C_1 > 0$ such that, for any $\mu \in [1, \lambda]$ and $f \in \mathcal{H}^{\mu-2}\left(\widehat{\Omega}\right)$, the solution $\widehat{u}$ to (10) satisfies*

$$\|\widehat{u}\|_{H^\mu(\widehat{\Omega})} \le C_1 \|f\|_{\mathcal{H}^{\mu-2}(\widehat{\Omega})}.$$

Lemma 10 *Let the Neumann problem on $\widehat{\Omega}$ be $H^\lambda\left(\widehat{\Omega}\right)$-regular for some $\lambda \in \left]\frac{3}{2}, 2\right]$ (cf. Assumption 9). Then, for any $\frac{3}{2} < \delta \le \mu \le \lambda$ and $f \in \mathcal{H}^{\mu-2}\left(\widehat{\Omega}\right)$, there holds*

$$\|\widehat{u}\|_{H^1(\widehat{\Omega}\backslash\Omega)} \le \tilde{C}\left(\sqrt{\varepsilon}\|f\|_{\mathcal{H}^{\delta-2}(\widehat{\Omega})} + \varepsilon^{\mu-1}\|f\|_{\mathcal{H}^{\mu-2}(\widehat{\Omega})}\right). \tag{17}$$

The constant $\tilde{C}$ depends continuously on C_1, δ, μ, λ, and c_w and, possibly, tends to infinity as $\delta, \mu, \lambda \to 3/2$.

Proof. Part I: By using a finite system of local charts and changes of variables, we can localize and rescale the estimate, so that it is sufficient to consider the case of a hypercube $\widetilde{\Omega} = (0,1)^d \cong B \times (0,1)$, where $B = (0,1)^{d-1}$ and the boundary is reduced to

$$\gamma := B \times \{0\}.$$

In this case, the above-mentioned transformation of the strip S_ε is contained in

$$\widetilde{S}_\varepsilon = B \times (0, C_{\mathrm{I}}\varepsilon),$$

for some $C_{\mathrm{I}} > 0$ which only depends on $\widehat{\Omega}$ and c_w (cf. (16)). The transformed function $\widehat{u}$ is denoted by $\widetilde{u}$.

First, we will show that there exists $C > 0$ such that, for all $1/2 < \kappa \le s \le 1$, there holds

$$\|v\|_{L^2(\widetilde{S}_\varepsilon)} \le C\left(\sqrt{\varepsilon}\|v\|_{H^\kappa(\widetilde{\Omega})} + \varepsilon^s\|v\|_{H^s(\widetilde{\Omega})}\right) \qquad \forall v \in H^s\left(\widetilde{\Omega}\right). \tag{18}$$

We will prove (18) by using finite element approximation theory on an auxiliary mesh. In order to avoid technicalities we assume that $(C_{\mathrm{I}}\varepsilon)^{-1} \in \mathbb{N}$. This allows us to define a conforming, uniform, simplicial mesh $\mathcal{G}_{\text{aux}}$, where all triangles are translations and rotations of the simplex

$$\left\{x \in (\mathbb{R}_{>0})^d : \sum_{i=1}^d x_i < C_{\mathrm{I}}\varepsilon\right\}.$$

Further, we may assume that there is a subset $\mathcal{G}_\varepsilon \subset \mathcal{G}_{\text{aux}}$ which defines a partitioning of $\widetilde{S}_\varepsilon \subset \widetilde{\Omega}$. The auxiliary finite element space S_{aux} is given by

$$S_{\text{aux}} := \left\{u \in C^0\left(\widetilde{\Omega}\right) \mid \forall\tau \in \mathcal{G}_{\text{aux}} : u|_\tau \in \mathbb{P}_1\right\}.$$

Now let $1/2 < s \leq 1$ and $v \in H^s\left(\widetilde{\Omega}\right)$. Then, it is well known (see [5, Section 4.8], [23], [24]) that there is a projection operator $P : H^s\left(\widetilde{\Omega}\right) \to S_{\text{aux}}$ such that

$$\|v - Pv\|_{H^r\left(\widetilde{S}_\varepsilon\right)} \leq \|v - Pv\|_{H^r\left(\widetilde{\Omega}\right)} \leq C_{\text{II}}\varepsilon^{t-r}\|v\|_{H^t\left(\widetilde{\Omega}\right)} \tag{19}$$

holds for all $0 \leq r \leq t \leq s$. The constant C_{II} depends only on s (since $\mathcal{G}_{\text{aux}}$ is a uniform grid, no mesh parameters enter the constant C_{II} in the approximation error estimates). We conclude that

$$\|v\|_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq \|v - Pv\|_{L^2\left(\widetilde{S}_\varepsilon\right)} + \|Pv\|_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq C_{\text{II}}\varepsilon^s\|v\|_{H^s\left(\widetilde{\Omega}\right)} + \|Pv\|_{L^2\left(\widetilde{S}_\varepsilon\right)}. \tag{20}$$

We will prove in Part III that, for any $1/2 < \kappa \leq s$, there holds

$$\|Pv\|_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq C_{\text{III}}\left(\sqrt{\varepsilon}\|Pv\|_{H^\kappa\left(\widetilde{\Omega}\right)} + \varepsilon^s\|Pv\|_{H^s\left(\widetilde{\Omega}\right)}\right), \tag{21}$$

where C_{III} only depends (continuously) on s, κ, and C_{II} and, possibly, deteriorates if $s \to 1/2$ or $\kappa \to 1/2$. From (19) we conclude for $k \in \{\kappa, s\}$

$$\|Pv\|_{H^k\left(\widetilde{\Omega}\right)} \leq \|Pv - v\|_{H^k\left(\widetilde{\Omega}\right)} + \|v\|_{H^k\left(\widetilde{\Omega}\right)} \leq (C_{\text{II}} + 1)\|v\|_{H^k\left(\widetilde{\Omega}\right)}. \tag{22}$$

The combination of (20) - (22) yields

$$\|v\|_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq C_{\text{IV}}\left(\sqrt{\varepsilon}\|v\|_{H^\kappa\left(\widetilde{\Omega}\right)} + \varepsilon^s\|v\|_{H^s\left(\widetilde{\Omega}\right)}\right), \tag{23}$$

where C_{IV} only depends on C_{II} and C_{III}.

Part II: Next, we will derive (17) from (23).

Applying estimate (23) to $\partial_i\widetilde{u} \in H^{\mu-1}\left(\widetilde{\Omega}\right)$, for all $i = 1, \cdots, d$, we obtain

$$\|\widetilde{u}\|_{H^1\left(\widetilde{S}_\varepsilon\right)} \leq C_{\text{V}}\left(\sqrt{\varepsilon}\|\widetilde{u}\|_{H^\delta\left(\widetilde{\Omega}\right)} + \varepsilon^{\mu-1}\|\widetilde{u}\|_{H^\mu\left(\widetilde{\Omega}\right)}\right),$$

for all $3/2 < \delta \leq \mu$ and C_{V} only depends on C_{IV}. The conclusion follows by the estimates

$$\|\widetilde{u}\|_{H^\delta(\widetilde{\Omega})} \leq C_1\|\widetilde{f}\|_{\mathcal{H}^{\delta-2}(\widetilde{\Omega})} \quad \text{and} \quad \|\widetilde{u}\|_{H^\mu(\widetilde{\Omega})} \leq C_1\|\widetilde{f}\|_{\mathcal{H}^{\mu-2}(\widetilde{\Omega})}.$$

Part III: In this part, we will establish (21).

Let $w \in S_{\text{aux}}$ and note that $w \in H^1\left(\widetilde{\Omega}\right)$. We employ the representation

$$w(y', y_d) = w(y', 0) + \int_0^{y_d}\frac{\partial w(y', t)}{\partial t}dt \qquad \forall y = (y', y_d) \in B \times (0, C_{\text{I}}\varepsilon).$$

Cauchy-Schwarz inequalities lead to

$$w^2(y', y_d) \leq 2w^2(y', 0) + 2C_{\text{I}}\varepsilon\int_0^{C_{\text{I}}\varepsilon}\left(\frac{\partial w(y', t)}{\partial t}\right)^2 dt.$$

Integrating over $\widetilde{S}_\varepsilon$ results in

$$\|w\|^2_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq 2\varepsilon\|w\|^2_{L^2(\gamma)} + 2(C_{\text{I}}\varepsilon)^2\|\nabla w\|^2_{L^2\left(\widetilde{S}_\varepsilon\right)}. \tag{24}$$

Since the trace operator $\gamma_0 : H^\kappa\left(\widetilde{\Omega}\right) \to L^2(\gamma)$ is continuous for any $\kappa > 1/2$, we obtain

$$\|w\|_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq C\left\{\sqrt{\varepsilon}\|w\|_{H^\kappa\left(\widetilde{\Omega}\right)} + \varepsilon\|\nabla w\|_{L^2\left(\widetilde{S}_\varepsilon\right)}\right\}. \tag{25}$$

where C only depends on C_{I} and κ and may deteriorate as $\kappa \to 1/2$. Finally, we employ an inverse inequality for the (uniform) mesh $\mathcal{G}_{\text{aux}}$ (cf. [5]) to obtain

$$\|\nabla w\|_{L^2\left(\widetilde{S}_\varepsilon\right)} \leq C\varepsilon^{s-1}\|w\|_{H^s\left(\widetilde{S}_\varepsilon\right)} \tag{26}$$

for any $s \in [0, 1]$. The combination of (25) and (26) yields the proof of Part III. ■

Theorem 11 *Let the assumptions in Lemma 10 be satisfied. Then, for any $\frac{3}{2} < \delta \leq \mu \leq \lambda$ and $f \in \mathcal{H}^{\mu-2}\left(\widehat{\Omega}\right)$, there holds*

$$\|e\|_{H^1(\Omega)} \leq C_0 \left(\sqrt{\varepsilon}\, \|f\|_{\mathcal{H}^{\delta-2}(\widehat{\Omega})} + \varepsilon^{\mu-1} \|f\|_{\mathcal{H}^{\mu-2}(\widehat{\Omega})}\right), \tag{27}$$

with $C_0 := C_{\mathrm{ext}}\tilde{C}$ and $\tilde{C}$ is as in Lemma 10.

Proof. Combine (15) with (17). ■

2.2 Error Analysis of the Galerkin Solution

The convergence analysis for composite finite element discretization is based on the quasi-optimality of the Galerkin discretization and transformed as usual to an estimate of the interpolation error. However, since the intersections $\tau \cap \Omega$ are neither shape regular nor affine equivalent to a reference element we cannot apply the standard interpolation estimates straightforwardly but have to employ an extension operator first.

First, we will introduce the constants which will appear in the error estimates.

Regularity on $\widehat{\Omega}$

We always assume that the Neumann problem on $\widehat{\Omega}$ is $H^\lambda\left(\widehat{\Omega}\right)$-regular for some $\lambda \in \left]\frac{3}{2}, 2\right]$ (cf. Assumption 9).

Bounds for the minimal extension operator

Recall that $\Omega^\star$ is the domain which is covered by the overlapping finite element mesh. Let $\mathfrak{E}_2 : H^2\left(\widehat{\Omega}\right) \to H^2\left(\Omega^\star\right)$ denote the minimal extension operator with norm

$$C_{\mathrm{ext}}^{\mathrm{I}} := \sup\left\{\|\mathfrak{E}_2 v\|_{H^2(\Omega^\star)} : v \in H^2\left(\widehat{\Omega}\right) \wedge \|v\|_{H^2(\widehat{\Omega})} = 1\right\} < \infty.$$

Note that Sobolev's embedding theorem implies, for $d = 1, 2, 3$,

$$\mathfrak{E}_2 : H^2\left(\widehat{\Omega}\right) \to C^0\left(\Omega^\star\right) \quad \text{and} \quad C_{\mathrm{ext}}^{\mathrm{II}} := \|\mathfrak{E}_2\|_{C^0(\Omega^\star) \leftarrow H^2(\widehat{\Omega})} < \infty.$$

We put $C_{\mathrm{ext}} := \max\left\{C_{\mathrm{ext}}^{\mathrm{I}}, C_{\mathrm{ext}}^{\mathrm{II}}\right\}$.

Bounds for the interpolation error

Let $\Omega^\star$ be the overlapping domain which is covered by the finite element mesh $\mathcal{G}$. Since the embedding $H^2\left(\Omega^\star\right) \hookrightarrow C^0\left(\Omega^\star\right)$ is continuous, the nodal interpolation $I_{\mathcal{G}} : H^2\left(\Omega^\star\right) \to S_{\mathcal{G}}^\star$ is well defined. It is well known that, for $m = 0, 1$, the constants $C_{\mathrm{apx},m}$, only depend on the minimal angles in the mesh $\mathcal{G}$

$$C_{\mathrm{apx},m} := h_{\mathcal{G}}^{m-2} \sup_{\substack{v \in H^2(\Omega^\star) \\ \|v\|_{H^2(\Omega^\star)} = 1}} \|v - I_{\mathcal{G}} v\|_{H^m(\Omega^\star)}.$$

We put $C_{\mathrm{apx}} := \max\left\{C_{\mathrm{apx},0}, C_{\mathrm{apx},1}\right\}$.

Theorem 12 *Let Assumption 9 be satisfied. Let $f \in \mathcal{H}^{\lambda-2}\left(\widehat{\Omega}\right)$. Then, for any $3/2 < \delta \leq \lambda$, the Galerkin solution $u_{\mathcal{G}}$ (cf. (4)) satisfies the error estimate*

$$\|u - u_{\mathcal{G}}\|_{H^1(\Omega)} \leq C_0 \sqrt{\varepsilon}\, \|f\|_{\mathcal{H}^{\delta-2}(\Omega)} + \left(C_0 \varepsilon^{\lambda-1} + C_5 h_{\mathcal{G}}^{\lambda-1}\right) \|f\|_{\mathcal{H}^{\lambda-2}(\Omega)}. \tag{28}$$

The constant C_5 only depends on C_1, C_{ext} and C_{apx}.

Proof. The continuity and ellipticity constants for the Neumann problem on Ω are 1, i.e.,

$$\begin{aligned} |a(u,v)| &\leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u, v \in H^1(\Omega), \\ a(u,u) &= \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

Let $\widehat{u}$ denote the solution of the extended problem (10). Hence, the quasi-optimality of the Galerkin discretization and Theorem 11 yield

$$\begin{aligned} \|u - u_{\mathcal{G}}\|_{H^1(\Omega)} &= \inf_{v \in S_{\mathcal{G}}} \|u - v\|_{H^1(\Omega)} \leq \|u - \widehat{u}\|_{H^1(\Omega)} + \inf_{v \in S_{\mathcal{G}}} \|\widehat{u} - v\|_{H^1(\Omega)} \\ &\leq C_0 \left(\sqrt{\varepsilon}\, \|f\|_{\mathcal{H}^{\delta-2}(\widehat{\Omega})} + \varepsilon^{\lambda-1} \|f\|_{\mathcal{H}^{\lambda-2}(\widehat{\Omega})}\right) + \inf_{v \in S_{\mathcal{G}}} \|\widehat{u} - v\|_{H^1(\Omega)}. \end{aligned}$$

It remains to estimate the infimum in the estimate above. First, we assume that the Neumann problem (10) on $\widehat{\Omega}$ is H^2-regular.

The infimum can be estimated by introducing

$$v_{\text{int}} = (I_{\mathcal{G}} \mathfrak{E}_2 \widehat{u})|_{\Omega}.$$

This leads to the estimate

$$\begin{aligned}\inf_{v \in S_{\mathcal{G}}} \|\widehat{u} - v\|_{H^1(\Omega)} &\leq \|\widehat{u} - v_{\text{int}}\|_{H^1(\Omega)} \leq \|\mathfrak{E}_2 \widehat{u} - I_{\mathcal{G}} \mathfrak{E}_2 \widehat{u}\|_{H^1(\Omega^\star)} \\ &\leq C_{\text{apx}} h_{\mathcal{G}} \|\mathfrak{E}_2 \widehat{u}\|_{H^2(\Omega^\star)} \leq C_{\text{ext}} C_{\text{apx}} h_{\mathcal{G}} \|\widehat{u}\|_{H^2(\Omega^\star)} \\ &\leq C_{\text{ext}} C_{\text{apx}} C_1 h_{\mathcal{G}} \|f\|_{L^2(\widehat{\Omega})}.\end{aligned}$$

The result for intermediate Sobolev spaces $H^\lambda(\Omega)$, $\lambda \in \left]\frac{3}{2}, 2\right]$, follows by interpolation applied to the operator $\mathcal{L}u := u - \mathcal{P}(u)$, where $\mathcal{P}(u)$ is the H^1-orthogonal projection of u onto $S_{\mathcal{G}}$. ■

Corollary 13 *Let the assumptions of Theorem 12 be satisfied. Assume that (3) holds. Then, for any* $3/2 < \delta \leq \lambda$, *the Galerkin solution* $u_{\mathcal{G}}$ *(cf. (4)) satisfies the error estimate*

$$\|u - u_{\mathcal{G}}\|_{H^1(\Omega)} \leq C_0 \sqrt{\varepsilon} \|f\|_{\mathcal{H}^{\delta-2}(\Omega)} + C_6 h_{\mathcal{G}}^{\lambda-1} \|f\|_{\mathcal{H}^{\lambda-2}(\Omega)}.$$

The constant C_6 *only depends on* C_5 *and* C_{res} *(cf. (3)).*

The Aubin-Nitsche duality argument allows to obtain error estimates with respect to weaker norms. In this light, we define the function $v_\varphi \in H^1(\Omega)$ as the unique solution of

$$a(v_\varphi, w) = (\varphi, w)_{L^2(\Omega)} \quad \forall w \in H^1(\Omega).$$

for given $\varphi \in \mathcal{H}^{-1}$.

Corollary 14 *Let the assumptions of Theorem 12 be satisfied. Assume that (3) holds. Let* $3/2 < \delta \leq \mu \leq \lambda$ *and* $\frac{3}{2} < \delta \leq s \leq \lambda$. *Then, the Galerkin solution* $u_{\mathcal{G}}$ *(cf. (4)) satisfies the error estimate*

$$\|u - u_{\mathcal{G}}\|_{H^{2-\mu}(\Omega)} \leq \left(C_0 \sqrt{\varepsilon} + C_6 h_{\mathcal{G}}^{\mu-1}\right) \left(C_0 \sqrt{\varepsilon} \|f\|_{\mathcal{H}^{\delta-2}} + C_6 h_{\mathcal{G}}^{s-1} \|f\|_{\mathcal{H}^{s-2}}\right).$$

Proof. Let $e = u - u_{\mathcal{G}}$. By duality we have for $3/2 < \mu \leq \lambda$:

$$\begin{aligned}\|e\|_{H^{2-\mu}(\Omega)} &= \sup_{\varphi \in \mathcal{H}^{-2+\mu} \backslash \{0\}} \frac{(e, \varphi)_{L^2(\Omega)}}{\|\varphi\|_{\mathcal{H}^{-2+\mu}}} = \sup_{\varphi \in \mathcal{H}^{-2+\mu} \backslash \{0\}} \frac{a(e, v_\varphi)}{\|\varphi\|_{\mathcal{H}^{-2+\mu}}} \\ &= \sup_{\varphi \in \mathcal{H}^{-2+\mu} \backslash \{0\}} \inf_{\tilde{v}_\varphi \in S_{\mathcal{G}}} \frac{a(e, v_\varphi - \tilde{v}_\varphi)}{\|\varphi\|_{\mathcal{H}^{-2+\mu}}} \leq \|e\|_{H^1(\Omega)} \sup_{\varphi \in H^{-2+\mu} \backslash \{0\}} \inf_{\tilde{v}_\varphi \in S_{\mathcal{G}}} \frac{\|v_\varphi - \tilde{v}_\varphi\|_{H^1(\Omega)}}{\|\varphi\|_{\mathcal{H}^{-2+\mu}}}.\end{aligned}$$

By choosing $\tilde{v}_\varphi$ as the Galerkin approximation of v_φ, we may apply Corollary 13 twice to obtain

$$\|e\|_{H^{2-\mu}(\Omega)} \leq \left(C_0 \sqrt{\varepsilon} + C_6 h_{\mathcal{G}}^{\mu-1}\right) \left(C_0 \sqrt{\varepsilon} \|f\|_{\mathcal{H}^{\delta-2}} + C_6 h_{\mathcal{G}}^{s-1} \|f\|_{\mathcal{H}^{s-2}}\right).$$

■

Corollary 15 *Let Assumption 9 be satisfied. Let* $f \in \mathcal{H}^{-1}\left(\widehat{\Omega}\right)$. *Assume that (3) holds. Let* $1 \leq \mu \leq \lambda$. *Then, the Galerkin solution* $u_{\mathcal{G}}$ *(cf. (4)) satisfies the error estimate*

$$\|u - u_{\mathcal{G}}\|_{H^{2-\mu}(\Omega)} \leq 2 \left(C_0 \sqrt{\varepsilon} + C_6 h_{\mathcal{G}}^{\mu-1}\right) \|f\|_{\mathcal{H}^{-1}(\widehat{\Omega})}.$$

The constant C *only depends on* C_0, C_1, C_{ext}, *and* C_{apx}.

Proof. Let $e = u - u_{\mathcal{G}}$. As in the proof of Corollary 14 one shows

$$\|e\|_{H^{2-\mu}(\Omega)} \leq \left(C_0 \sqrt{\varepsilon} + C_6 h_{\mathcal{G}}^{\mu-1}\right) \|e\|_{H^1(\Omega)}.$$

The ellipticity of the bilinear form $a(\cdot, \cdot)$ implies

$$\|e\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} + \|u_{\mathcal{G}}\|_{H^1(\Omega)} \leq 2 \|f\|_{\mathcal{H}^{-1}(\widehat{\Omega})}.$$

■

3 Multigrid Convergence

In this section, we will investigate the convergence of the multi-grid method following the general multi-grid convergence theory in [11]. However, the proofs require an approximation property for finite element spaces which might depend on the geometric details in a complicated way.

Here, we will prove the approximation property by combining the perturbation estimates with regularity estimates on the simplified domain $\widehat{\Omega}$. Since our focus in this paper is more on the approximation property of composite finite elements and less on the choice of an optimal smoother we restrict here to a damped Jacobi-type method as the smoothing iteration. For the system of linear equations $\mathbf{A}_\ell \mathbf{u}_\ell = \mathbf{f}_\ell$, it is given by

$$\mathbf{u}_\ell^{(i+1)} = \mathbf{u}_\ell^{(i)} - \mathbf{N}_\ell^{-1} \left(\mathbf{A}_\ell \mathbf{u}_\ell^{(i)} - \mathbf{f}_\ell \right), \tag{29}$$

where

$$\mathbf{N}_\ell^{-1} := \omega h_\ell^2 \mathbf{M}_\ell^{-1} \tag{30}$$

and $\mathbf{M}_\ell$ denotes the mass matrix:

$$(\mathbf{M}_\ell)_{x,y} := \left(\varphi_{x,\ell}|_\Omega, \varphi_{y,\ell}|_\Omega \right)_{L^2(\Omega)} \qquad \forall x, y \in \Theta_\ell.$$

The parameter $\omega > 0$ is a suitable damping parameter. We will prove the convergence in the framework of *geometric* multigrid methods (cf. [11]).

Remark 16 *We have chosen $\mathbf{N}_\ell$ as in (30) in order to simplify the analysis of the smoothing property as much as possible and to focus on the approximation property. For the practical realization one has to solve in each iteration step a linear system of the form*

$$\mathbf{M}_\ell \mathbf{x} = \mathbf{y} \tag{31}$$

Some aspects are discussed below:

1. *For the numerical experiments, we have always replaced $\mathbf{N}_\ell$ by the diagonal part of $\mathbf{A}_\ell$, i.e., $\mathbf{N}_\ell := \omega \operatorname{diag}\left[(\mathbf{A}_\ell)_{x,x} : x \in \Theta_\ell \right]$ and obtained convergence rates which are independent of the geometric details in the domain (see Section 4).*

2. *In standard cases, the condition number of the mass matrix is of order 1 and the solution of (31) requires only a small number of iteration steps which, in particular, is independent of $\dim \mathbf{M}_\ell$. This can be proved as long as the areas of the intersections $(\operatorname{supp} \varphi_{x,\ell}) \cap \Omega$ are of order h_ℓ^d.*

3. *The case that $(\operatorname{supp} \varphi_{x,\ell}) \cap \Omega$ is degenerate (i.e., much smaller than h_ℓ^d) for some $x \in \Theta_\ell$, is analyzed in [25] for some model problem and it was shown that the multigrid convergence is not affected by such scaling effects.*

The numerical solution of boundary value problems on complicated domains is a topic of vivid research. Our approach differs from techniques such as AMG ([19], [15], [4], [26]), agglomeration methods ([1], [3], [6], [2], [9], [7]), subspace correction methods ([14], [27], [28]) since the construction is based on the coarse scale discretization of the boundary value problem where the asymptotic convergence order is preserved on coarser grids. Hence, it can be used *not only* for constructing a spectral equivalent preconditioner for the fine scale equations but also for a low dimensional discretization of the PDE for a given prescribed (moderate) accuracy.

3.1 Smoothing and Approximation Property

We start with considering the two grid method. The iteration can be written as an affine map in the form $\mathbf{u}_L^{(i+1)} = \mathbf{K}_L^{TGM} \mathbf{u}_L^{(i)} + \mathbf{R}_L^{TGM} \mathbf{f}_L$ with the two-grid iteration matrix

$$\mathbf{K}_L^{TGM} := \mathbf{K}_L^{\nu_2} \left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1} \mathbf{A}_{L-1}^{-1} \mathbf{r}_{L-1,L} \right) \mathbf{A}_L \mathbf{K}_L^{\nu_1}$$

and the iteration matrix $\mathbf{K}_L := \mathbf{I}_L - \mathbf{N}_L^{-1} \mathbf{A}_L$ of the linear solver (9). For the Jacobi-type smoother (cf. (29)) we have

$$\mathbf{K}_L = \mathbf{I}_L - \omega h_L^2 \mathbf{M}_L^{-1} \mathbf{A}_L.$$

The iteration converges if and only if the spectral radius $\rho\left(\mathbf{K}_L^{TGM}\right)$ is smaller than one. The convergence proof is based on a multiplicative splitting of $\mathbf{K}_L^{TGM}$ and an estimate of the factors in appropriate norms. In this light, we introduce, for $\alpha \in [-1,1]$, a scale of norms $\|\cdot\|_{\alpha,\ell} : \mathbb{R}^{\Theta_\ell} \to \mathbb{R}$. Let

$$L_\ell : S_\ell \to S_\ell$$

be defined by

$$(L_\ell u, v)_{L^2(\Omega)} = a\,(u,v) \qquad \forall u, v \in S_\ell.$$

Remark 17 *The operator L_ℓ can be expressed by means of the system matrix $\mathbf{A}_\ell$, the mass matrix $\mathbf{M}_\ell$ and the prolongation P_ℓ as*

$$L_\ell = P_\ell \mathbf{M}_\ell^{-1} \mathbf{A}_\ell P_\ell^{-1}.$$

It is easy to see that L_ℓ is self-adjoint with respect to the $L^2(\Omega)$-scalar product and satisfies

$$(L_\ell u, u)_{L^2(\Omega)} = \|u\|_{H^1(\Omega)}^2 \qquad \forall u \in S_\ell.$$

Hence, powers of L_ℓ are well-defined for any real $\alpha \in \mathbb{R}$.

The operator L_ℓ allows to define a scale of norms on S_ℓ. For $\alpha \in [-1,1]$, we set

$$(u,v)_{\alpha,\ell} := (L_\ell^\alpha u, v)_{L^2(\Omega)} \quad \text{and} \quad \|u\|_{\alpha,\ell} := (u,u)_{\alpha,\ell}^{1/2}.$$

Remark 18 *Note that for $\alpha = 0, 1$ and $u \in S_\ell$, it holds $\|u\|_{0,\ell} = \|u\|_{L^2(\Omega)}$ and $\|u\|_{1,\ell} = \|u\|_{H^1(\Omega)}$.*

The discrete counterpart of the scalar product $(\cdot,\cdot)_{\alpha,\ell}$ and norm $\|\cdot\|_{\alpha,\ell}$ are given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\alpha,\ell} := (P_\ell \mathbf{u}, P_\ell \mathbf{v})_{\alpha,\ell} \quad \text{and} \quad |||\mathbf{u}|||_{\alpha,\ell} := \langle \mathbf{u}, \mathbf{u} \rangle_{\alpha,\ell}^{1/2}.$$

Throughout this section we assume that the Neumann problem on $\widehat{\Omega}$ is H^λ-regular for some $\lambda \in \,]3/2, 2]$. We will establish the multigrid convergence with respect to the $|||\cdot|||_{s-2,L}$-norm, where

$$s := \min\left\{\lambda, \frac{3}{2} + \frac{\nu}{2}\right\}. \tag{32}$$

The proof of the following lemma requires an inverse assumption. Since we assume that the difference Ω^{diff} (cf. (1)) has small measure $\varepsilon < C_{\text{res}} h_L^{1+\nu}$ (cf. (3)), the constant $C_{\text{inv}} > 0$ in

$$C_{\text{inv}} := h_\ell \sup_{u \in S_\ell \setminus \{0\}} \frac{\|u\|_{H^1(\Omega)}}{\|u\|_{L^2(\Omega)}} \tag{33}$$

should have moderate size (cf. [8, Corollary 1]).

Lemma 19 *Let the assumptions of Corollary 15 be satisfied. Let (3) be satisfied for some $\nu > 0$ and let C_{inv} be bounded independent of the refinement level ℓ. Then,*

$$\sup_{\mathbf{f} \in \mathbb{R}^{\Theta_\ell} \setminus \{0\}} \frac{\|P_\ell \mathbf{f}\|_{\mathcal{H}^{s-2}}}{|||\mathbf{f}|||_{s-2,\ell}} \le C_{\mathrm{P}}. \tag{34}$$

Proof. Let $R_\ell : S_\ell \to \mathbb{R}^{\Theta_\ell}$ be the adjoint to P_ℓ

$$\langle R_\ell u, \mathbf{v} \rangle_{0,\ell} = (u, P_\ell \mathbf{v})_{L^2(\Omega)} \qquad \forall u \in S_\ell \ \forall \mathbf{v} \in \mathbb{R}^{\Theta_\ell}. \tag{35}$$

Then, $R_\ell P_\ell = \mathbf{I}_\ell$ is the identity matrix and $\hat{P}_\ell := P_\ell (R_\ell P_\ell)^{-1}$ equals P_ℓ. Hence, the proof follows by using Lemma 25 and 26 and applying [11, Lemma 6.3.24(ii)]. ■

Corollary 20 *Let the assumptions of Lemma 19 be satisfied. Then*

$$\sup_{v \in S_\ell \setminus \{0\}} \frac{|||P_\ell^{-1} v|||_{2-s,\ell}}{\|v\|_{\mathcal{H}^{2-s}}} \le C_{\mathrm{P}}. \tag{36}$$

Proof. We have

$$P_\ell^{-1} u = P_\ell^\star u \quad \forall u \in S_\ell.$$

Consider the Banach spaces $S_\ell^\mu := \left(S_\ell, \|\cdot\|_{\mu,\ell}\right)$ and $\mathbf{H}_\ell^\mu := \left(\mathbb{R}^{\Theta_\ell}, |||\cdot|||_{\mu,\ell}\right)$. Then, the left-hand side in (36) is the operator norm of the adjoint operator

$$\left\|P_\ell^{-1}\right\|_{\mathbf{H}_\ell^{2-s} \leftarrow S_\ell^{2-s}} = \left\|P_\ell^\star\right\|_{\mathbf{H}_\ell^{2-s} \leftarrow S_\ell^{2-s}} = \left\|P_\ell\right\|_{S_\ell^{s-2} \leftarrow \mathbf{H}_\ell^{s-2}}.$$

Since the norm on the right-hand side equals the left-hand side in (34), the assertion follows. ■

The convergence proof for the two-grid method is split into the smoothing property

$$|||\mathbf{M}_L^{-1}\mathbf{A}_L\mathbf{K}_L^\nu|||_{s-2,L\leftarrow 2-s,L} \le C_S h_L^{2-2s}\eta(\nu) \tag{37}$$

with $\eta(\nu) \to 0$ as $\nu \to \infty$ and the approximation property

$$|||\left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1}\mathbf{A}_{L-1}^{-1}\mathbf{r}_{L-1,L}\right)\mathbf{M}_L|||_{2-s,L\leftarrow s-2,L} \le C_A h_L^{2s-2}.$$

Theorem 21 *If the parameter ω is small enough so that $\omega h_L^2 \rho(L_L) < 1$, then the smoothing property holds:*

$$|||\mathbf{M}_L^{-1}\mathbf{A}_L\mathbf{K}_L^\nu|||_{s-2,L\leftarrow 2-s,L} \le C_S h_L^{2-2s}\eta(\nu).$$

Proof. The definition of the norms imply

$$\begin{aligned} |||\mathbf{M}_L^{-1}\mathbf{A}_L\mathbf{K}_L^\nu|||_{s-2,L\leftarrow 2-s,L} &= \left\|L_L^{(s-2)/2} P_L \mathbf{M}_L^{-1}\mathbf{A}_L\mathbf{K}_L^\nu P_L^{-1} L_L^{(s-2)/2}\right\|_{0,L\leftarrow 0,L} \\ &= \left\|L_L^{s/2} P_L \mathbf{K}_L^\nu P_L^{-1} L_L^{(s-2)/2}\right\|_{0,L\leftarrow 0,L}. \end{aligned}$$

The power of the iteration matrix can be written in the form

$$P_L \mathbf{K}_L^\nu P_L^{-1} = \left(I - \omega h_L^2 L_L\right)^\nu$$

and, hence,

$$\begin{aligned} |||\mathbf{M}_L^{-1}\mathbf{A}_L\mathbf{K}_L^\nu|||_{s-2,L\leftarrow 2-s,L} &= \left\|L_L^{s/2}\left(I - \omega h_L^2 L_L\right)^\nu L_L^{(s-2)/2}\right\|_{0,L\leftarrow 0,L} \\ &= \left(\omega h_L^2\right)^{1-s}\left\|\left(\omega h_L^2 L_L\right)^{s-1}\left(I - \omega h_L^2 L_L\right)^\nu\right\|_{0,L\leftarrow 0,L} \\ &\le \omega^{1-s} h_L^{2-2s}\left(\eta_0\left(\frac{\nu}{s-1}\right)\right)^{s-1}, \end{aligned}$$

where

$$\eta_0(\nu) = \frac{\nu^\nu}{(\nu+1)^{\nu+1}} = \frac{1}{\mathrm{e}\,\nu} + O\left(\nu^{-2}\right).$$

Hence, the choice of s as in (32) implies that $\eta(\nu) := \omega^{1-s}\left(\eta_0\left(\frac{\nu}{s-1}\right)\right)^{s-1}$ tends to zero as $\nu \to \infty$. ■

For the approximation property, we employ the theory in [11, Chapter 6.3.1.3]. We assume that there is a constant C_s such that

$$h_{\ell-1} \le C_s h_\ell \qquad \forall 1 \le \ell \le L. \tag{38}$$

Theorem 22 *Let the assumptions of Corollary 14, Lemma 19, and (38) be satisfied. Then, for any $3/2 < \delta \le s$*

$$|||\left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1}\mathbf{A}_{L-1}^{-1}\mathbf{r}_{L-1,L}\right)\mathbf{M}_L\mathbf{f}|||_{2-s,L} \le C\left(\sqrt{\varepsilon} + h_L^{s-1}\right)\left(\sqrt{\varepsilon}\,|||\mathbf{f}|||_{\delta-2,L} + h_L^{s-1}\,|||\mathbf{f}|||_{s-2,L}\right),$$

where C only depends on C_0, C_6, and C_s (as in (38)).

Proof. Let $\mathbf{f} \in \mathbb{R}^{\Theta_L}$ and define $f_L \in S_L$ by

$$f_L := \sum_{x \in \Theta_L} \alpha_{L,x} f_x \quad \text{with} \quad \alpha_L := \mathbf{M}_L^{-1}\mathbf{f}.$$

Let u_L (resp. u_{L-1}) denote the composite finite element solution to problem (4) with $S_{\mathcal{G}}$ and f being replaced by S_L and f_L (resp. by S_{L-1} and f_L). The exact solution $u \in H^1(\Omega)$ for the right-hand side f_L is denoted by u. Then

$$\left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1}\mathbf{A}_{L-1}^{-1}\mathbf{r}_{L-1,L}\right)\mathbf{f} = P_L^{-1}\left(u_L - u_{L-1}\right).$$

Corollary 20 implies

$$\sup_{v \in S_L \setminus \{0\}} \frac{\left|\left|\left|P_L^{-1} v\right|\right|\right|_{2-s,L}}{\|v\|_{H^{2-s}(\Omega)}} \leq C_{\mathrm{P}}.$$

This and a triangle inequality yield

$$\left\|\left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1}\mathbf{A}_{L-1}^{-1}\mathbf{r}_{L-1,L}\right)\mathbf{f}\right\|_{2-s,L} \leq C_{\mathrm{P}}\left(\|u_L - u\|_{H^{2-s}(\Omega)} + \|u_{L-1} - u\|_{H^{2-s}(\Omega)}\right).$$

From the convergence estimates for the Galerkin solution and the regularity properties (cf. Corollary 14) we derive

$$\begin{aligned}
\|u_L - u\|_{H^{2-s}(\Omega)} &\leq \left(C_0\sqrt{\varepsilon} + C_6 h_L^{s-1}\right)\left(C_0\sqrt{\varepsilon}\,\|f\|_{\mathcal{H}^{\delta-2}} + C_6 h_L^{s-1}\,\|f\|_{\mathcal{H}^{s-2}}\right) \\
&= \left(C_0\sqrt{\varepsilon} + C_6 h_L^{s-1}\right)\left(C_0\sqrt{\varepsilon}\,\left\|P_L\left(\mathbf{M}_L^{-1}\right)\mathbf{f}\right\|_{\mathcal{H}^{\delta-2}} + C_6 h_L^{s-1}\,\left\|P_L\left(\mathbf{M}_L^{-1}\right)\mathbf{f}\right\|_{\mathcal{H}^{s-2}}\right) \\
&\leq C\left(\sqrt{\varepsilon} + h_L^{s-1}\right)\left(\sqrt{\varepsilon}\,\left|\left|\left|\mathbf{M}_L^{-1}\mathbf{f}\right|\right|\right|_{\delta-2,L} + h_L^{s-1}\,\left|\left|\left|\mathbf{M}_L^{-1}\mathbf{f}\right|\right|\right|_{s-2,L}\right),
\end{aligned}$$

where C depends on C_0, C_6, and C_{P}.

The estimate of the difference $u_{L-1} - u$ is completely analogously while h_L is replaced by h_{L-1}. However, the compatibility of consecutive step widths (cf (38)) leads to

$$\left\|\left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1}\mathbf{A}_{L-1}^{-1}\mathbf{r}_{L-1,L}\right)\mathbf{f}\right\|_{2-s,L} \leq C\left(\sqrt{\varepsilon} + h_L^{s-1}\right)\left(\sqrt{\varepsilon}\,\left|\left|\left|\mathbf{M}_L^{-1}\mathbf{f}\right|\right|\right|_{\delta-2,L} + h_L^{s-1}\,\left|\left|\left|\mathbf{M}_L^{-1}\mathbf{f}\right|\right|\right|_{s-2,L}\right).$$

Substituting $\mathbf{f}$ by $\mathbf{M}_L\mathbf{f}$ yields the assertion. ∎

Corollary 23 *Let the Assumptions of Theorem 22 be satisfied and assume (3). Then, the approximation property holds*

$$\left\|\left(\mathbf{A}_L^{-1} - \mathbf{p}_{L,L-1}\mathbf{A}_{L-1}^{-1}\mathbf{r}_{L-1,L}\right)\mathbf{M}_L\mathbf{f}\right\|_{2-s,L} \leq C h_L^{2s-2}\,\|\mathbf{f}\|_{s-2,L}.$$

Theorem 24 *Let the Assumptions of Theorems 21 and 22 be satisfied. Then, the norm of the two-grid operator can be estimated by*

$$\left\|\mathbf{K}_L^{TGM}\right\|_{s-2,L\leftarrow s-2,L} \leq C\eta(\nu)$$

where the function $\eta(\nu) \to 0$ is independent of h_L and tends to zero as $\nu \to \infty$.

Since estimate [11, (7.1.2)] holds with $\underline{C}_p = \overline{C}_p = 1$ and [11, (7.1.1)] follows from Theorem 21, the convergence of the W-cycle is implied by [11, Theorem 7.1.2].

In summary, we have proved that the multi-grid method on complicated domains converges robustly with respect to the area measure $0 < \varepsilon \leq C h_L^{1+\nu}$ under very weak geometric assumptions on the domains.

4 Numerical Experiments

We have performed numerical experiments to study the convergence behavior of the multigrid method based on composite finite elements for a Neumann problem on the complicated domain of the baltic sea (cf. Figure 5).

We have employed the V-cycle multigrid algorithm with 2 symmetric Gauß-Seidel smoothing steps. The stopping criterion is $\left\|\mathbf{A}_L\mathbf{u}^{(i)} - \mathbf{f}_L\right\| \leq 10^{-8}$. In Table 1 we display the number of iterations of our multigrid

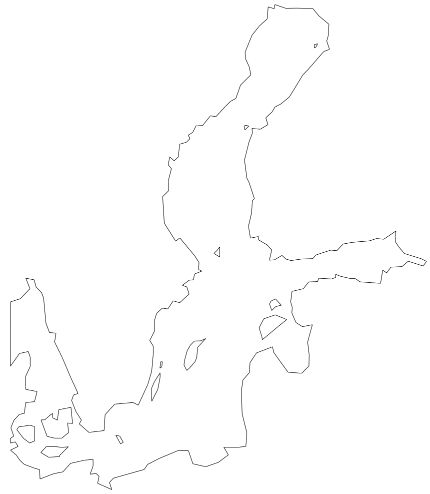


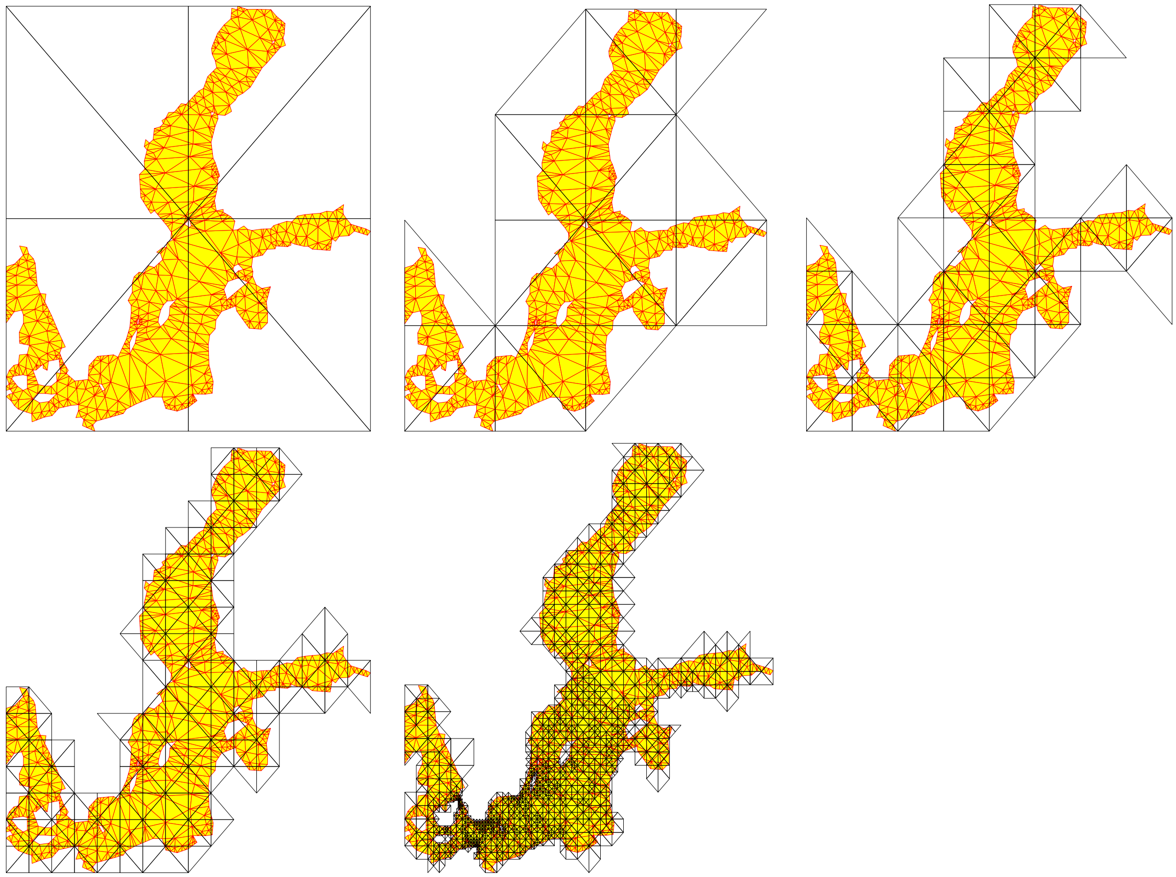
Figure 5: Ω is the two-dimensional surface of the baltic sea.

Level	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sharp$mg-iteration	6	7	12	16	13	20	22	17	18	17	16	15	13	13

Table 1: Number of iterations for the multigrid algorithm

algorithm as a function of the levels. Since these numbers are independent of the level, the convergence rates are small and independent of the refinement level.

Finally, we have depicted the mesh sequence which shows that the coarsest mesh has only 9 degree of freedom. The overlaps of triangles with the domain are of rather general shape and neither quasi-uniform or shape regular. Note that the underlying mesh which resolves Ω is only used for numerical integration and is not related to degrees of freedom.



A Proof of some Norm Equivalences

In this section, we will prove (34). We will employ [11, Lemma 6.3.24(ii)]. In this light, we will prove that [11, Lemma 6.3.22] holds also in our setting.

Recall the definitions of the finite element prolongation $P_\ell : \mathbb{R}^{\Theta_\ell} \to S_\ell \subset \mathcal{H}^1$ as in (7) and its adjoint $R_\ell : \mathcal{H}^{-1} \to \mathbb{R}^{\Theta_\ell}$ (cf. (35)). Let $\mathbf{L}_\ell := \mathbf{M}_\ell^{-1}\mathbf{A}_\ell$ and define the operator

$$X_\ell^\star := I - Q_\ell \quad \text{with} \quad Q_\ell := P_\ell \mathbf{L}_\ell^{-1} R_\ell L. \tag{39}$$

Lemma 25 *Let Q_ℓ be defined as in (39). Then*

$$\|Q_\ell\|_{\mathcal{H}^1 \leftarrow \mathcal{H}^1} \leq 1.$$

Proof. The continuity constant of the elliptic boundary value problem equals 1 and, hence,

$$\|L\|_{\mathcal{H}^{-1} \leftarrow \mathcal{H}^1} = 1.$$

The definition of the norm $|||\cdot|||_{1,\ell}$ implies

$$\sup_{\mathbf{u} \in \mathbb{R}^{\Theta_\ell} \backslash \{0\}} \frac{\|P_\ell \mathbf{u}\|_{\mathcal{H}^1}}{|||\mathbf{u}|||_{1,\ell}} = \sup_{f \in \mathcal{H}^{-1} \backslash \{0\}} \frac{|||R_\ell f|||_{-1,\ell}}{\|f\|_{\mathcal{H}^{-1}}} = 1.$$

Finally, the ellipticity constant for the bilinear form $a(\cdot,\cdot)$ equals 1 and this leads to (see [12, Lemma 6.5.3])

$$\sup_{\mathbf{f} \in \mathbb{R}^{\Theta_\ell} \backslash \{0\}} \frac{\left|\left|\left|\mathbf{L}_\ell^{-1}\mathbf{f}\right|\right|\right|_{1,\ell}}{|||\mathbf{f}|||_{-1,\ell}} \leq 1.$$

■

Lemma 26 *Let the assumptions of Corollary 15 be satisfied. Let (3) be satisfied for some $\nu > 0$. Then, for any $1 \leq \mu \leq \min\left\{\lambda, \frac{3+\nu}{2}\right\}$ and any $u \in \mathcal{H}^1$, we have*

$$\|(I - Q_\ell)\, u\|_{\mathcal{H}^{2-\mu} \leftarrow \mathcal{H}^1} \leq 2\,(C_0 + C_6)\, h^{\mu-1}\, \|u\|_{\mathcal{H}^1} \tag{40}$$

Proof. Note that

$$I - Q_\ell = \left(L^{-1} - P_\ell \mathbf{L}_\ell^{-1} R_\ell\right) L.$$

For $u \in \mathcal{H}^1$, we obtain by using Corollary 14 and the regularity of the boundary value problem

$$\left\|\left(L^{-1} - P_\ell \mathbf{L}_\ell^{-1} R_\ell\right) Lu\right\|_{\mathcal{H}^{2-\mu}} \leq 2\left(C_0\sqrt{\varepsilon} + C_6 h^{\mu-1}\right)\|Lu\|_{\mathcal{H}^{-1}} \leq 2\left(C_0\sqrt{\varepsilon} + C_6 h^{\mu-1}\right)\|u\|_{\mathcal{H}^1}.$$

Condition (3) implies the assertion. ■

Acknowledgement: Thanks are due to Profs. Monique Dauge and Wolfgang Hackbusch for fruitful discussions.

References

[1] R. Bank and R. Smith. An algebraic multilevel multigraph algorithm. *SIAM J. Sci. Comp.*, 23:1572–1592, 2002.

[2] R. Bank and J. Xu. A Hierarchical Basis Multi-Grid Method for Unstructured Grids. In W. Hackbusch and G. Wittum, editors, *Fast Solvers for Flow Problems, Proceedings of the Tenth GAMM-Seminar, Kiel.* Verlag Vieweg, 1995.

[3] R. Bank and J. Xu. An algorithm for coarsening unstructured meshes. *Numer. Math.*, 73:1–36, 1996.

[4] D. Braess. Towards algebraic multigrid for elliptic problems of second order. *Computing*, 55:379–393, 1995.

[5] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer Verlag, New York, 1994.

[6] T. F. Chan and B. F. Smith. Domain decomposition and multigrid algorithms for elliptic problems on unstructured meshes. *ETNA*, 2:171–182, 1994.

[7] T. Chan, J. Xu, and L. Zikatanov. An agglomeration multigrid method for unstructured grids. *Contemp. Math.*, 218:67–81, 1998.

[8] D. Feuchter, I. Heppner, S. Sauter, and G. Wittum. Bridging the Gap Between Geometric and Algebraic Multi-Grid Methods. *Comp. and Vis. in Sci.*, 6(1):1–13, 2003.

[9] M. Griebel and S. Knapek. A multigrid-homogenization method. In W. Hackbusch and G. Wittum, editors, *Modeling and Computation in Environmental Sciences*, pages 187–202, Braunschweig, 1997. Vieweg. Notes Numer. Fluid Mech. 59.

[10] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 21 of *Monographs and Studies in Mathematics.* Pitman, Boston–London–Melbourne, 1985.

[11] W. Hackbusch. *Multi-Grid Methods and Applications.* Springer Verlag, Berlin, 1985, 2nd edition 2003.

[12] W. Hackbusch, *Elliptic Differential Equations. Theory and numerical treatment*, volume 18 of *SCM.* Springer-Verlag, Berlin, 1992.

[13] W. Hackbusch and S. Sauter. Composite Finite Elements for Problems Containing Small Geometric Details. Part II: Implementation and Numerical Results. *Comput. Visual. Sci.*, 1(1):15–25, 1997.

[14] R. Kornhuber and H. Yserentant. Multilevel Methods for Elliptic Problems on Domains not Resolved by the Coarse Grid. *Contemporay Mathematics*, 180:49–60, 1994.

[15] J. Mandel, M. Brezina, and P. Vaněk. Energy optimization of algebraic multigrid bases. *Computing*, 62:205–228, 1999.

[16] V. Maz'ja. *Sobolev Spaces.* Springer-Verlag, Berlin, New York, 1985.

[17] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, 2000.

[18] O. Oleinik, A. Shamaev, and G. Yosifian. *Mathematical Problems in Elasticity and Homogenization.* North-Holland, Amsterdam, 1992.

[19] J. Ruge and K. Stüben. Algebraic multigrid. In S. McCormick, editor, *Multigrid Methods*, pages 73–130, Pennsylvania, 1987. SIAM Philadelphia.

[20] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton, University Press, Princeton, N.J., 1970.

[21] R. Warnke. *Fortsetzungsoperatoren auf perforierten Gebieten.* Diplomarbeit, Universität Kiel, 1997.

[22] S.A. Sauter and R. Warnke. Extension Operators and Approximation on Domains Containing Small Geometric Details. *East-West J. Numer. Math.*, 7(1):61-78, 1999.

[23] L.R. Scott and S. Zhang. Finite element interpolation of non-smooth functions satisfying boundary conditions. *Math. Comp*, 54:483-493, 1990.

[24] L.R. Scott and S. Zhang. Higher dimensional non-nested multigrid methods. *Math. Comp*, 58:457-466, 1992.

[25] N. Stahn. *Composite Finite Elements and Multigrid.* Dissertation, Universität Zürich, planned for 2005.

[26] P. Vaněk, J. Mandel, and M. Brezina. Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems. *Computing*, 56:179–196, 1996.

[27] J. Xu. The Auxiliary Space Method and Optimal Multigrid Preconditioning Techniques for Unstructured Grids. *Computing*, 56:215-235, 1996.

[28] H. Yserentant. Coarse grid spaces for domains with a complicated boundary. Numerical Algorithms 21 (1999), 387-392.