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Two-Scale Composite Finite Element Method for Dirichlet Problems on Complicated Domains

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Summary In this paper, we define a new class of finite elements for the discretization of problems with Dirichlet boundary conditions. In contrast to standard finite elements, the minimal dimension of the approximation space is independent of the domain geometry and this is especially advantageous for problems on domains with complicated micro-structures. For the proposed finite element method we prove the optimal-order approximation (up to logarithmic terms) and convergence estimates valid also in the cases when the exact solution has a reduced regularity due to re-entering corners of the domain boundary. Numerical experiments confirm the theoretical results and show the potential of our proposed method.

Key words two-scale composite finite elements – Dirichlet boundary conditions – complicated domain

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1 Introduction

The problem of numerically solving partial differential equations on complicated domains arises in many physical applications such as environmental modelling, porous media flows, modelling of complex technical engines and many others. In principle, this problem can be

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treated with the standard finite element method; however, the usual requirement

the finite element mesh has to resolve the domain boundary (1)

makes a coarse-scale discretization impossible. Every reasonable discretization will necessarily contain a huge number of unknowns being directly linked to the number of geometric details of the physical domain.

This is in sharp contrast to a flexible, problem-adapted, and goal-oriented discretization:

- The finite element discretization should allow the adaption to the characteristic (possibly singular) behavior of the exact solution without adding "too" many degrees of freedom but, e.g., by adapting the shape of the finite element functions to the behavior of the solution by introducing slave nodes.
- Starting from a very coarse discretization and a very crude approximation of constraints such as Dirichlet boundary conditions, an a-posteriori error estimation should be used to enrich the finite element space to improve the local accuracy.

In [4], [5] the Composite Finite Elements (CFE) have been introduced for coarse-level discretizations of boundary value problems with Neumann-type boundary conditions. The minimal number of unknowns in the method was independent of the number and size of geometric details. For functions in $H^k(\Omega)$, the approximation property was proven in an analogue generality as established for standard finite elements (see [4]).

In this paper, we will introduce composite finite elements for an adaptive approximation of Dirichlet boundary conditions. These finite elements can be interpreted as a generalization of standard finite elements by allowing the approximation of Dirichlet boundary conditions in a flexible adaptive manner. In this light, we will establish in this paper the approximation and convergence properties of these finite elements in the framework of an a-priori analysis.

In [8], we will introduce the combination of these finite element spaces with an a-posteriori error estimator in order to improve the approximation of Dirichlet boundary conditions in a problem-adapted way.

Related approaches in the literature can be found in [1], [6], [14]. In those papers, the efficient solution of the fine-scale discretization was the major goal and *not* the preservation of the asymptotic convergence order of the underlying discretization on coarser meshes, as in our two-scale approach.

We will introduce the composite finite element method for problems with Dirichlet boundary conditions via a two-scale discretization: One coarse scale H describes the approximation of the solution in the interior of the domain at a proper distance to the boundary and one fine scale h describes the local mesh size which is used for the approximation of Dirichlet boundary conditions.

As a model problem we consider the Poisson equation with homogeneous Dirichlet boundary condition

$$-\Delta u = f \quad \text{in } \Omega\,, \tag{2}$$
$$u = 0 \quad \text{on } \Gamma\,, \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ having a finite length. For the sake of simplicity, we assume that Ω is a polygonal domain, but it may still have a very complicated shape. We emphasis that the extension of the presented theory to general 2nd-order elliptic problems or three-dimensional problems is straightforward from the conceptional point of view.

The aim of this work is to set up a family of finite elements which possesses the optimal approximation property (up to logarithmic terms) for functions in $H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $s \in \left[\frac{1}{2}, 1\right]$. If we denote by N_Γ the number of line segments in Γ, the minimal number of unknowns in the standard FEM ranges between $\mathcal{O}(N_\Gamma)$ and $\mathcal{O}\left(N_\Gamma^2\right)$, depending on the mesh generator, which may exceed the memory capacity of modern computers. In this paper, we define a two-scale finite element space where the minimal dimension is independent of N_Γ; thus, the number of unknowns can be adapted to a given, possibly moderate accuracy requirement and is no more restricted by the geometric condition (1).

To achieve this goal, we relax the condition (1) by introducing a two-scale grid: The coarse scale grid $\mathcal{T}_H$ which contains the *degrees of freedom* and the fine scale grid $\mathcal{T}_h$ which adaptively resolves the boundary Γ and contains only *slave nodes* which are used to adapt the shape functions to the Dirichlet boundary conditions. For a triangle $\tau \in \mathcal{T}_H$, we denote its diameter by h_τ and the index H in $\mathcal{T}_H$ is the largest triangle diameter: $H := \max\{h_\tau : \tau \in \mathcal{T}\}$. The index h in $\mathcal{T}_h$ is the *smallest* diameter of triangles in $\mathcal{T}_h$.

Below, we will summarize the main features of the two-scale CFE-method. We will prove that

- the mesh width for the resolution of the boundary has to obey the relation $h = H^{\frac{3}{2}}$ in order to preserve always the asymptotic convergence rates with respect to H (independently of the precise

knowledge of the regularity). In contrast, for the standard FEM, the relation is more restrictive: $h = H^2$.
- The resolution condition (1) is replaced by the *overlap condition* for the mesh $\mathcal{T}_H$:

$$\Omega \subset \bigcup_{\tau \in \mathcal{T}_H} \tau$$

and hence, the minimal number of elements n in $\mathcal{T}_H$ is independent of the number and size of geometric details in Ω. The fine scale grid is concentrated locally at the boundary Γ. The additional points for the boundary resolution are slave nodes and employed only to incorporate the Dirichlet boundary conditions into the coarse scale discretization in a flexible way. Hence, the number of unknowns is $\mathcal{O}(H^{-2})$. In contrast, the number of freedoms for the standard FEM ranges from $\mathcal{O}(H^{-2} + |\Gamma| h^{-1})$ to $\mathcal{O}(h^{-2})$, depending on the mesh generator.
- The two-scale CFE-method allows to employ an a posteriori error indicator already on a very coarse discretization and to enrich the CFE-space and/or to improve the non-conforming approximation of the boundary conditions in a problem-adapted way. Current research is devoted to the combination of the a posteriori error estimator [8], which takes into account the approximation of Dirichlet boundary conditions.

The paper is organized as follows. We define the two-scale composite finite element space in Section 2 and prove the approximation error estimates in Section 3. Section 4 is devoted to the convergence analysis for the CFE solution of problem (2), (3). Numerical experiments are reported in Section 5 and give insights in the practical performance of the proposed method, especially in the constants of the theoretical convergence estimates.

In this paper, we will use the standard notation $\|\cdot\|_{s,\Omega}$ for the norm in the Sobolev space $H^s(\Omega)$, $s \geq 0$, and $|\cdot|_{k,\Omega}$ for the seminorm in $H^k(\Omega)$, $k = 1, 2$ (i.e. $|u|_{k,\Omega} = (\sum\limits_{|\alpha|=k} \|D^\alpha u\|^2_{0,\Omega})^{1/2}$).

In order to improve readability, we have collected below the most relevant notations, while their precise definitions will be given later in the text.

Notation 1

$\mathcal{T}_H, \Theta_H$	*Initial, overlapping coarse grid and corresponding set of vertices,*
$\mathcal{T}_\Gamma$	*subset of* $\mathcal{T}_H$*, which contains all near-boundary triangles,*
$\mathcal{T}_{H,h}, \Theta_{H,h}$	*two scale grid with corresponding set of grid points,*
$\mathcal{T}_H^{\text{in}}, \Theta_{\text{dof}}$	*inner grid of* $\mathcal{T}_{H,h}$ *with corresponding set of grid points (degrees of freedom),*
Θ_{slave}	*set of slave nodes* $\Theta_{\text{slave}} := \Theta_{H,h} \backslash \Theta_{\text{dof}}$;
x^Γ	*for* $x \in \Theta_{\text{slave}}$, $x^\Gamma \in \Gamma$ *has min. distance to* x,
Δ_x	*for* $x \in \Theta_{\text{slave}}$, $\Delta_x \in \mathcal{T}_H^{\text{in}}$ *has min. distance to* x,
τ	*(closed) triangle,*
$\mathbf{V}(\tau)$	*set of vertices of a triangle* τ.

2 The composite finite element space

The construction of the composite finite element (CFE) space is realized in three steps. We emphasize that all steps can be incorporated easily in any standard grid refinement algorithm.

Step 1: Overlapping two-scale grid
Let $\mathcal{T}_H = \{\tau_1, \tau_2, \ldots, \tau_n\}$ denote a conforming shape regular finite element mesh (in the sense of Ciarlet [3]) consisting of (closed) triangles with maximal diameter H.

Notation 2 *For any triangle* τ*, the set of vertices is denoted by* $\mathbf{V}(\tau)$*. The open interior of a (closed) triangle is denoted by* $\operatorname{int}(\tau)$*.*

The assumption on the grid conformity excludes the presence of hanging nodes in $\mathcal{T}_H$. Further, we assume that $\mathcal{T}_H$ is an overlapping grid, i.e.

$$\Omega \subset \bigcup_{\tau \in \mathcal{T}_H} \tau \quad \text{and} \quad \forall \tau \in \mathcal{T}_H : \operatorname{int}(\tau) \cap \Omega \neq \emptyset. \tag{4}$$

It is evident that, for any bounded domain, there exists a triangulation with very few elements which satisfies these conditions. In order to resolve the boundary (conditions) in an adaptive way, the triangles in a certain neighborhood of Γ will be refined. The width of this neighborhood is controlled by a parameter $c_{\text{dist}} > 0$. We employ a simple coloring algorithm which marks two “layers” of triangles about the boundary Γ provided the distance of such triangles from the boundary is not "too" far: $\operatorname{dist}(\tau,\Gamma) \le c_{\text{dist}} h_\tau$. The procedure requires as input the mesh $\mathcal{T}_H$ and the output is the near-boundary part $\mathcal{T}_\Gamma$ of the mesh. It is called by **mark_near_boundary_triangles** and defined by

procedure mark_near_boundary_triangles;
begin
  $\mathcal{T}_{\text{temp}} := \emptyset$; $\mathcal{T}_\Gamma := \emptyset$;
  for all $\tau \in \mathcal{T}_H$ **do**
    if $\operatorname{int}(\tau) \cap \Gamma \neq \emptyset$ **then** $\mathcal{T}_{\text{temp}} := \mathcal{T}_{\text{temp}} \cup \{\tau\}$;
  for all $\tau \in \mathcal{T}_{\text{temp}}$ **do**
    for all $t \in \mathcal{T}_H$ **with** $t \cap \tau \neq \emptyset$ **do**
      if $\operatorname{dist}(t,\Gamma) \le c_{\text{dist}} h_t$ **then** $\mathcal{T}_\Gamma := \mathcal{T}_\Gamma \cup \{t\}$;
end;

Next, the near-boundary triangles $\tau \in \mathcal{T}_\Gamma$ are refined adaptively towards Γ until the fine scale triangles $t \subset \tau$ satisfy the following condition

$$\operatorname{dist}(t,\Gamma) > 0 \vee \operatorname{stop}(t) = \text{true}, \tag{5}$$

where $\operatorname{stop}(\cdot)$ is an abstract stopping criterion which will be addressed in Remark 7 and Lemma 5 or replaced by an a-posteriori error estimation (see [8]).

For a triangle τ, let **refine**(τ) denote the set of four triangles which arise by connecting the midpoints of the edges in τ. The procedure **adapt_boundary** successively refines the near-boundary triangles, i.e., which violate condition (5). In order to keep the procedure local about the boundary, we employ an *active set* $\mathcal{T}_{\text{active}}$ which contains level-by-level the newly generated near-boundary triangles and is updated via an auxiliary set $\mathcal{T}_{\text{temp}}$. It is called by

$\mathcal{T}_{H,h} := \mathcal{T}_H$; $\mathcal{T}_{\text{temp}} := \mathcal{T}_\Gamma$; **adapt_boundary**;

and defined by

procedure adapt_boundary;
begin
  $\mathcal{T}_{\text{active}} := \{\tau \in \mathcal{T}_{\text{temp}} : \text{Condition (5) is violated}\}$; $\mathcal{T}_{\text{temp}} := \emptyset$;
  while $\mathcal{T}_{\text{active}} \neq \emptyset$ **do begin**
    for all $\tau \in \mathcal{T}_{\text{active}}$ **do begin**
      $\sigma_{\text{temp}} := \{t \in \textbf{refine}(\tau) : |t \cap \Omega| > 0\}$;
      $\mathcal{T}_{H,h} := \mathcal{T}_{H,h} \backslash \{\tau\} \cup \sigma_{\text{temp}}$;
      $\mathcal{T}_{\text{temp}} := \mathcal{T}_{\text{temp}} \cup \sigma_{\text{temp}}$;
    end;
    green_closure$(\mathcal{T})$;
    $\mathcal{T}_{\text{active}} := \{\tau \in \mathcal{T}_{\text{temp}} : \text{Cond. (5) is violated}\}$; $\mathcal{T}_{\text{temp}} := \emptyset$;

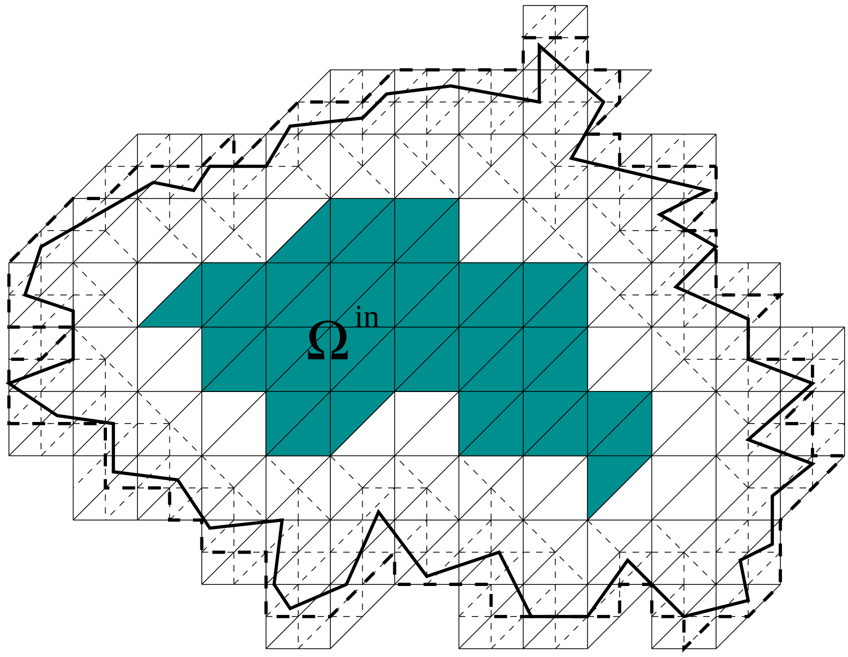

Fig. 1. Two-scale grid $\mathcal{T}_{H,h}$. The dark-shaded triangles form the inner triangulation $\mathcal{T}_H^{\text{in}}$ and contain the degrees of freedom. The near-boundary triangles are surrounded by dotted lines and contain the slave nodes as vertices. The solid line is the domain boundary.

end;
end;

Here, the procedure **green_closure** eliminates all hanging nodes in the actual triangulation $\mathcal{T}_{H,h}$. If a common triangle $\tau \in \mathcal{T}_{H,h} \cap \mathcal{T}_{\text{temp}}$ is subdivided by the procedure **green_closure**, we employ the convention that the triangle τ is replaced by the refined triangles not only in $\mathcal{T}_{H,h}$ but also in the set $\mathcal{T}_{\text{temp}}$.

For any $\tau \in \mathcal{T}_H$, we define the set of sons by

$$\text{sons}\,(\tau) := \{t \in \mathcal{T}_{H,h} : t \subset \tau\} \tag{6}$$

and denote its number by $n_\tau := \sharp\,\text{sons}\,(\tau)$.

As a result of this algorithm, we obtain a new conforming and shape regular grid that is more refined than $\mathcal{T}_H$ in the vicinity of Γ and does not differ from $\mathcal{T}_H$ in the interior of Ω (see Figure 1).

The two-scale nature of the grid $\mathcal{T}_{H,h}$ becomes apparent: In the interior of the domain, at some distance from Γ, the submesh

$$\mathcal{T}_H^{\text{in}} := \{t \in \text{sons}\,(\tau) : \tau \in \mathcal{T}_H \backslash \mathcal{T}_\Gamma\} \subset \mathcal{T}_{H,h}$$

is characterized by the coarse-scale mesh parameter H. (Note that $\mathcal{T}_H^{\text{in}}$ differ from $\mathcal{T}_H \backslash \mathcal{T}_\Gamma$ only by those triangles $t \in \mathcal{T}_H \backslash \mathcal{T}_\Gamma$ which are refined via the green-closure algorithm.)

In the neighborhood of Γ the two-scale mesh $\mathcal{T}_{H,h}$ is characterized by the fine-scale parameter $h := \min\{h_t : t \in \mathcal{T}_{H,h}\}$, $h \leq H$. Later we will see that, for the optimal convergence rate of the CFE solution, the parameters should obey the relation $h = \mathcal{O}(H^s)$ and the choice of s will be discussed in Section 4.

By choosing the stopping criterion (5) in an appropriate way, the near-boundary triangles satisfy

$$\operatorname{dist}(\tau, \Gamma) \leq c_{\text{dist}} h_\tau \qquad \forall \tau \in \mathcal{T}_{H,h} \backslash \mathcal{T}_H^{\text{in}}. \tag{7}$$

(More precisely, the stopping criterion must contain (7).)

Remark 1 For the constructed grid $\mathcal{T}_{H,h}$ we can distinguish two limiting cases.

1. The number n_τ of subtriangles in $\tau \in \mathcal{T}_\Gamma$ equals 1; it means that there is no subdivision of τ and the grid $\mathcal{T}_{H,h}$ simply coincides with the coarse-scale grid $\mathcal{T}_H$ ($h = \mathcal{O}(H)$ in this case). The method will be denoted as 'one-scale CFE-method', whereas in the case $h \ll H$ it is called 'two-scale CFE-method'.
2. The number n_τ is so large, that the domain Ω is fully resolved by the grid $\mathcal{T}_{H,h}$ (the full resolution of Ω can be achieved by applying the above mentioned refinement algorithm until the connectivity components $\tau \backslash \Gamma$ can be meshed by only few triangles; then, further subdivision of $t \in \operatorname{sons}(\tau)$ into these triangles leads to the grid exactly aligned with the boundary Γ); in this case, $h = \mathcal{O}(h_\Gamma)$ where h_Γ is the characteristic scale of Γ.

Step 2: Marking the degrees of freedom

Next, we will define the "free nodes" where the degrees of freedom will be located and the "slave nodes" where the function values are constraint. The degrees of freedom correspond to those vertices in the coarse mesh $\mathcal{T}_H$ – more precisely in the inner mesh $\mathcal{T}_H^{\text{in}}$ – having a proper distance to the boundary. Let Θ_H denote the set of *all* vertices in $\mathcal{T}_H$ and define

$$\Theta_{\text{dof}} := \left\{ x \in \mathbf{V}(\tau) : \tau \in \mathcal{T}_H^{\text{in}} \right\}.$$

All other nodes in $\mathcal{T}_{H,h}$ are slave nodes and the values of a composite finite element function is determined by its values at the nodes $x \in \Theta_{\text{dof}}$. In this light, the triangles and grid points which are generated by the procedure **adapt_boundary** do not increase the dimension of the finite element space but are used for *adapting* the shape of the finite element functions to the Dirichlet boundary conditions.

Step 3: Definition of an extrapolation operator

The degrees of freedom of the composite finite element space are located at the inner nodes Θ_{dof} and the values at the slave nodes of the two-scale mesh $\mathcal{T}_{H,h}$ are determined via a simple extrapolation method.

Let $\Theta_{H,h}$ denote the set of all vertices of the two-scale mesh $\mathcal{T}_{H,h}$. The set of slave nodes is given by

$$\Theta_{\mathrm{slave}} := \Theta_{H,h} \backslash \Theta_{\mathrm{dof}}.$$

For a slave node $x \in \Theta_{\mathrm{slave}}$, we determine a closest point x^{Γ} on the boundary Γ and a closest coarse grid triangle $\Delta_x \in \mathcal{T}_H^{\mathrm{in}}$.

Remark 2 For $x \in \Theta_{\mathrm{slave}}$, the computation of a closest boundary point x^{Γ} and a closest coarse grid triangle Δ_x can be performed efficiently by using the hierarchical structure of the two-scale mesh.

Let $\mathbf{u} : \Theta_{\mathrm{dof}} \to \mathbb{R}$ denote a grid function. For any $\tau \in \mathcal{T}_H$, there exists an uniquely determined linear function $u_\tau : \mathbb{P}_1\left(\mathbb{R}^2\right)$ which interpolates $\mathbf{u}$ in the vertices of τ. Here, and in the sequel, $\mathbb{P}_1\left(\mathbb{R}^2\right)$ denotes the space of bivariate polynomials on $\mathbb{R}^2$ of maximal degree 1. The values of the extension of $\mathbf{u}$ at a slave node $x \in \Theta_{\mathrm{slave}}$ is defined by

$$(\mathcal{E}\mathbf{u})_x := u_{\Delta_x}(x) - u_{\Delta_x}\left(x^{\Gamma}\right).$$

This relation defines an extrapolation operator $\mathcal{E} : \mathbb{R}^{\Theta_{\mathrm{dof}}} \to \mathbb{R}^{\Theta_{H,h}}$ for grid functions:

$$(\mathcal{E}\mathbf{u})_x := \begin{cases} \mathbf{u}_x & x \in \Theta_{\mathrm{dof}}, \\ u_{\Delta_x}(x) - u_{\Delta_x}\left(x^{\Gamma}\right) & x \in \Theta_{\mathrm{slave}}. \end{cases} \tag{8}$$

Let S denote the continuous, piecewise linear finite element space on the mesh $\mathcal{T}_{H,h}$

$$S := \left\{ u \in C^0\left(\Omega_{H,h}\right) \mid \forall \tau \in \mathcal{T}_{H,h} : u|_\tau \in \mathbb{P}_1 \right\},$$

where $\Omega_{H,h} := \operatorname{int}\left(\bigcup_{\tau \in \mathcal{T}_{H,h}} \tau\right)$. The composite finite element space is a subspace of S, where the values at the slave nodes are restricted by the extrapolation.

Definition 1 *The composite finite element space for the two-scale approximation of Dirichlet boundary conditions on the mesh $\mathcal{T}_{H,h}$ is*

$$S^{\mathrm{CFE}} := \left\{ u \in S \mid \exists \mathbf{u} \in \mathbb{R}^{\Theta_{\mathrm{dof}}} \quad \forall x \in \Theta_{H,h} : u(x) = (\mathcal{E}\mathbf{u})_x \quad \right\}.$$

Remark 3 From the viewpoint of the approximation quality of the composite finite element space, it is essential that the extrapolation from an inner triangle Δ_x to a slave node x is not performed over a "too" large distance. (Such a situation might appear if a slave node is located in a long outlet of the domain, far away from an inner triangle). If such situations arise, we simply modify the definition (8) by employing a control parameter $\eta_{\mathrm{ext}} > 0$ and using the generalized definition

$$(\mathcal{E}\mathbf{u})_x := \begin{cases} \mathbf{u}_x & x \in \Theta_{\mathrm{dof}}, \\ u_{\Delta_x}(x) - u_{\Delta_x}\left(x^{\Gamma}\right) & x \in \Theta_{\mathrm{slave}} \wedge \mathrm{dist}\,(x, \Delta_x) \leq \eta_{\mathrm{ext}} h_{\Delta_x}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Remark 4

1. Obviously, $S^{\mathrm{CFE}} \subset S$; since the dimension of S^{CFE} is determined only by the number of nodes in Θ_{dof}, it may be much smaller than the dimension of S, especially in the case of very complicated boundary Γ.
2. A composite finite element function $u \in S^{\mathrm{CFE}}$ is, in general, not affine inside of each triangle $\tau \in \mathcal{T}_H$ but continuously composed of affine pieces on triangles of $\mathcal{T}_{H,h}$. However, in the interior of the domain (i.e. on triangles $\tau \in \mathcal{T}_H^{\mathrm{in}}$) it is a standard finite element function being piecewise affine on these triangles.

Remark 5 The space S^{CFE} is, in general, non-conforming in the sense that the triangles in $\mathcal{T}_{H,h}$ might overlap the boundary Γ and, then, the functions from S^{CFE} satisfy the homogeneous boundary condition only approximately. However, as we will see in Section 4, a small error in the approximation of boundary conditions is harmless for the quasi-optimal (with respect to the coarse-scale parameter H) convergence rate of the CFE solution.

3 Approximation property

In this section we investigate the approximation property of the composite finite element space. The error estimates for composite finite elements will be based on the existence of an appropriate extension operator for the given domain Ω. It is known that, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, there exists a continuous, linear extension operator $\mathfrak{E} : H^k(\Omega) \to H^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, such that

$$\forall u \in H^k(\Omega): \quad \mathfrak{E}u|_\Omega \equiv u \quad \text{and} \quad \|\mathfrak{E}u\|_{H^k(\mathbb{R}^d)} \le C_{\text{ext}} \|u\|_{H^k(\Omega)}$$

with the constant C_{ext} depending only on k and Ω (cf. [11]). It is worth noting that, for domains containing a large number of holes and possibly a rough outer boundary, there exists an extension operator with the bounded norm C_{ext} independent of the number of the holes and of their sizes. For all details including the characterisation of the class of domain geometries, we refer to [10].

To derive the approximation error estimates, we will need a preparatory Lemma. Let τ denote an arbitrary triangle with diameter h_τ and mass center M_τ. For $c \ge 1$, we introduce the scaled version of τ by

$$T_c := \{M_\tau + c\,(y - M_\tau) : y \in \tau\}\,. \tag{10}$$

Lemma 1 (neighborhood property) *Let $u \in H^2(\mathbb{R}^2)$ and τ be an arbitrary triangle with diameter h_τ. Let $u_\tau \in \mathbb{P}_1\left(\mathbb{R}^2\right)$ denote the affine interpolation of u at the vertices of τ and let T_R be the scaled version of τ as in (10) for some $R \ge 1$ about the mass center of τ. For $m \in \{0,1\}$ and $1 \le p \le \infty$ with the exception $(m,p) \ne (1,\infty)$, we have the error estimate*

$$|u - u_\tau|_{W^{m,p}(T_R)} \le C\,(1+R)\,(Rh_\tau)^{1+\frac{2}{p}-m}\,|u|_{H^2(T_R)}\,, \tag{11}$$

where C only depends on the minimal angles of τ.

Proof For $R \ge 1$, we write T short for T_R. Obviously τ and T are congruent and the diameter of T satisfies $h_T = Rh_\tau$. For $u \in H^2\left(\mathbb{R}^2\right)$, let $\mathcal{I}_T u \in \mathbb{P}_1$ (resp. $\mathcal{I}_\tau u \in \mathbb{P}_1$) denote the affine function which interpolates u at the vertices of T (resp. τ). The projection property of $\mathcal{I}_\tau$ on $\mathbb{P}_1$ leads to

$$u - \mathcal{I}_\tau u = (I - \mathcal{I}_\tau)\,(u - \mathcal{I}_T u)\,, \tag{12}$$

where I is the identity. Hence,

$$\begin{aligned} |u - \mathcal{I}_\tau u|_{W^{m,p}(T)} \le{}& |u - \mathcal{I}_T u|_{W^{m,p}(T)} \\ &+ \left(\sup_{v \in C^0(T)\backslash\{0\}} \frac{|\mathcal{I}_\tau v|_{W^{m,p}(T)}}{\|v\|_{L^\infty(T)}}\right) \|u - \mathcal{I}_T u\|_{L^\infty(T)}\,. \end{aligned} \tag{13}$$

The estimates

$$\begin{aligned} |u - \mathcal{I}_T u|_{W^{m,p}(T)} &\le C h_T^{1+\frac{2}{p}-m}\,|u|_{H^2(T)} \quad \text{and} \\ \|u - \mathcal{I}_T u\|_{L^\infty(T)} &\le C h_T\,|u|_{H^2(T)} \end{aligned} \tag{14}$$

are well known (see, e.g. [3, Theorem 3.1.6]).

Next, we will estimate the supremum in (13). Let z_i, $1 \leq i \leq 3$, denote the vertices of τ with corresponding shape functions $b_i \in \mathbb{P}_1\left(\mathbb{R}^2\right)$ defined by $b_i\left(z_i\right) = 1$ and $b_i\left(z_j\right) = 0$ for $i \neq j$.

$$\begin{aligned} \left|\mathcal{I}_\tau v\right|_{W^{m,p}(T)} &= \left|\sum_{i=1}^{3} v\left(z_i\right) b_i\right|_{W^{m,p}(T)} \leq \max_{1\leq i\leq 3}\left|v\left(z_i\right)\right| \sum_{i=1}^{3}\left|b_i\right|_{W^{m,p}(T)} \\ &\leq \|v\|_{L^\infty(\tau)} \sum_{i=1}^{3}\left|b_i\right|_{W^{m,p}(T)}. \end{aligned} \tag{15}$$

Since b_i is affine, we obtain the estimate for all $y \in T$

$$\begin{aligned} \left|b_i(y)\right| = \left|b_i\left(M_\tau\right) + \left\langle \nabla b_i, y - M_\tau\right\rangle\right| &\leq 1 + \left|b_i\right|_{W^{1,\infty}(\tau)} \left\|y - M_\tau\right\| \\ &\leq 1 + C h_\tau^{-1} h_T \leq 1 + CR, \end{aligned}$$

where C only depends on the minimal angles in τ. Thus, for $m = 0$, we get

$$\left\|b_i\right\|_{L^p(T)} \leq (1 + CR)\, h_T^{2/p}.$$

The estimate for $m = 1$ is simpler since ∇b_i is constant and an inverse inequality leads to

$$\left|b_i\right|_{W^{1,p}(T)} \leq C h_\tau^{-1} h_T^{2/p} \leq C \frac{h_T}{h_\tau} h_T^{2/p-1} \leq CR h_T^{2/p-1}.$$

Taking into account (15), we have proven

$$\left|\mathcal{I}_\tau v\right|_{W^{m,p}(T)} \leq C\,(1 + CR)\, h_T^{2/p-m} \|v\|_{L^\infty(\tau)}. \tag{16}$$

The combination of (13)-(16) yields the assertion

$$\left|u - \mathcal{I}_\tau u\right|_{W^{m,p}(T)} \leq C\,(1 + R)\, h_T^{1+2/p-m} |u|_{H^2(T)}. \qquad \square$$

Now we are able to prove the main result concerning the approximation properties of the proposed composite finite elements. In order to avoid too many technicalities, we assume that there is a constant $\eta_{\text{ext}} > 0$ such that

$$\operatorname{dist}\left(x, \varDelta_x\right) \leq \eta_{\text{ext}} h_{\varDelta_x} \qquad \forall x \in \varTheta_{\text{slave}} \tag{17}$$

and, thus, definition (9) reduces to (8). With Condition (17) at hand one may deduce from (11) the estimate:

$$\left|u(x) - u_{\varDelta_x}(x)\right| \leq C h_T |u|_{H^2(T)} \qquad \forall x \in \varTheta_{\text{slave}}, \tag{18}$$

where T is the minimal scaled version of Δ_x (cf. (10)) such that $x \in T$. The constant C only depends on the minimal angle in Δ_x and the constant η_{ext}.

In the following, we define certain geometric constants which will enter the error estimates.

1. For a triangle τ and a point $x \in \mathbb{R}^2$, let $T_{x,\tau}$ denote the triangle T_R as in Lemma 1, where R is chosen as the minimal number such that x, τ are contained in T_R.
2. For $\tau \in \mathcal{T}_\Gamma$, let $\mathcal{T}_\tau^{\text{ext}} \subset \mathcal{T}_H^{\text{in}}$ denote the set of triangles in $\mathcal{T}_H^{\text{in}}$, which are employed for the extrapolation on $\Theta_{\text{slave}} \cap \tau$:
$$\mathcal{T}_\tau^{\text{ext}} := \{\Delta_z \mid \forall z \in \Theta_{\text{slave}} \cap \tau\}. \tag{19a}$$
The constant N_{ext} is defined by
$$N_{\text{ext}} := \max_{\tau \in \mathcal{T}_\Gamma} \sharp \mathcal{T}_\tau^{\text{ext}} \tag{19b}$$
and $N_{\text{ext}} \sim 1$ expresses the fact that only triangles in a local neighborhood of τ are employed for the extrapolation.
3. Let $\tau \in \mathcal{T}_\Gamma$. For $t \in \text{sons}(\tau)$ and any pair of vertices $x, y \in \mathbf{V}(t)$, let $Q_{t,x,y}$ denote the minimal rectangle, which contains x^Γ, y^Γ, and t with one side being parallel to $\overline{x^\Gamma y^\Gamma}$ (if $x^\Gamma = y^\Gamma$, the alignment condition is skipped).
Let Q_t denote the minimal rectangle which contains $\bigcup_{x,y \in \mathbf{V}(t)} Q_{t,x,y}$
and define the constant C_Q by
$$C_Q := \max_{\tau \in \mathcal{T}_\Gamma} \max_{t \in \text{sons}(\tau)} (\operatorname{diam} Q_t) / h_t. \tag{20}$$
Condition (7) implies that $C_Q = \mathcal{O}(1)$.

For $\tau \in \mathcal{T}_\Gamma$, the minimal ball which contains the set
$$\tau \cup \left(\bigcup_{x \in \Theta_{\text{slave}} \cap \tau} \left(T_{x,\Delta_x} \cup T_{x^\Gamma,\Delta_x} \right) \right) \cup \left(\bigcup_{t \in \text{sons}(\tau)} Q_t \right)$$

is denoted by B_τ. For $\tau \in \mathcal{T}_H^{\text{in}}$ we set $B_\tau = \tau$. The constant C_{uni}, defined by
$$C_{\text{uni}} := \max_{\tau \in \mathcal{T}_\Gamma} \max_{\substack{t \in \mathcal{T}_H \\ t \cap B_\tau \neq \emptyset}} \frac{\operatorname{diam} B_\tau}{h_t}, \tag{21}$$
describes the local quasi-uniformity of the initial overlapping mesh $\mathcal{T}_H$ near the boundary.

The approximation error estimates for the near-boundary triangles $\tau \in \mathcal{T}_\Gamma$ will be decomposed into a sum of error estimates on the sons, $t \in \operatorname{sons}(\tau)$. For each $t \in \operatorname{sons}(\tau)$, these estimates will involve the given function in the neighborhood Q_t of t. As a consequence, a quantity which measures the overlap of such neighborhoods will enter the error estimates. In this light we define, for $\tau \in \mathcal{T}_\Gamma$ and $t \in \operatorname{sons}(\tau)$, the set

$$\mathcal{T}_{\mathrm{ol}}(t) := \left\{ \tilde{t} \in \operatorname{sons}(\tau) : Q_{\tilde{t}} \cap t \neq \emptyset \right\}.$$

The number of elements in $\mathcal{T}_{\mathrm{ol}}(t)$ can be estimated by the following technical lemma.

Lemma 2 *For any $\tau \in \mathcal{T}_\Gamma$ and $t \in \operatorname{sons}(\tau)$, we have*

$$\sharp \mathcal{T}_{\mathrm{ol}}(t) \leq C \left(1 + \log\left(h_\tau / h_t\right)\right),$$

where C only depends on C_Q as in (20) and the shape regularity of the mesh.

Proof Fix $t \in \operatorname{sons}(\tau)$. For $R > 0$, let $B_t(R)$ denote the disc with radius $R > 0$ about the mass center of t. Obviously, there holds $\tau \subset B_t(h_\tau)$. Let L denote the smallest integer such that $2^{-L} h_\tau \leq 8 h_t$. (This implies $h_t \leq 2^{-2-L} h_\tau$ and $L \leq C\left(1 + \log\left(h_\tau / h_t\right)\right)$.) We introduce annular regions about t by

$$A_\ell := B_t\left(2^{-\ell} h_\tau\right) \backslash B_t\left(2^{-\ell-1} h_\tau\right) \qquad \ell = 0, 1, \ldots, L-1$$

(cf. Figure 2) and set $A_L := B_t\left(2^{-L} h_\tau\right)$. For $0 \leq \ell \leq L$, we define (non-disjoint) subsets $\mathcal{T}_{\mathrm{ol}}^{\ell}(t) \subset \mathcal{T}_{\mathrm{ol}}(t)$ by

$$\mathcal{T}_{\mathrm{ol}}^{\ell}(t) := \left\{ \tilde{t} \in \mathcal{T}_{\mathrm{ol}}(t) : \tilde{t} \cap A_\ell \neq \emptyset \right\}.$$

Obviously, we have $\mathcal{T}_{\mathrm{ol}}(t) = \bigcup_{\ell=0}^{L} \mathcal{T}_{\mathrm{ol}}^{\ell}(t)$. For $\ell < L$ we have

$$\operatorname{dist}\left(A_\ell, t\right) \geq 2^{-\ell-1} h_\tau - h_t \geq 2^{-\ell-2} h_\tau,$$

while $\tilde{t} \in \mathcal{T}_{\mathrm{ol}}^{\ell}(t)$ and $Q_{\tilde{t}} \cap t \neq \emptyset$ lead to $\operatorname{diam} Q_{\tilde{t}} \geq 2^{-\ell-2} h_\tau$. The definition of C_Q as in (20) yields the second estimate in

$$2^{-\ell-2} h_\tau \leq \operatorname{diam} Q_{\tilde{t}} \leq C_Q h_{\tilde{t}}.$$

The shape regularity of the triangles leads to the estimate

$$\left|\tilde{t}\right| \geq C C_Q^{-2} 2^{-2\ell-4} h_\tau^2$$

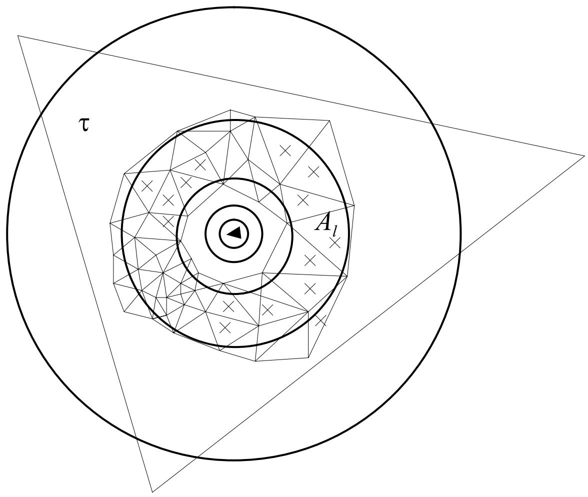

Fig. 2. Triangle $\tau \in \mathcal{T}_\Gamma$ and (black-shaded) son $t \in \text{sons}(\tau)$. The concentric annular regions A_ℓ contain triangles $\hat{t}$ (marked with $\times$), where the boxes $Q_{\tilde{t}}$ intersect t and, hence, belong to $\mathcal{T}_{\text{ol}}^{\ell}(t)$.

of the area of $\tilde{t}$. Since the area of A_ℓ is $3\pi h_\tau^2 2^{-2\ell-2}$, i.e., is of the same order as $\left|\tilde{t}\right|$, it is easy to see that $\sharp\mathcal{T}_{\text{ol}}^{\ell}(t) \leq C$, where C only depends on the shape regularity of the triangles and the constant C_Q. Hence

$$\sum_{\ell=0}^{L-1} \sharp\mathcal{T}_{\text{ol}}^{\ell}(t) \leq CL.$$

It remains to investigate $\sharp\mathcal{T}_{\text{ol}}^{L}(t)$. First, we will show that each $\tilde{t} \in \mathcal{T}_{\text{ol}}^{L}(t)$ satisfies $h_{\tilde{t}} \geq ch_t$. Let

$$U_t := \left\{\tilde{t} \in \mathcal{T}_{\text{ol}}^{L}(t) : \tilde{t} \cap t \neq \emptyset\right\}.$$

The shape regularity of the mesh $\mathcal{T}_{H,h}$ implies that $h_{\tilde{t}} \geq c_1 h_t$ holds for all $\tilde{t} \in U_t$. Now consider $\tilde{t} \in \mathcal{T}_{\text{ol}}^{L}(t) \backslash U_t$. Again from the shape regularity of the mesh $\mathcal{T}_{H,h}$ we conclude $\text{dist}\left(\tilde{t}, t\right) \geq c_2 h_t$. The condition $\tilde{t} \in \mathcal{T}_{\text{ol}}^{L}(t)$ implies $Q_{\tilde{t}} \cap t \neq \emptyset$ and, by taking into account the previous estimate, $\text{diam}\, Q_{\tilde{t}} \geq c_3 h_t$. From the definition of the constant C_Q we conclude

$$c_3 h_t \leq \text{diam}\, Q_{\tilde{t}} \leq C_Q h_{\tilde{t}}.$$

The shape regularity of the mesh directly implies for the area of $\tilde{t}$

$$\left|\tilde{t}\right| \geq c_4 h_t^2 \geq c_4 2^{-6-2L} h_\tau^2.$$

Since the area of A_L is $\pi 2^{-2L} h_\tau^2$, i.e., of the same order as the area of $\tilde{t}$ the number $\sharp\mathcal{T}_{\text{ol}}^{L}(\tau)$ is bounded by constant depending only on the shape regularity of the mesh and the constant C_Q. $\square$

In order to measure the cardinality of the set $\mathcal{T}_{\mathrm{ol}}(t)$ globally we introduce $C_{\mathrm{ol}}^{\mathrm{I}}$ as the minimal constant such that,

$$\sharp\mathcal{T}_{\mathrm{ol}}(\tau) \leq C_{\mathrm{ol}}^{\mathrm{I}} \max_{t\in\mathrm{sons}(\tau)} \left(1+\log\left(h_\tau/h_t\right)\right) =: \widetilde{\log}\left(h_\tau/h_\tau^{\min}\right) \quad \forall\tau\in\mathcal{T}_\Gamma,$$

holds, where $h_\tau^{\min} := \min_{t\in\mathrm{sons}(\tau)} h_t$. For $\tau\in\mathcal{T}_H^{\mathrm{in}}$, we put $\widetilde{\log}\left(h_\tau/h_\tau^{\min}\right) := 1$. The global analogue is

$$\widetilde{\mathrm{Log}}\left(H/h\right) := \max\left\{\widetilde{\log}\left(h_\tau/h_\tau^{\min}\right) : \tau\in\mathcal{T}_\Gamma\right\}.$$

Related to the constant C_{uni} is the second overlap constant $C_{\mathrm{ol}}^{\mathrm{II}}$ defined by

$$C_{\mathrm{ol}}^{\mathrm{II}} := \max_{t\in\mathcal{T}_\Gamma} \sharp\left\{\tau\in\mathcal{T}_H : \left|B_\tau\cap t\right|>0\right\}.$$

Theorem 1 *Let $u\in H_0^1(\Omega)\cap H^2(\Omega)$ and let assumptions (7) and (17) be satisfied. Then, there exists $u^{\mathrm{CFE}}\in S^{\mathrm{CFE}}$ such that*

$$\sqrt{\sum_{t\in\mathrm{sons}(\tau)} \|u-u^{\mathrm{CFE}}\|_{m,t}^2} \leq C h_\tau^{2-m}\widetilde{\log}^{m/2}\left(h_\tau/h_\tau^{\min}\right)|u|_{2,B_\tau} \ \forall\tau\in\mathcal{T}_H, \tag{22}$$

$$\left\|u-u^{\mathrm{CFE}}\right\|_{m,\Omega} \leq C H^{2-m}\widetilde{\mathrm{Log}}^{m/2}\left(H/h\right)\|u\|_{2,\Omega}, \tag{23}$$

where $m=0,1$ and u – in the neighborhood B_τ of the triangle $\tau\in\mathcal{T}$ – is identified with its extension $\mathfrak{E}u$. The constant C only depends on the minimal angles in the triangulation $\mathcal{T}_{H,h}$ and C_{dist}, η_{ext}, C_{ext}, N_{ext}, C_{uni}, $C_{\mathrm{ol}}^{\mathrm{I}}$, $C_{\mathrm{ol}}^{\mathrm{II}}$.

Proof For $u\in H_0^1(\Omega)\cap H^2(\Omega)$, we define the grid function $\mathbf{u}:\Theta_{\mathrm{dof}}\to\mathbb{R}$ by $\mathbf{u}_x := u(x)$, $x\in\Theta_{\mathrm{dof}}$. Let the extension operator $\mathcal{E}$ be as in (8) and let u^{CFE} be the $\mathbb{P}_1$-nodal interpolant of $\mathcal{E}\mathbf{u}$ on $\mathcal{T}_{H,h}$. We identify u with its extension $\mathfrak{E}u$.

We will show that u^{CFE} satisfies the estimates stated in the theorem.

1. Local estimate:

For any $\tau\in\mathcal{T}_H^{\mathrm{in}}$, the function $\left.u^{\mathrm{CFE}}\right|_\tau$ is the affine interpolant on τ of the values $\left(u(x)\right)_{x\in\mathbf{V}(\tau)}$ and the estimate (22) is the standard interpolation estimate (see, e.g., [3]).

Next, we consider $\tau\in\mathcal{T}_\Gamma$. Recall the definition of the set of sons as in (6).

For any $t\in\mathrm{sons}(\tau)$, we can write

$$\|u-u^{\mathrm{CFE}}\|_{m,t} \leq \|u-\mathcal{I}_t u\|_{m,t} + \|\mathcal{I}_t u - u^{\mathrm{CFE}}\|_{m,t}, \tag{24}$$

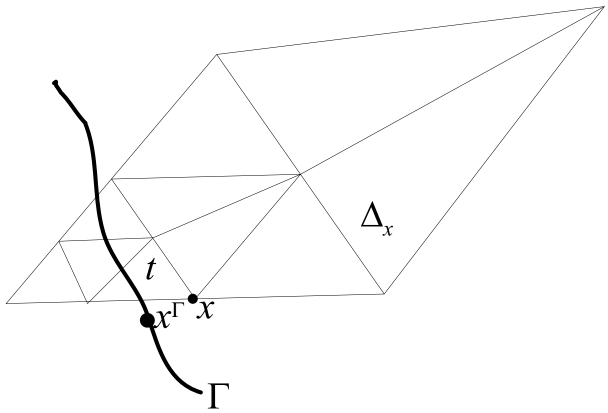

Fig. 3. Slave node x, closest boundary point x^Γ, and closest inner triangle Δ_x.

where, as in (12), $\mathcal{I}_t$ is the Lagrange linear interpolation operator on t, $\mathcal{I}_t : C^0(t) \to \mathbb{P}_1(t)$. For the first term on the right-hand side of (24) we have the standard interpolation estimate

$$\|u - \mathcal{I}_t u\|_{m,t} \leq C h_t^{2-m} |u|_{2,t}, \tag{25}$$

where h_t is the diameter of t. For the second term, we use the inverse estimate (see, e.g. [2, Section 4.5]):

$$\|\mathcal{I}_t u - u^{\mathrm{CFE}}\|_{m,t} \leq C h_t^{1-m} \|\mathcal{I}_t u - u^{\mathrm{CFE}}\|_{L^\infty(t)}. \tag{26}$$

Now we notice that $\|\mathcal{I}_t u - u^{\mathrm{CFE}}\|_{L^\infty(t)} = \max_{x \in \mathbf{V}(t)} \left|\mathcal{I}_t u(x) - u^{\mathrm{CFE}}(x)\right|$. Then, from (26) we obtain

$$\|\mathcal{I}_t u - u^{\mathrm{CFE}}\|_{m,t} \leq C h_t^{1-m} \max_{x \in \mathbf{V}(t)} \left|u(x) - u^{\mathrm{CFE}}(x)\right|. \tag{27}$$

We have

$$u^{\mathrm{CFE}}(x) = \begin{cases} u(x) & \text{if } x \in \Theta_{\mathrm{dof}}, \\ u_{\Delta_x}(x) - u_{\Delta_x}\left(x^\Gamma\right) & \text{if } x \in \Theta_{\mathrm{slave}}, \end{cases}$$

where Δ_x and x^Γ are as in (8). As before, the function $u_{\Delta_x} \in \mathbb{P}_1\left(\mathbb{R}^2\right)$ denotes the unique affine function which interpolates the values of u at the vertices of Δ_x. The case $x \in \Theta_{\mathrm{dof}}$ is trivial. For the other case, $x \in \Theta_{\mathrm{slave}}$, we can write (see Figure 3)

$$\left|u(x) - u^{\mathrm{CFE}}(x)\right| \leq |u(x) - u_{\Delta_x}(x)| + \left|u\left(x^\Gamma\right) - u_{\Delta_x}\left(x^\Gamma\right)\right|, \tag{28}$$

using the fact that $u\left(x^{\Gamma}\right) = 0$. Since $\operatorname{dist}(x, \Delta_x) \leq \eta_{\text{ext}} h_{\Delta_x}$ (cf. (17)) and $\operatorname{dist}(x, x^{\Gamma}) \leq (C_{\text{dist}} + 1)\, h_t$ (cf. (7)), we may infer that both terms on the right-hand side of (28) can be estimated by (18) and we obtain

$$\left|u(x) - u^{\text{CFE}}(x)\right| \leq C h_T \left|u\right|_{2,T} \quad \forall x \in \mathbf{V}(t),$$

where T is a triangle with diameter $h_T \sim (h_{\Delta_x} + h_t)$ which contains x, x^{Γ}, and Δ_x.

In combination with (27), we get

$$\left\|\mathcal{I}_t u - u^{\text{CFE}}\right\|_{m,t} \leq C h_t^{1-m} C h_T \left|u\right|_{2,T}. \tag{29}$$

A summation over all $t \in \operatorname{sons}(\tau)$ yields:

$$\sum_{t \in \operatorname{sons}(\tau)} \left\|\mathcal{I}_t u - u^{\text{CFE}}\right\|_{m,t}^2 \leq C h_T^2 \left|u\right|_{2,T}^2 \sum_{t \in \operatorname{sons}(\tau)} h_t^{2-2m}. \tag{30}$$

The shape regularity of the triangles implies $h_t^2 \sim |t|$ and, for $m = 0$, we obtain

$$\sum_{t \in \operatorname{sons}(\tau)} h_t^2 \leq C \sum_{t \in \operatorname{sons}(\tau)} |t| \leq C\,|\tau| \leq C h_\tau^2.$$

Plugging this estimate into (30) and employing (21) yields

$$\sqrt{\sum_{t \in \operatorname{sons}(\tau)} \left\|\mathcal{I}_t u - u^{\text{CFE}}\right\|_{0,t}^2} \leq C h_\tau^2 \left|u\right|_{2,B_\tau}.$$

For $m = 1$, this estimate becomes too pessimistic since $\sum_{t \in \operatorname{sons}(\tau)} h_t^{2-2m}$ in (30) equals $\sharp\,(\operatorname{sons}\tau)$ and this number cannot be, in general, bounded in terms of h_T.

Hence, for $m = 1$, we refine our analysis as follows. Let $t \in \operatorname{sons}(\tau)$ and let $z \in \mathbf{V}(t)$ be an arbitrary chosen vertex of t.

Then, the function $u^{\text{CFE}}\big|_t$ can be written in the form

$$\begin{aligned} u^{\text{CFE}}\big|_t &= \sum_{x \in \mathbf{V}(t)} \left(u_{\Delta_x}(x) - u_{\Delta_x}\left(x^{\Gamma}\right)\right) b_{x,t} = u_{\Delta_z} - u_{\Delta_z}\left(z^{\Gamma}\right) \\ &\quad + \sum_{x \in \mathbf{V}(t)} \Bigg\{ \underbrace{\left\{u_{\Delta_x}(x) - u_{\Delta_x}\left(x^{\Gamma}\right)\right\} - \left\{u_{\Delta_z}(x) - u_{\Delta_z}\left(x^{\Gamma}\right)\right\}}_{d_1(x)} \\ &\quad - \underbrace{\left(u_{\Delta_z}\left(x^{\Gamma}\right) - u_{\Delta_z}\left(z^{\Gamma}\right)\right)}_{d_2(x)} \Bigg\}\, b_{x,t}. \end{aligned}$$

As before, for any triangle T, the function $u_T \in \mathbb{P}_1\left(\mathbb{R}^2\right)$ is the unique affine interpolation of the values of u at the vertices $\mathbf{V}(T)$. Further, $u_{\Delta_z}\left(z^\Gamma\right)$ is the function on t with constant value $u_{\Delta_z}\left(z^\Gamma\right)$ and $b_{x,t}$ is the finite element basis function on t corresponding to the vertex x. Note that $d_1(x)$ can be rewritten as

$$d_1(x) = \left\langle \nabla u_{\Delta_x} - \nabla u_{\Delta_z}, x - x^\Gamma \right\rangle .$$

Thus,

$$\nabla\left(u^{\mathrm{CFE}} - u\right)\Big|_t = \nabla\left(u_{\Delta_z} - u\right)\big|_t + \sum_{x \in \mathbf{V}(t)} \left\{d_1(x) - d_2(x)\right\} \nabla b_{x,t} \quad (31)$$

and we estimate all three terms separately.

For the first term in (31) we employ Lemma 1 and obtain (recall the definition of $\mathcal{T}_\tau^{\mathrm{ext}}$ and N_{ext} as in (19))

$$\begin{aligned}
\sum_{t \in \mathrm{sons}(\tau)} \left\|\nabla\left(u_{\Delta_z} - u\right)\right\|_{L^2(t)}^2 &= \sum_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \sum_{\substack{t \in \mathrm{sons}(\tau) \\ T = \Delta_z}} \left\|\nabla\left(u_T - u\right)\right\|_{L^2(t)}^2 \\
\leq \sum_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \left\|\nabla\left(u_T - u\right)\right\|_{L^2(\tau)}^2 &\leq N_{\mathrm{ext}} \left\|\nabla\left(u_T - u\right)\right\|_{L^2(\tau)}^2 \\
\leq C h_\tau^2 \left|u\right|_{H^2(B_\tau)}^2 .
\end{aligned}$$

Next, we will consider the term in (31) related to d_2:

$$\sum_{x \in \mathbf{V}(t)} \left(u_{\Delta_z}\left(x^\Gamma\right) - u_{\Delta_z}\left(z^\Gamma\right)\right) \nabla b_{x,t}.$$

The case $x^\Gamma = z^\Gamma$ is trivial and we assume from now on that $z^\Gamma \neq x^\Gamma$. Condition (7) yields

$$\left\|x^\Gamma - z^\Gamma\right\| \leq \left\|x^\Gamma - x\right\| + \left\|x - z\right\| + \left\|z - z^\Gamma\right\| \leq C h_t. \quad (32)$$

By a rotation of the coordinate system we may assume $x^\Gamma = \left(x_1^\Gamma, 0\right)$, $z^\Gamma = \left(z_1^\Gamma, 0\right)$, and t, x^Γ and z^Γ are contained in the minimal axes-parallel rectangle $Q_{t,x,z} = (a_1, b_1) \times (a_2, b_2)$ with $\operatorname{diam} Q_{t,x,z} \leq C h_t$. We employ $u\left(x^\Gamma\right) = u\left(z^\Gamma\right) = 0$ and Hölder's inequality to obtain

$$\begin{aligned}
&\left|u_{\Delta_z}\left(x^\Gamma\right) - u_{\Delta_z}\left(z^\Gamma\right)\right|^2 \\
&= \left|u_{\Delta_z}\left(x^\Gamma\right) - u_{\Delta_z}\left(z^\Gamma\right) - \left(u\left(x^\Gamma\right) - u\left(z^\Gamma\right)\right)\right|^2 \\
&= \left|\int_{x_1^\Gamma}^{z_1^\Gamma} \partial_1\left(u_{\Delta_z} - u\right) ds\right|^2 \leq \left|z_1^\Gamma - x_1^\Gamma\right| \int_{x_1^\Gamma}^{z_1^\Gamma} \left|\partial_1\left(u_{\Delta_z} - u\right)\right|^2 ds \\
&\leq C h_t \int_{a_1}^{b_1} \left|\partial_1\left(u_{\Delta_z} - u\right)\right|^2 ds.
\end{aligned}$$

Integration over $Q_{t,x,z}$ yields, along with an inverse inequality for $\nabla b_{x,t}$,

$$\begin{aligned}
&\left\| \left(u_{\Delta_z}\left(x^\Gamma\right) - u_{\Delta_z}\left(z^\Gamma\right)\right) \nabla b_{x,t} \right\|^2_{L^2(Q_{t,x,z})} \\
&\leq C h_t^{-2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left| u_{\Delta_z}\left(x^\Gamma\right) - u_{\Delta_z}\left(z^\Gamma\right) \right|^2 dx_2 dx_1 \\
&\leq C h_t^{-1} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_1}^{b_1} \left| \partial_1 \left(u_{\Delta_z} - u\right)\right|^2 ds dx_2 dx_1 \\
&\leq C \left| u_{\Delta_z} - u \right|^2_{H^1(Q_{t,x,z})}.
\end{aligned} \tag{33}$$

Thus, the combination of (33) with (31) yields

$$\begin{aligned}
&\sum_{t \in \mathrm{sons}(\tau)} \left\| \sum_{x \in \mathbf{V}(t)} \left(u_{\Delta_z}\left(x^\Gamma\right) - u_{\Delta_z}\left(z^\Gamma\right)\right) \nabla b_{x,t} \right\|^2_{L^2(t)} \\
&= \sum_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \sum_{\substack{t \in \mathrm{sons}(\tau) \\ T = \Delta_z}} \left\| \sum_{x \in \mathbf{V}(t)} \left(u_T\left(x^\Gamma\right) - u_T\left(z^\Gamma\right)\right) \nabla b_{x,t} \right\|^2_{L^2(t)} \\
&\leq C \sum_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \sum_{\substack{t \in \mathrm{sons}(\tau) \\ T = \Delta_z}} \sum_{x \in \mathbf{V}(t)} \left| u_T - u \right|^2_{H^1(Q_{t,x,z})} \\
&\leq C \sum_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \sum_{\substack{t \in \mathrm{sons}(\tau) \\ T = \Delta_z}} \left| u_T - u \right|^2_{H^1(Q_t)} \\
&\leq C \widetilde{\log}\left(h_\tau / h_\tau^{\min}\right) \sum_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \left| u_T - u \right|^2_{H^1(B_\tau)} \\
&\leq C \widetilde{\log}\left(h_\tau / h_\tau^{\min}\right) N_{\mathrm{ext}} \max_{T \in \mathcal{T}_\tau^{\mathrm{ext}}} \left| u_T - u \right|^2_{H^1(B_\tau)} \\
&\leq C \widetilde{\log}\left(h_\tau / h_\tau^{\min}\right) h_\tau^2 \left| u \right|^2_{H^2(B_\tau)}.
\end{aligned}$$

Finally, we will estimate the term in (31) related to d_1. A triangle inequality in combination with condition (7) and an inverse inequality for the basis functions yields

$$\begin{aligned}
\left| \sum_{x \in \mathbf{V}(t)} d_1(x) \nabla b_{x,t} \right| &= \left| \sum_{x \in \mathbf{V}(t)} \left\langle \nabla u_{\Delta_x} - \nabla u_{\Delta_z}, x - x^\Gamma \right\rangle \nabla b_{x,t} \right| \\
&\leq C \max_{x \in \mathbf{V}(t)} \left\| \nabla \left(u_{\Delta_x} - u_{\Delta_z}\right) \right\|,
\end{aligned}$$

where the gradients on the right-hand side are constant vectors in $\mathbb{R}^2$. Thus,

$$
\begin{aligned}
&\sum_{t\in\mathrm{sons}(\tau)} \left\| \sum_{x\in\mathbf{V}(t)} d_1(x)\,\nabla b_{x,t} \right\|^2_{L^2(t)} \\
&\le C \sum_{t\in\mathrm{sons}(\tau)} \sum_{x\in\mathbf{V}(t)} \left\|\nabla\left(u_{\Delta_x}-u_{\Delta_z}\right)\right\|^2 |t| \\
&\le C \sum_{T,\tilde{T}\in\mathcal{T}_\tau^{\mathrm{ext}}} \left\|\nabla\left(u_T-u_{\tilde{T}}\right)\right\|^2 \sum_{\substack{t\in\mathrm{sons}(\tau)\\ \Delta_z=\tilde{T}}} \sum_{\substack{x\in\mathbf{V}(t)\\ \Delta_x=T}} |t| \\
&\le 3C \sum_{T,\tilde{T}\in\mathcal{T}_\tau^{\mathrm{ext}}} \left\|\nabla\left(u_T-u_{\tilde{T}}\right)\right\|^2 |\tau| \\
&\le \tilde{C} N_{\mathrm{ext}}^2 h_\tau^2 \max_{T,\tilde{T}\in\mathcal{T}_\tau^{\mathrm{ext}}} \left\|\nabla\left(u_T-u_{\tilde{T}}\right)\right\|^2 .
\end{aligned}
$$

The estimate

$$
\begin{aligned}
\left\|\nabla\left(u_T-u_{\tilde{T}}\right)\right\|^2 &= \frac{1}{|B_\tau|}\left\|\nabla\left(u_T-u_{\tilde{T}}\right)\right\|^2_{L^2(B_\tau)} \\
&\le C h_\tau^{-2}\left\{\left\|\nabla\left(u_T-u\right)\right\|^2_{L^2(B_\tau)} + \left\|\nabla\left(u_{\tilde{T}}-u\right)\right\|^2_{L^2(B_\tau)}\right\} \\
&\le \tilde{C}\,|u|^2_{H^2(B_\tau)}
\end{aligned}
$$

follows from the neighborhood property (Lemma 1) and finishes the proof of the local estimate.

2. Global estimate:

The global estimate (23) follows immediately from the local one:

$$
\begin{aligned}
\left\|u-u^{\mathrm{CFE}}\right\|^2_{m,\Omega} &\le \sum_{\tau\in\mathcal{T}_H}\sum_{t\in\mathrm{sons}\,\tau} \left\|u-u^{\mathrm{CFE}}\right\|^2_{m,t} \\
&\le C\widetilde{\mathrm{Log}}^m(H/h)\sum_{\tau\in\mathcal{T}_H} h_\tau^{2(2-m)}\|\mathcal{E}u\|^2_{2,B_\tau} \\
&\le CC_{\mathrm{ol}}^{\mathrm{II}}\widetilde{\mathrm{Log}}^m(H/h)\,H^{2(2-m)}\sum_{t\in\mathcal{T}_H}\|\mathcal{E}u\|^2_{2,t} \\
&\le CC_{\mathrm{ol}}^{\mathrm{II}}C_{\mathrm{ext}}^2\widetilde{\mathrm{Log}}^m(H/h)\,H^{2(2-m)}\|u\|^2_{2,\Omega}. \qquad\square
\end{aligned}
$$

Theorem 1 concerns the basic approximation property of the composite finite element space S^{CFE} in the case when the approximated function (we think of the exact solution to our problem) u belongs to $H^2(\Omega)$. However, especially when the polygonal boundary Γ is complicated, it is very likely for the exact solution of the Dirichlet problem to have a lower regularity owing to possible re-entrant corners of the boundary. Thus, we need some generalization of Theorem 1 for the case $u \in H^{1+s}(\Omega)$, $0 \le s \le 1$ (in fact, it would be sufficient to consider $1/2 \le s \le 1$).

First, we need the following result from the interpolation theory of Sobolev spaces.

Lemma 3 *Let Ω be a domain with Lipschitz boundary. Let $\mathcal{L}$ be a linear operator mapping $H^{m_0}(\Omega)$ to $H^{k_0}(\Omega)$ and, also, $H^{m_1}(\Omega)$ to $H^{k_1}(\Omega)$, where m_0, m_1, k_0, k_1 are arbitrary real numbers.*
Then, $\mathcal{L}$ maps $H^{(1-\theta)m_0+\theta m_1}(\Omega)$ to $H^{(1-\theta)k_0+\theta k_1}(\Omega)$ and, moreover,

$$\begin{aligned}&\|\mathcal{L}\|_{H^{(1-\theta)m_0+\theta m_1}(\Omega)\to H^{(1-\theta)k_0+\theta k_1}(\Omega)}\\ &\le \|\mathcal{L}\|^{1-\theta}_{H^{m_0}(\Omega)\to H^{k_0}(\Omega)}\cdot\|\mathcal{L}\|^{\theta}_{H^{m_1}(\Omega)\to H^{k_1}(\Omega)}\end{aligned}$$

for all $\theta \in (0,1)$.

Proof See Proposition (14.1.5) and Theorem (14.2.7) in [2]. □

Now we can prove the generalized approximation property of the space S^{CFE}.

Theorem 2
Let $u \in H^1_0(\Omega)\cap H^{1+s}(\Omega)$, $0 \le s \le 1$. Then, there exists $u^{\mathrm{CFE}} \in S^{\mathrm{CFE}}$ such that

$$\|u-u^{\mathrm{CFE}}\|_{m,\Omega} \le CH^{1+s-m}\widetilde{\mathrm{Log}}^{sm/2}(H/h)\|u\|_{1+s,\Omega}\,, \tag{34}$$

where $m = 0, 1$.

Proof Let $m \in \{0,1\}$ and Let $\mathcal{L}_m u := u - \mathcal{P}^{\mathrm{CFE}}_m(u)$, where $\mathcal{P}^{\mathrm{CFE}}_m(u)$ is the H^m-orthogonal projection of u onto S^{CFE}. Evidently, $\mathcal{L}_m$ is a linear operator, as the projection in Hilbert spaces is a linear operation.

It follows from Theorem 1 that $\mathcal{L}_m$ maps $H^2(\Omega)$ to $H^m(\Omega)$ ($m = 0, 1$) and

$$\|\mathcal{L}_m\|_{H^2(\Omega)\to H^m(\Omega)} \le CH^{2-m}\widetilde{\mathrm{Log}}^{m/2}(H/h)\,.$$

At the same time, $\mathcal{L}_m$ also maps $H^m(\Omega)$ to $H^m(\Omega)$ and

$$\|\mathcal{L}_m\|_{H^m(\Omega)\to H^m(\Omega)} \le 1\,,$$

since $\|u-\mathcal{P}^{\mathrm{CFE}}_m(u)\|_{m,\Omega} \le \|u-0\|_{m,\Omega} = \|u\|_{m,\Omega}$.

Then, according to Lemma 3, $\mathcal{L}_m$ maps $H^{1+s}(\Omega)$, $0 \le s \le 1$, to $H^m(\Omega)$ and

$$\|\mathcal{L}_m\|_{H^{1+s}(\Omega)\to H^m(\Omega)} \le CH^{1+s-m}\widetilde{\mathrm{Log}}^{sm/2}(H/h)\,. \qquad \square$$

Remark 6 The approximation theorems do not pose any restriction on the fine-scale parameter h. In fact, the approximation property of the composite finite element space S^{CFE} holds also when the grid $\mathcal{T}_{H,h}$ coincides with the coarse grid $\mathcal{T}_H$, i.e. in the case $h = \mathcal{O}(H)$.

4 Convergence estimates for the composite finite element solution

The given Dirichlet problem (2), (3) can be recast in the following variational form: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)\,, \tag{35}$$

where

$$a(u,v) := \int_\Omega \langle \nabla u, \nabla v\rangle \; dx\,, \quad (f,v) := \int_\Omega f v\, dx\,. \tag{36}$$

We assume that $f \in L^2(\Omega)$; then, problem (35) has a unique solution.

The approximation of (35) with the composite finite elements leads to the discrete problem: Find $u^{\mathrm{CFE}} \in S^{\mathrm{CFE}}$ such that

$$a(u^{\mathrm{CFE}}, v) = (f,v) \quad \forall v \in S^{\mathrm{CFE}}\,. \tag{37}$$

The unique solvability of problem (37) for any mesh width $H \in (0,1)$ would immediately follow from the Lax-Milgram lemma, if we can show the *uniform coercivity* of the bilinear form $a(\cdot,\cdot)$ on S^{CFE}, i.e.

$$\exists \gamma > 0 \quad \text{s.t.} \quad \gamma \|v\|_{1,\Omega}^2 \le a\,(v,v) \quad \forall v \in S^{\mathrm{CFE}}\,, \tag{38}$$

with the constant γ independent of the mesh parameters H and h.

We prove this result with the help of the following two Lemmas.

Lemma 4 *Let Ω be a bounded domain with Lipschitz boundary Γ. Then, there exists a positive constant C depending only on Ω such that*

$$\|u\|_{1,\Omega} \le C \left(|u|_{1,\Omega} + \left| \int_\Gamma u\, ds \right| \right) \quad \forall u \in H^1(\Omega)\,.$$

Proof See, e.g., Lemma (10.2.20) in [2].

The uniform coercivity will rely on the local mesh width of the triangles $t \in \mathcal{T}_{H,h}$, which intersect the boundary Γ. In this light, for $\tau \in \mathcal{T}_\Gamma$, we introduce the set $\mathrm{sons}_\Gamma(\tau) := \{t \in \mathrm{sons}\,(\tau) : |t \cap \Gamma| > 0\}$; here $|t \cap \Gamma|$ is the length of $t \cap \Gamma$.

Lemma 5 *Suppose that inside of each element $\tau \in \mathcal{T}_\Gamma$ the following conditions are satisfied:*

$$|\tau \cap \Gamma| \leq C h_\tau^{\beta_\tau}\,, \tag{39}$$
$$h_t \leq C h_\tau^{\alpha_\tau} \quad \forall t \in \operatorname{sons}_\Gamma(\tau)\,, \tag{40}$$

with some parameters $\beta_\tau > 0$ and $\alpha_\tau \geq 1$.
Then, we have

$$\|v\|_{L^2(\Gamma)} \leq C|v|_{1,\Omega} \quad \forall v \in S^{\mathrm{CFE}}\,, \tag{41}$$

and if, for all $\tau \in \mathcal{T}_\Gamma$, it holds $\alpha_\tau \geq \max\{1, 2 - \frac{\beta_\tau}{2}\}$, we have

$$\|v\|_{L^2(\Gamma)} \leq CH|v|_{1,\Omega} \quad \forall v \in S^{\mathrm{CFE}}\,, \tag{42}$$

where the constant C is independent of v and the mesh parameters H and h.

Proof We have for any $v \in S^{\mathrm{CFE}}$

$$\begin{aligned}\|v\|^2_{L^2(\Gamma)} &= \sum_{\tau \in \mathcal{T}_\Gamma} \sum_{t \in \operatorname{sons}_\Gamma(\tau)} \|v\|^2_{L^2(t \cap \Gamma)} \\ &\leq \sum_{\tau \in \mathcal{T}_\Gamma} \sum_{t \in \operatorname{sons}_\Gamma(\tau)} |t \cap \Gamma|\, \|v\|^2_{L^\infty(t)}\,.\end{aligned} \tag{43}$$

In order to evaluate $\|v\|_{L^\infty(t)}$, we note that $\|v\|_{L^\infty(t)} = \max_{x \in \mathbf{V}(t)} |v(x)|$. According to the definition of the space S^{CFE}, for any node $x \in \Theta_{\mathrm{slave}}$, we have

$$v(x) = v_{\Delta_x}(x) - v_{\Delta_x}(x^\Gamma),$$

where $\Delta_x \in \mathcal{T}_H^{\mathrm{in}}$ is a nearest inner triangle and $v_{\Delta_x} \in \mathbb{P}_1\left(\mathbb{R}^2\right)$ the analytic (i.e. affine) extension of $v|_{\Delta_x}$ onto $\mathbb{R}^2$. This implies

$$v(x) = \left\langle \nabla v_{\Delta_x}, x - x^\Gamma \right\rangle\,. \tag{44}$$

Next, we fix a triangle $\tau \in \mathcal{T}_\Gamma$. Since $x \in t \in \mathcal{T}_{H,h} \setminus \mathcal{T}_H^{\mathrm{in}}$ we have $|x - x^\Gamma| \leq h_t$ (cf. (7)) and, from (44),

$$|v(x)| \leq h_t \|\nabla v\|_{L^\infty(\Delta_x)}\,.$$

By using an inverse inequality and the local quasi-uniformity (cf. (21)), we get

$$|v\,(x)| \leq C\frac{h_t}{h_\tau}\|\nabla v\|_{L^2(\Delta_x)}\,. \tag{45}$$

Now we denote by $T_\tau \in \mathcal{T}_\tau^{\text{ext}}$ the triangle characterized by

$$\|\nabla v\|_{L^2(T_\tau)} = \max_{T \in \mathcal{T}_\tau^{\text{ext}}} \|\nabla v\|_{L^2(T)} .$$

Then, from (45) we obtain the estimate

$$\|v\|_{L^\infty(t)} \leq C \frac{h_t}{h_\tau} \|\nabla v\|_{L^2(T_\tau)} \quad \forall t \in \text{sons}_\Gamma(\tau) .$$

This estimate and (43) imply

$$\|v\|^2_{L^2(\Gamma)} \leq C \sum_{\tau \in \mathcal{T}_\Gamma} \|\nabla v\|^2_{L^2(T_\tau)} \sum_{t \in \text{sons}_\Gamma(\tau)} |t \cap \Gamma| \left(\frac{h_t}{h_\tau}\right)^2 .$$

Using the assumption (40) and, then, (39), we derive with

$$\delta := \min\{2\alpha_\tau + \beta_\tau - 2 : \tau \in \mathcal{T}_\Gamma\}$$

the estimate

$$\begin{aligned}\|v\|^2_{L^2(\Gamma)} &\leq C \sum_{\tau \in \mathcal{T}_\Gamma} h_\tau^{2\alpha_\tau + \beta_\tau - 2} \|\nabla v\|^2_{L^2(T_\tau)} \\ &\leq C H^\delta \sum_{\tau \in \mathcal{T}_\Gamma} \|\nabla v\|^2_{L^2(T_\tau)} \leq C C_{\text{ol}}^{\text{II}} H^\delta \, \|\nabla v\|^2_{L^2(\Omega)} ,\end{aligned}$$

which immediately yields (41) (since, for all $\tau \in \mathcal{T}_\Gamma$, we assumed $\beta_\tau > 0$, $\alpha_\tau \geq 1$) and (42), if $\alpha_\tau \geq \max\{1,\, 2 - \frac{\beta_\tau}{2} : \tau \in \mathcal{T}_\Gamma\}$. □

Remark 7

1. The condition $\alpha_\tau \geq 1$ in (40) is always satisfied, as $h_t \leq h_\tau \leq H$ holds for all $t \in \text{sons}(\tau)$.
2. The condition $\beta_\tau > 0$ in (39) is obvious, if Γ has a finite length, but it is possible to show that, in fact, $\beta_\tau \geq 1$ for all $\tau \in \mathcal{T}_\Gamma$. To sketch the idea, we consider the quasi-uniform case, where the diameter of all triangles in $\mathcal{T}_H$ are of order H. We argue as follows: Let n_Γ be the number of elements in $\mathcal{T}_\Gamma$; it is clear that n_Γ is not less than $\mathcal{O}(H^{-1})$, i.e., in general, $n_\Gamma = \mathcal{O}(H^{-\beta})$, where $\beta \geq 1$; since $|\Gamma|/n_\Gamma =:$ average_length$(\tau \cap \Gamma)$ and the length of Γ is independent of H, we obtain average_length$(\tau \cap \Gamma) = \mathcal{O}(|\Gamma| \, H^\beta)$ with $\beta \geq 1$.
   In this light, and if we assume that the length $|\Gamma|$ is moderately bounded, condition (39) in Lemma 5 is satisfied with $\beta_\tau \geq 1$. Hence, if we choose (40) with $\alpha_\tau = \max\{1, 2 - \beta_\tau/2\} \leq 3/2$ as the stopping criterion in (5), the estimate (42) will always hold true.

Now we are able to prove the uniform coercivity of the bilinear form $a(\cdot,\cdot)$ on S^{CFE}.

Theorem 3 *Let the assumptions of Lemma 5 be satisfied. The bilinear form $a(\cdot,\cdot)$ defined in (36) is uniformly coercive on S^{CFE}, i. e. , (38) holds with the constant γ independent of H and h.*

Proof For any function $v \in S^{\mathrm{CFE}}$, we have from Lemma 4

$$\|v\|_{1,\Omega} \le C_1 \left(|v|_{1,\Omega} + \left| \int_\Gamma v\, ds \right| \right), \tag{46}$$

since $S^{\mathrm{CFE}} \subset H^1(\Omega)$, and from Lemma 5

$$\|v\|_{L^2(\Gamma)} \le C_2 |v|_{1,\Omega} \tag{47}$$

with the constants C_1 and C_2 independent of v, H and h. Noticing that

$$\left| \int_\Gamma v\, ds \right| \le |\Gamma|^{1/2} \|v\|_{L^2(\Gamma)},$$

where $|\Gamma|$ is the length of Γ, and combining (46) and (47), we obtain (38) with the constant $\gamma = \frac{1}{C_1^2(1+|\Gamma|^{1/2}C_2)^2}$. □

To analyze the rate of convergence of the composite finite element solution u^{CFE} to the exact solution u, we need the following abstract Lemma.

Lemma 6 *Let V and V_h be subspaces of a Hilbert space W. Assume that $a(\cdot,\cdot)$ is a continuous bilinear form on W which is coercive on V_h, with respective continuity and coercivity constants K and γ. Let $u \in V$ solve*

$$a(u,v) = F(v) \quad \forall v \in V,$$

where $F \in W'$. Let $u_h \in V_h$ solve

$$a(u_h,v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Then

$$\|u-u_h\|_W \le \left(1+\frac{K}{\gamma}\right) \inf_{v_h\in V_h} \|u-v_h\|_W + \frac{1}{\gamma} \sup_{w_h\in V_h\setminus\{0\}} \frac{|a(u-u_h,w_h)|}{\|w_h\|_W}.$$

Proof See Lemma (10.1.1) in [2]. □

In our case, W is the space $H^1(\Omega)$, V is $H^1_0(\Omega)$, V_h is S^{CFE}, and the continuity constant K equals 1. Lemma 6 shows that the error in the energy norm consists of two parts: the approximation error and the error stemming from the non-conformity (i.e. from the violation of the Galerkin orthogonality), since, in general, $S^{\mathrm{CFE}} \not\subseteq H_0^1(\Omega)$.

Using this Lemma we can prove the main result on the convergence of the CFE solution u^{CFE}.

Theorem 4 *Let the exact solution* u *to problem (35) belong to* $H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \leq s \leq 1$. *Let the conditions of Lemma 5 be satisfied with* $\alpha_\tau = \max\{1, 2 - \frac{\beta_\tau}{2} : \tau \in \mathcal{T}_\Gamma\}$. *Then, for sufficiently small* H, *there holds*

$$\|u - u^{\mathrm{CFE}}\|_{1,\Omega} \leq C H^s \, \widetilde{\mathrm{Log}}^{s/2}(H/h) \|u\|_{1+s,\Omega} \, .$$

Proof The approximation error can be immediately estimated by virtue of Theorem 2 as

$$\inf_{v \in S^{\mathrm{CFE}}} \|u - v\|_{1,\Omega} \leq C H^s \, \widetilde{\mathrm{Log}}^{s/2}(H/h) \|u\|_{1+s,\Omega} \, . \tag{48}$$

To estimate the non-conformity error, we first note that, for all $w \in S^{\mathrm{CFE}} \subset H^1(\Omega)$, we have

$$\begin{aligned} a(u - u^{\mathrm{CFE}}, w) = a(u, w) - (f, w) &= \int_\Omega (-\Delta u) w \, dx + \int_\Gamma \frac{\partial u}{\partial n} w \, ds \\ &\quad - \int_\Omega f w \, dx = \int_\Gamma \frac{\partial u}{\partial n} w \, ds \, . \end{aligned}$$

Thus, using the Cauchy-Schwarz inequality, we get

$$|a(u - u^{\mathrm{CFE}}, w)| \leq \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma)} \|w\|_{L^2(\Gamma)}$$

and, with the trace theorem,

$$|a(u - u^{\mathrm{CFE}}, w)| \leq C \|u\|_{3/2,\Omega} \|w\|_{L^2(\Gamma)} \quad \forall w \in S^{\mathrm{CFE}} \, .$$

Combining the latter inequality and (42) of Lemma 5, we derive the estimate for the non-conformity error:

$$\sup_{w \in S^{\mathrm{CFE}} \setminus \{0\}} \frac{|a(u - u^{\mathrm{CFE}}, w)|}{\|w\|_{1,\Omega}} \leq C H \|u\|_{3/2,\Omega} \, . \tag{49}$$

The result of the Theorem follows from Lemma 6, (48) and (49). $\square$

Remark 8

1. Since we assume $f \in L^2(\Omega)$, the regularity of the exact solution $u \in H^{1+s}(\Omega)$, $1/2 \leq s \leq 1$, is typical for the two-dimensional Dirichlet problem on a polygonal domain. The maximal possible regularity $u \in H^2(\Omega)$ may deteriorate to $u \in H^{3/2}(\Omega)$ because of the boundary's re-entrant corners whose angles are close to 2π.
2. In the situation described in Remark 7, the conditions of Lemma 5 are always satisfied and (42) holds true, if the stopping criterion in (5) is chosen such that the smallest triangles in $\mathcal{T}_{H,h}$, which are used for the resolution of the boundary, satisfy $h \leq CH^{3/2}$. Thus, the latter condition is sufficient to obtain the quasi-optimal error-estimate of Theorem 4.

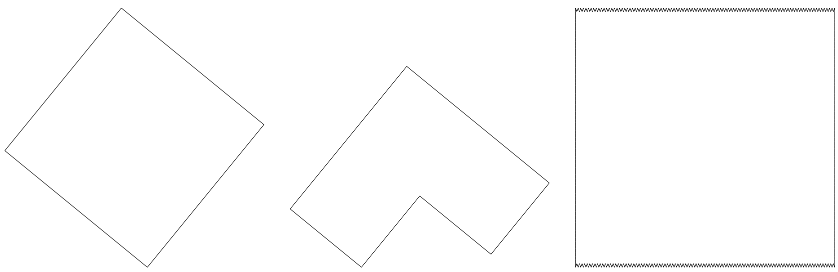

Fig. 4. Unit square, L-shape and zic-zac domain.

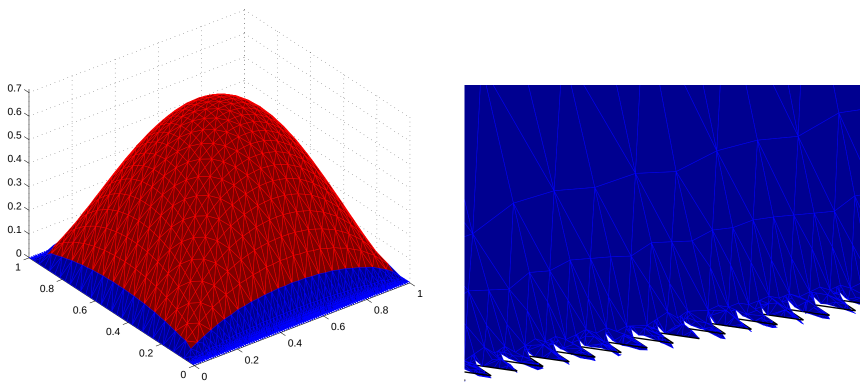

Fig. 5. Left: Two-scale CFE-solution on zic-zac-domain. Right: Zoom into boundary region.

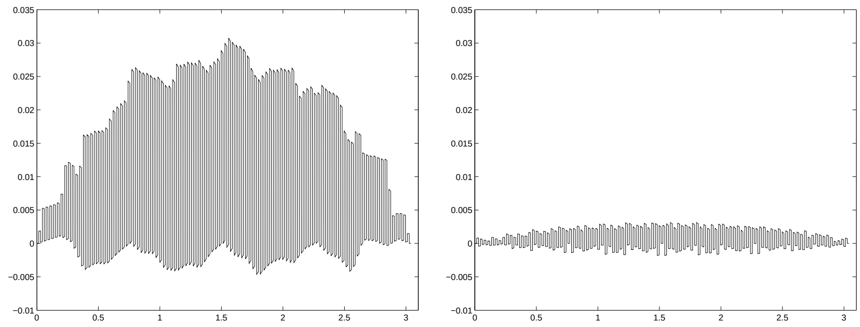

Fig. 6. Trace of CFE_1- (left) and CFE_2-solution (right) on part of the boundary of zic-zac domain Ω.

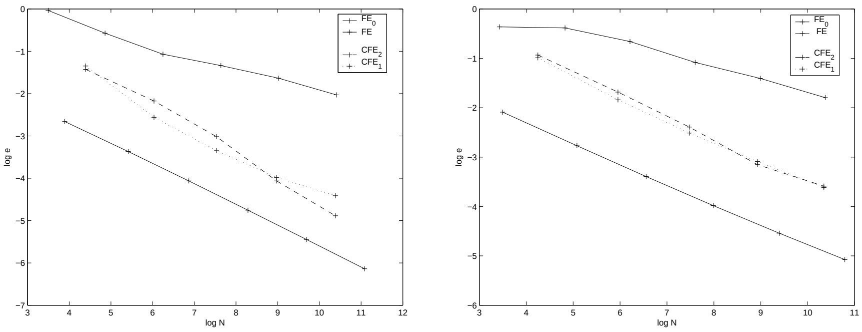

Fig. 7. H^1-error versus N = #dof: Square (left), L-shape domain (right). Observe that only CFE_2 converges at optimal rate as the standard finite element method. The ratio between the relative error of CFE_2 to FE, depending on N, is about 2-3. The ratio between the error of FE_0 to CFE_2 rises up to 17 on the finest mesh.

5 Numerical experiments

Typical applications for the CFE-method are boundary value problems on domains with a large number of geometric details. As an illustrative example we have chosen the Poisson model problem on a domain with 200 re-entering corners (cf. Fig. 4, right). However, we emphasize that our approach is by no means restricted to a periodic or regular distribution of the geometric details. Figure 5 shows the two-scale CFE solution and the non-conforming, adaptive approximation of the Dirichlet boundary conditions.

Figure 6 displays the (one-dimensional) traces of the one-scale and the two-scale solution along the zic-zac boundary (Remark 1 1.) and illustrates how the error in the approximation of the boundary conditions is reduced by using the two-scale approach.

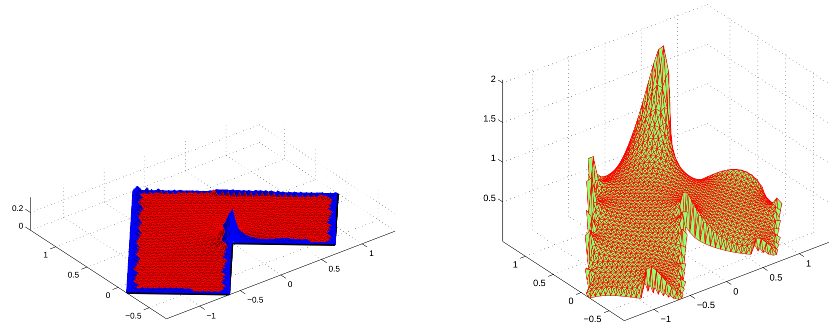

Fig. 8. Error of CFE_2 (1769 dof) and FE_0 (2002 dof) on L-shape domain.

# dof	CFE_1	# dof	CFE_2	# dof	FE	# dof	FE_0
81	0.25975	81	0.24007	225	0.03447	129	0.56466
417	0.07730	417	0.11373	961	0.01721	517	0.34355
1869	0.03516	1869	0.04891	3969	0.00862	2073	0.26244
7893	0.01869	7893	0.01720	16129	0.00431	8277	0.19477
32313	0.01213	32313	0.00755	65025	0.00216	33097	0.13132

Table 1. Ω: Unit square. Relative H^1-error on different levels and corresp. number of dof.

# dof	CFE_1	# dof	CFE_2	# dof	FE	# dof	FE_0
70	0.37232	70	0.39336	161	0.06271	125	0.68137
386	0.15888	386	0.18581	705	0.03359	499	0.51745
1769	0.08113	1769	0.09143	2945	0.01866	2002	0.33884
7557	0.04547	7557	0.04273	12033	0.01066	8009	0.24497
31138	0.02687	31138	0.02762	48641	0.00625	32026	0.16600

Table 2. Ω: L-shape domain. Relative H^1-error and corresp. number of dof.

In previous sections we proved convergence for the CFE-method. We showed, that – up to logarithmic terms – the error of the CFE-solution u^{CFE} behaves like

$$\|u - u^{\mathrm{CFE}}\|_{1,\Omega} \leq C\, H^s\, \|u\|_{1+s,\Omega}\,,$$

provided that $u \in H_0^1(\Omega) \cap H^s(\Omega)$, $s \in [\frac{1}{2}, 1]$. In order to validate the sharpness of the theoretical estimates and the size of the constants therein we have performed various parameter tests. We consider the problem (2), (3) on the unit square (full regularity) resp. L-shape domain (reduced regularity). In order to study the effect of an overlapping mesh and the approximation of the Dirichlet boundary conditions we have chosen the initial mesh such that the intersections of triangles with the boundary are of general shape.

The reference solutions are

$$u_{\mathrm{s}}(x) = (1 - x_1^2)\,(1 - x_2^2) \qquad \text{(Unit square)}$$
$$u_{\mathrm{L}}(x) = u_{\mathrm{s}}(x) \cdot r^{2/3} \sin \frac{2}{3}\varphi \qquad \text{(L-shape domain)}$$

and the right-hand sides are chosen accordingly. We have taken these examples to determine the convergence rates *systematically.* We emphasize that the typical applications for the two-scale CFE-method are very complicated domains (cf. Fig. 4, right). We have compared the convergence rates of the one- and the two-scale method with the standard FEM in order to study the quantitative convergence behaviour. As a further alternative we have considered the following method: Replace the physical domain Ω by the domain

$$\Omega_H \;=\; \bigcup_{\tau \in \mathcal{T}_H} \tau\,,$$

covered by the overlapping coarse scale mesh and apply standard FEM. The distance from the artificial boundary $\partial\Omega_H$ to the physical boundary Γ is in general of order $\mathcal{O}(H)$, especially if the length scales of the geometric details vary continuously over a large range. Theory predicts a suboptimal convergence rate 1/2. We compare the following methods:

1. One-scale Composite Finite Element Method (CFE_1),
2. Two-scale Composite Finite Element Method (CFE_2),
3. Standard FEM (FE),
4. Standard FEM on overlapping domain Ω_H (FE_0).

The errors are listed in Table 1, 2, and depicted in Figure 7, 8.

Observations

– CFE_2 (cf. Fig. 7 dashed line) shows optimal convergence and distinguishes from FE (lower solid line) just by a constant factor of about 2 - 3. CFE_1 (dotted line) asymptotically has suboptimal convergence rates. The energy-error, e. g. for $N = 32313$, differs by a factor 1.6 for the unit square (cf. Tab. 1).
– The experiments support the theoretically predicted suboptimal convergence rate of FE_0 (upper solid line) (Figure 7).

– Since the CFE basis functions have larger support near to the boundary, which can be interpreted as a discretization with slightly increased mesh width, the error distribution concentrates at the boundary Γ (Figure 8, left). A remarkable detail is that the CFE error is concentrated sharply at the domain boundary and does not significantly pollute the accuracy in the interior of Ω. In contrast, the error of FE_0 is larger not only at the boundary but smeared into the whole domain Ω. (Fig. 8, right). Note that the construction of the extrapolation process ensures that – depending on the regularity of the exact solution – the optimal convergence rate is preserved. The FE_0- and the CFE_1-meshes are identical and the two-scale mesh of CFE_2 differs just by the introduction of additional *slave* nodes. The computational complexity of CFE_1 and CFE_2 however is the same because the numerical integration is carried out in both cases on the two-scale mesh. For an efficient algorithmic realization we refer to [5] and [7].

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