Composite Finite Elements and Multigrid Lecture Notes of the Zürich Summerschool 02.

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Abstract

These lecture notes comprise the talks on composite finite elements and multigrid methods given at the Zürich Summerschool 02.

The theory of composite finite elements and various applications have been developed since the end of the ninetieth and reviewed in this paper. Basic knowledge in the theory of finite elements and multigrid methods is required.

1 Introduction

In many physical applications such as environmental modelling, lightweight constructions, or modelling of engines with very complicated shape, the assumptions

- that the coefficients of the governing equations are periodic are often violated, hence, the application of analytic homogenisation for periodic media is not possible,
- that the mesh for the computations is generated by a refinement process starting from a mesh with very few unknowns is unrealistic.

Let us mention various scales and parameters (there are certainly much further ones) in the applications which we have in mind:

a. The size and distribution of geometrical scales in the physical domain and in the coefficients of the PDE. Since these scales vary continuously (depending on the physical model) from very large sizes (e.g., in oceanography) to very small scales (e.g., for small islands in the sea or complicated local structure of the sea shore) one has to distinguish between relevant information

   – which are explicitly contained in the concrete deterministic model (e.g., geometric scales contained in the chosen (idealised) geometric approximation to the true physical domain),
   – which are available only as statistical information (will not be discussed in these notes).

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b. Ranges of parameters, e.g., lower and upper bounds, which appear in the governing equations.

c. The accuracy requirements for the numerical method.

d. The available computing power.

Although, we do not claim that we have (or are close to having) a general solution procedure for this abstract problem class it is important to understand a concrete model problem always in an appropriate larger (physical) picture.

In the following, we will formulate a class of model problems and explain the ranges of parameters for which we want to develop a numerical solution method. This class allows (a) domains with very complicated geometric structures, e.g., the geometric length scales may vary over orders of magnitude, (b) coefficients in the differential operator which are piecewise smooth on subregions, while these subregions might have very complicated geometric structure.

Our model problem consists of the most simple model which is adequate to develop the new numerical approaches and allows the conceptual generalisation to more general applications.

2 Model Problem

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz-domain with boundary $\Gamma := \partial\Omega$. We assume that Γ is partitioned into two disjoint, open and measurable subsets Γ_D, Γ_N such that $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$.

Let $H^1(\Omega)$ denote the Sobolev space of all functions in $L^2(\Omega)$ which have weak derivatives in $L^2(\Omega)$. We define the Sobolev space $V := H^1_D(\Omega)$ as

$$H^1_D(\Omega) := \left\{ u \in H^1(\Omega) : u = 0 \text{ in the sense of traces} \right\}$$

and the bilinear form $b : V \times V \to \mathbb{R}$ by

$$b(u, v) := \int_\Omega \left(\langle a \nabla u, \nabla v \rangle + uv \right) dx.$$

We assume that the coefficient matrix $a = (a_{i,j})_{i,j=1}^d \in \mathbb{R}^{d\times d}$ satisfies $a_{i,j} \in L^\infty(\Omega)$ and, for any $x \in \Omega$, is positive definite, more precisely, is symmetric and satisfies

$$0 < a_{\min} := \inf_{x\in\Omega} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle a(x)\, v, v\rangle}{\langle v, v\rangle} \leq \sup_{x\in\Omega} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \frac{\langle a(x)\, v, v\rangle}{\langle v, v\rangle} =: a_{\max} < \infty \tag{2.1}$$

Moreover, we assume that there exists open, disjoint subdomains $(\omega_i)_{i=1}^q \subset \Omega$ such that the restrictions $a_i := a|_{\omega_i}$, $1 \leq i \leq q$, are smooth, e.g., $a_i \in C^\infty(\omega_i)$ for simplicity, and

$$\Omega = \operatorname{int} \overline{\bigcup_{i=1}^q \omega_i},$$

where $\operatorname{int}(M)$ denotes the interior of a set $M \subset \mathbb{R}^d$. The union of the boundaries $\partial\omega_i$ forms the skeleton

$$\gamma := \bigcup_{i=1}^q \partial\omega_i. \tag{2.2}$$

For given right-hand side $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$ and $g_D \in H^{1/2}(\Gamma_D)$, we are seeking $u \in H^1(\Omega)$ with $u = g_D$ on Γ_D such that

$$b(u,v) = (f,v)_{L^2(\Omega)} + (g_N,v)_{L^2(\Gamma_N)} \qquad \forall v \in H^1_D(\Omega). \tag{2.3}$$

This variational problem arises, e.g., from the following strong formulation. Let us first introduce some notation. For a function $\zeta : \Omega \to \mathbb{R}$, we denote the restriction $\zeta|_{\omega_i}$ by ζ_i. If we assume that γ is orientable (in this case, crossing points in γ are forbidden), we may associate an oriented normal field $n \in L^\infty(\gamma)$ to $\gamma \cup \Gamma_N$ which is oriented to the exterior of Ω at Γ_N. Thus, for a function in $w \in \Pi_{i=1}^q (C^2(\omega_i) \cap C^0(\overline{\omega_i}))$ we may define the jump $[w]_\gamma \in C^0(\gamma)$ of w by

$$[w]_\gamma(x) := \lim_{\varepsilon \to +0} (w(x+\varepsilon n) - w(x-\varepsilon n))$$

for $x \in \gamma$ a.e. The strong formulation corresponding to problem (2.3) is given by seeking $u \in \Pi_{i=1}^q (C^2(\omega_i) \cap C^0(\overline{\omega_i}))$ such that

$$\begin{array}{rl} -a_i \Delta u_i = f_i & \text{in } \omega_i, \quad 1 \le i \le q, \\ [u]_\gamma = [\partial u/\partial \tilde{n}]_\gamma = 0 & \\ u = g_D & \text{on } \Gamma_D, \\ \partial u/\partial \tilde{n} = g_N & \text{on } \Gamma_N, \end{array} \tag{2.4}$$

where $\tilde{n} = an$ and $\partial u/\partial \tilde{n} = \langle an, \nabla u \rangle$ denotes the conormal derivative of u at γ.

2.1 Standard Finite Elements

Standard discretisations of problem (2.3) require a triangulation which resolves the interface, i.e., γ can be parametrised smoothly by some edges of the triangulation. We introduce the following notations along with some assumptions.

Let $\mathcal{T} := \{\tau_1, \ldots, \tau_N\}$ denote a triangulation of Ω (implying that Ω is a polygonal domain) which is regular in the sense of Ciarlet [1]. We assume that the Dirichlet portion Γ_D of the boundary is exactly matched by the union of some edges in $\mathcal{T}$. The space of continuous, piecewise linear finite elements on $\mathcal{T}$ is denoted by $\mathcal{S}$ and the subspace containing all function in $\mathcal{S}$ with zero-boundary conditions at Γ_D by $\mathcal{S}_D$. Let $\tilde{g}_D$ denote an approximation of g_D which is the trace of some function in $\mathcal{S}$.

The Galerkin finite element discretisation is given by seeking $u_\mathcal{T} \in \mathcal{S}$ with $u_\mathcal{T} = \tilde{g}_D$ such that

$$b(u_\mathcal{T}, v) = (f,v)_{L^2(\Omega)} + (g_N,v)_{L^2(\Gamma_N)} \qquad \forall v \in H^1_D(\Omega).$$

Next, we will investigate the convergence of this discretisation method and introduce the relevant parameters. Let $h_\tau := \operatorname{diam} \tau$ and $h := h_\mathcal{T} := \max_{\tau \in \mathcal{T}} h_\tau$. The shape regularity of the triangles is measured by

$$C_{sr} := \max_{\tau \in \mathcal{T}} h_\tau / \rho_\tau,$$

where ρ_τ denotes the radius of the maximal inscribed sphere in τ.

Theorem 2.1 *Let the boundary Γ and γ be smooth. Assume that either Γ_D or Γ_N is the empty set. For sufficiently smooth data f and g_D resp. g_N the exact solution satisfies $u \in H^1_D(\Omega) \cap \Pi_{i=1}^q H^2(\omega_i)$, $A\nabla u \in H^1(\Omega)$ and $u \in H^{3/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.*

For a proof we refer, e.g., to [7, Sec. 10.1.1]. Note that the constants in the regularity estimates depend on the structure of the geometry and the coefficient function a, i.e, on Ω and ω_i. They depend on the ratio $a_{\max}/a_{\min}$ as well.

Some consequences are listed below:

1. If the triangulation overlap the interface γ, the Galerkin error in the energy norm can be estimated from above by $Ch^{1/2-\varepsilon}$ for any $\varepsilon > 0$ and this is suboptimal compared to the convergence for globally smooth solutions $u \in H^2(\Omega)$, which is Ch.

2. The condition that the triangulation has to resolve the boundary of the domain and the interface implies that the minimal number of triangles for the discretisation of the problem is related to the number and size of the *geometric details* in the problem.

3. For problems, where the minimal number of triangles is huge such that no further refinement is possible (e.g., due to computer limitations), a straightforward application of multigrid methods is not obvious since no coarse discretisation is available.

4. For model problems (see, e.g., [9]), it can be proved that if the number of holes in the domain increases while their distances decreases (say ε is a measure of the minimal distance between two distinct holes), then the regularity estimate is of the form

$$\|u\|_{H^{1+s}(\Omega)} \leq C\varepsilon^{-s} \|f\|_{L^2(\Omega)},$$

   and the convergence is spoiled by ε, behaves like $(h/\varepsilon)^s$. (This implies that the step size, e.g., has to be chosen as $h = \varepsilon^2$ so that the Galerkin energy error is smaller than $C\varepsilon$.)

3 Composite Finite Elements

In this chapter we introduce the concept of composite finite elements. The principal idea is that these new finite elements allow to adapt hierarchically the *shape* of the finite element functions to the characteristic behaviour of the solution. Thus, they allow the discretisation of problems with complicated structures with very few unknowns. The adaptation process depends on the local structure of the problem and has to be developed for different classes of applications separately.

3.1 Representation of the data

We assume that Ω is a polygonal domain with possibly a huge number of vertices. The skeleton γ of the inclusions ω_i, $1 \leq i \leq q$, is as in (2.2). We assume that the geometry of the problem is resolved by a triangulation $\mathcal{G} = \{\tau_1, \ldots, \tau_{N_g}\}$ consisting of open, disjoint triangles with

$$\Omega = \operatorname{int} \overline{\bigcup_{i=1}^{N_g} \tau_i}.$$

The set of (open) triangle edges in $\mathcal{G}$ is denoted by $\mathcal{E}$ and we assume that there exist subsets $\mathcal{E}_D$, $\mathcal{E}_N$, $\mathcal{E}_\gamma$ such that

$$\Gamma_D = \bigcup_{e \in \mathcal{E}_D} e, \qquad \Gamma_N = \bigcup_{e \in \mathcal{E}_N} e, \qquad \gamma = \bigcup_{e \in \mathcal{E}_\gamma} e. \tag{3.1}$$

Remark 3.1 *(a) Condition (3.1) can be relaxed. By using curved finite elements as, e.g., isoparametric finite elements it is sufficient to assume that the boundary pieces Γ_D, Γ_N, and γ are smooth images of some edges in the triangulation.*

(b) For very complicated domains the assembling of a triangulation is a non-trivial task. For two-dimensional problems, the problem of triangulation of polygons is solved and software packages are available for this purpose. For three-dimensional domains the problem is much harder and is the topic of current research in various group.

(c) We emphasize that $\mathcal{G}$ is not necessarily the computational grid for the definition of the finite element space but is considered as a representation of the data on the computer. In this light, there is no conformity requirement and the triangulation may contain hanging nodes. However, in certain applications, the grid $\mathcal{G}$ and the computational grid $\mathcal{T}$ will be chosen identically.

3.2 Pure Neumann Problem

We start with the most simple model problem by considering the pure Neumann problem, i.e., $\Gamma = \Gamma_N$ and $\Gamma_D = \emptyset$, and smooth diffusion coefficient, $a \in C^\infty(\Omega)$. To reduce technicalities we assume $a \equiv 1$ and consider the problem of finding $u \in V := H^1(\Omega)$ such that

$$\int_\Omega (\langle \nabla u, \nabla v \rangle + uv)\, dx = F(v) \qquad \forall v \in V, \tag{3.2}$$

where the functional $F \in V'$ is related to a strong formulation (cf. (2.4)) by

$$F(v) = (f, v)_{L^2(\Omega)} + (g_N, v)_{L^2(\Gamma)}.$$

We distinguish between the following two applications.

1. The goal is to compute a Galerkin approximation to (3.2) including the discretisation process, while we assume that the geometry of the problem is represented as described in Section 3.1.

2. A finite element discretisation of problem (3.2) is *given*, i.e., a triangulation which resolves the geometry, a system of linear equations with possibly huge dimension, and a link between algebraic degrees of freedom (solution vector) and the corresponding finite element function. The goal in this application is to develop a method for solving the linear system efficiently.

For these two cases, the definition of composite finite element spaces differs slightly.

3.2.1 Composite finite elements for the discretisation of the pure Neumann problem

For the discretisation of the pure Neumann problem on complicated domains with, possibly, very few unknowns, it is necessary to relax the condition that the computational grid resolves the domain. We replace this condition by the overlap conditions

$$\Omega \subset \overline{\bigcup_{\tau\in\mathcal{T}} \tau} \quad \text{and} \quad \forall \tau \in \mathcal{T} : |\tau \cap \Omega| > 0, \tag{3.3}$$

where, for a measurable set $M \subset \mathbb{R}^d$, the notation $|M|$ denotes the d-dimensional measure of M.

It is evident that, for any bounded domain, there exists a triangulation $\mathcal{T}$ with very few elements which satisfies this condition.

Remark 3.2 *We emphasize that the following extension of the variational formulation (3.2) to the larger domain* $\Omega_{\mathcal{T}} := \operatorname{int} \bigcup_{\tau\in\mathcal{T}} \overline{\tau}$ *would result in a much too* ***large*** *discretisation error and we will not employ the* ***standard*** *finite element space* $\mathcal{S}(\Omega_{\mathcal{T}})$ *on* $\Omega_{\mathcal{T}}$ *for the discretisation.*

Find $u^\star \in \mathcal{S}(\Omega_{\mathcal{T}})$ *such that*

$$\int_{\Omega_{\mathcal{T}}} (\langle \nabla u^\star, \nabla v\rangle + uv)\, dx = \int_{\Omega_{\mathcal{T}}} f^\star v dx + \int_{\partial\Omega_{\mathcal{T}}} g_N^\star v ds \qquad \forall v \in \mathcal{S}(\Omega_{\mathcal{T}}),$$

where $f^\star$ *denotes some extension of* f *onto the larger domain* $\Omega_{\mathcal{T}}$ *and* $g_N^\star$ *denotes some Neumann data on* $\partial\Omega_{\mathcal{T}}$ *which should correspond to the given Neumann data* g_N *on* $\partial\Omega$*. The approximation* $u - u^\star|_\Omega$*, in general, will lead to* ***large*** *approximation errors due to the replacement of the original domain* Ω *by* $\Omega_{\mathcal{T}}$ *(cf. [1, Sec. 4.4]).*

Definition 3.3 *The composite finite element space* $\mathcal{S}^{CFE}$ *for problem (3.2) is given by restricting the functions in* $\mathcal{S}(\Omega_{\mathcal{T}})$ *to the domain* Ω *:*

$$\mathcal{S}^{CFE} := \{ u|_\Omega : u \in \mathcal{S}(\Omega_{\mathcal{T}}) \}.$$

The functions in $\mathcal{S}^{CFE}$ are affine on the intersections $\tau \cap \Omega$. Hence, the reason for using the word "composite" in the notation of the finite element space is not obvious. This will become more clear in Section 3.2.2 (cf. Remark 3.13).

The finite element discretisation to problem (3.2) takes the form: Find $u \in \mathcal{S}^{CFE}$ such that

$$\int_\Omega (\langle \nabla u, \nabla v\rangle + uv)\, dx = (f, v)_{L^2(\Omega)} + (g_N, v)_{L^2(\Gamma)} \qquad \forall v \in \mathcal{S}^{CFE}. \tag{3.4}$$

Although the definition of the composite finite element space looks rather simple, its efficient implementation is not quite obvious. To highlight the difficulties we introduce the basis representation of (3.4).

Let $(b_i)_{i=1}^n$ denote the standard nodal finite element basis in $\mathcal{S}(\Omega_{\mathcal{T}})$. A basis in $\mathcal{S}^{CFE}$ is given by the restrictions

$$b_i^{CFE} := b_i|_\Omega, \qquad 1 \le i \le n.$$

Hence, every function in $v \in \mathcal{S}^{CFE}$ has the representation

$$v = \sum_{i=1}^{n} \mathbf{v}_i b_i^{CFE} \tag{3.5}$$

with a coefficient vector $\mathbf{v} = (\mathbf{v}_i)_{i=1}^n \in \mathbb{R}^n$. The equation in (3.5) defines a mapping $P : \mathbb{R}^n \to \mathcal{S}^{CFE}$ by $v = P\mathbf{v}$. The inverse mapping is denoted by $R : \mathcal{S}^{CFE} \to \mathbb{R}^n$, $Rv = \mathbf{v}$.

By replacing u in (3.4) by the basis representation and testing with the basis functions lead to the system of linear equations

$$\mathbf{B}\mathbf{v} = \mathbf{r}$$

with

$$\left.\begin{aligned} \mathbf{B}_{i,j} &:= \textstyle\int_\Omega \left(\langle \nabla b_i, \nabla b_j \rangle + b_i b_j\right) dx \\ \mathbf{r}_i &:= (f, b_i)_{L^2(\Omega)} + (g_N, b_i)_{L^2(\Gamma)} \end{aligned}\right\} \qquad 1 \le i, j \le n.$$

As usual the matrix $\mathbf{B}$ and $\mathbf{r}$ is assembled via element matrices and vectors which, for composite finite elements, are of the form

$$\left.\begin{aligned} \mathbf{B}_{i,j}^\tau &= \int_{\tau\cap\Omega} \left(\langle \nabla b_i, \nabla b_j \rangle + b_i b_j\right) dx, \\ \mathbf{f}_i^\tau &:= \int_{\tau\cap\Omega} f b_i dx + \int_{\tau\cap\Gamma} g_N b_i dx, \\ \mathbf{g}_i^e &:= \textstyle\int_{e\cap\Gamma} g_N b_i. \end{aligned}\right\} \qquad \forall i, j : x_i, x_j \text{ are vertices in } \tau. \tag{3.6}$$

Recall that the triangles are open and the third integral corresponds to the case that the intersection of a triangle edge with the boundary Γ has positive measure. Note that the third integral cannot be avoided by replacing in the second integral $\tau \cap \Gamma$ by $\overline{\tau} \cap \Gamma$ since some of the edges appear twice in the summation over all triangles and some might appear only once.

The representation of the element matrices and vectors in (3.6) highlights the difficulty for the realisation of composite finite elements: The development of efficient numerical quadrature methods for the computation of the integral over the intersections $\tau \cap \Omega$, $\tau \cap \Gamma$, and $e \cap \Gamma$.

The following subsection is concerned with the efficient implementation of composite finite elements.

Implementation In this subsection, we will introduce quadrature methods for the efficient computation of the integrals (3.6).

Let Ω denote a polygonal domain with N_P vertices. According to the general assumptions in Section 3.2.1 we assume that a triangulation $\mathcal{G} = \left\{\tau_1, \ldots, \tau_{N_g}\right\}$ for the description of the geometry of the domain is given as described in 3.1. If the triangulation $\mathcal{G}$ contains large triangles in the interior of Ω and is graded towards the geometric details at the boundary the assumption $N_P \sim N_g$ is realistic.

In a typical numerical discretisation process, not only one discretisation has to be generated but a sequence of discretisations with increasingly higher resolution. Our goal is to set up an algorithm which generates such a sequence of discretisations related to a sequence of overlapping grids $(\mathcal{T}_\ell)_{\ell=0}^L$ with a computational and memory complexity of $O\left(N_g + \sum_{\ell=0}^L N_\ell\right)$, where N_ℓ denotes the number of elements in $\mathcal{T}_\ell$.

Since, in our applications, $N_\ell \ll N_g$ such a complexity cannot be achieved if for every grid $\mathcal{T}_\ell$, all elements in $\mathcal{G}$ are touched since then, the complexity would be LN_g.

Remark 3.4 *In many applications, the number of elements in $\mathcal{T}_\ell$ is increased, at least, by a fixed number $C_{ref} > 1$, i.e.,*

$$N_\ell \geq C_{ref} N_{\ell-1} \qquad \forall \ell = 1, 2, \ldots, L.$$

In this case, $\sum_{\ell=0}^{L} N_\ell = \sum_{\ell=0}^{L} N_L C_{ref}^{-\ell} \leq \frac{C_{ref}}{C_{ref}-1} N_L$ and, hence, $N_g + \sum_{\ell=0}^{L} N_\ell \leq O\left(N_g + N_L\right)$ holds.

The algorithm starts with a very coarse triangulation $\mathcal{T}_0$. The principal idea for the evaluation of the integrals in (3.6) is to subdivide the triangles in $\mathcal{T}_0$ and to use a composite quadrature rule. The subdivision is driven by the criterion: "A triangle t of the actual subdivision will be **further** refined if the intersection $t \cap \Omega$ **cannot** be triangulated by very few triangles." For the efficiency of the algorithm it is essential that any triangle (and the information generated on this triangle) which is generated for this *quadrature* procedure can be re-used if the *discretisation* process will arrive at that triangle.

Next we will explain the details of the algorithm. The essential ingredients are (a) a refinement pattern for a triangle and (b) a criterion for stopping the subdivision process.

Definition 3.5 *For any triangle τ, the procedure **refine** subdivides τ into two disjoint triangles τ_1, τ_2 by connecting the midpoint of a longest edge in τ with the opposite vertex. Formally this is expressed by **refine**$(\tau) = \operatorname{sons}(\tau) = \{\tau_1, \tau_2\}$.*

Definition 3.6 *For any triangle τ, the intersection $\tau \cap \Omega$ is **simple** if $|\tau \cap \Omega| > 0$ and $\overline{\tau \cap \Omega}$ can be subdivided into at most three triangles. The set of these triangles is denoted by $\mathcal{G}(\tau)$.*

By using the function **refine**(τ), it is straightforward to generate, for each triangle, $\tau \in \mathcal{T}_0$ a tree[1] $\mathbb{T}_\tau$ with the following properties

1. τ is the root of the tree,
2. the leaves in the tree consist of triangles which are simple. The set of leaves is denoted by $\mathcal{L}_\tau$.
3. Each element in the tree is a triangle t with the property that either t is a leaf or there exists a minimal subset $sons(t) \subset \mathbb{T}_\tau \setminus \{t\}$ such that the elements of $sons(t)$ are disjoint triangles satisfying

$$\overline{t \cap \Omega} = \left(\bigcup_{t' \in sons(t)} \overline{t' \cap \Omega} \right) \subset \left(\bigcup_{t' \in sons(t)} \overline{t'} \right) \subset \overline{t}. \tag{3.7}$$

The important property of this tree structure is that the integral over a triangle can be written as a recursion

$$\int_{t \cap \Omega} w(x)\, dx = \begin{cases} \displaystyle\sum_{t' \in sons(t)} \int_{t' \cap \Omega} w(x)\, dx & \text{if } t \in \mathbb{T}_\tau \setminus \mathcal{L}_\tau, \\ \displaystyle\sum_{t' \in \mathcal{G}(t)} \int_{t' \cap \Omega} w(x)\, dx & \text{if } t \in \mathcal{L}_\tau, \text{ i.e., } t \text{ is simple.} \end{cases} \tag{3.8}$$

[1]Usually a tree is a graph (V, E) with vertices V and edges E having a certain structure. Here the structure will be given by the sons of the vertices, while V is identified with $\mathbb{T}$.

To obtain an efficient multilevel quadrature scheme we will employ the hierarchy of basis function in $\mathcal{S}^{CFE}$. For a triangle τ with vertices $P_{\tau,i}$, $1 \le i \le 3$, we denote by $(b_{\tau,i})_{i=1}^{3}$ the set of affine functions on τ with

$$b_{\tau,i}\left(P_{\tau,j}\right) := \begin{cases} 1 & i = j \\ 0 & i \neq 0 \end{cases} \qquad 1 \le i,j \le 3. \tag{3.9}$$

The Lagrange property of the basis functions (3.9) implies that, for any $t \subset \tau$, we have

$$b_{\tau,i} = \sum_{j=1}^{3} \mathbf{p}_{i,j}^{\tau,t} b_{t,j}, \tag{3.10}$$

where the local prolongation matrix $\mathbf{p}^{\tau,t} \in \mathbb{R}^{3\times 3}$ is defined by

$$\mathbf{p}_{i,j}^{\tau,t} := b_{\tau,i}\left(P_{t,j}\right) \qquad 1 \le i,j \le 3.$$

The local element matrix $\mathbf{B}^{\tau} \in \mathbb{R}^{3\times 3}$ (cf. 3.6) has the representation

$$\mathbf{B}_{i,j}^{\tau} = \int_{\tau\cap\Omega} \left(\langle \nabla b_{\tau,i}, \nabla b_{\tau,j}\rangle + b_{\tau,i} b_{\tau,j}\right) dx.$$

By combining the functional hierarchy (3.10) with the tree structure (3.8) we obtain

$$\mathbf{B}^{t} = \begin{cases} \displaystyle\sum_{t' \in sons(t)} \mathbf{p}^{t,t'} \mathbf{B}^{t'} \left(\mathbf{p}^{t,t'}\right)^{\intercal} & \text{if } t \in \mathbb{T}_{\tau} \backslash \mathcal{L}_{\tau}, \\ \displaystyle\sum_{t' \in \mathcal{G}(t)} \mathbf{p}^{t,t'} \mathbf{B}^{t'} \left(\mathbf{p}^{t,t'}\right)^{\intercal} & \text{if } t \in \mathcal{L}_{\tau}. \end{cases} \tag{3.11}$$

The algorithm for computing the element matrices $\mathbf{B}^{\tau}$ for all coarse grid triangles $\tau \in \mathcal{T}_0$ is structured as follows:

1. For all $\tau \in \mathcal{T}_0$: Generate and store the tree $\mathbb{T}_{\tau}$.
2. For all $\tau \in \mathcal{T}_0$, $t \in \mathcal{L}_{\tau}$ and $t' \in \mathcal{G}(t)$: Compute and store the element matrices $\mathbf{B}^{t}$.
3. For all $\tau \in \mathcal{T}_0$ and $t \in \mathbb{T}_{\tau}$: Compute and store $\mathbf{B}^{t}$ according to the recursion (3.11).

Remark 3.7 *The computational and memory complexity of this algorithm is* $O\left(\sum_{\tau\in\mathcal{T}_0}\sum_{t\in\mathbb{T}_{\tau}}\sum_{t'\in\mathcal{G}(t)} 1\right)$. *Under mild assumptions, one can prove that* $\sum_{\tau\in\mathcal{T}_0}\sum_{t\in\mathbb{T}_{\tau}}\sum_{t'\in\mathcal{G}(t)} 1 \sim N_P$, *i.e., the complexity is proportional to the number of details in the geometry of* Ω.

Remark 3.8 *The computation of the remaining integrals in (3.6) can be performed in a similar fashion.*

One important property of the algorithm is its perfect hierarchical structure. If the initial mesh $\mathcal{T}_0$ is refined for *improving* the discretisation, the refinement can be chosen such that all triangles $t \in \mathcal{T}_{\ell}$, $\ell \ge 1$, have the property that

- either $t \subset \Omega$ and the computation of the element matrix is straightforward,
- or $t \in \mathbb{T}_{\tau}$ for some $\tau \in \mathcal{T}_0$ and the corresponding element matrix $\mathbf{B}^{t}$ was already computed and stored during the *quadrature* computation for the element matrix $\mathbf{B}^{\tau}$.

Thus, the amount of work for generating the discretisation for finer triangulations $\mathcal{T}_{\ell}$, $1 \le \ell \le L$, is *as for standard finite elements.*

Note that this algorithm is applicable without any conceptual changes to three-dimensional problems, higher order elements, quadrilaterals elements, or isoparametric elements.

3.2.2 Composite finite elements for the solution of the system of linear equations for the pure Neumann problem

In this subsection, we will consider another important application for composite finite elements. In many commercial computer codes, a mesh generator is employed which generates directly a mesh $\mathcal{T}$ for the discretisation. If the domain is complicated, such meshes are, typically, unstructured, contain a huge number of triangles and -due to restrictions of computer power- cannot be refined furthermore. Then, the system of linear equations is assembled and the *goal* is to solve this system efficiently. In this section we will develop a variant of composite finite elements for solving this problem efficiently.

Notation: $\mathcal{T}$ denotes a given unstructured mesh and

$$\mathbf{B}\mathbf{u} = \mathbf{r} \tag{3.12}$$

denotes the corresponding system of linear equations which has to be solved efficiently.

Note that, in the situation described above, the given mesh $\mathcal{T}$ serves also as the description of the domain.

Our aim is to solve equation (3.12) by a multi-grid method. Since no hierarchy of coarse discretisations is at hand, the essential step for the application of multi-grid methods is the construction of such a hierarchy from the equations (3.12) by an appropriate agglomeration process. We will show that composite finite elements can be used for this coarsening process.

Abstract Multi-grid Method based on Galerkin products We begin we recapitulating the abstract form of a multi-grid method where the intergrid transfer (prolongation/restriction) is defined by Galerkin products.

Let $(\mathcal{S}_\ell)_{\ell=0}^L$ denote a family of finite element spaces which is nested

$$\mathcal{S}_0 \subset \ldots \subset \mathcal{S}_L.$$

The dimension of $\mathcal{S}_\ell$ is denoted by n_ℓ and the basis by $(b_{\ell,i})_{i=1}^{n_\ell}$. For $1 \leq \ell \leq L$, let $\mathbf{p}_{\ell,\ell-1}$ denote the basis representation of the injection $\iota_{\ell,\ell-1} : \mathcal{S}_{\ell-1} \to \mathcal{S}_\ell$, $\iota_{\ell,\ell-1}u = u$, i.e.,

$$\left(\sum_{i=1}^{n_\ell} \mathbf{w}_i b_{\ell,i} = \sum_{i=1}^{n_{\ell-1}} \mathbf{v}_i b_{\ell-1,i}\right) \Longleftrightarrow \mathbf{w} = \mathbf{p}_{\ell,\ell-1}\mathbf{v}.$$

We assume that the system of linear equations on the finest level $\mathcal{S}_L$ is *given*:

$$\mathbf{B}_L\mathbf{u}_L = \mathbf{r}_L.$$

In a first step, a sequence of matrices $\mathbf{B}_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is computed via the *Galerkin product*

$$\mathbf{B}_{\ell-1} := \mathbf{p}_{\ell,\ell-1}^\intercal \mathbf{B}_\ell \mathbf{p}_{\ell,\ell-1}, \qquad \ell = L, L-1, \ldots, 1. \tag{3.13}$$

Next, for $1 \leq \ell \leq L$, a smoother $\mathfrak{S}_\ell$ has to be chosen for linear systems with coefficient matrix $\mathbf{B}_\ell$. The application of ν smoothing steps to the system

$$\mathbf{B}_\ell \mathbf{v}_\ell = \mathbf{r}_\ell$$

is denoted by $\mathbf{v}_\ell^{(\nu)} := \mathfrak{S}_\ell^{(\nu)}(\mathbf{v}_\ell, \mathbf{r}_\ell)$.

The following multigrid algorithm requires as an input only

1. the fine grid system, i.e., $\mathbf{B}_L$, $\mathbf{r}_L$,

2. the prolongation matrices $\mathbf{p}_{\ell,\ell-1}$, $1 \le \ell \le L$,

3. the smoothing procedure $\mathfrak{S}_\ell^{(\nu)}$,

4. the parameters ν_1, $\nu_2 \in \mathbb{N}$ specifying the number of pre- and postsmoothing steps and a parameter $\gamma \in \{1, 2\}$ controlling whether a V- or a W-cycle is employed (for details we refer to [6]).

Algorithm 3.9 *The multi-grid algorithm is called by*

$$\mathbf{u}_L := \mathbf{0}; \qquad \boldsymbol{mg}\,(\mathbf{u}_L, \mathbf{r}_L, L)\,;$$

and defined by

procedure mg$(\mathbf{u}_\ell, \mathbf{r}_\ell, \ell)\,;$
begin
  if $(\ell = 0)$ **then** $\mathbf{u}_\ell := \mathbf{B}_\ell^{-1}\mathbf{r}_\ell$ **else begin**
    $\mathbf{u}_\ell := \mathfrak{S}_\ell^{(\nu_1)}\,(\mathbf{u}_\ell, \mathbf{r}_\ell)\,;$
    $\mathbf{d}_\ell := \mathbf{B}_\ell\mathbf{u}_\ell - \mathbf{r}_\ell;$
    $\mathbf{d}_{\ell-1} := \mathbf{p}_{\ell,\ell-1}^{\intercal}\mathbf{d}_\ell;$
    $\mathbf{v}_{\ell-1} := \mathbf{0};$
    for $j := 1$ **to** γ **do mg**$(\mathbf{v}_{\ell-1}, \mathbf{d}_{\ell-1}, \ell - 1)\,;$
    $\mathbf{u}_\ell := \mathbf{u}_\ell - \mathbf{p}_{\ell,\ell-1}\mathbf{v}_{\ell-1};$
    $\mathbf{u}_\ell := \mathfrak{S}_\ell^{(\nu_2)}\,(\mathbf{u}_\ell, \mathbf{f}_\ell)\,;$
  end;
end;

The crucial step for our application is the definition of the prolongation matrices $\mathbf{p}_{\ell,\ell-1}$. We will not discuss here special choices of smoothers here. Instead, we will refer to [5], [4], where it is proved that, even for very complicated problems, standard smoothers as the Jacobi or Gauß-Seidel method yield grid-independent convergence rates.

Construction of the prolongation matrices via composite finite elements As the input of the construction procedure, we assume that a mesh $\mathcal{T}$ and a corresponding finite element discretisation of the Neumann problem is given in the form of a system of linear equations, i.e., by the matrix $\mathbf{B}$ and the vector $\mathbf{r}$ in (3.12).

In a first step of the algorithm we will generate a sequence of meshes $(\mathcal{T}_\ell)_{\ell=0}^L$ which satisfy the conditions (3.3) and form a tree in the sense that, for any $\ell < L$ and any $\tau \in \mathcal{T}_\ell$, there exists a set of sons, $\mathit{sons}\,(\tau) \subset \mathcal{T}_{\ell+1}$, satisfying condition (3.7). (Note that we do not require condition (3.7) for the pair $(\mathcal{T}_L, \mathcal{T})$. Such a tree structure can be easily generated recursively by specifying a refinement pattern and a stopping criterion. The refinement pattern may be chosen as in Definition 3.5 while other refinement patterns such as, e.g., connecting midpoint of edges, might be chosen as well.

The stopping criterion in this situation should be such that the *finest* auxiliary grid $\mathcal{T}_L$ is slightly coarser than the given grid but has a similar triangle distribution as $\mathcal{T}$.

Definition 3.10 *For any triangle τ the function* ***stop***(τ) *has the value "true" if τ contains at most 4 triangles of $\mathcal{T}$ and is "false" otherwise.*

For details of an efficient implementation we refer to [3].

The composite finite element spaces are defined by specifying an appropriate coarsening process. The essential step is to define $\mathcal{S}_L^{CFE}$ corresponding to the finest *auxiliary* grid $\mathcal{T}_L$.We assume that $\Omega = \operatorname{int} \overline{\bigcup_{\tau\in\mathcal{T}} \tau}$ and introduce, for $0 \le \ell \le L$, the domains covered by the triangulation $\mathcal{T}_\ell$ by

$$\Omega_\ell := \operatorname{int} \overline{\bigcup_{\tau\in\mathcal{T}_\ell} \tau}.$$

Since the grids $\mathcal{T}_\ell$ satisfies (3.3) we have

$$\Omega \subset \Omega_L \subset \Omega_{L-1} \subset \ldots \subset \Omega_0. \tag{3.14}$$

The set of nodal points of $\mathcal{T}$ are denoted by $\Theta = \{x_i : 1 \le i \le n\}$ and, for $0 \le \ell \le L$, the corresponding sets of grid points for the auxiliary grids $\mathcal{T}_\ell$ by $\Theta_\ell = \{x_{\ell,1}, x_{\ell,2}, \ldots, x_{\ell,n_\ell}\}$. Let $\mathcal{S}_\ell$ (resp. $\mathcal{S}$) denote the *standard* finite element space on the grid $\mathcal{T}_\ell$ (resp. on $\mathcal{T}$).

In view of the nestedness of the domains (3.14), we introduce interpolation operators $I_L : \mathcal{S}_L \to \mathcal{S}_\mathcal{T}$ and, for $1 \le \ell \le L$, operators $I_{\ell,\ell-1} : \mathcal{S}_{\ell-1} \to \mathcal{S}_\ell$ by

$$I_L u := \sum_{i=1}^{n} u(x_i)\, b_i \quad \text{and} \quad I_{\ell,\ell-1} u := \sum_{i=1}^{n_\ell} u(x_{\ell,i})\, b_{\ell,i}. \tag{3.15}$$

The composition of the interpolation operator over several grids yields, for $0 \le \ell \le L$, the iterated interpolation $I_\ell : \mathcal{S}_\ell \to \mathcal{S}_\mathcal{T}$

$$I_\ell := I_L \circ I_{L,L-1} \circ I_{L-1,L-2} \circ \ldots \circ I_{\ell+1,\ell}. \tag{3.16}$$

Definition 3.11 *The composite finite element space $\mathcal{S}_\ell^{CFE}$ is the range of $\mathcal{S}_\ell$ under the mapping I_ℓ :*

$$\mathcal{S}_\ell^{CFE} := \{I_\ell u : u \in \mathcal{S}_\ell\}.$$

Remark 3.12 *The definition of the composite finite element spaces implies the nestedness*

$$\mathcal{S}_0^{CFE} \subset \mathcal{S}_1^{CFE} \subset \ldots \subset \mathcal{S}_L^{CFE} \subset \mathcal{S}_\mathcal{T}.$$

Remark 3.13 *The last inclusion in (3.7) implies*

$$I_{\ell,\ell-1} u = u|_{\Omega_\ell} \qquad \forall u \in S_{\ell-1}. \tag{3.17}$$

Since the interpolation $I_L : \mathcal{S}_L \to \mathcal{S}_\mathcal{T}$ is on non-nested grids $\mathcal{T}_L$, $\mathcal{T}$, property (3.17) does not hold in general for I_L. Instead, we have

$$I_\ell u = I_L\left(u|_{\Omega_L}\right) \qquad \forall u \in S_\ell.$$

Hence, a finite element function in $\mathcal{S}_\ell^{CFE}$ is not affine on triangles of $\mathcal{T}_\ell$ but composed of continuous, piecewise linear pieces on triangles of $\mathcal{T}$. This property motivates the word "composite" in the notation "composite finite elements".

For the algorithmic realisation of the composite finite element spaces, we introduce the basis representation of definition (3.15). Let $\mathbf{v} \in \mathbb{R}^{n_\ell - 1}$. The corresponding finite element function $v \in \mathcal{S}_{\ell-1}$ is

$$v = \sum_{i=1}^{n_{\ell-1}} \mathbf{v}_i b_{\ell-1,i}.$$

The interpolation $I_{\ell,\ell-1}v$ has the representation

$$I_{\ell,\ell-1}v = \sum_{i=1}^{n_{\ell-1}} \mathbf{v}_i I_{\ell,\ell-1} b_{\ell-1,i} = \sum_{i=1}^{n_{\ell-1}} \mathbf{v}_i \sum_{j=1}^{n_\ell} b_{\ell-1,i}\left(x_{\ell,j}\right) b_{\ell,j}.$$

In this light, we introduce the global prolongation matrix $\mathbf{p}_{\ell,\ell-1} \in \mathbb{R}^{n_\ell \times n_{\ell-1}}$ by

$$\left(\mathbf{p}_{\ell,\ell-1}\right)_{i,j} := b_{\ell-1,j}\left(x_{\ell,i}\right), \qquad 1 \le i \le n_\ell,\, 1 \le j \le n_{\ell-1}.$$

This definition implies

$$\left(I_{\ell,\ell-1}v = \sum_{i=1}^{n_\ell} \mathbf{w}_i b_{i,\ell}\right) \Longleftrightarrow \mathbf{w} = \mathbf{p}_{\ell,\ell-1}\mathbf{v}.$$

The prolongation matrix $\mathbf{p}_L$ for the grids $\mathcal{T}_L$ and $\mathcal{T}$ is defined in an analogous way.

Thus, we have defined prolongation matrices for the pure Neumann problem which allows the application of the multigrid method (cf. Algorithm 3.9).

For the computation of the agglomerated system matrices $\mathbf{B}_\ell$ via Galerkin products (3.13), it is essential to realise the prolongation locally. The following remark concerns the sparsity pattern of $\mathbf{p}_{\ell,\ell-1}$.

Definition 3.14 *For a nodal point $x_{\ell,i}$ of the grid $\mathcal{T}_\ell$, we associate a coarser triangle $\tau_{\ell-1}\left(x_{\ell,i}\right) \in \mathcal{T}_{\ell-1}$ by the condition*

$$x_{\ell,i} \in \overline{\tau_{\ell-1}\left(x_{\ell,i}\right)}. \tag{3.18}$$

If this choice is non-unique, we fix one of all possible triangles.

Remark 3.15 *The prolongation matrix is sparse in the sense*

$$\left(\mathbf{p}_{\ell,\ell-1}\right)_{i,j} \neq 0 \Longleftrightarrow j \text{ is a vertex of } \tau_{\ell-1}\left(x_{\ell,i}\right).$$

Thus, a matrix row in $\mathbf{p}_{\ell,\ell-1}$ contains at most three non-zero entries.

Hence, the prolongation $\mathbf{p}_{\ell,\ell-1}$ can be realised locally by
(a) associating to any point $x_{\ell,i}$ one coarse grid triangle $\tau_{\ell-1}\left(x_{\ell,i}\right)$ satisfying (3.18).
(b) linking the vertices of $\tau_{\ell-1}\left(x_{\ell,i}\right)$ to $x_{\ell,i}$ and writing the prolongation weight

$$\left(\mathbf{p}_{\ell,\ell-1}\right)_{i,j} = b_{\ell,i}\left(x_j\right), \qquad \text{for all vertices } x_j \text{ of } \tau_{\ell-1}\left(x_{\ell,i}\right)$$

into this link.

The Galerkin product can be evaluated by standard procedures for the multiplication of sparse matrices.

Remark 3.16 *The construction for the local realisation of the prolongation matrices can be applied verbatim to the prolongation matrix $\mathbf{p}_L$ for the grids $\mathcal{T}_L$ and $\mathcal{T}$.*

3.3 Pure Dirichlet problem

In this subsection, we will introduce composite finite elements for Dirichlet boundary conditions, i.e., $\Gamma_D = \Gamma$ and $\Gamma_N = \emptyset$.

We start with the most simple model problem, i.e., $g_D = 0$ and restrict to smooth diffusion coefficient, $a \in C^\infty(\Omega)$. To reduce technicalities we assume $a \equiv 1$ and consider the problem of finding $u \in V_0 := H_0^1(\Omega)$ such that

$$\int_\Omega (\langle \nabla u, \nabla v \rangle + uv)\, dx = F(v) \qquad \forall v \in V_0, \tag{3.19}$$

where the function $F \in V_0'$ is related to a strong formulation (cf. (2.4)) by

$$F(v) = (f, v)_{L^2(\Omega)}.$$

The essential boundary conditions imply that the composite finite element space as in Definition 3.3 is not contained in V_0. One possibility to circumvent this problem is to introduce the boundary conditions as Lagrange multipliers in the bilinear form. We choose the other possibility, i.e., to modify Definition 3.3 to handle Dirichlet boundary conditions since it will turn out that this modification is quite simple, while the use of Lagrange multipliers has further implications (the elliptic problem becomes a saddle point problem; line integral have to evaluated; the boundary has to be meshed and this additional mesh has to satisfy some compatibility conditions compared to the mesh of Ω).

The conceptual idea is the same as for Neumann boundary conditions. We start with the standard, conforming finite element space on an overlapping mesh and adapt in a hierarchical way over a sequence of meshes the functions to the required Dirichlet boundary conditions on the true mesh.

The construction of composite finite elements for problems with Dirichlet boundary conditions consists of four steps.

Phase a: Overlapping grid

Let $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_n\}$ denote a finite element grid consisting of (open) triangles without hanging nodes. We assume that $\mathcal{T}$ is an overlapping grid, i.e., satisfying (3.3). The vertices of $\mathcal{T}$ are denoted by $\Theta = \{x_i : 1 \le i \le n\}$ and the vertices of a triangle τ by $\mathbf{V}(\tau)$.

Phase b: Marking the degrees of freedom

Next, we will define the "free nodes" where the degrees of freedom will be located and the "slave nodes" where the function values are constraint. For a nodal point $x_i \in \mathcal{T}$, we introduce the triangle neighbourhood of x_i by

$$\mathcal{T}_i := \{\tau \in \mathcal{T} : x_i \in \overline{\tau}\}.$$

Let

$$\Theta^\Gamma := \bigcup_{\tau \in \mathcal{T} : |\tau \cap \Gamma| > 0} \mathbf{V}(\tau) \quad \text{and} \quad \mathcal{T}^\Gamma := \bigcup_{i \in \Theta^\Gamma} \mathcal{T}_i.$$

The complements are $\Theta^{in} := \Theta \backslash \Theta^\Gamma$, $\mathcal{T}^{in} := \mathcal{T} \backslash \mathcal{T}^\Gamma$, and we introduce the "interior" domain

$$\Omega^{in} := \operatorname{int}\left(\bigcup_{\tau \in \mathcal{T}^{in}} \overline{\tau} \right).$$

The degrees of freedom are associated with the nodes in Θ^{in}.

Phase c: Definition of an extrapolation operator

Let $\mathcal{S}^{in}$ denote the standard finite element space for the domain Ω^{in} (without boundary conditions) and the triangulation $\mathcal{T}^{in}$, i.e.,

$$\mathcal{S}^{in} := \left\{ u \in C^0\left(\Omega^{in}\right) \mid \forall \tau \in \mathcal{T}^{in} : u|_{\tau} \in \mathbb{P}_1 \right\},$$

where $\mathbb{P}_1$ is the space of affine functions and $\mathcal{S}$ is the standard finite element space for the domain Ω

$$\mathcal{S} := \left\{ u \in C^0\left(\Omega\right) \mid \forall \tau \in \mathcal{T} : u|_{\tau} \in \mathbb{P}_1 \right\}.$$

Next, we will define an extrapolation operator $\mathcal{E} : \mathcal{S}^{in} \to \mathcal{S}$. First we have to introduce three notations.

1. For a function $v \in \mathcal{S}^{in}$ and any $\tau \in \mathcal{T}^{in}$, let $v_{\tau}^{\star} : \mathbb{R}^2 \to \mathbb{R}$ denote the analytic extension, i.e., extension by "itself", of $v|_{\tau}$ to $\mathbb{R}^2$.

2. For a point $x \in \mathbb{R}^2$ let $\tau_x \in \mathcal{T}^{in}$ denote a triangle with minimal distance from x.

3. For a point $x \in \mathbb{R}^2$, let $x^{\Gamma} \in \Gamma$ denotes a point with minimal distance from x.

Choose a control parameter $\eta > 0$ which controls for a slave node $x \in \Theta^{\Gamma}$ the distance to its associated triangle $\tau_x \in \mathcal{T}^{in}$.

Let $v \in \mathcal{S}^{in}$. It is sufficient to specify the values of $\mathcal{E}v$, $v \in \mathcal{S}^{in}$, at the nodal points Θ, since the function $\mathcal{E}v$ is the unique nodal interpolation of these values. For any $x \in \Theta$, we define

$$\left(\mathcal{E}v\right)(x) := \begin{cases} v\left(x\right) & \text{if } x \in \Theta^{in}, \\ v_{\tau_x}^{\star}\left(x\right) - v_{\tau_x}^{\star}\left(x^{\Gamma}\right) & \text{if } x \in \Theta^{\Gamma} \quad \text{and} \quad \operatorname{dist}\left(x, \tau_x\right) \leq \eta \operatorname{diam} \tau_x, \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

Phase d: Definition of composite finite elements with homogeneous boundary conditions

Next, we will use a hierarchy of grids to gradually adapt the finite element space $\mathcal{S}$ to the homogeneous boundary conditions.

In order to indicate the level in the grid hierarchy, we denote the grid $\mathcal{T}$ by $\mathcal{T}_{\ell}$ and add the index ℓ to all quantities related to the grid $\mathcal{T}$, e.g., write $b_{\ell,i}^{CFE}$, Θ_{ℓ}^{in}, Θ_{ℓ}^{Γ}, $\mathcal{T}_{\ell}^{in}$, $\mathcal{T}_{\ell}^{\Gamma}$, S_{ℓ}^{CFE}, $\mathcal{E}_{\ell}$ instead of b_i^{CFE}, Θ^{in}, Θ^{Γ}, $\mathcal{T}^{in}$, $\mathcal{T}^{\Gamma}$, S^{CFE}, $\mathcal{E}_{\ell}$. We use the notation "basic grid" for $\mathcal{T}_{\ell}$.

We assume that we have generated a sequence $(\mathcal{T}_{\lambda})_{\lambda=\ell}^{L}$ of grids having the following tree-like structure:

- for all $\ell < \lambda \leq L$ and all $\tau \in \mathcal{T}_{\lambda}$ there exists a *father triangle* $t \in \mathcal{T}_{\lambda-1}$ with $\tau \subset t$. Vice versa τ is a *son* of t and the set of all sons of t is denoted by $sons\left(t\right)$.

The near-boundary and inner grids $\mathcal{T}_{\lambda}^{in}$, $\mathcal{T}_{\lambda}^{\Gamma}$ and corresponding set of vertices Θ_{λ}^{in} and $\Theta_{\lambda}^{\Gamma}$ are defined analogously as for the basic grid $\mathcal{T}$.

The composite finite element space, with basic grid $\mathcal{T}_{\ell}$ and boundary resolution grid $\mathcal{T}_L$ is defined in a recursive way.

We will employ for this purpose a modification operator $\mathcal{M}_{\lambda+1,\lambda} : \mathcal{S}_\lambda \to \mathcal{S}_{\lambda+1}$ which adapts a (standard) finite element function $u \in \mathcal{S}_\lambda$ in a neighbourhood of the boundary to the boundary conditions. Again, it suffices to define, for $u \in \mathcal{S}_\lambda$, the function $(\mathcal{M}_{\lambda+1,\lambda}u) \in \mathcal{S}_{\lambda+1}$ in the nodal points $x \in \Theta_{\lambda+1}$ by

$$(\mathcal{M}_{\lambda+1,\lambda}u)(x) := \begin{cases} u(x) - u\left(x^\Gamma\right) & x \in \Theta_{\lambda+1}^\Gamma, \\ u(x) & x \in \Theta_{\lambda+1}^{in}. \end{cases}$$

The composite operator $\mathcal{M}_{\lambda,\ell} : \mathcal{S}_\ell \to \mathcal{S}_\lambda$ is given for $\ell \leq \lambda \leq L$ by

$$\mathcal{M}_{\lambda,\ell} := \mathcal{M}_{\lambda,\lambda-1} \circ \mathcal{M}_{\lambda-1,\lambda-2} \circ \ldots \circ \mathcal{M}_{\ell+1,\ell}.$$

We emphasize that the space

$$\left\{\mathcal{M}_{L,\ell}\mathcal{E}_\ell u : u \in \mathcal{S}_\ell^{in}\right\} \tag{3.21}$$

is still non-conforming. Triangles in $\mathcal{T}_L$ still might overlap the boundary Γ and the functions in (3.21), in general, satisfy the homogenous boundary conditions only in an approximate sense. To overcome this problem, we apply the same technique as for Neumann boundary conditions by assuming that the triangulation $\mathcal{T}_L$ is fine enough such that all intersection $\tau \cap \Omega$ are simple, i.e., can be meshed by at most three triangles (cf. Definition 3.6). For $\tau \in \mathcal{T}_L$, the submesh which resolves $\tau \cap \Omega$ is, again, denoted by $\mathcal{T}(\tau)$ and we put

$$\mathcal{T}_{L+1} = \bigcup_{\tau \in \mathcal{T}} \mathcal{T}(\tau). \tag{3.22}$$

Since the mesh $\mathcal{T}_{L+1}$ resolves the domain the application of the operator $\mathcal{M}_{L+1,\ell}\mathcal{E}_\ell$ to $\mathcal{S}_\ell^{in}$ results in a finite element space which satisfies the homogenous boundary conditions.

Remark 3.17 *Note that the application of $\mathcal{M}_{L+1,L}$ to a function $u \in \mathcal{S}_L$ yields*

$$(\mathcal{M}_{L+1,L}u)(x_i) = \begin{cases} u(x_i) & \text{if } x_i \text{ is a vertex in } \mathcal{T}_{L+1} \text{ and } x_i \notin \Gamma, \\ 0 & \text{if } x_i \text{ is a vertex in } \mathcal{T}_{L+1} \text{ and } x_i \in \Gamma. \end{cases}$$

Definition 3.18 *The composite finite element space for Dirichlet boundary conditions is given by*

$$\mathcal{S}_\ell^{CFE} := \left\{\mathcal{M}_{L+1,\ell}\mathcal{E}_\ell u : u \in \mathcal{S}_\ell^{in}\right\}.$$

Remark 3.19 *The composite finite elements $\mathcal{S}_\ell^{CFE}$ for Dirichlet boundary conditions are, in general, not piecewise affine on the triangles in $\mathcal{T}_\ell$ but continuously composed of piecewise affine pieces on triangles in $\mathcal{T}_{L+1}$. However, in the **interior** of the domain (at proper distance from the boundary) they are the standard finite elements, being piecewise affine on triangles in $\mathcal{T}_\ell$.*

Although this definition of composite finite elements might look, at first glance, rather complicated, we will see in the next subsection that their algorithmic realisation is as simple as for the Neumann problem.

3.3.1 Assembling of the system matrix

The system matrix will be generated in a hierarchical way. In this paragraph, we will describe a procedure to assemble the element matrix for the basic grid $\mathcal{T}_\ell$ and finest grid $\mathcal{T}_{L+1}$.

In the first step, we will realise the modification operator $\mathcal{M}_{\ell,\ell-1} : \mathcal{S}_{\ell-1} \to \mathcal{S}_\ell$ in a hierarchical, algorithmic way. For $\tau \in \mathcal{T}_L$ and $t \in \mathcal{T}(\tau)$, we define the 3×3 local modification matrices by

$$\left(\mathbf{p}^{\tau,t}\right)_{i,j} := \begin{cases} b_{\tau,i}(x_j) & \text{if } x_j \notin \Gamma, \\ 0 & \text{if } x_j \in \Gamma \end{cases} \qquad \forall x_j \in \mathbf{V}(t), \forall x_i \in \mathbf{V}(\tau).$$

In the case of $\tau \in \mathcal{T}_\ell$, $0 \le \ell < L$, and $t \in sons(\tau)$, the 3×3 local modification matrices are defined by

$$\left(\mathbf{p}^{\tau,t}\right)_{i,j} := \begin{cases} b_{\tau,i}(x_j) & \text{if } x_j \in \Theta_{\ell+1}^{in} \\ b_{\tau,i}(x_j) - b_{\tau,i}\left(x_j^\Gamma\right) & \text{if } x_j \in \Theta_{\ell+1}^{\Gamma} \end{cases} \qquad \forall x_j \in \mathbf{V}(t), \forall x_i \in \mathbf{V}(\tau).$$

By using the local modification matrices we can generate the linear system for the space

$$\{\mathcal{M}_{L+1,\ell} u : u \in \mathcal{S}_\ell\} \tag{3.23}$$

by a recursion which is of the same *form* as (3.11). Note that the space $\mathcal{S}_\ell^{CFE}$ is a subspace of the space (3.23).

Let $\tau \in \mathcal{T}_\ell$ and denote by $\mathbb{T}_\tau$ the tree with root τ as introduced in Subsection 3.2.1. The leaves of $\mathbb{T}_\tau$ are denoted by $\mathcal{L}_\tau$. Then,

$$\mathbf{B}^t = \begin{cases} \displaystyle\sum_{t' \in sons(t)} \mathbf{p}^{t,t'} \mathbf{B}^{t'} \left(\mathbf{p}^{t,t'}\right)^\intercal & \text{if } t \in \mathbb{T}_\tau \backslash \mathcal{L}_\tau, \\ \displaystyle\sum_{t' \in \mathcal{G}(t)} \mathbf{p}^{t,t'} \mathbf{B}^{t'} \left(\mathbf{p}^{t,t'}\right)^\intercal & \text{if } t \in \mathcal{L}_\tau. \end{cases} \tag{3.24}$$

The system matrix corresponding to the space (3.23) can be assembled from the element matrices $\mathbf{B}^\tau$ by well-known finite element methodology.

In this way, the element matrices $\mathbf{B}^t$ are computed for all $\tau \in \mathcal{T}_\ell$ and $t \in \mathbb{T}_\tau$ and the global system matrix $\tilde{\mathbf{B}}_\ell$ is assembled for the space (3.23).

In the next step, the system matrix corresponding to the composite finite element space will be assembled. We only need a local basis representation of the extrapolation process (3.20) which will denoted by $\mathbf{E}_\ell : \mathbb{R}^{n_\ell^{in}} \to \mathbb{R}^{n_\ell}$, where $n_\ell^{in} := \dim S_\ell^{in}$. The system matrix for $\mathcal{S}_\ell^{CFE}$ is obtained by

$$\mathbf{B}_\ell := \mathbf{E}_\ell^\intercal \tilde{\mathbf{B}}_\ell \mathbf{E}_\ell. \tag{3.25}$$

The extrapolation process increases the supports of the basis functions which are close to the boundary. The algorithm is structured as follows:

1. For all slave nodes $x \in \Theta_\ell^\Gamma$, generate a link to a closest triangle $\tau_x \in \mathcal{T}_\ell^{in}$ and a closest boundary point x^Γ.

2. For any triangle $\tau \in \mathcal{T}_\ell^{in}$, generate the set $\Theta_\tau := \left\{j : x_j \in \Theta_\ell^\Gamma \wedge \tau = \tau_{x_j}\right\}$.

3. For $x_i \in \Theta_\ell^{in}$ initialise the set (pattern) $\mathfrak{P}_i := \{i\}$.

4. For any triangle $\tau \in \mathcal{T}_\ell^{in}$ and for all vertices $x_i \in \tau$, update $\mathfrak{P}_i \leftarrow \mathfrak{P}_i \cup \Theta_\tau$.

This algorithm computes the support of $\mathcal{E}_\ell b_{\ell,i}$ according to

$$\mathcal{E}_\ell b_{\ell,i} = \sum_{j \in \mathfrak{P}_i} \left(\mathcal{E}_\ell b_{\ell,i}\right)(x_j)\, b_{\ell,j},$$

where $b_{\ell,j}$ is the standard finite element basis function for $x_j \in \Theta_\ell$. This representation leads to the definition of the entries of the extrapolation matrix:

$$\left(\mathbf{E}_\ell\right)_{i,j} := \begin{cases} \left(\mathcal{E} b_{\ell,i}\right)(x_j) & \text{if } j \in \mathfrak{P}_i, \\ 0 & \text{otherwise.} \end{cases}$$

We briefly summarise the generation process for assembling the system matrix for composite finite elements for homogeneous Dirichlet boundary conditions.

(a) Generate the element matrix for the finest mesh $\mathcal{T}_{L+1}$.

(b) Agglomerate these element matrixes by the recursion (3.24)

(c) Assemble the corresponding system matrix for the space (3.23).

(d) Generate the global extrapolation matrix $\mathbf{E}_\ell$ in a sparse format.

(e) Employ the representation (3.25) to generate the system matrix for the composite finite element space $\mathcal{S}_\ell^{CFE}$ for homogeneous Dirichlet data.

Remark 3.20 *The generation of the right-hand side is performed along the same lines as described above.*

Remark 3.21 *The generalisation to the different applications which are described in Subsections 3.2.1 and 3.2.2 is completely analogous as for the Neumann problem.*

3.3.2 Inhomogeneous Dirichlet boundary conditions

The case of inhomogeneous Dirichlet boundary conditions can be treated as follows. Consider the problem of finding $u \in H^1(\Omega)$ with $u = g_D$ on Γ such that

$$b(u,v) := \int_\Omega \left(\langle \nabla u, \nabla v \rangle + uv\right) dx = \int_\Omega f v dx \qquad \forall v \in H_0^1(\Omega).$$

Assume that an extension of g_D to a function $g \in H^1(\Omega)$ is known. This leads to the ansatz $u = u_0 + g$, where $u_0 \in H_0^1(\Omega)$ is the unique solution of: Find $u_0 \in H_0^1(\Omega)$ such that

$$b(u_0, v) = \int_\Omega f v dx - b(g, v) \qquad \forall v \in H_0^1(\Omega).$$

Thus, the original problem is transferred to a homogeneous problem and the composite finite element space of the previous section can be employed for its discretisation.

Due to the possibly very complicated boundary, the construction of the trace lifting $g_D \to g$ is not obvious. We will see that the function g can be constructed by an analogous adaption scheme as has been developed for the modification operator $\mathcal{M}$ in Definition 3.18.

In order to reduce technicalities we assume that the given data g_D is continuous and, hence, point evaluations are defined. (This can be generalised by replacing the point evaluations by suitable integral averages).

The function g is constructed iteratively. Let $\mathcal{S}_\ell$ denote the composite finite element space on the overlapping mesh $\mathcal{T}_\ell$. Define the operator $\mathcal{M}^{g_D}_{\lambda+1,\lambda} : \mathcal{S}_\lambda \to \mathcal{S}_{\lambda+1}$ by

$$\left(\mathcal{M}^{g_D}_{\lambda+1,\lambda} u\right)(x) := \begin{cases} g_D\left(x^\Gamma\right) + u(x) - u\left(x^\Gamma\right) & x \in \Theta^\Gamma_{\lambda+1}, \\ u(x) & x \in \Theta^{in}_{\lambda+1}, \end{cases}$$

where $\Theta^\Gamma_{\lambda+1}$, $\Theta^{in}_{\lambda+1}$, x^Γ are as explained in Subsection 3.3. The composition leads to

$$\mathcal{M}^{g_D}_{L+1,\ell} := \mathcal{M}^{g_D}_{L+1,L} \circ \mathcal{M}^{g_D}_{L,L-1} \circ \ldots \circ \mathcal{M}^{g_d}_{\ell} : \mathcal{S}_\ell \to \mathcal{S}_{L+1},$$

where $\mathcal{S}_{L+1}$ is the standard finite element space for the grid $\mathcal{T}_{L+1}$ which resolves the domain Ω.

Definition 3.22 *The approximate trace lifting is given by the function $g_\ell \in \mathcal{S}_{L+1}$ defined by*

$$g_\ell := \mathcal{M}^{g_D}_{L+1,\ell}\mathbf{0}, \qquad \text{where } \mathbf{0} : \Omega_\ell \to \mathbb{R} \text{ is the zero function.}$$

Remark 3.23 *Since $\mathcal{T}_{L+1}$ resolves the domain, the trace of the function g_ℓ interpolates the given function g_D along the boundary on the edges of $\mathcal{T}_{L+1}$ which lie on the boundary.*

3.3.3 Mixed boundary conditions

The case of mixed boundary conditions can be treated by a combination of the modification technique (cf. $\mathcal{M}_{L+1,\ell}$) described for the Dirichlet boundary conditions and the simple restriction employed for the Neumann boundary conditions.

Formally, one has only to modify the definition of the "inner" grid (resp. "inner" grid points) by exchanging the boundary Γ in the corresponding definitions by the boundary Γ_D and apply the construction of Dirichlet boundary conditions verbatim. Since the portion of the overlapping grid which covers the Neumann boundary conditions will not be influenced by the modification operator $\mathcal{M}_{L+1,\ell}$ according to the new definition of Θ^Γ, the composite finite element functions in a neighbourhood of the Neumann boundary are still the simple restrictions of the overlapping space. In a neighbourhood of the Dirichlet boundary the functions are adapted as for the pure Dirichlet problem.

3.4 Composite Finite Elements for Problems with Discontinuous Coefficients

The development of composite finite elements for elliptic problems with discontinuous coefficients is a topic of vivid research. It is beyond the scope of these notes to develop the fully hierarchical construction of composite finite elements for such problems. For details, we refer to [13]. Here we will restrict the algorithmic formulation and the error analysis to a one-dimensional model problem.

Let $\Omega = (0,1)$ and consider the problem of finding $u \in V_0 = H^1_0(\Omega)$ such that

$$\int_0^1 au'v'dx = \int_0^1 fvdx \qquad \forall v \in V_0. \tag{3.26}$$

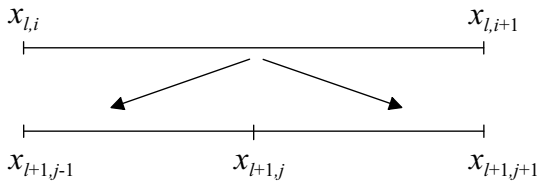

Figure 1: Triangle $\tau = (x_{\ell,i}, x_{\ell,i+1})$ with two sons τ_1 and τ_2.

We assume that a is piecewise constant and positive. Let $(\mathcal{T}_\ell)_{\ell=0}^L$ denote a sequence of meshes consisting of disjoint, open intervals which satisfy

$$\Omega = \operatorname{int} \overline{\bigcup_{\tau \in \mathcal{T}} \tau}.$$

The meshes are assumed to be nested in the sense that, for any $0 \leq \ell < L$ and $\tau \in \mathcal{T}_\ell$, there exists a subset $sons\,(\tau) \subset \mathcal{T}_{\ell+1}$ with

$$\tau = \operatorname{int} \overline{\bigcup_{t \in sons(\tau)} t}.$$

Furthermore, we assume that the finest mesh $\mathcal{T}_L$ resolves the structure of the coefficients in the sense that the restrictions $a|_\tau$ are constant for all $\tau \in \mathcal{T}_L$. We emphasize that this condition is only imposed for the finest mesh $\mathcal{T}_L$, while the coarsest mesh may overlap a huge number of jump points (i.e., points where the coefficient function a is discontinuous).

Remark 3.24 *The condition that $\mathcal{T}_L$ resolves the structure of a can be relaxed by the concept of "simple" triangles as introduced in 3.6.*

The composite finite element spaces $\mathcal{S}_\ell^{CFE}$ are defined via a recursion from the finest level L to the coarsest level 0.

- The finest level:

  Since $\mathcal{T}_L$ resolves the structure of the coefficients, we simply put

  $$\mathcal{S}_L^{CFE} := \mathcal{S}_L,$$

  where, for $0 \leq \ell \leq L$, we denote by $\mathcal{S}_\ell$ the standard finite element space on the mesh $\mathcal{T}_\ell$ with Dirichlet boundary conditions. The nodal basis of $\mathcal{S}_\ell$ is be denoted by $(b_{\ell,i})_{i=1}^{n_\ell}$ while the nodal basis for composite finite element spaces will be denoted by $\left(b_{\ell,i}^{CFE}\right)_{i=1}^{n_\ell}$.

- Recursion from $\ell = L, L-1, \ldots, 0$.

  Assume that the composite finite element space $\mathcal{S}_{\ell+1}^{CFE}$ is defined and that the stiffness matrix $\mathbf{B}^{\ell+1}$ is already generated. Next, we will define an operator $\mathcal{M}_\ell : \mathcal{S}_\ell \to \mathcal{S}_{\ell+1}^{CFE}$ which adapts standard finite element functions to the characteristic behaviour of the solution at the discontinuities of the coefficients.

  The following construction is illustrated in 1. Consider an interval $\tau = (x_{\ell,i}, x_{\ell,i+1}) \in \mathcal{T}_\ell$. For simplicity, we assume that τ has two sons $\tau_1 = (x_{\ell+1,j-1}, x_{\ell+1,j})$, $\tau_2 = (x_{\ell+1,j}, x_{\ell+1,j+1})$.

We assume that the space $\mathcal{S}_{\ell+1}^{CFE}$ with basis $\left(b_{\ell+1,j}^{CFE}\right)_{j=1}^{n_{\ell+1}}$ is already generated and the corresponding system matrix $\mathbf{B}^{\ell+1}$ as well.

Let $u \in \mathcal{S}_\ell$. We employ the ansatz

$$\mathcal{M}_\ell u|_\tau = \sum_{k=j-1}^{j+1} \beta_k \, b_{\ell+1,k}^{CFE}\big|_\tau . \tag{3.27}$$

The condition for determining the coefficients β_k, $k \in \{j-1, j, j+1\}$, are

- Continuity: $\beta_{j-1} = u\left(x_{\ell,i}\right)$, $\beta_{j+1} = u\left(x_{\ell,i+1}\right)$.
- $\mathcal{M}_\ell u|_\tau$ has to satisfy the homogeneous differential equation with boundary conditions $u\left(x_{\ell,i}\right)$, $u\left(x_{\ell,i+1}\right)$, i.e.,

$$\int_\tau a \left(\mathcal{M}_\ell u|_\tau\right)' b'_{\ell+1,j} dx = 0.$$

We employ ansatz (3.27) to obtain

$$\beta_{j-1}\mathbf{B}_{j-1,j}^{\ell+1} + \beta_j \mathbf{B}_{j,j}^{\ell+1} + \beta_{j+1}\mathbf{B}_{j,j+1}^{\ell+1} = 0,$$

i.e.,

$$\beta_j := -\frac{\mathbf{B}_{j-1,j}^{\ell+1} u\left(x_{\ell+1,j-1}\right) + \mathbf{B}_{j,j+1}^{\ell+1} u\left(x_{\ell+1,j+1}\right)}{\mathbf{B}_{j,j}^{\ell+1}}.$$

The application of $\mathcal{M}_\ell$ to the basis functions $(b_{\ell,i})_{i=1}^{n_\ell}$ leads to the composite finite element basis on level ℓ:

$$\begin{aligned}
b_{\ell,i}^{CFE}\big|_\tau &:= b_{\ell+1,j-1}^{CFE}\big|_\tau - \frac{\mathbf{B}_{j-1,j}^{\ell+1}}{\mathbf{B}_{j,j}^{\ell+1}} b_{\ell+1,j}^{CFE} && \text{for } x_{\ell,i} = x_{\ell+1,j-1} \text{ and } \tau = \left(x_{\ell,i}, x_{\ell,i+1}\right), \\
b_{\ell,i+1}^{CFE}\big|_\tau &:= b_{\ell+1,j+1}^{CFE}\big|_\tau - \frac{\mathbf{B}_{j,j+1}^{\ell+1}}{\mathbf{B}_{j,j}^{\ell+1}} b_{\ell+1,j}^{CFE} && \text{for } x_{\ell,i+1} = x_{\ell+1,j+1} \text{ and } \tau = \left(x_{\ell,i}, x_{\ell,i+1}\right), \\
b_{\ell,k}^{CFE}\big|_\tau &:= 0 && \text{for } k \notin \{i, i+1\}.
\end{aligned}$$

Definition 3.25 *The composite finite element space is given by*

$$\mathcal{S}_\ell^{CFE} := \operatorname{span}\left\{b_{\ell,i}^{CFE} : 1 \le i \le n_\ell\right\}.$$

Remark 3.26 *Any function $u \in S_\ell^{CFE}$ satisfies $u \in C^0\left(\overline{\Omega}\right)$ and*

$$\begin{aligned}
&(au')' = 0 && \text{on every } \tau \in \mathcal{T}_L, \\
&[au']_P = 0 && \text{for any } P \in \Theta_L \backslash \Theta_\ell, \\
&u(0) = u(1) = 0.
\end{aligned} \tag{3.28}$$

Remark 3.27 *The computation of the system matrix for the composite finite element space $\mathcal{S}_\ell^{CFE}$ can be generated via Galerkin products (cf. Subsection 3.2.2). The prolongation matrix $\mathbf{p}_{\ell,\ell-1}$ arising in this product is the basis representation of the modification operator $\mathcal{M}_\ell$. The realisation of the Galerkin product can be performed locally in an analogous way as explained in Subsection 3.2.2 for the Neumann problem on non-nested grids.*

*Note that the complexity for the construction of the sequence of stiffness matrices by local Galerkin products is much lower compared to solving the equation (3.28) directly (without a recursion) for any (coarse) triangle $\tau \in \mathcal{T}_\ell$ which might contain a huge number of discontinuities of the coefficient. (Solving **one** problem of dimension n_L, typically, is more time-consuming than solving n_L problems of dimension $O(1)$).*

4 Approximation property

This section is devoted to the approximation property of composite finite elements. First, we will recapitulate some tools from the theory of the approximation property for standard finite elements.

The proof of the approximation property for composite finite elements relies on the existence of a certain extension operator which will be introduced in the second subsection.

Afterwards, we will prove the approximation property for composite finite elements for the different applications introduced in the previous sections.

4.1 Approximation property for standard finite elements

In this subsection, we will recall some basic tools for proving the approximation property for standard finite elements.

We restrict here to estimate the error in the energy norm, i.e., H^1-norm, and assume that the solution satisfies $u \in H^2(\Omega)$. However, we emphasize that, with some more technique, the results can be generalised to the case that the solution is in $H^{1+s}(\Omega)$ for some $0 \leq s \leq 1$ and to the measure the error in the L^2-norm. Some remarks concerning such generalisations are included at the relevant places.

The norm in $H^k(\Omega)$ is denoted by $\|\cdot\|_{k,\Omega}$. The semi-norm containing only the derivatives of highest order is denoted by $|\cdot|_{k,\Omega}$.

Let $\mathcal{T}$ denote a triangulation of a polygonal domain Ω. We recall the constant C_{sr} measuring the shape-regularity of the mesh $\mathcal{T}$:

$$C_{sr} := \sup_{\tau \in \mathcal{T}} \frac{h_\tau}{\rho_\tau},$$

where $h_\tau := \operatorname{diam} \tau$ and ρ_τ denotes the radius of the largest inscribed ball of τ. The maximal stepsize is denoted by $h := \max_{\tau \in \mathcal{T}} h_\tau$ and the finite element space on $\mathcal{T}$ by $\mathcal{S}$ (cf. Section 2.1).

Theorem 4.1 *There exists a linear mapping $P : H^2(\Omega) \to \mathcal{S}$ such that the global error estimate*

$$\|u - Pu\|_{1,\Omega} \leq C_a h \|u\|_{2,\Omega} \qquad \forall u \in H^2(\Omega)$$

holds. For any $\tau \in \mathcal{T}$, the local estimate

$$\|u - Pu\|_{1,\tau} \leq C_a h_\tau \|u\|_{2,\tau} \qquad \forall u \in H^2(\Omega) \tag{4.1}$$

holds. The constant C_a in these estimates only depend on C_{sr}.

Remark 4.2 *(a) The assumptions $u \in H^2(\Omega)$ and $d \leq 3$ imply that P can be chosen as the local nodal interpolation.*

(b)If $u \in H^s(\Omega)$ for $1 \leq s \leq 2$, the nodal interpolant is not always well defined. Instead, one may choose P as the "Clément interpolation operator" as introduced in [2]. The norm on the right-hand side in (4.1) has to replaced by $\|\cdot\|_{s,U_\tau}$, where U_τ is some triangle neighbourhood of τ.

(c) Analogous error estimates can be derived for the error with respect to the L^2-norm.

4.2 Extension operators

The error estimates for composite finite elements will be based on the existence of appropriate extension operators. The norm of such operators will enter the error estimates.

Let $\Omega \subset \mathbb{R}^d$ denote a Lipschitz domain. In this section, we will define an extension operator $\mathfrak{E} : H^k(\Omega) \to H^k(\mathbb{R}^d)$ so that the norm

$$\|\mathfrak{E}\|_{H^k(\mathbb{R}^d) \leftarrow H^k(\Omega)} := \sup_{u \in H^k(\Omega)\setminus\{0\}} \frac{\|\mathfrak{E}u\|_{H^k(\mathbb{R}^d)}}{\|u\|_{H^k(\Omega)}} =: C_k < \infty.$$

is moderately bounded for a large class of domains, which may contain a huge number of geometric details. To reduce technicalities we focus here on some characteristic examples and refer to [11] and [9] for proofs and general considerations.

Assumption 4.3 *Ω is a Lipschitz domain which arises by removing, from a Lipschitz domain $\Omega^\star$, a set of "holes" $\mathcal{H} = \{\omega_1, \dots, \omega_J\}$, where a hole is a simple connected domain $\omega \subset \mathbb{R}^d$, i.e.,*

$$\Omega = \Omega^\star \setminus \overline{\bigcup_{\omega \in \mathcal{H}} \omega}.$$

The set of holes $\mathcal{H}$ satisfies the separation condition if there exists a constant c_{sep} such that

$$\operatorname{dist}(\omega_j, \omega_k) \geq c_{sep} \max\{\operatorname{diam}\omega_j, \operatorname{diam}\omega_k\} \tag{4.2a}$$

$$\operatorname{dist}(\omega_j, \partial\Omega^\star) \geq c_{sep} \operatorname{diam}\omega_j \tag{4.2b}$$

for all distinct $1 \leq j, k \leq J$.

Condition (4.2b) implies that the holes do not hit the boundary $\partial\Omega^\star$ and, thus, the boundary of Ω is not too complicated. We emphasize that condition (4.2b) can be relaxed allowing that the holes might intersect the boundary as well (cf. [11]). We impose here both condition in (4.2) to reduce technicalities.

The extension operator $\mathfrak{E}$ is constructed in two steps.

1. Let $u \in H^k(\Omega)$ for some $k \in \mathbb{N}_0$. Since Ω and $\Omega^\star$ are Lipschitz domains it is well known that there exists an extension operator $\mathfrak{E}^{Stein} : H^k(\Omega) \to H^k(\mathbb{R}^d)$. Put $u_1^\star := \mathfrak{E}^{Stein}u \in H^1(\mathbb{R}^d)$.

2. The local Ritz-projections $u_0^\star$ of $u_1^\star$ on holes $\omega \in \mathcal{H}$ are subtracted to end up with $u^\star := u_1^\star - u_0^\star =: \mathfrak{E}u$. The details of this construction along with illustrating examples will be discussed in this section.

Theorem 4.4 (Stein) *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a continuous extension operator $\mathfrak{E}^{Stein} : H^k(\Omega) \to H^k(\Omega^\star)$.*

For a proof, we refer to [12]. Theorem 4.4 does not imply that the operator norm of $\mathfrak{E}^{Stein}$ is moderately bounded (the bound might be very large for domains with a huge number of small geometric details).

Definition 4.5 *The Ritz-projection* $\mathfrak{R} : H^k(\Omega^\star) \to H_0^k(\Omega^\star)$ *is given for* $v_1^\star \in H^k(\Omega^\star)$ *by* $v_2^\star := \mathfrak{R} v_1^\star$ *where* $v_2^\star \in H_0^k(\Omega)$ *is the solution of*

$$\int_{\Omega^\star} \sum_{|\alpha|=k} D^\alpha v_2^\star D^\alpha w = \int_{\Omega^\star} \sum_{|\alpha|=k} D^\alpha v_1^\star D^\alpha w \, dx \qquad \text{for all } w \in H_0^k(\Omega^\star).$$

Here, $\alpha \in \mathbb{N}_0^d$ *is a multi-index and* $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots, \partial_d^{\alpha_d}$ *is the multidimensional derivative.*

Now, we have all ingredients for defining the extension operator $\mathfrak{E}_D$.

Definition 4.6 *The extension operator* $\mathfrak{E} : H^k(\Omega) \to H^k\left(\mathbb{R}^k\right)$ *is given by the composition*

$$\mathfrak{E} := \mathfrak{E}_\infty^{Stein} (I - \mathfrak{R}) \, \mathfrak{E}^{Stein} \tag{4.3}$$

where $\mathfrak{E}_\infty^{Stein} : H^k(\Omega^\star) \to H^k\left(\mathbb{R}^d\right)$ *denotes Stein's extension operator from the domain* $\Omega^\star$ *to* $\mathbb{R}^d$.

Theorem 4.7 *The extension operator* $\mathfrak{E} : H^k(\Omega) \to H^k\left(\mathbb{R}^d\right)$ *has bounded norm*

$$C_k := \|\mathfrak{E}\|_{H^k\left(\mathbb{R}^d\right) \leftarrow H^k(\Omega)} < \infty.$$

The constant C_k *depends on* c_{sep}, $\Omega^\star$, k, *and the shape of the normalised holes:*

$$\hat{\omega}_j := \left\{ x/h_{\omega_j} : x \in \omega_j \right\} \qquad 1 \le j \le J,$$

where h_{ω_j} *denotes the diameter of* ω_j.

Remark 4.8 *If all holes are, e.g., either balls or squares, Theorem 4.7 implies that the* H^1- *and* H^2-*norm of the extension operator can be bounded by a constant which depend only on* c_{sep} *and* $\Omega^\star$.

The following example shows that the separation condition (4.2) is not necessary in order to bound the norm of the minimal extension operator by a moderate constant.

Example 4.9 *Let* $\Omega^\star = (-1,1)^3$ *and, for* $j = 1, 2$, $\omega_j = B_j \times (-1,1)$. *Here,* B_j *denotes the open disc with radius* ε *about the points* $(\pm 2\varepsilon, 0)^\intercal$. *Then, the norm of the minimal extension operator* $\mathfrak{E} : H^1(\Omega^\star \backslash \overline{\omega_1 \cup \omega_2}) \to H^1(\Omega^\star)$ *is bounded uniformly as* $\varepsilon \to 0$.

4.3 Approximation Property for Composite Finite Elements

In this section, we will develop the approximation property for composite finite elements.

4.3.1 Approximation Property for Composite Finite Elements for the Pure Neumann Problem

We begin with the proof of the approximation property for the pure Neumann problem and the composite finite element spaces as in Definition 3.3.

Theorem 4.10 *Let $u \in H^2(\Omega)$. Assume that there exists an extension operator $\mathfrak{E} : H^2(\Omega) \to H^2(\mathbb{R}^d)$ with norm $C_{ext} < \infty$. Then, there exists $u_{\mathcal{T}} \in \mathcal{S}^{CFE}$ such that the global error estimate*

$$\|u - u_{\mathcal{T}}\|_{1,\Omega} \leq C_a C_{ext} h \|u\|_{2,\Omega} \tag{4.4a}$$

holds. The function $u_{\mathcal{T}}$ satisfies, for any $\tau \in \mathcal{T}$, the local estimate

$$\|u - u_{\mathcal{T}}\|_{1,\tau\cap\Omega} \leq C_a h_\tau \|u^\star\|_{2,\tau}. \tag{4.4b}$$

The constant C_a in (4.4) is as in Theorem 4.1.

Proof. Let $u \in H^2(\Omega)$ and put $u^\star := \mathfrak{E}u \in H^2(\Omega)$. The approximant $u^\star_{\mathcal{T}} \in \mathcal{S}$ is defined by

$$u^\star_{\mathcal{T}} = Pu^\star$$

with P as in Theorem 4.1. We will show, that the function $u_{\mathcal{T}} := u^\star_{\mathcal{T}}|_\Omega \in \mathcal{S}^{CFE}$ satisfies the required estimates. We obtain the second estimate from

$$\|u - u_{\mathcal{T}}\|_{1,\tau\cap\Omega} \leq \|u^\star - u^\star_{\mathcal{T}}\|_{1,\tau} \leq C_a h_\tau \|u^\star\|_{2,\tau}$$

with the constant C_a as in (4.1). By summing the squares of these terms over all $\tau \in \mathcal{T}$, we obtain

$$\|u - u_{\mathcal{T}}\|_{1,\Omega} \leq C_a h \|u^\star\|_{2,\mathbb{R}^d} \leq C_a C_{ext} h \|u\|_{2,\Omega}.$$

■

In the next step, we generalise this estimate for the composite finite element space as in Definition 3.11. We recall the definition of the given mesh $\mathcal{T}$ and the finest auxiliary mesh $\mathcal{T}_L$ as in Subsection 3.2.2. The essential difference to the previous case is that the "given" grid $\mathcal{T}$ and the finest auxiliary grid $\mathcal{T}_L$ are non-nested. First, we have to introduce some notation. For $\tau \in \mathcal{T}_L$, define $\mathcal{T}^1_\tau \subset \mathcal{T}$ as a minimal subset such that

$$\tau \subset \Omega^1_\tau \quad \text{with} \quad \Omega^1_\tau := \operatorname{int} \overline{\bigcup_{t\in\mathcal{T}^1_\tau} t}. \tag{4.5}$$

A further layer around Ω^1_τ is defined as a minimal subset $\mathcal{T}^2_\tau \subset \mathcal{T}_L$ such that

$$\Omega^1_\tau \subset \Omega^2_\tau \quad \text{with} \quad \Omega^2_\tau := \operatorname{int} \overline{\bigcup_{\tau'\in\mathcal{T}^2_\tau} \tau'}. \tag{4.6}$$

We introduce a constant measuring the local quasi-uniformity of the mesh by

$$C_{qu} := \max\left\{ \max_{\tau\in\mathcal{T}_L} \frac{\sum_{\tau'\in\mathcal{T}^2_\tau} h_{\tau'}}{h_\tau}, \max_{\tau\in\mathcal{T}_L} \max_{\tau'\in\mathcal{T}^2_\tau} \frac{h_\tau}{h_{\tau'}} \right\}. \tag{4.7}$$

Finally, we will need a constant measuring the overlap of the subdomain Ω^2_τ and set

$$C_\sharp := \max\left\{ \operatorname{card} \mathcal{T}^1_\tau : \tau \in \mathcal{T}_L \right\}.$$

As a prerequisite, we need the stability of the interpolation operator on non-nested grids.

Lemma 4.11 *Let $\mathcal{T}_L$, $\mathcal{T}$ denote two triangulations as in Subsection 3.2.2. Then, for any $\tau \in \mathcal{T}_L$, there holds*

$$\|I_L u\|_{1,\tau} \leq \hat{C} \|u\|_{1,\Omega_\tau^2} \qquad \forall u \in \mathcal{S}_L,$$

where $\hat{C}$ depends only on C_{sr}, C_{qu}, and $C_\sharp$.

Proof. Let $u \in \mathcal{S}_L$ and consider $\tau \in \mathcal{T}_L$. Our goal is to estimate

$$\|I_L u\|_{1,\tau} \leq C \|u\|_{1,\Omega_\tau^2} \ ,$$

where Ω_τ^2 is as in (4.6).

Let $\mathcal{T}_\tau^1 \subset \mathcal{T}$ be as in (4.5) and consider an arbitrary triangle $t \in \mathcal{T}_\tau^1$ with vertices P_i, $1 \leq i \leq 3$. The nodal interpolant I_L of u on t is given by

$$I_L u|_t := \sum_{i=1}^{3} u_i b_{t,i} \quad \text{with} \quad u_i := u(P_i).$$

Explicit calculations results in

$$|I_L u|_{1,t}^2 = \frac{1}{4|t|} \begin{pmatrix} u_1 - u_3 \\ u_2 - u_1 \end{pmatrix}^\intercal \begin{bmatrix} \|P_2 - P_1\|^2 & \langle P_2 - P_1, P_3 - P_1 \rangle \\ \langle P_2 - P_1, P_3 - P_1 \rangle & \|P_3 - P_1\|^2 \end{bmatrix} \begin{pmatrix} u_1 - u_3 \\ u_2 - u_1 \end{pmatrix}.$$

Hence,

$$|I_L u|_{1,t}^2 \leq C_1 \left((u_1 - u_3)^2 + (u_2 - u_1)^2 \right), \tag{4.8}$$

where the constant C_1 only depends on C_{sr}. Let $\mathcal{T}_\tau^2 \subset \mathcal{T}_L$ be as in (4.6). Consequently, we can choose a sequence $\left(\tau'_{i_j}\right)_{j=1}^q \subset \mathcal{T}_\tau^2$ such that

$$P_1 \in \overline{\tau'_{i_1}}, \quad P_3 \in \overline{\tau'_{i_q}}, \quad \overline{\tau'_{i_j}} \cap \overline{\tau'_{i_{j+1}}} \neq \emptyset \qquad \forall 1 \leq j \leq q-1.$$

Since $\tau'_{i_j} \in \mathcal{T}_L$, the gradient of u is constant on every τ'_{i_j} and denoted by g_j. Hence,

$$u_1 - u_3 = \langle g_1, P_1 - Q_1 \rangle + \langle g_2, Q_1 - Q_2 \rangle + \ldots + \langle g_{q-1}, Q_{q-2} - Q_{q-1} \rangle + \langle g_q, Q_{q-1} - P_3 \rangle,$$

with some $Q_j \in \overline{\tau'_{i_j}} \cap \overline{\tau'_{i_{j+1}}}$, $1 \leq j \leq q-1$. Hence, in view of (4.6) and (4.7), we get

$$|u_1 - u_3| \leq |u|_{W^{1,\infty}(\Omega_\tau^2)} \sum_{j=1}^{q} h_{\tau'_{i_j}} \leq C_{qu} h_\tau |u|_{W^{1,\infty}(\Omega_\tau^2)}.$$

In the same fashion we get the estimate for the difference $|u_1 - u_2|$ and in combination with (4.8) we arrive at

$$|I_L u|_{1,t} \leq \sqrt{2C_1} C_{qu} h_\tau |u|_{W^{1,\infty}(\Omega_\tau^2)}.$$

Let $\tilde{\tau} \in \mathcal{T}_\tau^2$ be a triangle with $|u|_{W^{1,\infty}(\Omega_\tau^2)} = |u|_{W^{1,\infty}(\tilde{\tau})}$. The equivalency of norms in finite dimensional spaces and a scaling argument ensures the existence of a constant C_p, depending only on C_{sr}, such that

$$|u|_{W^{1,\infty}(\Omega_\tau^2)} = |u|_{W^{1,\infty}(\tilde{\tau})} \leq C_p h_{\tilde{\tau}}^{-1} |u|_{1,\tilde{\tau}} \leq C_p h_{\tilde{\tau}}^{-1} |u|_{1,\Omega_\tau^2} \leq C_p C_{qu} h_\tau^{-1} |u|_{1,\Omega_\tau^2}.$$

Altogether we have proved the local stability w.r.t. the H^1-seminorm of the nodal interpolation

$$|I_L u|_{1,t} \leq \sqrt{2C_1} C_{qu}^2 C_p \, |u|_{1,\Omega_\tau^2} .$$

The local L^2-stability of the nodal interpolation follows from

$$\begin{aligned} \|I_L u\|_{0,t} &= \left(\sum_{i=1}^{3} |u_i| \right) \max_{1 \leq i \leq 3} \|b_{\tau,i}\|_{0,\tau} \leq C_2 C_p h_\tau \, \|u\|_{L^\infty(\Omega_\tau^2)} \\ &\leq C_2 C_p C_{qu} \, \|u\|_{0,\Omega_\tau^2} , \end{aligned}$$

where C_2 only depends on C_{sr}. Consequently

$$\|I_L u\|_{1,t} \leq \tilde{C} \, \|u\|_{1,\Omega_\tau^2} \quad \text{with} \quad \tilde{C} := C_p C_{qu} \sqrt{1 + 2C_1 C_2 C_{qu}^2 C_p}.$$

It remains to sum this estimate over all triangles $t \in \mathcal{T}_\tau^1$. This yields

$$\|I_L u\|_{1,\tau}^2 \leq \sum_{t \in \mathcal{T}_\tau^1} \|I_L u\|_{0,t}^2 \leq \tilde{C}^2 \sum_{t \in \mathcal{T}_\tau^1} \|u\|_{1,\Omega_\tau^2}^2 = \tilde{C}^2 C_\sharp \, \|u\|_{1,\Omega_\tau^2}^2 .$$

■

The following Lemma shows that the result in Lemma 4.11 carries over to the multilevel interpolation I_ℓ between $\mathcal{T}_\ell$ and $\mathcal{T}$. For $\tau \in \mathcal{T}_\ell$ we introduce the subset $\mathcal{T}_\tau^3 \subset \mathcal{T}_L$ by

$$t \in \mathcal{T}_\tau^3 \Longleftrightarrow t \subset \tau \tag{4.9}$$

and the subset $\mathcal{T}_\tau^4 \subset \mathcal{T}_L$ such that

$$\mathcal{T}_\tau^4 := \bigcup_{t \in \mathcal{T}_\tau^3} \mathcal{T}_t^2.$$

The domain covered by $\mathcal{T}_\tau^4$ and τ is

$$\Omega_\tau^4 := \operatorname{int} \left(\overline{\tau} \cup \overline{\bigcup_{t \in \mathcal{T}_\tau^4} t} \right) .$$

The local stability of the nodal interpolation on the grid $\mathcal{T}_\ell$ will depend on the overlap constant

$$C_{ov} := \max_{0 \leq \ell \leq L} \max_{\tau \in \mathcal{T}_\ell} \max_{\tilde{\tau} \in \Omega_\tau^4} \sum_{t \in \mathcal{T}_\tau^3 : \tilde{\tau} \subset \Omega_t^3} 1.$$

Corollary 4.12 *Let $\mathcal{T}_\ell$ and $\mathcal{T}$ be as in Subsection 3.2.2. Then, for any $u \in \mathcal{S}_\ell$ and any $\tau \in \mathcal{T}_\ell$, we have*

$$\|I_\ell u\|_{1,\tau} \leq C_{stab} \, \|u\|_{1,\Omega_\tau^4} ,$$

where the constant C_{stab} only depends on C_{sr}, C_{qu}, $C_\sharp$, and C_{ov}.

Proof. Let $u \in \mathcal{S}_\ell$ and consider $\tau \in \mathcal{T}_\ell$. Let $\mathcal{T}_\tau^3 \subset \mathcal{T}_L$ denote the subset as in (4.9). Then,

$$\|I_\ell u\|_{1,\tau}^2 \leq \|u\|_{1,\tau}^2 + \sum_{t \in \mathcal{T}_\tau^3} \|I_\ell u\|_{1,t}^2 .$$

Applying Lemma 4.11 to the terms $\|u\|_{1,t}$ results in

$$\begin{aligned}\|I_\ell u\|_{1,\tau}^2 &\leq \|u\|_{1,\tau}^2 + \hat{C}^2 \sum_{t\in\mathcal{T}_\tau^3} \|u\|_{1,\Omega_t^2}^2 \leq \|u\|_{1,\tau}^2 + \hat{C}^2 \sum_{\tilde{\tau}\in\mathcal{T}_\tau^4} \|u\|_{1,\tilde{\tau}}^2 \left(\sum_{t\in\mathcal{T}_\tau^3:\tilde{\tau}\subset\Omega_t^3} 1 \right) \\ &\leq \|u\|_{1,\tau}^2 + \hat{C}^2 C_{ov} \|u\|_{1,\Omega_\tau^4}^2 \leq \left(1 + \hat{C}^2 C_{ov}\right) \|u\|_{1,\Omega_\tau^4}^2 .\end{aligned}$$

■

Next, we will prove the local and global approximation property of composite finite elements. Again, we have to employ certain neighbourhoods for triangles in $\mathcal{T}_\ell$ and impose a certain overlap condition. For $\tau \in \mathcal{T}_\ell$, let $T(\tau)$ denote a minimal equal-sided triangle that contains Ω_τ^4. For $\tau \in \mathcal{T}_\ell$, let $\mathcal{T}_{\ell,\tau} \subset \mathcal{T}_\ell$ denote a minimal subset such that

$$T(\tau) \cup \Omega_\tau^4 \subset \Omega_{\ell,\tau} \quad \text{with} \quad \Omega_{\ell,\tau} := \operatorname{int} \overline{\bigcup_{t\in\mathcal{T}_{\ell,\tau}} t}. \tag{4.10}$$

We need to control the triangle diameters in $\mathcal{T}_{\ell,\tau}$ and $T(\tau)$ and introduce

$$C_{qu}^{II} := \max_{0\leq\ell\leq L} \max_{\tau\in\mathcal{T}_\ell} \max \left\{ \frac{h_{T(\tau)}}{h_\tau}, \max_{t\in\mathcal{T}_{\ell,\tau}} \frac{h_t}{h_\tau} \right\}.$$

The overlap constant C_{ov}^{II} is defined by

$$C_{ov}^{II} := \max_{0\leq\ell\leq L} \max_{t\in\mathcal{T}_\ell} \sum_{\tau\in\mathcal{T}_\ell:t\subset\Omega_{\ell,\tau}} 1.$$

Theorem 4.13 *Let $\mathcal{T}$ and $(\mathcal{T}_\ell)_{\ell=0}^L$ be as in Subsection 3.2.2 and let $u \in H^2(\Omega)$. Assume that there exists an extension operator $\mathfrak{E} : H^2(\Omega) \to H^2(\mathbb{R}^d)$ with norm $C_{ext} < \infty$. Then, there exists $u_\ell \in \mathcal{S}_\ell^{CFE}$ such that the global error estimate*

$$\|u - u_\ell\|_{1,\Omega} \leq C_1 h_\ell \|u\|_{2,\Omega} \quad \textit{with} \quad h_\ell := \max_{\tau\in\mathcal{T}_\ell} h_\tau \tag{4.11a}$$

holds. The function u_ℓ satisfies, for any $\tau \in \mathcal{T}_\ell$, the local estimate

$$\|u - u_\ell\|_{1,\tau\cap\Omega} \leq C_2 h_\tau \|u^\star\|_{2,\Omega_{\ell,\tau}} . \tag{4.11b}$$

The constant C_2 only depends on C_{sr}, C_{qu}^I, C_{qu}^{II}, $C_\sharp$, C_{ov}, C_{ov}^{II} while C_1, in addition, depends on C_{ext}.

Proof. Let $u \in H^2(\Omega)$ and put $u^\star := \mathfrak{E}u$. Define the function $\tilde{u}_\ell^\star \in \mathcal{S}_\ell$ by $\tilde{u}_\ell^\star := P_\ell u^\star$ with P_ℓ as in Theorem 4.1. Let

$$u_\ell^\star := I_\ell^\star \tilde{u}_\ell^\star := \sum_{i=1}^n \tilde{u}_\ell^\star(x_i)\, b_i \in \mathcal{S}. \tag{4.12}$$

The definition of the interpolation operator I_ℓ as in (3.16) implies $u_\ell := u_\ell^\star|_\Omega \in \mathcal{S}_\ell^{CFE}$. We will prove that u_ℓ satisfies (4.11) and consider a triangle $\tau \in \mathcal{T}_\ell$. The nodal interpolant of $u^\star$ on $T(\tau)$ (cf. (4.10)) is given by

$$P_{T(\tau)} u^\star = \sum_{i:P_i \text{ is a vertex of } T(\tau)} u^\star(P_i)\, b_{T(\tau),i}.$$

With a slight abuse of notation we denote the analytic extension (extension by itself) of $P_{T(\tau)}u^\star$ to an affine function on $\mathbb{R}^d$ again by $P_{T(\tau)}u^\star$.

The error on $\tau \cap \Omega$ now can be estimated by

$$\|u - u_\ell\|_{1,\tau\cap\Omega} \leq \|u^\star - u_\ell^\star\|_{1,\tau} \leq \left\|u^\star - P_{T(\tau)}u^\star\right\|_{1,\tau} + \left\|P_{T(\tau)}u^\star - u_\ell^\star\right\|_{1,\tau}. \tag{4.13}$$

Since $P_{T(\tau)}$ is the nodal interpolant of $u^\star$, we can use (4.1) to obtain for the first summand on the right-hand side in (4.13) the estimate

$$\left\|u^\star - P_{T(\tau)}u^\star\right\|_{1,\tau} \leq \left\|u^\star - P_{T(\tau)}u^\star\right\|_{1,T(\tau)} \leq C_a h_{T(\tau)} \|u^\star\|_{2,T(\tau)} \leq C_a C_{qu}^{II} h_\tau \|u^\star\|_{2,T(\tau)}. \tag{4.14}$$

For the second term in (4.13), we use $u_\ell^\star = I_\ell^\star \tilde{u}_\ell^\star = I_L^\star \tilde{u}_\ell^\star$ (cf. (4.12)) and $I_L P_{T(\tau)}u^\star = P_{T(\tau)}u^\star$ (since $P_{T(\tau)}u^\star$ is globally affine) to obtain

$$\left\|P_{T(\tau)}u^\star - u_\ell^\star\right\|_{1,\tau} \leq \left\|I_L P_{T(\tau)}u^\star - I_L\tilde{u}_\ell^\star\right\|_{1,\tau} \leq \left\|I_L\left(P_{T(\tau)}u^\star - P_\ell u_\ell^\star\right)\right\|_{1,\tau}.$$

Note that the difference function $P_{T(\tau)}u^\star - P_\ell u_\ell^\star$ is in $\mathcal{S}_\ell$ and, thus, Corollary 4.12 implies

$$\left\|P_{T(\tau)}u^\star - u_\ell^\star\right\|_{1,\tau} \leq C_{stab}\left\|P_{T(\tau)}u^\star - P_\ell u_\ell^\star\right\|_{1,\Omega_\tau^4}.$$

Hence,

$$\left\|P_{T(\tau)}u^\star - u_\ell^\star\right\|_{1,\tau} \leq C_{stab}\left\{\left\|P_{T(\tau)}u^\star - u^\star\right\|_{1,\Omega_\tau^4} + \|u^\star - P_\ell u_\ell^\star\|_{1,\Omega_\tau^4}\right\}. \tag{4.15}$$

Using (4.14) and (4.1) we get for the first term on the right-hand side above

$$\begin{aligned}\left\|P_{T(\tau)}u^\star - u^\star\right\|_{1,\Omega_\tau^4} &\leq \left\|P_{T(\tau)}u^\star - u^\star\right\|_{1,T(\tau)} \leq C_a h_{T(\tau)} \|u^\star\|_{1,T(\tau)} \\ &\leq C_a C_{qu}^{II} h_\tau \|u^\star\|_{1,\Omega_{\ell,\tau}}.\end{aligned}$$

For the second term on the right-hand side in (4.15), we apply Theorem 4.10 to obtain

$$\|u^\star - P_\ell u_\ell^\star\|_{1,\Omega_\tau^4} \leq C_a C_{qu}^{II} h_\tau \|u^\star\|_{2,\Omega_{\ell,\tau}}.$$

Summarising, we have proved the local error estimate

$$\|u - u_\ell\|_{1,\tau\cap\Omega} \leq C_2 h_\tau \|u^\star\|_{1,\Omega_{\ell,\tau}} \quad \text{with} \quad C_2 := C_a C_{qu}^{II}\left(1 + 2C_{stab}\right).$$

In order to get the global estimate we sum over all triangles $\tau \in \mathcal{T}_\ell$ and obtain

$$\begin{aligned}\|u - u_\ell\|_{1,\Omega}^2 &\leq C_2^2 h_\ell^2 \sum_{\tau\in\mathcal{T}_\ell} \|u^\star\|_{2,\Omega_{\ell,\tau}}^2 \leq C_2^2 h_\ell^2 \sum_{t\in\mathcal{T}_\ell} \|u^\star\|_{2,t}^2 \left(\sum_{\tau\in\mathcal{T}_\ell : t\subset\Omega_{\ell,\tau}} 1\right) \\ &\leq C_2^2 C_{ov}^{II} h_\ell^2 \|u^\star\|_{2,\mathbb{R}^d}^2 \leq C_2^2 C_{ov}^{II} C_{ext}^2 h_\ell^2 \|u\|_{2,\Omega}^2.\end{aligned}$$

■

4.3.2 Approximation Property for Composite Finite Elements for the Pure Dirichlet Problem

The approximation property for composite finite elements for the pure Dirichlet problem is more technical due to the hierarchical modification operator. It is beyond the scope of these notes to prove the approximation property for the Dirichlet problem. Instead, we refer to [10], [8] for slightly different construction. A paper concerning the approximation property for the Dirichlet problem for the composite finite element space as in Definition 3.18 is in preparation and will be available on the homepage http://www.math.unizh.ch/compmath/ soon.

4.3.3 Approximation Property for Composite Finite Elements for Discontinuous Coefficients

In this subsection, we will prove the approximation property for composite finite elements for elliptic problems with discontinuous coefficients. As in Subsection 3.4, we restrict to the one-dimensional model problem (3.26).

Let $u \in H_0^1(\Omega)$ denote the exact solution. The function $u_\ell \in \mathcal{S}_\ell^{CFE}$ for which we will prove the approximation property is the nodal interpolant

$$u_\ell := \sum_{i=1}^{n_\ell} u(x_{\ell,i})\, b_{\ell,i}^{CFE}. \tag{4.16}$$

Our goal is to prove the estimate

$$\|ae_\ell'\|_{0,\Omega} \leq Ch_\ell \left\|(au')'\right\|_{0,\Omega}, \tag{4.17}$$

where $e_\ell = u - u_\ell$. The regularity requirement $(au')' \in L^2(\Omega)$ is satisfied provided the right-hand side in (3.26) is in $L^2(\Omega)$ since, in this case, $(au')' = f$ in Ω holds. The constant in (4.17) is independent of the structure and the upper/lower bounds of the coefficient function a.

For the proof of (4.17) we will need the following lemma. Let $\sigma = ae'$ and recall that, on any fine grid triangle $t \in \mathcal{T}_L$, the coefficient a is constant: $a_t := a|_t$.

Lemma 4.14 *Let $e, \sigma \in H^1(\Omega)$. For any $\tau \in \mathcal{T}_\ell$ and $t \in \mathcal{T}_L$ with $t = (A, B) \subset \tau$, we have*

$$\int_t \sigma^2 dx = -\int_A^B \sigma'(x) \left(\int_A^x \sigma(s)\, ds\right) dx + \sigma(B) \int_A^B \sigma dx. \tag{4.18}$$

Proof. By using the notations which were introduced above we get

$$\begin{aligned}
\int_t \sigma^2 dx &= \int_A^B \sigma a_t e' dx = -\int_A^B \sigma' a_t e dx + \sigma(B)\, a_t e(B) - \sigma(A)\, a_t e(A) \\
&= -\int_A^B \sigma'(x) \left(\int_A^x \sigma(s)\, ds + a_t e(A)\right) dx + \sigma(B)\, a_t e(B) - \sigma(A)\, a_t e(A) \\
&= -\int_A^B \sigma'(x) \left(\int_A^x \sigma(s)\, ds\right) dx - (\sigma(B) - \sigma(A))\, a_t e(A) + \sigma(B)\, a_t e(B) \\
&\quad - \sigma(A)\, a_t e(A) \\
&= -\int_A^B \sigma'(x) \left(\int_A^x \sigma(s)\, ds\right) dx + \sigma(B)\, a_t (e(B) - e(A)) \\
&= -\int_A^B \sigma'(x) \left(\int_A^x \sigma(s)\, ds\right) dx + \sigma(B) \int_A^B \sigma dx.
\end{aligned}$$

■

To estimate the quantity $\sigma(B)$ we will prove that $\sigma(B)$ has a root in $\overline{\tau}$. Recall that, due to Sobolev's embedding theorem, σ is continuous.

Lemma 4.15 *The function σ has a root in $\overline{\tau}$, for any $\tau \in \mathcal{T}_\ell$.*

Proof. We define the functional $F_\tau \in (C^0(\overline{\tau}))'$ by

$$F_\tau(g) := \int_\tau a^{-1}(x)\, g(x)\, dx.$$

Since $0 < a(x) < \infty$, the functional F_τ is monotone and, hence, the implication

$$(F_\tau(g) = 0) \Rightarrow (g \text{ has a root in } \overline{\tau})$$

holds. Explicit calculations lead to

$$F_\tau(\sigma) = \int_\tau a^{-1}(x)\, a(x)\, e'(x)\, dx = e(B) - e(A).$$

Since u_ℓ is the interpolant of u (cf. (4.16)), the error vanishes at the nodal points and this implies $F_\tau(\sigma) = 0$. ∎

In view of Lemma 4.15, the value $\sigma(B)$ in (4.18) is given by

$$\sigma(B) = \int_\xi^B \sigma'(s)\, ds, \tag{4.19}$$

where ξ denote a root of σ in $\overline{\tau}$. The combination of these relations leads to the proof of the following theorem.

Theorem 4.16 *Let $u \in H_0^1(\Omega)$ and $au' \in H^1(\Omega)$. The nodal interpolant u_ℓ (cf. (4.16)) satisfies the error estimates*

$$\left\| a(u - u_\ell)' \right\|_{L^2(\tau)} \le 2h_\tau \left\| (au')' \right\|_{L^2(\tau)}, \tag{4.20a}$$

$$\left\| a(u - u_\ell)' \right\|_{L^2(\Omega)} \le 2h \left\| (au')' \right\|_{L^2(\Omega)}. \tag{4.20b}$$

Proof. Let $\tau \in \mathcal{T}_\ell$. The combination of Lemma 4.14, equation (4.19) and the Cauchy-Schwarz inequality leads to

$$\begin{aligned}
\|\sigma\|_{0,\tau}^2 = \sum_{\substack{t \in \mathcal{T}_L \\ t \subset \tau}} \|\sigma\|_{0,t}^2 &\le \sum_{\substack{t \in \mathcal{T}_L \\ t \subset \tau}} \left(\|\sigma'\|_{0,t}\, h_t\, \|\sigma\|_{0,t} + \sqrt{h_\tau h_t}\, \|\sigma'\|_{0,\tau}\, \|\sigma\|_{0,t} \right) \\
&\le \sqrt{\left(\sum_{\substack{t \in \mathcal{T}_L \\ t \subset \tau}} \|\sigma'\|_{0,t}^2\, h_t^2 \right) \left(\sum_{\substack{t \in \mathcal{T}_L \\ t \subset \tau}} \|\sigma\|_{0,t}^2 \right)} + \sqrt{h_\tau}\, \|\sigma'\|_{0,\tau} \sum_{\substack{t \in \mathcal{T}_L \\ t \subset \tau}} \sqrt{h_t}\, \|\sigma\|_{0,t} \\
&\le h_\tau\, \|\sigma'\|_{0,\tau}\, \|\sigma\|_{0,\tau} + \sqrt{h_\tau}\, \|\sigma'\|_{0,\tau} \sqrt{\sum_{\substack{t \in \mathcal{T}_L \\ t \subset \tau}} h_t}\, \|\sigma\|_{0,\tau} \\
&= 2h_\tau\, \|\sigma'\|_{0,\tau}\, \|\sigma\|_{0,\tau}.
\end{aligned}$$

Remark 3.26 implies

$$\sigma' = (ae')' = (au')'.$$

Thus, by dividing the inequality above by $\|\sigma\|_{0,\tau}$ leads to

$$\|\sigma'\|_{0,\tau} \le 2h_\tau \left\| (au')' \right\|_{0,\tau}$$

and this is the local approximation property. The global approximation property follows by summing over all intervals $\tau \in \mathcal{T}_\ell$.

∎

References

[1] P. Ciarlet. *The finite element method for elliptic problems*. North-Holland, 1987.

[2] P. Clément. Approximation by Finite Element Functions using Local Regularization. *RAIRO, Sér. Rouge Anal. Numér.*, R-2:77–84, 1975.

[3] D. Feuchter, I. Heppner, S. Sauter, and G. Wittum. Bridging the gap between geometric and algebraic multigrid methods (extended version). Technical report, Universität Zürich, 2002. Preprint no. 01-2002.

[4] N. Frauböse and S. Sauter. Composite Finite Elements and Multigrid. Part II: Convergence Theory in 2-d. *in preparation*.

[5] N. Frauböse and S. Sauter. Composite Finite Elements and Multigrid. Part I: Convergence Theory in 1-d. In W. Hackbusch and U. Langer, editors, *(Electronic) Proceedings of the 17th GAMM-Seminar Leipzig on Construction of Grid Generation Algorithms*, MPI Leipzig, 2001. Preprint no. 11-01, Inst. f. Math, Univ. Zürich, 2001, www.mis.mpg.de/conferences/gamm/2001/GAMM_2001_Proceedings_1.html.

[6] W. Hackbusch. *Multi-Grid Methods and Applications*. Springer Verlag, Berlin, 1985.

[7] W. Hackbusch. *Elliptic Differential Equations*. Springer Verlag, 1992.

[8] W. Hackbusch and S. Sauter. Composite Finite Elements for problems with complicated boundary. Part III: Essential boundary conditions. Technical report, Universität Kiel, 1997.

[9] O. Oleinik, A. Shamaev, and G. Yosifian. *Mathematical Problems in Elasticity and Homogenization*. North-Holland, Amsterdam, 1992.

[10] S. Sauter. *Vergröberung von Finite-Elemente-Räumen*. Universität Kiel, Germany, 1997. Habilitationsschrift.

[11] S. Sauter and R. Warnke. Extension operators and approximation on domains containing small geometric details. *East-West J. Numer. Math.*, 7(1):61–78, 1999.

[12] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton, University Press, Princeton, N.J., 1970.

[13] R. Warnke. *Schnelle Löser für Elliptische Randwertprobleme mit Springenden Koeffizienten*. PhD thesis, Universität Zürich, planned for 2002.