Sparse Convolution Quadrature for Time Domain Boundary Integral Formulations of the Wave Equation

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Abstract

Many important physical applications are governed by the wave equation. The formulation as time domain boundary integral equations involves retarded potentials. For the numerical solution of this problem we employ the convolution quadrature method for the discretization in time and the Galerkin boundary element method for the space discretization.

We introduce a simple a-priori cutoff strategy where small entries of the system matrix are replaced by zero. The threshold for the cutoff is determined by an a-priori analysis which will be developed in this paper.

This method reduces the computational complexity for solving time domain integral equations from $O\left(M^2 N \log^2 N\right)$ to $O\left(M^{1+s} N \log^2 N\right)$ for some $s \in [0,1[$, where N denotes the number of time steps and M is the dimension of the boundary element space.

1 Introduction

Boundary value problems governed by the wave equation

$$\partial_t^2 u - \Delta u = f$$

arise in many physical applications such as electromagnetic wave propagation or the computation of transient acoustic waves. Since such problems are typically formulated in unbounded domains, the method of integral equations is an elegant tool to transform this partial differential equation to an integral equation on the bounded surface of the scatterer.

Although this approach goes back to the early 1960s (cf. [13]), the development of fast numerical methods for integral equations in the field of hyperbolic problems is still in its infancies compared to the vast of fast methods for elliptic boundary integral equations (cf. [27] and references therein). Existing numerical discretization methods include collocation methods with some stabilisation techniques (cf. [3] , [4], [8], [9], [10], [25], [26]) and Laplace-Fourier methods coupled with Galerkin boundary elements in space ([2], [6], [11], [15]). Numerical experiments can be found, e.g., in [16].

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In [12], a fast version of the *marching-on-in-time* (MOT) method is presented which is based on a suitable plane wave expansion of the arising potential which reduces the storage and computational costs.

The convolution quadrature method for the time discretization has been developed in [20], [21], [22], [23]. It provides a straightforward way to obtain a stable time stepping scheme using the Laplace transform of the kernel function.

In this paper, we employ the convolution quadrature method for the time discretization and a Galerkin boundary element method in space. We present a simple cutoff strategy where the densely populated Galerkin matrices related to each time step are replaced by sparse versions where a substantial portion of the stiffness matrix is replaced by zero.

The remainder of the paper is structured into five sections. In Section 2, we briefly introduce the formulation of the wave equation as an integral equation and recall its stability properties. Section 3 is devoted to the convolution quadrature method for the time discretization and the boundary element method for the space discretization. We introduce our a-priori cutoff strategy to replace small matrix entries by zero and discuss some algorithmic aspects. In Section 4 , we analyse the effect of the perturbation introduced by the cutoff strategy and prove the convergence of the corresponding solution. In Section 5, we discuss the complexity of our method. We show that the dependence of the computational complexity of the new method on the number M of unknowns in space is reduced from M^2 to M^{1+s} where s is some number between 0 and 1. Finally, in Section 6, we summarise the results and give an outlook on future research.

We emphasise that this paper paves the way to introduce and analyse further perturbations in the space-time discretization. Forthcoming papers will be devoted to panel clustering techniques for the retarded potential boundary integral equation which will further reduce the dependence of the computational complexity on M. Another paper will be concerned with efficient quadrature methods for approximating the remaining matrix entries.

2 Integral Formulation of the Wave Equation

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain with boundary Γ. In this paper, we present efficient methods for numerically solving the homogeneous wave equation

$$\partial_t^2 u = \Delta u \qquad \text{in } \Omega \times (0,T) \tag{2.1a}$$

with initial conditions

$$u(\cdot,0) = \partial_t u(\cdot,0) = 0 \qquad \text{in } \Omega \tag{2.1b}$$

and boundary conditions

$$u = g \quad \text{on } \Gamma \times (0,T) \tag{2.1c}$$

on a time interval $(0,T)$ for some $T > 0$. For its solution, we employ an ansatz as a *single layer potential*

$$u(x,t) = \int_0^t \int_\Gamma k(x-y,t-\tau)\phi(y,\tau)d\Gamma_y d\tau\,, \quad (x,t) \in \Omega \times (0,T)\,, \tag{2.2}$$

where $k(z,t)$ is the fundamental solution of the wave equation,

$$k(z,t) = \frac{\delta(t-\|z\|)}{4\pi\|z\|}\,, \tag{2.3}$$

$\delta(t)$ being the Dirac delta distribution. The ansatz (2.2) satisfies the homogeneous equation (2.1a) and the initial conditions (2.1b). The extension $x \to \Gamma$ is continuous and hence, the unknown density ϕ in (2.2) is determined via the boundary conditions (2.1c), $u(x,t) = g(x,t)$. This results in the boundary integral equation for ϕ,

$$(V\phi)(x,t) := \int_0^t \int_\Gamma k(x-y, t-\tau)\phi(y,\tau) d\Gamma_y d\tau = g(x,t) \qquad \forall (x,t) \in \Gamma \times (0,T). \tag{2.4}$$

Existence and uniqueness results for the solution of the continuous problem are proven in [22]. To recall them, we introduce appropriate norms and spaces. We define the Sobolev space $H^s(\Gamma)$, $s \geq 0$, in the usual way (see, e.g., [17] or [24]). The range of s for which $H^s(\Gamma)$ is defined may be limited, depending on the global smoothness of the surface Γ. Throughout, we let $[-k, k]$ denote the range of Sobolev indices for which we will prove the inverse estimates (where k is a positive integer), and we assume that $H^s(\Gamma)$ is defined for all $s \in [-k, k]$, with the negative order spaces defined by duality in the usual way. (For example, if Γ is a Lipschitz manifold, then $-1 \leq s \leq 1$ and if Ω is a domain with C^∞ boundary then $s \in \,]-\infty, \infty[$). The norm is denoted by $\|\cdot\|_{H^s(\Gamma)}$.

For real r and $s \in [-k, k]$, the anisotropic Sobolev space $H^r(\mathbb{R}; H^s(\Gamma))$ is given by

$$H^r(\mathbb{R}; H^s(\Gamma)) := \left\{ g : \Gamma \times \mathbb{R} \to \mathbb{R} : (1 + |\omega|)^{2r} \|\mathcal{F}g(\cdot, \omega)\|^2_{H^s(\Gamma)} < \infty \right\},$$

where $\mathcal{F}$ denotes the Fourier transform with respect to the time variable $t \in \mathbb{R}$. The norm in this space is given by

$$\|f\|^2_{H^r(\mathbb{R}; H^s(\Gamma))} := \int_{-\infty}^{\infty} (1 + |\omega|)^{2r} \|\mathcal{F}f(\cdot, \omega)\|^2_{H^s(\Gamma)} d\omega.$$

The space $H_0^r(0, T; H^s(\Gamma))$ is defined by

$$\begin{aligned} H_0^r(0, T; H^s(\Gamma)) := \Big\{ g : [0, T] \times \Gamma \to \mathbb{R} \quad &: \quad g = g^\star|_{[0,T]} \\ &\text{for some } g^\star \in H^r(\mathbb{R}, H^s(\Gamma)) \text{ with } g^\star \equiv 0 \text{ on }]-\infty, 0[\Big\} \end{aligned}$$

and the norm $\|\cdot\|_{H_0^r(0,T;H^s(\Gamma))}$ is given by

$$\begin{aligned} \|g\|^2_{H_0^r(0,T;H^s(\Gamma))} := \min \Big\{ &\|g^\star\|_{H_0^r(\mathbb{R};H^s(\Gamma))} : g^\star \in H^r(\mathbb{R}, H^s(\Gamma)) \\ &\text{with } g = g^\star|_{[0,T]} \text{ and } g^\star \equiv 0 \text{ on }]-\infty, 0[\Big\}. \end{aligned}$$

Theorem 2.1 *Let $g \in H^{r+2}(0, T; H^{1/2}(\Gamma))$ for some $r \in \mathbb{R}$. Then, (2.4) has a unique solution $\phi \in H^r(0, T; H^{-1/2}(\Gamma))$, with*

$$\|\phi\|_{H_0^r(0,T;H^{-1/2}(\Gamma))} \leq C_T \|g\|_{H_0^{r+2}(0,T;H^{1/2}(\Gamma))}.$$

For $r > 5/2$, the pointwise estimate

$$\|\phi(\cdot, t)\|_{H^{-1/2}(\Gamma)} \leq C_T \|g\|_{H_0^{r+2}(0,T;H^{1/2}(\Gamma))}$$

holds for all $t \in [0, T]$.

For a proof, we refer to [2, Prop. 3] resp. [22, (2.23), (2.24)].

3 Numerical Discretization

3.1 Time Discretization via Convolution Quadrature

For the time discretization, we employ the convolution quadrature approach which has been developed by Lubich in [20], [21], [22], [23]. We do not recall the theoretical frame work here but directly apply the approach to the wave equation.

We split the time interval $[0,T]$ into $N+1$ time steps of equal length $\Delta t = T/N$ and compute an approximate solution at the discrete time levels $t_n = n\Delta t$. The continuous convolution operator V is replaced by the discrete convolution operator,

$$\left(V^{\Delta t}\phi_{\Delta t}\right)_n(x) := \sum_{j=0}^{n}\int_\Gamma \omega_{n-j}^{\Delta t}(x-y)\phi_{\Delta t}^{j}(y)d\Gamma_y\,, \tag{3.1}$$

for $n = 1,\ldots,N$. The convolution weights $\omega_n^{\Delta t}(x)$ will be defined below (see (3.6)). The semidiscrete problem is given by

$$\left(V^{\Delta t}\phi_{\Delta t})\right)_n(x) = g_{\Delta t}^{n}(x)\,, \qquad n = 1,\ldots,N, \quad x\in\Gamma, \tag{3.2}$$

where $g_{\Delta t}^{n}(x)$ is some approximation to $g(x,t_n)$, or $g(x,t_n)$ itself.

Following the approach in [20], [21], [22], the convolution quadrature method is based on a linear multistep method which, for an ordinary differential equation $u'(t) = f(u(t))$, can be formulated as

$$\sum_{j=0}^{k}\alpha_j u^{n+j-k} = \Delta t\sum_{j=0}^{k}\beta_j f(u^{n+j-k})\,, \tag{3.3}$$

where $u^n \approx u(t_n)$. Let

$$\gamma(\zeta) := \frac{\sum_{j=0}^{k}\alpha_j\zeta^{k-j}}{\sum_{j=0}^{k}\beta_j\zeta^{k-j}}$$

be the quotient of the generating polynomials of the linear multistep method (3.3).

Definition 3.1 *The convolution weights $\omega_n^{\Delta t}(x-y)$ of the convolution quadrature method (3.2) are given by the coefficients of the power series of the Laplace transform $\hat{k}(z,\gamma(\zeta)/\Delta t) = (4\pi\|z\|)^{-1}\exp\left(-\frac{\gamma(\zeta)}{\Delta t}\|z\|\right)$ of (2.3), i.e.,*

$$\hat{k}\left(z,\frac{\gamma(\zeta)}{\Delta t}\right) = \sum_{n=0}^{\infty}\omega_n^{\Delta t}(z)\zeta^n. \tag{3.4}$$

We employ the second order accurate, A-stable BDF2 scheme which is given by

$$\alpha_0^{BDF2} = \frac{3}{2},\quad \alpha_1^{BDF2} = -2,\quad \alpha_0^{BDF2} = \frac{1}{2},\quad \beta_0^{BDF2} = 1,$$

i.e.,

$$\gamma^{BDF2}(\zeta) = \frac{1}{2}\left(\zeta^2 - 4\zeta + 3\right)\,. \tag{3.5}$$

Because the kernel function only depends on the distance $d = \|x-y\|$, we write $\hat{k}(d,\cdot)$ and $\omega_n^{\Delta t}(d)$ short for $\hat{k}(x-y,\cdot)$ and $\omega_n^{\Delta t}(x-y)$. The coefficients of the power series (3.4) can be obtained by the Taylor expansion of $\hat{k}(d, \frac{\gamma(\zeta)}{\Delta t})$ about $\zeta = 0$,

$$\omega_n^{\Delta t}(d) = \frac{1}{n!} \left. \frac{\partial^n \hat{k}(d, \frac{\gamma(\zeta)}{\Delta t})}{\partial \zeta^n} \right|_{\zeta=0} = \frac{1}{n!} \frac{1}{4\pi d} \left. \frac{\partial^n e^{-\frac{\gamma(\zeta)}{\Delta t} d}}{\partial \zeta^n} \right|_{\zeta=0} .$$

Using the formula for multiple differentiation of composite functions (see, e.g., [14]), we obtain the explicit representation

$$\omega_n^{\Delta t}(d) = \frac{1}{n!} \frac{1}{4\pi d} \left(\frac{d}{2\Delta t} \right)^{n/2} e^{-\frac{3d}{2\Delta t}} H_n \left(\sqrt{\frac{2d}{\Delta t}} \right) , \tag{3.6}$$

where H_n are the Hermite polynomials.

3.2 Space Discretization. Galerkin Boundary Element Methods

In the previous section, we have derived the semidiscrete problem: For $n = 1, 2, \ldots, N$, find $\phi_{\Delta t}^n \in H^{-1/2}(\Gamma)$ such that

$$\sum_{j=0}^{n} \int_\Gamma \omega_{n-j}^{\Delta t}(x-y) \phi_{\Delta t}^j(y) d\Gamma_y = g_{\Delta t}^n(x), \qquad n = 1, \ldots, N , \quad x \in \Gamma. \tag{3.7}$$

For the space discretization, we employ a Galerkin boundary element method. Let $\mathcal{G}$ be a regular (in the sense of Ciarlet [5]) boundary element mesh on Γ consisting of shape regular, possibly curved triangles. For a triangle $\tau \in \mathcal{G}$, the (regular) pullback to the reference triangle $\widehat{\tau} := \operatorname{conv}\left\{ \binom{0}{0}, \binom{1}{0}, \binom{0}{1} \right\}$ is denoted by $\chi_\tau : \widehat{\tau} \to \tau$. The space of piecewise constant, discontinuous functions is

$$S_{-1,0} := \{ u \in L^\infty(\Gamma) \quad : \quad \forall \tau \in \mathcal{G} : u|_\tau \in \mathbb{P}_0 \} ,$$

and, alternatively, we consider the space of continuous, piecewise linear functions

$$S_{0,1} := \left\{ u \in C^0(\Gamma) \quad : \quad \forall \tau \in \mathcal{G} : (u \circ \chi_\tau)|_\tau \in \mathbb{P}_1 \right\}$$

for the space discretization. As a basis for $S_{-1,0}$, we choose the characteristic functions for the panels $\tau \in \mathcal{G}$, while the basis for $S_{0,1}$ consists of the standard hat functions, lifted to the surface Γ. The general notation is S for the boundary element space and $(b_i)_{i=1}^M$ for the basis. The mesh width is given by

$$h := \max_{\tau \in \mathcal{G}} h_\tau, \quad \text{where} \quad h_\tau := \operatorname{diam} \tau.$$

For the space-time discrete solution at time t_n we employ the ansatz

$$\phi_{\Delta t,h}^n(y) = \sum_{i=1}^{M} \phi_{n,i} b_i(y) , \tag{3.8}$$

where $\phi_n = (\phi_{n,i})_{i=1}^M \in \mathbb{R}^M$ are the nodal values of the discrete solution at time step t_n. The collection of these solution vectors is denoted by $\overrightarrow{\phi}_n := (\phi_i)_{i=0}^n \in \mathbb{R}^{(n+1)M}$. Note that we always include ϕ_0 in this vector, although it is always 0.

For the Galerkin boundary element method, we replace $\phi_{\Delta t}^j$ in (3.7) by some $\phi_{\Delta t,h}^j \in S$ and impose the integral equation not pointwise but in a weak form: Find $\phi_{\Delta t,h}^n \in S$ of the form (3.8) such that

$$\sum_{j=0}^{n} \sum_{i=1}^{M} \phi_{j,i} \int_\Gamma \int_\Gamma \omega_{n-j}^{\Delta t}(x-y) b_i(y) b_k(x) d\Gamma_y d\Gamma_x = \int_\Gamma g_{\Delta t}^n(x) b_k(x) d\Gamma_x \tag{3.9}$$

for all $1 \leq k \leq M$ and $n = 1, \ldots, N$. This can be written as a linear system

$$\sum_{j=0}^{n} \mathbf{A}_{n-j} \phi_j = \mathbf{g}_n \,, \qquad n = 1, \ldots, N \,, \tag{3.10}$$

with

$$(\mathbf{A}_n)_{k,i} := \int_\Gamma \int_\Gamma \omega_n^{\Delta t}(x-y) b_i(y) b_k(x) d\Gamma_y d\Gamma_x \,,$$

and

$$(\mathbf{g}_n)_k = \int_\Gamma g_{\Delta t}^n(x) b_k(x) d\Gamma_x \,.$$

3.3 Algorithmic Realization

The linear systems in (3.10) can be written in the compact block form

$$\overrightarrow{\mathbf{A}}_N \overrightarrow{\phi}_N := \overrightarrow{\mathbf{g}}_n, \tag{3.11}$$

where, for $0 \leq n \leq N$, the block matrix $\overrightarrow{\mathbf{A}}_n \in \mathbb{R}^{(n+1)M} \times \mathbb{R}^{(n+1)M}$ and the vector $\overrightarrow{\mathbf{g}}_n \in \mathbb{R}^{(n+1)M}$ are defined by

$$\overrightarrow{\mathbf{A}}_n := \begin{pmatrix} \mathbf{A}_0 & \mathbf{0} & \cdots & & & \mathbf{0} \\ \mathbf{A}_1 & \mathbf{A}_0 & \ddots & & & \vdots \\ \mathbf{A}_2 & \mathbf{A}_1 & \ddots & & & \\ \vdots & \mathbf{A}_2 & \ddots & & \ddots & \\ & & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{A}_n & \cdots & & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 \end{pmatrix} \quad \text{and} \quad \overrightarrow{\mathbf{g}}_n := \begin{pmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_n \end{pmatrix}. \tag{3.12}$$

The matrices $\mathbf{A}_j$ have dimension $M \times M$ and are fully populated. The straightforward procedure for solving this system is given by the following recursion.

For $n = 1, 2, \ldots$, one computes

$$\mathbf{w}_n := \mathbf{g}_n - \sum_{i=0}^{n-1} \mathbf{A}_{n-i} \phi_i \tag{3.13}$$

and then solves the system

$$\mathbf{A}_0 \phi_n = \mathbf{w}_n. \tag{3.14}$$

This naive procedure requires $O\left(N^2M^2\right)$ operations. If we assume that a fast iterative procedure is employed which solves (3.14) in $O\left(M^2\right)$ operations, the total amount of work is given by

$$\underbrace{O\left(N^2M^2\right)}_{(3.13)} + \underbrace{O\left(NM^2\right)}_{(3.14)}.$$

The computational costs for (3.13) can be decreased by using the following algorithm, described in [18]. The procedure depends on a (small) control parameter r.

For $0 \leq n \leq N$, we introduce the block matrix $\overrightarrow{\mathbf{B}}_n \in \mathbb{R}^{(n+1)M} \times \mathbb{R}^{(n+1)M}$ by

$$\overrightarrow{\mathbf{B}}_n = \begin{pmatrix} \mathbf{A}_{n+1} & \mathbf{A}_n & \dots & \mathbf{A}_2 & \mathbf{A}_1 \\ \mathbf{A}_{n+2} & \ddots & & & \mathbf{A}_2 \\ \vdots & & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \mathbf{A}_n \\ \mathbf{A}_{2n+1} & \dots & & \mathbf{A}_{n+2} & \mathbf{A}_{n+1} \end{pmatrix}$$

Algorithm 3.2 (Recursive solver for block tridiagonal system)

Comment: Main program

begin

$\overrightarrow{\mathbf{w}}_N := \overrightarrow{\mathbf{g}}_N$; $\overrightarrow{\phi}_N := \mathbf{0}$; ***solve_triangular***$(0, N)$;

end;

Comment: *The recursive subroutine* ***solve_triangular*** *is defined as follows.*

procedure solve_triangular$(a, b :$ integer$)$;

begin

if $b - a \leq r - 1$ ***then solve***[1]

$$\overrightarrow{\mathbf{A}}_{b-a} \left(\phi_n\right)_{n=a}^{b} = \left(\mathbf{w}_n\right)_{n=a}^{b} \tag{3.15}$$

else begin

$m := \left\lceil \frac{b+a}{2} \right\rceil$;

solve_triangular$(a, m-1)$;

evaluate

$$\left(\mathbf{w}_n\right)_{n=m}^{b} = \left(\mathbf{w}_n\right)_{n=m}^{b} - \overrightarrow{\mathbf{B}}_{b-a} \left(\phi_n\right)_{n=a}^{b}; \tag{3.16}$$

solve_triangular(m, b)

end end;

Some remarks are important for the performance of this algorithm.

1. The solution of the block tridiagonal system (3.15) for computing $\left(\phi_n\right)_{n=a}^{b}$ is solved by backward substitution in a straightforward manner. The computational cost is bounded by $\mathcal{O}\left(r^2M^2\right)$ operations.

2. For the evaluation of the convolution $\overrightarrow{\mathbf{B}}_{b-a} \left(\phi_n\right)_{n=a}^{b}$ in (3.16), the discrete Fourier transform (see, e.g., [19]) should be employed. The total cost for evaluating (3.16) is $\mathcal{O}\left(M^2 \left(b-a\right) \log\left(b-a\right)\right)$.

[1]Notation: $\left(\phi_n\right)_{n=a}^{b} := \left(\phi_a, \phi_{a+1}, \dots, \phi_b\right)^{\mathsf{T}}$.

3. The **procedure solve_triangular** calls itself two times with dimension cut in half. The total computational cost sums up $\mathcal{O}\left(M^2 N \log^2 N\right)$ (cf. [19]).

The *goal* in this paper is to develop a sparse approximation of the matrices $\mathbf{A}_j$ so that the total amount of work is reduced to

$$O\left(M^{1+s} N \log^2 N\right) \tag{3.17}$$

with some $0 \leq s < 1$.

3.4 Sparse approximation of the matrices $\mathbf{A}_n$

We recall the definition of the matrix $\mathbf{A}_n$,

$$(\mathbf{A}_n)_{i,j} = \int_{\mathrm{supp}(b_i)} \int_{\mathrm{supp}(b_j)} \omega_n^{\Delta t}(x-y) b_i(x) b_j(y) d\Gamma_y d\Gamma_x \,, \tag{3.18}$$

where $\mathrm{supp}(b_i)$ denotes the support of the basis function b_i. The matrices $\mathbf{A}_n$ are full matrices. However, it turns out that a substantial part of the matrix entries is small and can be replaced by 0. In Section 4.3 we determine, depending on a tolerance $\varepsilon > 0$, the interval $I_{n,\varepsilon}^{\Delta t} := [t_n - c_{n,\varepsilon}^{\Delta t}, t_n + c_{n,\varepsilon}^{\Delta t}] \cap [0, \operatorname{diam}\Gamma]$ such that

$$|\omega_n^{\Delta t}(d)| \leq \frac{\varepsilon}{4\pi d}\,, \qquad \forall d \notin I_{n,\varepsilon}^{\Delta t}\,. \tag{3.19}$$

The result of this analysis yields

$$c_{n,\varepsilon}^{\Delta t} = 3\sqrt{\Delta t}\sqrt{t_n} \log \frac{1}{\varepsilon}. \tag{3.20}$$

Let $\mathcal{P}_\varepsilon \subset \{1, \ldots, M\} \times \{1, \ldots, M\}$ be defined by

$$\mathcal{P}_\varepsilon := \left\{(i,j) \,:\, \exists\, (x,y) \in \operatorname{supp} b_i \cap \operatorname{supp} b_j : \|x-y\| \in I_{n,\varepsilon}^{\Delta t}\right\}. \tag{3.21}$$

This induces a sparse approximation $\tilde{\mathbf{A}}_n$ by

$$(\tilde{\mathbf{A}}_n)_{i,j} := \begin{cases} (\mathbf{A}_n)_{i,j} & \text{if } (i,j) \in \mathcal{P}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{3.22}$$

The space-time discretization with sparse matrix approximations is given by replacing the matrices $\mathbf{A}_n$ in (3.12) by the sparse versions (3.22) and plugging the corresponding solution $\left(\tilde{\phi}_0, \tilde{\phi}_1, \ldots, \tilde{\phi}_N\right)^\intercal$ into the basis representation

$$\tilde{\phi}_{\Delta t,h}^n := \sum_{i=1}^{M} \tilde{\phi}_{n,i} b_i. \tag{3.23}$$

4 Convergence Analysis

The convergence analysis consists of three parts. In Section 4.1, the analysis of the space-time discretization without sparse matrix approximation is given. The sparse approximation of the matrices $\mathbf{A}_n$ induces a perturbation in the space discretization and in Section 4.2, we analyse the effect of such perturbations on the overall discretization error. The size of the perturbation depends on the smallness of the function $\omega_n^{\Delta t}$ outside the interval $I_{n,\varepsilon}^{\Delta t}$. In Section 4.3, we determine the interval $I_{n,\varepsilon}^{\Delta t}$ such that the arising perturbation error is in balance with the overall discretization error.

4.1 Error Estimates for the space-time discretization without sparse matrix approximation

For the semidiscrete solution $\phi^n_{\Delta t}$ of (3.2), the following theorem holds [22].

Theorem 4.1 *For smooth, compatible data g, for $0 \leq \Delta t \leq \Delta t^*$, Δt^* arbitrary, the error satisfies*

$$\left(\Delta t \sum_{n=0}^{N} \|\phi^n_{\Delta t}(\cdot) - \phi(\cdot, t_n)\|^2_{H^{-1/2}(\Gamma)} \right)^{1/2} \leq C_{\Delta t^*} \Delta t^2 \|g\|_{H^5_0(0,T;H^{1/2}(\Gamma))} .$$

The A-stability of the linear multistep method is inherited to the convolution quadrature method, i.e., all Δt^* are permitted in the above estimate.

Let $\left(\phi^n_{k,h}\right)_{n=0}^{N}$ be the sequence of solutions of (3.9) at the time levels t_n, $n = 0, 1, \ldots, N$. We quote from [22] the convergence theorem.

Theorem 4.2 *For smooth compatible data g, the fully discrete method (3.9) (Galerkin in space, operational quadrature in time) is unconditionally stable and the solution $\phi^n_{\Delta t,h} \in S_{m-1,m}$, $0 \leq n \leq N$, $m \in \{0, 1\}$, satisfies the error estimate*

$$\left\| \phi(\cdot, t_n) - \phi^n_{\Delta t,h}(\cdot) \right\|_{H^{-1/2}(\Gamma)} \leq C_g \left(\Delta t^2 + h^{m+3/2} \right).$$

As an immediate consequence of this theorem, we see that the spatial and temporal errors are balanced if

$$\Delta t^2 \sim h^{m+3/2}. \tag{4.1}$$

4.2 Perturbations in the space discretization

In this subsection, we study the influence of replacing the matrices $\mathbf{A}_n$ by the sparse approximation $\tilde{\mathbf{A}}_n$. Our perturbation analysis is based on the theory which was developed in [22]. For this, we introduce the time continuous, space discrete problem which is given by: Find $\phi_h : [0, T] \to S$ such that

$$\int_0^t \int_\Gamma \int_\Gamma k(x - y, t - \tau) \phi_h(y, \tau) \psi_h(x) \, d\Gamma_y d\Gamma_x d\tau = \int_\Gamma g(x, t) \psi_h(x) \, d\Gamma_x \qquad \forall \psi_h \in S. \tag{4.2}$$

Recall the definition of the one-sided Laplace transform

$$\hat{f}(s) := (\mathcal{L} f)(s) := \int_0^\infty e^{-st} f(t) \, dt.$$

(Convention: If a function depends on space and time variables, the Laplace transform is always applied to the time variable.) Applying this transformation to (4.2) and using the rule for the Laplace transform of convolutions, we obtain (cf. [22])

$$\int_\Gamma \int_\Gamma \hat{k}(x - y, s) \hat{\phi}_h(y, s) \psi_h(x) \, d\Gamma_y d\Gamma_x = \int_\Gamma \hat{g}(x, s) \psi_h(x) \, d\Gamma_x \qquad \forall \psi_h \in S \quad \forall s \in I_\sigma, \tag{4.3}$$

where $I_\sigma := \{\sigma + \mathrm{i}\, u : u \in \mathbb{R}\}$ for some $\sigma > 0$. The Laplace transform of k is given by

$$\hat{k}(z, s) = \frac{e^{-s\|z\|}}{4\pi \|z\|},$$

and $\hat{\phi}_h$ is the Laplace transform of ϕ_h. For $s \in I_\sigma$, we define the operator $V_h(s) : S \to S$ by

$$(V_h(s)\varphi_h, \psi_h)_{L^2(\Gamma)} := \int_\Gamma \int_\Gamma \hat{k}(x-y,s)\varphi_h(y)\psi_h(x)\, d\Gamma_y d\Gamma_x \quad \forall \varphi_h, \psi_h \in S.$$

Let $P_h : H^{1/2}(\Gamma) \to S$ denote the orthogonal projection, i.e.,

$$(P_h f, \psi_h)_{L^2(\Gamma)} := (f, \psi_h)_{L^2(\Gamma)} \quad \forall \psi_h \in S.$$

With these notations at hand, the time continuous, space discrete problem (4.3) can be written in the compact form: Find $\hat{\phi}_h : I_\sigma \to S$ such that

$$\left(V_h(s)\,\hat{\phi}_h(s), \psi_h\right)_{L^2(\Gamma)} = (P_h \hat{g}(\cdot, s), \psi_h)_{L^2(\Gamma)} \quad \forall \psi_h \in S \quad \forall s \in I_\sigma.$$

The time discretization can be described by replacing s in $V_h(s)$ by $\gamma\left(e^{-s\Delta t}\right)/\Delta t$: Find $\hat{\phi}_{\Delta t,h} : I_\sigma \to S$ such that

$$\left(V_{\Delta t,h}(s)\,\hat{\phi}_{\Delta t,h}(s), \psi_h\right)_{L^2(\Gamma)} = (P_h \hat{g}(\cdot, s), \psi_h)_{L^2(\Gamma)} \quad \forall \psi_h \in S \quad \forall s \in I_\sigma, \tag{4.4}$$

where $V_{\Delta t,h}(s) := V_h\left(\gamma\left(e^{-s\Delta t}\right)/\Delta t\right)$.

Remark 4.3 *The solution $\phi^n_{\Delta t,h}$ at time step $t_n = n\Delta t$ (cf. (3.8)) can be written by means of the inverse Laplace transform as*

$$\phi^n_{\Delta t,h} = \left(\mathcal{L}^{-1}\hat{\phi}_{\Delta t,h}\right)(t_n).$$

Next, we express the solution $\tilde{\phi}^n_{\Delta t,h}$ of (3.23) in a similar fashion. Our cutoff strategy is based on the approximation of the coefficients $\omega^{\Delta t}_n(d)$ in the power series

$$\hat{k}(d, \frac{\gamma(\zeta)}{\Delta t}) = \sum_{n=0}^{\infty} \omega^{\Delta t}_n(d)\,\zeta^n$$

by

$$\tilde{\omega}^{\Delta t}_n(d) := \begin{cases} \omega^{\Delta t}_n(d) & d \in I^{\Delta t}_{n,\varepsilon}, \\ 0 & d \notin I^{\Delta t}_{n,\varepsilon}. \end{cases}$$

Let

$$\begin{aligned} G(d,s) &:= \hat{k}\left(d, \frac{\gamma\left(e^{-s\Delta t}\right)}{\Delta t}\right) = \textstyle\sum_{n=0}^{\infty} \omega^{\Delta t}_n(d)\, e^{-s\Delta t n}, \\ \tilde{G}(d,s) &:= \textstyle\sum_{n=0}^{\infty} \tilde{\omega}^{\Delta t}_n(d)\, e^{-s\Delta t n}. \end{aligned} \tag{4.5}$$

For $s \in I_\sigma$, let $\tilde{V}_{\Delta t,h}(s) : S \to S$ be the operator defined by

$$\left(\tilde{V}_{\Delta t,h}(s)\varphi_h, \psi_h\right)_{L^2(\Gamma)} := \int_\Gamma \int_\Gamma \tilde{G}(\|x-y\|, s)\,\varphi_h(y)\psi_h(x)\, d\Gamma_y d\Gamma_x \quad \forall \varphi_h, \psi_h \in S.$$

Consider the problem: Find $\hat{\tilde{\phi}}_{\Delta t,h}(s) \in S$ such that

$$\left(\tilde{V}_{\Delta t,h}(s)\,\hat{\tilde{\phi}}_{\Delta t,h}(s), \psi_h\right)_{L^2(\Gamma)} = (P_h \hat{g}(\cdot, s), \psi_h)_{L^2(\Gamma)} \quad \forall \psi_h \in S \quad \forall s \in I_\sigma. \tag{4.6}$$

Then the solution $\tilde{\phi}^n_{\Delta t,h}$ of (3.23) can be expressed by means of the inverse Laplace transform

$$\tilde{\phi}^n_{\Delta t,h} := \left(\mathcal{L}^{-1}\widehat{\tilde{\phi}}_{\Delta t,h}\right)(t_n) .$$

By combining (4.4) and (4.6) we see that the Laplace transform of the error $e_{\Delta t,h} := \tilde{\phi}_{\Delta t,h} - \phi_{\Delta t,h}$ satisfies

$$\left(V_{\Delta t,h}(s)\,\hat{e}_{\Delta t,h}(s), \psi_h\right)_{L^2(\Gamma)} = \left(\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right)\widehat{\tilde{\phi}}_{\Delta t,h}(s), \psi_h\right)_{L^2(\Gamma)} \qquad \forall \psi_h \in S, \forall s \in I_\sigma.$$

This leads to the estimate

$$\|\hat{e}_{\Delta t,h}(s)\|_{H^{-1/2}(\Gamma)} \le \left\|V^{-1}_{\Delta t,h}(s)\right\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} \left\|\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right)\widehat{\tilde{\phi}}_{\Delta t,h}(s)\right\|_{H^{1/2}(\Gamma)} \quad (4.7)$$

for all $s \in I_\sigma$.

In order to estimate the terms in (4.7), we need the following estimate of $\|V^{-1}(s)\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)}$ (cf. [22, (2.20)]): Let $\sigma > 0$. Then, there exists $M(\sigma)$, such that

$$\|V^{-1}(s)\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} \le M(\sigma)|s|^2 \quad \forall\ \mathrm{Re}(s) > \sigma . \qquad (4.8)$$

Lemma 4.4 *Let the time discretization be based on convolution quadrature with the BDF2 scheme. Then, for $\sigma > 0$ there exists $c_\sigma > 0$ independent of the discretization parameters $\Delta t, h$ such that*

$$\left\|V^{-1}_{\Delta t,h}(s)\,P_h\right\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} \le c_\sigma \frac{1}{\Delta t^2} \qquad \forall s \in I_\sigma . \qquad (4.9)$$

Proof. From [22, (5.17)] we deduce the estimate

$$\begin{aligned}\left\|V^{-1}_{\Delta t,h}(s)\,P_h\right\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} &= \left\|V_h^{-1}\left(\gamma\left(e^{-s\Delta t}\right)/\Delta t\right)P_h\right\|_{H^{-1/2}(\Gamma)\leftarrow H^{1/2}(\Gamma)} \\ &\le M(\sigma_0)\left|\frac{\gamma\left(e^{-s\Delta t}\right)}{\Delta t}\right|^2 \qquad \forall s \in I_\sigma \qquad (4.10)\end{aligned}$$

for σ_0 such that $\mathrm{Re}\left(\frac{\gamma(e^{-s\Delta t})}{\Delta t}\right) > \sigma_0\ \forall s \in I_\sigma$. σ_0 can be chosen independent of Δt. The estimate now follows due to the boundedness of $|\gamma(e^{-s\Delta t})|$. ■

Next, we turn to the second factor in the right-hand side of (4.7). For the following lemma, we need an inverse inequality which holds for our boundary element spaces (cf. [7]), while the constant depends on the quasiuniformity of the mesh. Let $C_{\mathrm{inv}} > 0$ denote the smallest constant such that

$$\|\psi_h\|_{L^2(\Gamma)} \le C_{\mathrm{inv}} h^{-1/2}\,\|\psi_h\|_{H^{-1/2}(\Gamma)} \qquad \forall \psi_h \in S \qquad (4.11)$$

holds.

Lemma 4.5 *Let the time discretization be based on convolution quadrature with the BDF2 scheme. Then*

$$\left\|\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right)\widehat{\tilde{\phi}}_{\Delta t,h}(s)\right\|_{H^{1/2}(\Gamma)} \le \frac{c_\Delta \varepsilon h^{-1}}{1 - e^{-\sigma\Delta t}}\left\|\widehat{\tilde{\phi}}_{\Delta t,h}(s)\right\|_{H^{-1/2}(\Gamma)} \quad \forall s \in I_\sigma . \quad (4.12)$$

The constant c_Δ is associated with the Laplace operator and C_{inv} and is independent of the discretization parameters Δt and h .

Proof. For any $\phi_h \in S$, the difference $\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right)\hat{\phi}_h(s)$ can be written in the form

$$\left\|\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right)\hat{\phi}_h(s)\right\|_{H^{1/2}(\Gamma)} = \sup_{\substack{\varphi_h \in S\backslash\{0\} \\ \|\varphi_h\|_{H^{-1/2}(\Gamma)}=1}} \left|\int_\Gamma\int_\Gamma \delta(\|x-y\|)\,\hat{\phi}_h(y,s)\,\varphi_h(x)\,d\Gamma_y d\Gamma_x\right|,$$

where (cf. (4.5))

$$\delta(d) := \sum_{n=0}^{\infty}\left(\omega_n^{\Delta t}(d) - \tilde{\omega}_n^{\Delta t}(d)\right)e^{-s\Delta t n}.$$

From the construction of our cutoff strategy (cf. (3.19)) we deduce

$$|\delta(d)| \le \frac{\varepsilon}{4\pi d}\sum_{n=0}^{\infty} e^{-\sigma\Delta t n} = \frac{\varepsilon}{4\pi d\left(1-e^{-\sigma\Delta t}\right)}.$$

By using the well known L^2-continuity of the single layer potential for the Laplacian, we obtain

$$\begin{aligned}
\left\|\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right)\hat{\phi}_h(s)\right\|_{H^{1/2}(\Gamma)} &\le \frac{\varepsilon}{1-e^{-\sigma\Delta t}} \sup_{\substack{\varphi_h \in S\backslash\{0\} \\ \|\varphi_h\|_{H^{-1/2}(\Gamma)}=1}} \int_\Gamma\int_\Gamma \frac{\left|\hat{\phi}_h(y,s)\right| |\varphi_h(x)|}{4\pi\|x-y\|} d\Gamma_y d\Gamma_x \\
&\le \frac{C\varepsilon h^{-1/2}}{1-e^{-\sigma\Delta t}}\|\phi_h\|_{L^2(\Gamma)} \\
&\le \frac{c_\Delta \varepsilon h^{-1}}{1-e^{-\sigma\Delta t}}\|\phi_h\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

■

Finally, we investigate the existence and boundedness of the solution $\tilde{\phi}_{\Delta t,h}$. We do not employ the possible smoothness of $\tilde{\phi}_{\Delta t,h}$ with respect to time since only the constants in the convergence and complexity estimates would be improved but not the rates.

Lemma 4.6 *Let the time discretization be based on convolution quadrature with the BDF2 scheme. Then given* $\sigma > 0$, *for all cutoff parameters* ε *in (3.22) with* $0 < \varepsilon < \frac{1-e^{-\sigma\Delta t}}{2c_\Delta c_\sigma} h\Delta t^2$, *the solution* $\tilde{\phi}_{\Delta t,h}$ *in (3.23) exists and satisfies the stability estimate*

$$\left\|\widehat{\tilde{\phi}}_{\Delta t,h}(s)\right\|_{H^{-1/2}(\Gamma)} \le 2c_\sigma \Delta t^{-2}\|\hat{g}(s)\|_{H^{1/2}(\Gamma)} \quad \forall s \in I_\sigma.$$

Proof. We start with the splitting

$$\tilde{V}_{\Delta t,h}(s) = V_{\Delta t,h}(s)(I - X(s)) \quad \text{with} \quad X(s) := V_{\Delta t,h}^{-1}(s)\left(V_{\Delta t,h}(s) - \tilde{V}_{\Delta t,h}(s)\right).$$

Lemmata 4.4 and 4.5 imply

$$\|X(s)\|_{H^{-1/2}(\Gamma)\leftarrow H^{-1/2}(\Gamma)} \le c_\Delta c_\sigma \frac{1}{\Delta t^2}\frac{\varepsilon h^{-1}}{1-e^{-\sigma\Delta t}}.$$

By choosing $0 < \varepsilon < \frac{1-e^{-\sigma\Delta t}}{2c_\Delta c_\sigma} h\Delta t^2$, we obtain $\|X(s)\|_{H^{-1/2}(\Gamma)\leftarrow H^{-1/2}(\Gamma)} < 1/2$ uniformly for all $s \in I_\sigma$. This directly implies the stability estimate

$$\left\|\tilde{V}_{\Delta t,h}^{-1}(s)P_h\right\|_{H^{1/2}(\Gamma)\leftarrow H^{-1/2}(\Gamma)} \le 2\left\|V_{\Delta t,h}^{-1}(s)P_h\right\|_{H^{1/2}(\Gamma)\leftarrow H^{-1/2}(\Gamma)} \le 2c_\sigma \Delta t^{-2}.$$

■

The combination of Lemmata 4.4, 4.5, and 4.6 leads to the convergence estimate of the solution $\tilde{\phi}_{\Delta t,h}$.

Theorem 4.7 *Let the time discretization be based on convolution quadrature with the BDF2 scheme. We assume that the exact solution $\phi(\cdot,t)$ is in $H^{m+1}(\Gamma)$ for any $t \in [0,T]$. Then for all cutoff parameters ε in (3.22) with $0 < \varepsilon < \frac{1-e^{-\sigma\Delta t}}{2c_\Delta c_\sigma} h\Delta t^2$, the solution $\tilde{\phi}_{\Delta t,h}$ in (3.23) exists and satisfies the error estimate*

$$\left\| \tilde{\phi}^n_{\Delta t,h} - \phi(\cdot,t_n) \right\|_{H^{-1/2}(\Gamma)} \leq C_g(t_n) \left(\varepsilon \Delta t^{-5} + \Delta t^2 + h^{m+3/2} \right) .$$

where C_g depends on the right hand side g and on σ.

Proof. We employ the splitting

$$\tilde{\phi}^n_{\Delta t,h} - \phi(t_n) = e^n_{\Delta t,h} + \left(\phi^n_{\Delta t,h} - \phi(t_n) \right) .$$

The estimate [22, Theorem 5.4] implies, for the second summand,

$$\left\| \phi^n_{\Delta t,h} - \phi(t_n) \right\|_{H^{-1/2}(\Gamma)} \leq C h^{m+3/2}.$$

The first summand can be estimated by combining Lemmata 4.4, 4.5, and 4.6

$$\begin{aligned} \left\| \hat{e}_{\Delta t,h}(s) \right\|_{H^{-1/2}(\Gamma)} &\leq 2c_\sigma^2 c_\Delta h^{-1} \Delta t^{-4} \frac{\varepsilon}{1-e^{-\sigma\Delta t}} \left\| \hat{g}(\cdot,s) \right\|_{H^{1/2}(\Gamma)} \\ &\leq C_\sigma \varepsilon h^{-1} \Delta t^{-5} \left\| \hat{g}(\cdot,s) \right\|_{H^{1/2}(\Gamma)} \quad \forall s \in I_\sigma. \end{aligned}$$

From this, the estimate of the perturbation $\tilde{\phi}_{\Delta t,h} - \phi_{\Delta t,h}$ in the original time space follows from the Laplace inversion formula. ■

Corollary 4.8 *Let the assumptions as in Theorem 4.7 be satisfied. Let*

$$\Delta t^2 \sim h^{m+3/2}$$

and choose

$$\varepsilon \sim h^{7m/2+25/4}.$$

Then the solution $\tilde{\phi}^n_{\Delta t,h}$ exists and converges with optimal rate

$$\left\| \tilde{\phi}^n_{\Delta t,h} - \phi(\cdot,t_n) \right\|_{H^{-1/2}(\Gamma)} \leq C_g(t_n) h^{m+3/2} \sim C_g(t_n) \Delta t^2.$$

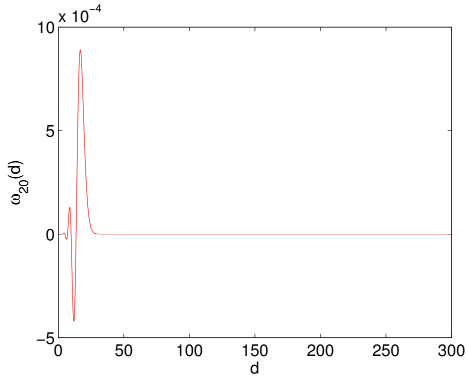
4.3 Approximation of ω_n by cutoff

In this section, we analyse the approximation of the convolution functions

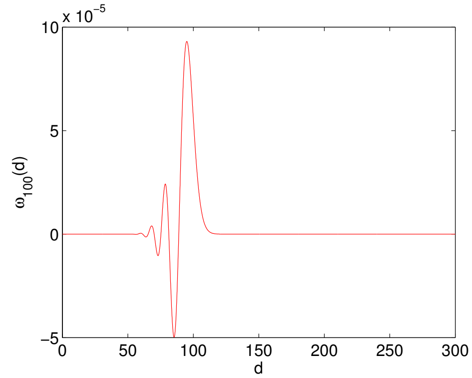
$$\omega_n^{\Delta t}(d) = \frac{1}{n!} \frac{\partial^n}{\partial \zeta^n} \frac{e^{-\gamma(\zeta)\frac{d}{\Delta t}}}{4\pi d} \Bigg|_{\zeta=0} ,$$

where

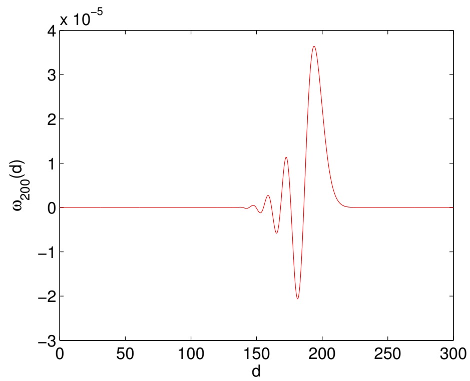
$$\gamma(\zeta) = \frac{1}{2}(\zeta^2 - 4\zeta + 3) .$$



(a) $n = 20$



(b) $n = 100$



(c) $n = 200$

Figure 1: The convolution weights $\omega_n^{\Delta t}(d)$ for $\Delta t = 1$ and different values of n.

Recall the explicit formula as in (3.6)

$$\omega_n^{\Delta t}(d) = \frac{1}{n!}\frac{1}{4\pi d}\left(\frac{d}{2\Delta t}\right)^{n/2} e^{-\frac{3d}{2\Delta t}} H_n\left(\sqrt{\frac{2d}{\Delta t}}\right), \tag{4.13}$$

where H_n are the Hermite polynomials. For $n = 0$, we have

$$\omega_0^{\Delta t}(d) = \frac{e^{-\frac{3}{2}\frac{d}{\Delta t}}}{4\pi d},$$

with a singularity at $d = 0$ and, for $n = 1$,

$$\omega_1^{\Delta t}(d) = \frac{1}{\Delta t}\frac{e^{-\frac{3}{2}\frac{d}{\Delta t}}}{2\pi}.$$

In Figure 1, we plot $\omega_n^{\Delta t}(d)$ for $\Delta t = 1$ and different n. The convolution functions are approximately scaled and translated versions of each other. To find an estimate for $\omega_n^{\Delta t}(d)$, we employ the ansatz

$$|\omega_n^{\Delta t}(d)| \leq \frac{1}{4\pi d}\sigma_n \Omega_n\left(f_n^{\Delta t}(d)\right),$$

with some scaling factors σ_n, some translation functions $f_n^{\Delta t}(d)$, and a function $\Omega_n(x)$ that converges towards a function $\Omega(x)$ as $n \to \infty$.

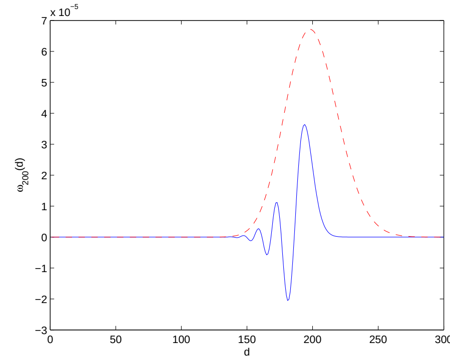


Figure 2: Comparison of $\omega^1_{100}(d)$ (solid line) and $\frac{1}{4\pi d}\sigma_n\Omega_n$ (dashed line)

Lemma 4.9 *For $n \geq 1$, let*

$$\Omega_n(x) = \left(\frac{x}{\sqrt{n}} + 1\right)^{n/2} e^{-\frac{x\sqrt{n}}{2}} \quad \text{and} \quad \sigma_n = \frac{k}{(2\pi n)^{1/4}}$$

with $k \approx 1.086435$. Then

$$|\omega_n^{\Delta t}(d)| \leq \frac{1}{4\pi d}\sigma_n\Omega_n\left(\frac{d - t_n}{\sqrt{\Delta t}\sqrt{t_n}}\right).$$

Proof. To obtain an estimate for $|\omega_n^{\Delta t}(d)|$, we use the following estimate, see [1, (22.14.17)],

$$|H_n(x)| < e^{x^2/2} k 2^{n/2}\sqrt{n!}$$

with $k \approx 1.086435$. Applying this to (4.13) yields

$$|\omega_n^{\Delta t}(d)| \leq \frac{k}{4\pi d}\frac{\left(\frac{d}{\Delta t}\right)^{n/2} e^{-\frac{d}{2\Delta t}}}{\sqrt{n!}}.$$

For $n \geq 1$, Stirling's formula leads to $\frac{1}{\sqrt{n!}} \leq \frac{\left(\frac{e}{n}\right)^{n/2}}{(2\pi n)^{1/4}}$, and we obtain

$$|\omega_n^{\Delta t}(d)| \leq \frac{k}{4\pi d}\frac{e^{n/2}\left(\frac{d}{t_n}\right)^{n/2}}{(2\pi n)^{1/4}} e^{-\frac{d}{2\Delta t}} = \frac{k}{4\pi d}\frac{1}{(2\pi n)^{1/4}}\Omega_n\left(\frac{d - t_n}{\sqrt{\Delta t}\sqrt{t_n}}\right).$$

∎

Lemma 4.10 *There holds*

$$\lim_{n\to\infty}\Omega_n(x) = e^{-x^2/4}.$$

Proof. The logarithm of Ω_n can be written as

$$\begin{aligned}
\log\Omega_n(x) &= \frac{n}{2}\log\left(1 + \frac{x}{\sqrt{n}}\right) - \frac{x\sqrt{n}}{2} \\
&= \frac{n}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{x}{\sqrt{n}}\right)^k - \frac{x\sqrt{n}}{2} \\
&= -\frac{1}{4}x^2 + \sum_{k=3}^{\infty}\frac{1}{3}\frac{(-1)^{k+1}}{2k}\left(\frac{x^k}{n^{\frac{k}{2}-1}}\right)
\end{aligned}$$

from which we conclude that

$$\lim_{n\to\infty} \log \Omega_n(x) = -\frac{1}{4}x^2$$

holds. Thus the statement of the lemma follows. ∎

Remark 4.11 *$\Omega_n(x)$ is decreasing for increasing n. Thus, $\Omega_n(x) \geq e^{-x^2/4}$.*

In Section 3.4, we have introduced a sparse approximation of $\mathbf{A}_n$ by replacing $\omega_n^{\Delta t}(d)$ by zero outside an interval $I_{n,\varepsilon}^{\Delta t} = [t_n - c_{n,\varepsilon}^{\Delta t}, t_n + c_{n,\varepsilon}^{\Delta t}]$. To determine $I_{n,\varepsilon}^{\Delta t}$ such that

$$|\omega_n^{\Delta t}(d)| \leq \frac{\varepsilon}{4\pi d} \quad \forall d \notin I_{n,\varepsilon}^{\Delta t},$$

we first seek an interval $\tilde{I}_{n,\varepsilon}$, such that

$$\Omega_n(x) \leq C\varepsilon \quad \forall x \notin \tilde{I}_{n,\varepsilon}. \tag{4.14}$$

Simple analysis show that Ω_n has one maximum at $x = 0$ and is strictly monotonously increasing for $x < 0$ and strictly monotonously decreasing for $x > 0$. Since $\Omega_n(x)$ is decreasing with increasing $n \geq 1$ a sufficient condition for $\Omega_n(x) \leq C\varepsilon$ is $\Omega_1(x) \leq \varepsilon$. This leads to the condition

$$\Omega_1(x) = \sqrt{x+1}e^{-\frac{1}{2}x} \leq C\varepsilon.$$

If we choose

$$\tilde{c} = 3\log\frac{1}{\varepsilon}, \tag{4.15}$$

inequality (4.14) is satisfied for all $x \notin \tilde{I}_{n,\varepsilon} := [-\tilde{c}, \tilde{c}]$ with $C = \sqrt{3}e^{-1/3}$.

Lemma 4.12 *Let $n \geq 1$ and $c_{n,\varepsilon}^{\Delta t} = \sqrt{\Delta t}\sqrt{t_n}\tilde{c}$ with $\tilde{c}$ as in (4.15). For $I_{n,\varepsilon}^{\Delta t} := [t_n - c_{n,\varepsilon}^{\Delta t}, t_n + c_{n,\varepsilon}^{\Delta t}]$, there holds*

$$|\omega_n^{\Delta t}(d)| \leq \frac{\varepsilon}{4\pi d} \quad \forall d \notin I_{n,\varepsilon}^{\Delta t}.$$

For $n = 0$ and $I_{0,\varepsilon}^{\Delta t} := \left[0, \frac{2}{3}\Delta t \log\frac{1}{\varepsilon}\right]$ there holds

$$|\omega_0^{\Delta t}(d)| \leq \frac{\varepsilon}{4\pi d} \quad \forall d \notin I_{n,\varepsilon}^{\Delta t}.$$

Proof. We have

$$|\omega_n^{\Delta t}(d)| \leq \frac{k}{4\pi d}\frac{1}{(2\pi n)^{1/4}}\Omega_n(\frac{d-t_n}{\sqrt{\Delta t}\sqrt{t_n}}) \leq \frac{\varepsilon}{4\pi d}$$

since $\frac{d-t_n}{\sqrt{\Delta t}\sqrt{t_n}} \notin [-\tilde{c}, \tilde{c}]$. For $n = 0$, we have

$$\omega_0^{\Delta t}(d) = \frac{e^{-\frac{3}{2}\frac{d}{\Delta t}}}{4\pi d}$$

and the condition $d \geq \frac{2}{3}\Delta t \log\frac{1}{\varepsilon}$ implies

$$\frac{e^{-\frac{3}{2}\frac{d}{\Delta t}}}{4\pi d} \leq \frac{\varepsilon}{4\pi d}.$$

∎

5 Complexity Estimates

Next, we determine the storage requirements for the matrices $\tilde{\mathbf{A}}_n$. For the boundary element mesh we assume that the dimension of the boundary element space satisfies

$$c_1 h^{-2} \leq M \leq C_1 h^{-2}. \tag{5.1}$$

A further assumption is related to the surface Γ and the mesh $\mathcal{G}$. We assume that there is a moderate constant C such that for any $1 \leq i \leq M$, the subset

$$\mathcal{P}_i := \{j \in \{1, \ldots, M\} : (i,j) \in \mathcal{P}_\varepsilon\} \ ,$$

with $\mathcal{P}_\varepsilon$ as in (3.21), satisfies

$$\sharp \mathcal{P}_i \leq C \max \left\{ 1, \frac{\sqrt{\Delta t} t_n^{3/2}}{h^2} \log M \right\}. \tag{5.2}$$

This assumption can be derived from two assumptions, namely, that the area of

$$R_{i,n} := \left\{ y \in \Gamma \ : \ \exists x \in \operatorname{supp} b_i : \|x - y\| \in I_{n,\varepsilon}^{\Delta t} \right\}$$

satisfies $|R_{i,n}| \leq C\sqrt{\Delta t} t_n^{3/2} |\log(\varepsilon)|$ and that $ch^2 \leq \operatorname{supp} b_j \leq Ch^2$.

Theorem 5.1 *The number of nonzero entries in the sparse approximation* $\tilde{\mathbf{A}}_n$ *is bounded from above*

- *for piecewise constant boundary elements by*

$$C t_n^{3/2} M^{1+\frac{13}{16}} \log M \, . \tag{5.3a}$$

For the first time steps, $t_n = q\Delta t$, *where* $q = O(\log M)$, *we obtain the improved upper bound*

$$C M^{1+\frac{1}{4}} \log^{3/2} M \, . \tag{5.3b}$$

- *For piecewise linear boundary elements, the number of nonzero entries is bounded by*

$$C t_n^{3/2} M^{1+\frac{11}{16}} \log M \, . \tag{5.3c}$$

For the first time steps, $t_n = q\Delta t$, *where* $q = O(\log M)$, *the improved upper bounded is*

$$CM \, . \tag{5.3d}$$

Proof. The number of nonzero matrix entries in $\tilde{\mathbf{A}}$ can be estimated by using (5.2)

$$\sum_{i=1}^{M} \sharp \mathcal{P}_i \leq CM \max \left\{ 1, \sqrt{\Delta t} t_n^{3/2} h^{-2} \log M \right\}.$$

The relation (4.1) allows to substitute $\sqrt{\Delta t}$ and the combination with (5.1) yields

$$\sum_{i=1}^{M} \sharp \mathcal{P}_i \leq CM \max \left\{ 1, t_n^{3/2} M^{\frac{13}{16} - \frac{1}{8} m} \log M \right\}.$$

The improved estimates follow by using the relation $(\Delta t)^{3/2} \leq CM^{-\frac{3}{8}m-\frac{9}{16}}$. ■

Note that the solution of (3.11) requires that N linear systems of the form

$$\mathbf{A}_0 \phi_n = r.h.s.$$

have to be solved. If the dimension M is large, iterative methods have to be employed for this purpose which require a matrix-vector multiplication in each iteration step. In this light, the improved estimates (5.3b), (5.3d) of the number of nonzero matrix entries for $\tilde{\mathbf{A}}_0$ accelerate this solution process.

6 Conclusions

In this paper, we have followed the convolution quadrature approach by Lubich and combined it with a Galerkin BEM for solving the retarded potential boundary integral formulation of the wave equation. We have presented a simple a-priori cutoff strategy where the number of matrix elements which have to be computed is substantially reduced and a significant portion of the matrix is replaced by zero. A perturbation analysis established the stability of the perturbed problem.

In two forthcoming papers, we will develop a variant of the panel clustering method for the wave equation in order to further reduce the storage and computational costs. In addition, efficient quadrature methods will be introduced and the effect of these additional perturbations will be analysed. The analysis will be based on the perturbation analysis developed in Section 4.2.

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