OPTIMAL PANEL-CLUSTERING IN THE PRESENCE OF ANISOTROPIC MESH REFINEMENT

I.G. GRAHAM [‡], L. GRASEDYCK[†], W. HACKBUSCH[†], AND S.A. SAUTER[§]

Abstract. In this paper we propose and analyse a new enhanced version of the panel-clustering algorithm for discrete boundary integral equations on polyhedral surfaces in 3D, which is designed to perform efficiently even when the meshes contain the highly stretched elements needed for efficient discretisation when the solution contains edge singularities. The key features of our algorithm are: (i) the employment of partial analytic integration in the direction of stretching, yielding a new kernel function on a one dimensional manifold where the influence of the high aspect ratios in the stretched elements is removed and (ii) the introduction of a generalised admissibility condition with respect to the partially integrated kernel which ensures that certain stretched clusters which are inadmissible in the classical sense now become admissible. In the context of a model problem, we prove that our algorithm yields an accurate (up to discretisation error) matrix-vector multiplication which requires $O(N \log^\kappa N)$ operations, where N is the number of degrees of freedom and κ is small and independent of the aspect ratio of the elements. We also show that the classical admissibility condition leads to a sub-optimal clustering algorithm for these problems. A numerical experiment shows that the theoretical estimates can be realised in practice. The generalised admissibility condition can be viewed as a simple addition to the classical method which may be useful in general when stretched meshes are present.

Key words. Boundary elements, panel-clustering, anisotropic meshes, matrix compression

AMS subject classifications. 65N12, 65N22, 65N38, 65R20

1. Introduction. The boundary integral method is a well-known method for solving several important classes of boundary-value problems. Since boundary integral operators are non-local, their straightforward discretisation via the boundary element method leads to dense linear systems, which, for large applications, have to be solved iteratively. The storage requirement and the cost of a matrix-vector multiplication scale quadratically with the number of unknowns N and this is a major bottleneck in the solution process.

The panel-clustering algorithm ([18, 14, 15, 20, 16, 24, 25]) employs separable polynomial expansions of the kernel function in the far field (i.e. for matrix entries corresponding to well-separated pairs of freedoms) to obtain an approximate matrix for which the complexity of storage and matrix-vector multiplication is $O(N \log^\kappa N)$, for moderate κ, but for which the asymptotic accuracy of the resulting numerical solution (as the mesh is refined) remains unchanged. The fast multipole method is closely related [23], [19].

To perform a thorough analysis of the panel-clustering algorithm we are required to address both the stability/consistency of the approximation and the complexity of the resulting matrix-vector multiplication. For quasi-uniform meshes this analysis can be found, e.g., in [18], [16], [26]. For more general classes of meshes, including certain degenerate meshes, a stability/consistency analysis was recently given in [9]. However the complexity analysis has so far not been extended beyond the quasi-uniform case.

In this paper we focus on a class of degenerate meshes which are typically used in connection with the h-version of the Galerkin boundary element method on domains with non-smooth boundaries. Because the solution being sought features anisotropic edge singularities, it can be shown (cf. [21, 6]) that meshes which are *algebraically graded* towards edges and corners allow a much better approximation of the solution than quasi-uniform meshes. These graded meshes are neither quasi-uniform nor shape-regular: Elements which are near edges but away from corners have an aspect ratio which tends to infinity as $N \to \infty$. Unfortunately, for such meshes, the standard panel-clustering algorithm will have sub-optimal complexity, due to the fact that the near-field becomes too large in the vicinity of the stretched elements. (This is explained in §§5,7 below and was also the subject of previous unpublished work [8]).

In this paper we present a new version of the panel-clustering method which does not suffer from this defect. In order for the fundamental idea to be made clear, we will present the algorithm for general operators of order $> 1/2$ on polyhedra, discretised with piecewise constant elements. To keep the paper at reasonable length, the proofs will be limited to the case of the single layer potential for the Laplace operator. However, we also indicate throughout how the method and, in principle, the proofs can be extended to more general situations which are encountered in practical applications. The results of this paper complement those in [10], where the

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conditioning of Galerkin matrices on anisotropic meshes is analysed, along with simple preconditioners which remove ill-conditioning due to the anisotropy.

The paper is organised as follows. In Section 2, we briefly review the boundary integral equations, the graded meshes and corresponding boundary element spaces having optimal approximation property. In Section 3, we introduce the new (generalised) version of the panel-clustering method for stretched meshes and the algorithms for its efficient realisation. The key idea is the introduction of a generalised admissibility condition which is based on the partial analytic integration of the kernel function on stretched elements in the direction parallel to the nearest edge. Here we make the (natural) assumption that the degenerate elements are parallel to the edge in question. Section 4 is devoted to the error analysis of the generalised panel-clustering approximation. Specifically we show that the approximate system yields a solution which satisfies the same asymptotic error estimate as the true Galerkin solution. In Section 5, we will show that the complexity of the new panel-clustering algorithm is of order $N \log^{\kappa} N$ for moderate κ. We will also show that the *standard* panel-clustering method for the stretched meshes considered here has complexity $N^{1+\delta}$, where $\delta \in (0, 1/2)$ and depends on the grading of the mesh. (In an extreme case δ approaches 1/2.) In §6, explicit representations are presented for the partially integrated kernel function which is employed in §3, along with the properties of the partially integrated kernel which are required for the proofs in §§4, 5. Numerical experiments are reported in §7.

2. Boundary Elements on Anisotropically Refined Meshes. Let Γ be the surface of a closed bounded Lipschitz polyhedron in $\mathbb{R}^3$. Linear elliptic boundary-value problems with constant coefficients in the domain interior or exterior to Γ may be reformulated as boundary integral equations

$$(\lambda I + \mathcal{K})u(\mathbf{x}) := \lambda u(\mathbf{x}) + \int_\Gamma k(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) ds_{\mathbf{y}} = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{2.1}$$

where $\lambda \in \mathbb{R}$, k is a known kernel function and f is a known right-hand side. We will assume (as is true in very many cases) that for some $\mu \in \mathbb{R}$, the corresponding weak form:

$$\text{Find } u \in H^\mu(\Gamma) \text{ s.t. } a(u, v) := ((\lambda I + \mathcal{K})u, v) = (f, v) \quad \text{for all } v \in H^\mu(\Gamma) \tag{2.2}$$

constitutes a well-posed variational principle in the Sobolev space $H^\mu(\Gamma)$. (In some cases the energy space must be chosen as a suitable closed subspace of $H^\mu(\Gamma)$.) The continuous extension of the $L^2(\Gamma)$-scalar product $(\cdot, \cdot)$ to the duality pairing $H^{-\mu}(\Gamma) \times H^{\mu}(\Gamma)$ in (2.2) is again denoted by $(\cdot, \cdot)$. Typical examples include all the standard boundary integral formulations of Laplace's equation and the Helmholtz equation (see, e.g. [17, 26]). While the algorithm which we discuss in this paper is applicable to general boundary integral equations we will present proofs for the following particular case:

EXAMPLE 2.1. *The Laplace single layer integral equation is of the form* (2.1) *with* $\lambda = 0$ *and* $k(\mathbf{x}, \mathbf{y}) = 1/(4\pi \|\mathbf{x} - \mathbf{y}\|)$. *The formulation* (2.2) *holds with* $\mu = -1/2$ *and the corresponding bilinear form is symmetric, bounded and coercive in* $H^{-1/2}(\Gamma)$.

To approximate (2.2) we assume that we have a mesh $\mathcal{T}$ of elements (here called *panels*) $\tau \in \mathcal{T}$. Each panel may be a triangle or quadrilateral and is assumed to be the image of a unit triangle or quadrilateral under an affine map. On this mesh we can introduce a space of piecewise polynomial functions $\mathcal{S} \subset H^\mu(\Gamma)$. The Galerkin method then seeks an approximate solution $U \in \mathcal{S}$ by solving

$$a(U, V) = (f, V) \qquad \text{for all } V \in \mathcal{S}. \tag{2.3}$$

Suppose $\mathbf{u}$ is the vector of coefficients of U with respect to a basis $\{\phi_i : i \in \mathcal{I}\}$, for some index set $\mathcal{I}$ with cardinality N. Then we have to solve the $N \times N$ linear system:

$$(\lambda M + K)\mathbf{u} = \mathbf{f}, \tag{2.4}$$

with $M_{i,j} = (\phi_i, \phi_j)$, $K_{i,j} = (\mathcal{K}\phi_j, \phi_i)$ and $f_i = (f, \phi_i)$. The panel-clustering algorithm is used to approximate the dense matrix K. To be more precise, the panel-clustering is best thought of, *not* as an approximation of the *matrix entries* of K but as a data-sparse representation of the associated linear operator $K : \mathbb{R}^{\mathcal{I}} \to \mathbb{R}^{\mathcal{I}}$.

It is well-known (cf. [21, 26]) that the exact solution u exhibits a characteristic singular behaviour in the *normal direction* near edges of Γ and in the *radial direction* near corners of Γ. Moreover, u is smooth in

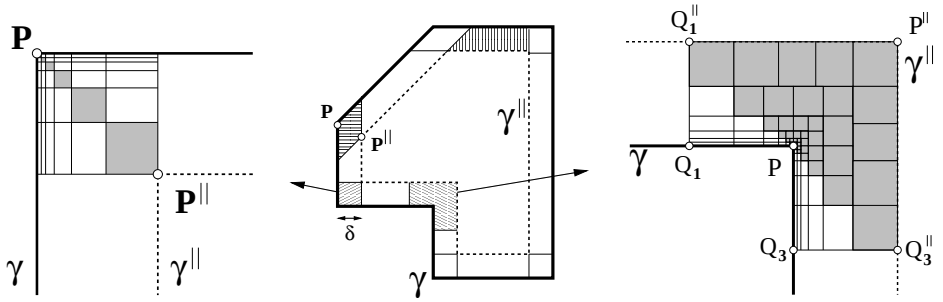


FIG. 2.1. *The face* Γ *(middle) is partitioned into segments with convex corners (left), concave corners (right) and the segments parallel to* γ *in between. Grey panels are shape-regular.*

the interior of the plane faces of Γ and also in the *tangential* direction relative to the edges. This suggests that it may be advantageous to employ graded meshes whose elements have the following properties (as the meshes are refined): (i) they become increasingly small in the normal direction near the edges but are larger in the tangential direction; (ii) they become (uniformly) small when lying close to corners and, (iii) they are quasi-uniform when bounded away from the corners and edges.

We briefly describe the construction of suitable meshes. In general we are considering polyhedral surfaces, but without loss of generality we can explain the meshing algorithm by restricting to the case when Γ is a single planar polygon, with polygonal boundary $\gamma = \partial\Gamma$. For sufficiently small $\delta > 0$ we insert a polygon $\gamma^{\|}$ into Γ which is parallel to γ and has distance δ from γ (cf. Figure 2.1 (middle)) [1]. The portion of Γ between γ and $\gamma^{\|}$ we call "the δ-strip". We assume that δ is sufficiently small so that, for any vertex P in γ there exists a corresponding vertex $P^{\|}$ in $\gamma^{\|}$.

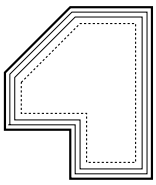
If P is a convex vertex of Γ we extend both segments of $\gamma^{\|}$ which meet at $P^{\|}$ so that they hit γ.
If P is a concave vertex of Γ we insert two lines into the δ-strip which are orthogonal to γ and a distance δ along γ from P.

If δ is chosen sufficiently small, this procedure subdivides Γ into (i) rhombus-shaped regions of the δ-strip near convex corners, (ii) L-shaped regions of the δ-strip near concave corners, (iii) trapezia-shaped regions near edges but away from corners and (iv) a polygonal inner region bounded by $\gamma^{\|}$.

The mesh on Γ depends on two parameters: $N \in \mathbb{N}$ (the number of elements in Γ) and $g \geq 1$ (the grading parameter which determines the concentration of panels towards edges). We shall study the convergence and complexity of our method as $N \to \infty$, with g fixed (although, in general, g may depend on the integral equation being solved, the polynomial degree of the elements and the angles at the corners of Γ). Since some part of the mesh on Γ will be defined using tensor product constructions, we also introduce a coordinate-wise discretisation parameter $n \in \mathbb{N}$ such that $N \sim n^2$. The maximum mesh diameter will be proportional to n^{-1}.

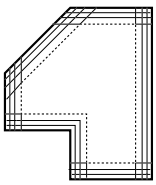
We will use the symbol "$\lesssim$" to compare two quantities $A \lesssim B$ if there exists a constant $C > 0$ which is independent of N such that $A \leq CB$.

ALGORITHM 2.2 (Construction of the mesh). *The meshing is applied to the four different types of region separately after the introduction of the* δ*-strip.*

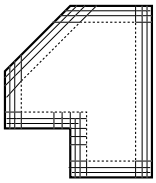


Step 1: Edge-parallel mesh lines.
For $n \in \mathbb{N}$*, insert further polygons* $\gamma_\ell^{\|}$*,* $1 \leq \ell \leq n-1$*, into* Γ *which are parallel to* γ *and have distance* $\delta\,(\ell/n)^g$ *from* γ*. For any vertex* P_ℓ *of* γ*, there exists a corresponding vertex* $P_\ell^{\|}$ *in* $\gamma_\ell^{\|}$*.*

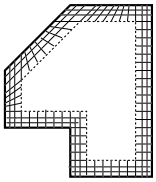
[1] To be more precise, for each segment γ_i in γ there is a corresponding segment $\gamma_i^{\|}$ in $\gamma^{\|}$ so that the infinite lines through γ_i and $\gamma_i^{\|}$ are parallel and are a distance δ apart.



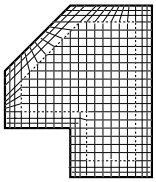
Step 2: Rectangular meshes on rhombi near convex vertices.
The straight line segments which meet at each $P_\ell^\parallel$ are prolonged until they hit γ. This produces a locally rectangular (non shape-regular) mesh on each near-vertex rhombus which is graded towards the outer edges of the rhombus (cf. Figure 2.1 (left)). The panels on the diagonal (towards the corner) are congruent to the near-vertex rhombus, i.e., they are shape-regular.



Step 3: Rectangular meshes on L-shaped regions at concave corners.
We consider an L-shaped region as in Figure 2.1 (right). In the two triangles $Q_1PQ_1^\parallel$ and $PQ_3Q_3^\parallel$ we use the same mesh as in a convex corner: graded towards the corner. Elements on the two lines $Q_1^\parallel P$ and $Q_3^\parallel P$ are then shape-regular. In the remaining part of the L-shaped region we use a shape-regular mesh that can be either conforming or non-conforming. We show the non-conforming case in Figure 2.1 (right).



Step 4: Quadrilateral meshes on near-edge trapezia.
Consider a near-edge trapezium with vertices $ABCD$ so that $\overline{AB} \subset \gamma$ and $\overline{CD} \subset \gamma^\parallel$. Subdivide $\overline{AB}$ into n equidistant segments and $\overline{CD}$ into n equidistant segments and connect the opposite grid points in $\overline{AB}$ with those in $\overline{CD}$ by lines.



Step 5: Refinement of the inner region. The inner region is finally partitioned with a quasi-uniform mesh consisting of triangles and quadrilaterals of diameter $O\left(n^{-1}\right)$ such that the mesh points on $\gamma^\parallel$ already defined above coincide with the mesh points of the inner mesh at $\gamma^\parallel$. The quadrilaterals can be subdivided into triangles if required.

REMARK 2.3. *The insertion of the lines as described in Algorithm 2.2, Step 4, subdivides the trapezium with vertices A, B, C, D into n smaller trapezia which we call* **containers**. *Each container Q consists of n parallel panels and has one and only one edge which lies in γ (namely the edge $\overline{AB}$). The extension of this edge to an infinite line is called base line and denoted by γ_Q.*

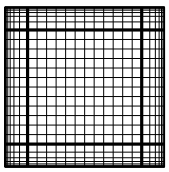
Since we will need it later, we now give the following explicit example.

EXAMPLE 2.4. *Let $\Gamma = [0,1] \times [0,1]$ and define the rectangular mesh:*

$$\mathcal{T} := \{[p_{i-1}, p_i] \times [p_{j-1}, p_j] \ : \ i,j = 1, \ldots, 3n\}, \ \text{where}$$

$$p_i := \begin{cases} \delta(\frac{i}{n})^g \ , & 0 \ \le i \ \le n \\ \delta + (1-2\delta)(\frac{i-n}{n}) \ , & n \ \le i \ \le 2n \\ 1 - \delta(\frac{3n-i}{n})^g \ , & 2n \ \le i \ \le 3n \end{cases} \quad \text{and}$$

$$\delta := 1/(g+2).$$



(The inner square $\gamma^\parallel$ is denoted by the thick lines in the figure .)

Recalling that $N \sim n^2$, we see that near edges but away from corners, we have degenerate elements with length $\sim N^{-1/2}$, and width $\sim \rho_\tau \sim N^{-g/2}$, i.e. aspect ratio growing with $N^{(g-1)/2}$ as $N \to \infty$. The reason for the specific choice of δ is that it ensures that adjacent elements on each side of any one of the thick lines have approximately the same sizes (as $N \to \infty$).

Now let $\mathcal{S}$ denote the space of piecewise constant functions with respect to the mesh on Γ. The analysis of the use of such graded meshes for approximating boundary integral equations on polyhedra can be found, for example, in [21, 22, 6, 7, 11]. In particular (see, e.g. [7, Theorem 1.4]) it is known that the Galerkin solution $U \in \mathcal{S}$ (see (2.3)) to the model problem in Example 2.1 on a Lipschitz polyhegral surface (for sufficiently smooth data f) satisfies:

$$\|u - U\|_{H^{-1/2}(\Gamma)} \ \lesssim \ N^{-3/4} \qquad \text{when} \ \ g > 3 \ . \tag{2.5}$$

Note that (2.5) constitutes an optimal estimate in terms of number of degrees of freedom N, since in the case of quasi-uniform meshes on smooth surfaces, we expect $O(h^{3/2})$ convergence in the $H^{-1/2}$ norm, and the mesh diameter $h \sim N^{-1/2}$. Suboptimal convergence is attained for $1 \leq g \leq 3$. (This result also extends to some higher order elements [21].)

The central problem which we are concerned with in this paper is the construction of a data-sparse representation of the dense stiffness matrix K appearing in (2.4) which has optimal complexity but for which the convergence rates (2.5) are preserved.

3. Panel-Clustering for Graded Meshes. The basic idea of the panel-clustering algorithm is to approximate the kernel k in (2.1) by a (short) separable expansion when $\mathbf{x}$ and $\mathbf{y}$ belong to sufficiently well-separated sets. We briefly recall the essential ingredients of the panel-clustering method before we explain the modifications for stretched meshes.

From now on, the mesh $\mathcal{T}$ on Γ will be written $\mathcal{T} = \{\tau_i \ : i \in \mathcal{I}\}$, where $\mathcal{I}$ is a suitable index set. For each i, let ξ_i denote the incenter (center of the largest inscribed circle), let h_i denote the diameter and let ρ_i denote the diameter of the largest inscribed circle in the panel τ_i.

3.1. Classical Panel-Clustering (Isotropic Admissibility).

Definition 3.1 (Cluster). *A cluster t is a subset of $\mathcal{I}$. If t is a cluster, the corresponding subdomain of Γ is $\Gamma_t := \bigcup_{i\in t} \tau_i$.*

The clusters are collected in a hierarchical cluster tree $T_{\mathcal{I}}$.

Definition 3.2 (Cluster Tree). *A tree $T_{\mathcal{I}}$ is a cluster tree if the following conditions are satisfied.*

1. *The nodes in $T_{\mathcal{I}}$ are clusters.*
2. *The root of $T_{\mathcal{I}}$ is $\mathcal{I}$.*
3. *The leaves of $T_{\mathcal{I}}$ are denoted by $\mathcal{L}(T_{\mathcal{I}})$ and the tree hierarchy is given by a father/son relation: For each interior node $t \in T_{\mathcal{I}} \setminus \mathcal{L}(T_{\mathcal{I}})$, the set of sons of t,* $\operatorname{sons}(t)$*, is the minimal subset in $\mathcal{T}_{\mathcal{I}} \setminus \{t\}$ such that*

$$t = \dot{\bigcup_{s\in \operatorname{sons}(t)}} s$$

holds. Vice versa, the father of any $s \in \operatorname{sons}(t)$ is t.

The standard (geometrically regular) construction of the cluster tree $T_{\mathcal{I}}$ is as follows. Choose a bounding box

$$\tilde{\mathcal{B}} := [a_1, b_1) \times [a_2, b_2) \times [a_3, b_3) \supset \Gamma.$$

Then subdivide $\tilde{\mathcal{B}}$ by bisecting a longest side, thus obtaining two sons $\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2$, e.g.,

$$\tilde{\mathcal{B}}_1 = [a_1, b_1') \times [a_2, b_2) \times [a_3, b_3), \quad \tilde{\mathcal{B}}_2 = [a_1', b_1) \times [a_2, b_2) \times [a_3, b_3),$$

where $a_1' = b_1' := (a_1 + b_1)/2$. Finally, applying the same bisection process recursively, starting with $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$, yields an infinite tree T_{box} with root $\tilde{\mathcal{B}}$. Letting $\tilde{\mathcal{B}}_j$ denote a typical box in this tree, we can define a corresponding cluster by (cf. Figure 3.1)

$$t(\tilde{\mathcal{B}}_j) := \{i \in \mathcal{I} \mid \xi_i \in \tilde{\mathcal{B}}_j\}.$$

This yields an infinite cluster tree with root $t(\tilde{\mathcal{B}})$. We construct a finite cluster tree by deleting (not constructing) sons of clusters below a minimal cardinality $n_{\min}$, e.g. $n_{\min} := 8$. The cardinality of the cluster tree for the meshes introduced in §2, as well as the complexity of its construction will be estimated in §5. (Note that for the special case that Γ is a flat polygon we may choose $\tilde{\mathcal{B}}$ as a two-dimensional bounding box.)

For a cluster, $t \subset \mathcal{I}$, let $\mathcal{B}_t$ denote the minimal bounding box (of the type from above) of Γ_t.

Conventionally, in order to obtain the approximation of the matrix K in (2.4), the kernel function k is approximated on $\Gamma_t \times \Gamma_s$, where (t, s) is a pair of clusters which satisfy the following condition.

Definition 3.3 (Isotropic Admissibility Condition). *Choose an isotropic admissibility parameter $\eta^{\text{iso}} > 0$. A pair of clusters $(t, s) \in T_{\mathcal{I}} \times T_{\mathcal{I}}$ is isotropically (η^{iso}-) admissible if*

$$\max\{\operatorname{diam}\mathcal{B}_t,\ \operatorname{diam}\mathcal{B}_s\} \leq \eta^{\text{iso}} \operatorname{dist}(\mathcal{B}_t,\ \mathcal{B}_s). \tag{3.1}$$

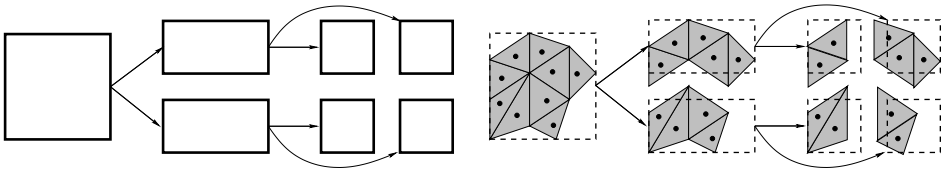


FIG. 3.1. *Left: Tree of regular boxes $\tilde{\mathcal{B}}$ obtained by bisection. Right: Tree of corresponding clusters $t = t(\tilde{\mathcal{B}})$. The black dots mark the corresponding incenters ξ_i of the panels $\tau_i \in \mathcal{T}$.*

The standard choice for the parameter in (3.1) is $\eta^{\mathrm{iso}} := 1$.

The admissibility condition (3.1) allows us to define a non-overlapping partition of $\mathcal{I} \times \mathcal{I}$ into admissible pairs of clusters (called block clusters) and non-admissible pairs of (small) clusters. This partition is formed by the leaves of the block cluster tree $T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}$ which we now define.

DEFINITION 3.4 (Block Cluster Tree $T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}$). *Let $T_{\mathcal{I}}$ be a cluster tree. Then the block cluster tree $T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}$ based on the isotropic admissibility condition (3.1) is defined by the root $(\mathcal{I},\mathcal{I})$ and the father-son relation for $\mathbf{b} = (t,s) \in T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}$:*

$$\mathrm{sons}\,(\mathbf{b}) := \begin{cases} \emptyset & \text{if } (t,s) \text{ is isotropically } \eta^{\mathrm{iso}}\text{-admissible,} \\ \mathrm{sons}(t) \times \mathrm{sons}(s) & \text{otherwise.} \end{cases}$$

The practical construction of the block cluster tree $T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}$ from the root to the leaves is straightforward (cf. [18], [16]). The leaves $\mathcal{L}(T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}})$ consist of two different types of block clusters,

$$\mathcal{L}^{\mathrm{iso}}_{\mathrm{far}} := \{\mathbf{b} \in \mathcal{L}(T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}) \ : \ \mathbf{b} \text{ is isotropically } \eta^{\mathrm{iso}}\text{-admissible } \}, \quad \mathcal{L}^{\mathrm{iso}}_{\mathrm{near}} := \mathcal{L}(T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}}) \setminus \mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}.$$

These are known as the nearfield block clusters ($\mathcal{L}^{\mathrm{iso}}_{\mathrm{near}}$) and the farfield block clusters ($\mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}$). For each block cluster $\mathbf{b} = (t,s)$ we denote the corresponding domain by $\Gamma_{\mathbf{b}} := \Gamma_t \times \Gamma_s$. The set of all subdomains $P^{\mathrm{iso}} = \{\Gamma_{\mathbf{b}} \mid \mathbf{b} \in \mathcal{L}(T^{\mathrm{iso}}_{\mathcal{I}\times\mathcal{I}})\}$ is a partition of the tensor surface $\Gamma \times \Gamma$. The partition P^{iso} consists of two different types of subdomains, the farfield $P^{\mathrm{iso}}_{\mathrm{far}}$ and the nearfield $P^{\mathrm{iso}}_{\mathrm{near}}$:

$$P^{\mathrm{iso}}_{\mathrm{far}} := \left\{\Gamma_{\mathbf{b}} \ : \ \mathbf{b} \in \mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}\right\}, \quad P^{\mathrm{iso}}_{\mathrm{near}} := P^{\mathrm{iso}} \setminus P^{\mathrm{iso}}_{\mathrm{far}}.$$

On each admissible block $\Gamma_{\mathbf{b}} \in P^{\mathrm{iso}}_{\mathrm{far}}$ the kernel function k is approximated by a degenerate expansion of the form

$$\tilde{k}_{\mathbf{b}}(\mathbf{x},\mathbf{y}) = \sum_{\nu=1}^{M} \Phi_{\mathbf{b},\nu}(\mathbf{x})\Psi_{\mathbf{b},\nu}(\mathbf{y}), \qquad (\mathbf{x},\mathbf{y}) \in \Gamma_t \times \Gamma_s. \tag{3.2}$$

REMARK 3.5. *For the kernel function of the single layer potential for the Laplace problem allowable expansion functions in (3.2) include, e.g., polynomial interpolation, multipole expansions, and Taylor expansion. One can show (for the proofs we refer, e.g., to [26], [1]) that, for sufficiently small η^{iso}, there exist constants $0 < \sigma < 1$ (depending on η^{iso}) and $0 < C < \infty$ with the following property: For all isotropically admissible blocks $\mathbf{b} = (t,s) \in \mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}$ the Taylor expansion, Čebyšev interpolation, or multipole expansion of order m satisfies*

$$\left|k(\mathbf{x},\mathbf{y}) - \tilde{k}_{\mathbf{b}}(\mathbf{x},\mathbf{y})\right| \ \le \ C\sigma^m \left(\mathrm{dist}(\mathcal{B}_t,\mathcal{B}_s)\right)^{-1} \quad \forall (\mathbf{x},\mathbf{y}) \in \Gamma_t \times \Gamma_s. \tag{3.3}$$

The number of terms M in (3.2) depends on m through $M\,(m) = m^q$ for some $q \in \{2,3\}$.

For interpolation and multipole expansions, (3.2) can be generalised so that the expansion functions only depend on the clusters and not on the blocks.

In the following, we briefly will sketch the algorithmic realisation in the case of the Laplace single layer potential in Example 2.1 discretised by the Galerkin BEM with piecewise constant boundary elements.

For $\mathbf{b} = (t,s) \in \mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}$ we define the matrices known as the "farfield coefficients" by

$$L^{\mathbf{b}}_{i,\nu} := \int_{\tau_i} \Phi_{\mathbf{b},\nu}(\mathbf{x})ds_{\mathbf{x}}, \quad R^{\mathbf{b}}_{\nu,j} := \int_{\tau_j} \Psi_{\mathbf{b},\nu}(\mathbf{y})ds_{\mathbf{y}}, \quad i \in t,\ j \in s,\ \nu \in \{1,\ldots,M\},$$

and the nearfield matrix $\mathbf{V}^{\mathrm{near}}$ with entries

$$\mathbf{V}^{\mathrm{near}}_{i,j} := \begin{cases} \int_{\tau_i\times\tau_j} k(\mathbf{x},\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}} & \text{if } i\in t, j\in s, (t,s)\in\mathcal{L}^{\mathrm{iso}}_{\mathrm{near}} \\ 0 & \text{otherwise.} \end{cases}$$

The number of columns of $L^{\mathbf{b}}$ and the number of rows of $R^{\mathbf{b}}$ is equal to M. We have in mind that M will be small compared to N (typically $M\sim\log^q N$ for some $q\in\{2,3\}$). For the Laplace single layer potential on the screen (Example 2.1) $M\sim\log^2 N$ (see [1]).

The bilinear form

$$a(U,V)=\int_{\Gamma\times\Gamma} k(\mathbf{x},\mathbf{y})U(\mathbf{x})V(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}} \qquad \forall U,V\in S$$

is split into the blockwise bilinear forms

$$a(U,V)=\sum_{\mathbf{b}\in\mathcal{L}^{\mathrm{iso}}_{\mathrm{near}}\cup\mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}} \underbrace{\int_{\Gamma_{\mathbf{b}}} k(\mathbf{x},\mathbf{y})U(\mathbf{x})V(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}}}_{=:a_{\mathbf{b}}(U,V)}.$$

The panel-clustering approximation $\tilde{a}(U,V)$ of the bilinear form $a(U,V)$ is then obtained by replacing the blockwise bilinear form $a_{\mathbf{b}}(U,V)$ in the farfield blocks $\mathbf{b}=(t,s)\in\mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}$ by

$$\begin{aligned}\tilde{a}_{\mathbf{b}}(U,V) &:= \int_{\Gamma_{\mathbf{b}}} \tilde{k}_{\mathbf{b}}(\mathbf{x},\mathbf{y})V(\mathbf{x})U(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}} \\ &= \sum_{\nu=1}^{M}\int_{\Gamma_{\mathbf{b}}} \Phi_{\mathbf{b},\nu}(\mathbf{x})\Psi_{\mathbf{b},\nu}(\mathbf{y})V(\mathbf{x})U(\mathbf{y})ds_{\mathbf{x}}ds_{\mathbf{y}} \\ &= \sum_{i\in t}\sum_{j\in s}\left(L^{\mathbf{b}}R^{\mathbf{b}}\right)_{i,j} V(\xi_i)U(\xi_j). \end{aligned} \tag{3.4}$$

The final equality follows because here we have restricted to piecewise constant elements.

The nearfield matrix $\mathbf{V}^{\mathrm{near}}$ contains all contributions from the nearfield blocks $\mathbf{b}\in\mathcal{L}^{\mathrm{iso}}_{\mathrm{near}}$. It is sparse since $P^{\mathrm{iso}}_{\mathrm{near}}$ covers only a small part of the tensor surface $\Gamma\times\Gamma$. The farfield coefficients $L^{\mathbf{b}}$ and $R^{\mathbf{b}}$ for blocks $\mathbf{b}\in\mathcal{L}^{\mathrm{iso}}_{\mathrm{far}}$ can be computed by standard quadrature techniques, since the degenerate expansion functions are smooth in the farfield. They contain only $M(\#t+\#s)$ entries instead of the $\#t\#s$ entries of the full block $t\times s$.

In [9], the stability and convergence of the standard panel-clustering method is analysed for general classes of meshes, including the stretched meshes discussed in §2. It is shown there that the mesh degeneracy does not affect the accuracy of the approximation. However it turns out that the standard algorithm is not optimal in terms of complexity, as the following example shows.

EXAMPLE 3.6. *Consider the Laplace single layer equation from Example 2.1 and consider piecewise constant elements on the meshes on the unit square in Example 2.4. In order to obtain optimal convergence rates we have to choose $g>3$ (see (2.5)).*

We fix an arbitrary index $n+1\le i\le 2n$. All panels $\tau_{i,j}:=[p_{i-1},p_i]\times[p_{j-1},p_j]$ satisfy

$$p_i-p_{i-1}=(1-2\delta)n^{-1}=\left(\frac{g}{g+2}\right)n^{-1}>\frac{3}{5}n^{-1}.$$

On the other hand, for all indices $j=1,\ldots,n^{1-1/g}+1$ we have

$$p_{j-1}=\delta(j-1)^g n^{-g}\le\delta n^{g-1}n^{-g}\le\frac{1}{5}n^{-1}<\frac{3}{5}n^{-1},$$

so that for all $j,\ell\in J:=\{1,\ldots,n^{1-1/g}+1\}$

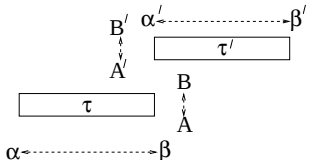
$$\mathrm{dist}(\tau_{i,j},\tau_{i,\ell})<\frac{3}{5}n^{-1}.$$

Thus any pair of clusters (t, s), where both t and s contain a panel $\tau_{i,j}$ with $j \in J$, cannot be isotropically admissible when the parameter η^{iso} is chosen such that $\eta^{iso} \leq 1$. Therefore, in the rows of the matrix K corresponding to the indices $\{(i,j) \mid i = n+1, \ldots, 2n,\ j \in J\}$ there are at least $\#J = n^{1-1/g}$ nearfield entries that must be computed and stored. There are $n^{2-1/g}$ such rows and so we have to store at least $n^{3-2/g} = \mathcal{O}(N^{3/2-1/g})$ entries, i.e., the (storage) complexity of conventional panel clustering is at least this. As g increases, the complexity approaches $\mathcal{O}(N^{3/2})$.

3.2. Anisotropic Admissibility. The reason for the sub-optimal performance of the panel-clustering method for stretched meshes is the isotropic admissibility condition. The nearfield of the long thin elements which lie close to edges of Γ becomes too large.

This problem is overcome by a modified panel-clustering algorithm which will be presented next.

3.2.1. The Case of the Screen. To illustrate the underlying idea we first consider the screen domain Γ from Example 2.4. Let two panels parallel to the x-axis be of the form



$$\begin{aligned} \tau &= (\alpha, \beta) \times (A, B), \\ \tau' &= (\alpha', \beta') \times (A', B') \end{aligned} \qquad (3.5)$$

with $\alpha < \beta$, $\alpha' < \beta'$, $A < B < A' < B'$.

Suppose also that

$$\max\{\operatorname{diam}(\tau), \operatorname{diam}(\tau')\} \ \geq\ \max\{\beta - \alpha, \beta' - \alpha'\} \ >\ \eta^{\mathrm{iso}}(A' - B) \ =\ \eta^{\mathrm{iso}} \operatorname{dist}(\tau, \tau')\ , \qquad (3.6)$$

but at the same time

$$\max\{B - A, B' - A'\} \ \leq\ \eta^{\mathrm{iso}}\ (A' - B). \qquad (3.7)$$

Clearly (3.6) indicates that this pair of panels is inadmissible in the sense of condition (3.1). The problem is that the long sides (with length $\beta - \alpha$ and $\beta' - \alpha'$) are much longer than the distance between the elements. On the other hand if somehow the lengths of the long sides could be ignored then these elements would become admissible by virtue of (3.7). The essence of our new method is the observation that, provided the elements are parallel, analytic partial integration may be performed, leading to a new admissibility criterion which admits pairs satisfying (3.7) but not (3.1).

Continuing with the example (3.5), assuming U and V are piecewise constant functions and using the notation $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$, we obtain

$$\begin{aligned} \int_\tau \int_{\tau'} k(\mathbf{x}, \mathbf{y}) V(\mathbf{x}) U(\mathbf{y}) \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} &= V(\xi) U(\xi') \int_A^B \int_{A'}^{B'} \int_\alpha^\beta \int_{\alpha'}^{\beta'} k((x_1, x_2), (y_1, y_2)) \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} \\ &= V(\xi) U(\xi') \int_A^B \int_{A'}^{B'} k^{\mathrm{ad}}(x_2, y_2) \mathrm{d}y_2 \mathrm{d}x_2\ , \end{aligned}$$

where ξ and ξ' are the incenters of τ, τ' and $k^{\mathrm{ad}}(x_2, y_2) := \int_\alpha^\beta \int_{\alpha'}^{\beta'} k((x_1, x_2), (y_1, y_2)) \mathrm{d}y_1 \mathrm{d}x_1$ is the antiderivative of k. We will show that the twofold integration over the *long* directions (α, β), (α', β') leads to a kernel function $k^{\mathrm{ad}} : (A, B) \times (A', B') \to \mathbb{R}$ which is admissible for the one-dimensional intervals (A, B), (A', B') via the one-dimensional admissibility condition (3.7).

The idea can be carried over to pairs of clusters (instead of pairs of panels), provided they fulfil the following anisotropic admissibility condition.

Definition 3.7 (Anisotropic Admissibility Condition for the Screen). *Choose $\eta^{\mathrm{aniso}} > 0$. A pair of clusters $(t, s) \in T_{\mathcal{I}} \times T_{\mathcal{I}}$ is anisotropically (η^{aniso}-) admissible if all panels τ_i, $i \in t$, and τ_j, $j \in s$, are of the form (in some suitable coordinate system, cf. Figure 3.2):*

$$\tau_i = (A_i^t, B_i^t) \times (\alpha^t, \beta^t), \quad \tau_j = (A_j^s, B_j^s) \times (\alpha^s, \beta^s), \qquad (3.8)$$

or

$$\tau_i = (\alpha^t, \beta^t) \times (A_i^t, B_i^t), \quad \tau_j = (\alpha^s, \beta^s) \times (A_j^s, B_j^s) \qquad (3.9)$$

and

$$\max\{|B^t - A^t|, |B^s - A^s|\} \;\le\; \eta^{\text{aniso}} \operatorname{dist}(\mathcal{B}_t, \mathcal{B}_s), \tag{3.10}$$

where $A^t := \min\{A_i^t : i \in t\}$, $B^t := \max\{B_i^t : i \in t\}$ *and* A^s, B^s *are defined analogously.*

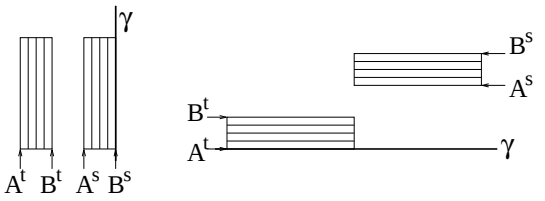


FIG. 3.2. *Examples of pairs* (t, s) *of anisotropically admissible clusters.*

Let t and s, $\mathbf{b} := (t, s)$, be two clusters with corresponding panels of the form (3.8) or (3.9). The evaluation of the blockwise bilinear form $a_{\mathbf{b}}(U, V)$ for two piecewise constant finite element functions U and V can be written in the form

$$\begin{aligned} a_{\mathbf{b}}(U,V) &= \int_{\Gamma_{\mathbf{b}}} k(\mathbf{x},\mathbf{y}) V(\mathbf{x}) U(\mathbf{y}) \mathrm{d}x\mathrm{d}y \\ &= \int_{\alpha^t}^{\beta^t} \int_{\alpha^s}^{\beta^s} \sum_{i\in t} \sum_{j\in s} \int_{A_i^t}^{B_i^t} \int_{A_j^s}^{B_j^s} V(x_1,x_2) U(y_1,y_2) k((x_1,x_2),(y_1,y_2)) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &= \sum_{i\in t} \sum_{j\in s} V(\xi_i) U(\xi_j) \int_{A_i^t}^{B_i^t} \int_{A_j^s}^{B_j^s} k_{\mathbf{b}}^{\mathrm{ad}}(x_2,y_2) \mathrm{d}y_2 \mathrm{d}x_2, \end{aligned}$$

where $k_{\mathbf{b}}^{\mathrm{ad}}$, as before, is the antiderivative of k,

$$k_{\mathbf{b}}^{\mathrm{ad}}(x_2,y_2) = \int_{\alpha^t}^{\beta^t} \int_{\alpha^s}^{\beta^s} k((x_1,x_2),(y_1,y_2)) \mathrm{d}y_1 \mathrm{d}x_1. \tag{3.11}$$

In Section 4 we will prove that for an anisotropically admissible block $\mathbf{b} = (t, s)$ the integrated kernel $k_{\mathbf{b}}^{\mathrm{ad}}$ can be approximated by a degenerate kernel $\tilde{k}_{\mathbf{b}}^{\mathrm{ad}}$ of the form

$$\tilde{k}_{\mathbf{b}}^{\mathrm{ad}}(x_2,y_2) = \sum_{\nu=1}^{m} \Phi_{\mathbf{b},\nu}(x_2) \Psi_{\mathbf{b},\nu}(y_2). \tag{3.12}$$

The bilinear form $a_{\mathbf{b}}(U, V)$ is then approximated by

$$\begin{aligned} \tilde{a}_{\mathbf{b}}(U,V) &:= \sum_{i\in t} \sum_{j\in s} V(\xi_i)\, U(\xi_j) \int_{A_i^t}^{B_i^t} \int_{A_j^s}^{B_j^s} \sum_{\nu=1}^{m} \Phi_{\mathbf{b},\nu}(x_2) \Psi_{\mathbf{b},\nu}(y_2) \mathrm{d}y_2 \mathrm{d}x_2 \\ &= \sum_{i\in t} \sum_{j\in s} V(\xi_i)\, U(\xi_j) \sum_{\nu=1}^{m} \underbrace{\int_{A_i^t}^{B_i^t} \Phi_{\mathbf{b},\nu}(x_2) \mathrm{d}x_2}_{=:L_{i,\nu}^{\mathbf{b}}} \underbrace{\int_{A_j^s}^{B_j^s} \Psi_{\mathbf{b},\nu}(y_2) \mathrm{d}y_2}_{=:R_{\nu,j}^{\mathbf{b}}} \\ &= \sum_{i\in t} \sum_{j\in s} (L^{\mathbf{b}} R^{\mathbf{b}})_{i,j} V(\xi_i) U(\xi_j), \end{aligned} \tag{3.13}$$

where $L^{\mathbf{b}} \in \mathbb{R}^{\#t \times m}$ and $R^{\mathbf{b}} \in \mathbb{R}^{m \times \#s}$. Since it will turn out that m can be chosen proportional to $\log(N)$, the storage and matrix by vector complexity is $\mathcal{O}(\log(N)(\#t + \#s))$ instead of the quadratic complexity for the standard representation.

Combining the isotropic and anisotropic admissibilities, we arrive at a final block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$ described as follows.

DEFINITION 3.8 (Block Cluster Tree $T_{\mathcal{I}\times\mathcal{I}}$). *Let $T_{\mathcal{I}}$ be a cluster tree. Then the block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$ based on the admissibility conditions (3.1), (3.10) is defined by the root $(\mathcal{I},\mathcal{I})$ and the father-son relation for $\mathbf{b}=(t,s)\in T_{\mathcal{I}\times\mathcal{I}}$:*

$$\operatorname{sons}(\mathbf{b}) := \begin{cases} \emptyset & \text{if } (t,s) \text{ is isotropically or anisotropically admissible,} \\ \operatorname{sons}(t)\times\operatorname{sons}(s) & \text{otherwise.} \end{cases}$$

The leaves $\mathbf{b}\in\mathcal{L}(T_{\mathcal{I}\times\mathcal{I}})$ of the block cluster tree form a partition of $\mathcal{I}\times\mathcal{I}$ and the corresponding domains $\Gamma_{\mathbf{b}}$ yield a partition of the tensor surface $\Gamma\times\Gamma$. In contrast to the isotropic case discussed in Definition 3.4, three different types of blocks will appear in this case:

- $\mathcal{L}_{\text{far}}^{II} := \{\mathbf{b}\in\mathcal{L}(T_{\mathcal{I}\times\mathcal{I}}) : \mathbf{b} \text{ is anisotropcally, but not isotropically, admissible}\}$
  These blocks allow an approximation by $\tilde{a}_{\mathbf{b}}$ given in (3.13) and involve the integrated kernel $\tilde{k}_{\mathbf{b}}^{\text{ad}}$.
- $\mathcal{L}_{\text{far}}^{I} := \{\mathbf{b}\in\mathcal{L}(T_{\mathcal{I}\times\mathcal{I}}) : \mathbf{b} \text{ is isotropically admissible}\}$
  These are treated by the standard techniques (3.4).
- $\mathcal{L}_{\text{near}} := \mathcal{L}(T_{\mathcal{I}\times\mathcal{I}})\backslash\mathcal{L}_{\text{far}}, \qquad \mathcal{L}_{\text{far}} := \mathcal{L}_{\text{far}}^{II}\cup\mathcal{L}_{\text{far}}^{I}$.
  For these blocks we use the original bilinear form $a_{\mathbf{b}}$.

For the block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$ we will prove in Section 5 that the nearfield matrix $\mathbf{V}^{\text{near}}$ corresponding to entries from nearfield blocks $\mathbf{b}\in\mathcal{L}_{\text{near}}$ is sparse (i.e., $\mathcal{O}(N)$ entries). This means that the generalised panel-clustering approximation

$$\tilde{a}(U,V) := \sum_{\mathbf{b}\in\mathcal{L}_{\text{near}}} a_{\mathbf{b}}(U,V) + \sum_{\mathbf{b}\in\mathcal{L}_{\text{far}}} \tilde{a}_{\mathbf{b}}(U,V) \tag{3.14}$$

can be evaluated in $\mathcal{O}(N\log(N)M)$ operations, where $M\approx\log^2 N$. In Section 4 we prove the consistency of the panel-clustering approximation, i.e. that the complexity of $\mathcal{O}(N\log^3 N)$ is attained without damaging the accuracy of the underlying boundary element method.

3.2.2. The Case of a Polyhedral Surface. The anisotropic admissibility condition (Definition 3.7) can be extended to the case of polyhedral surfaces Γ which are composed of faces with polygonal boundaries. Recall the definition of the containers Q and their base line γ_Q as in Remark 2.3.

DEFINITION 3.9 (Anisotropic Admissibility Condition for Polyhedral Surfaces). *A pair of clusters (t,s) is anisotropically admissible with respect to $\gamma\in\mathcal{E}$, if there exist containers Q_t, Q_s with the same base line $\gamma_{Q_t}=\gamma_{Q_s}$ such that*

$$\Gamma_t\subset Q_t \quad \textit{and} \quad \Gamma_s\subset Q_s$$

and

$$\max\{H_t,H_s\} \;\le\; \eta^{\text{aniso}}\;\operatorname{dist}(\mathcal{B}_t,\mathcal{B}_s),$$

where for $i=s,t$

$$H_i := \sup_{\mathbf{x},\mathbf{y}\in\Gamma_i}\{\|\mathbf{x}-\mathbf{y}\| : (\mathbf{x}-\mathbf{y})\perp(\gamma_{Q_i})\}.$$

Analogously to the screen case, the integration parallel to $\gamma_{Q_s}=\gamma_{Q_i}$ is carried out analytically leading again to an antiderivative of the kernel function which can be approximated efficiently by panel clustering with respect to the remaining (orthogonal) variables.

To be more precise, let (t,s) be a pair of clusters which is anisotropically admissible. Let Q_t, Q_s denote the containers for t, s which have the common base line $\gamma_Q=\gamma_{Q_t}=\gamma_{Q_s}$. Note that Q_t and Q_s may be subsets on the same face of Γ or subsets of two adjoining faces of Γ. We choose a local coordinate system such that the x_1- axis points in the direction of γ_Q and t lies in the (x_1,x_2)-plane. Because of the meshing algorithm, we have

$$t = \left\{(x_1,x_2,0)^{\intercal}\in\mathbb{R}^3 \mid A_t\le x_2\le B_t \text{ and } \alpha_t(x_2)\le x_1\le\beta_t(x_2)\right\}, \tag{3.15}$$

where $A_t < B_t$ are constants and α_t, β_t are affine functions which satisfy $\alpha_t(x_2) < \beta_t(x_2)$ for all $A_t \leq x_2 \leq B_t$. Analogously s is described by

$$s = \{(w_1, w_2 \cos\varphi, w_2 \sin\varphi)^\intercal \mid A_s \leq w_2 \leq B_s \text{ and } \alpha_s(w_2) \leq w_1 \leq \beta_s(w_2)\}.$$

Again, $A_s < B_s$ are constants and α_s, β_s are affine functions which satisfy $\alpha_s(w_2) < \beta_s(w_2)$. The angle $\varphi \in [0, 2\pi[$ is fixed and denotes the angle between Γ_t and Γ_s. The antiderivative of the kernel function is then defined (analogously to (3.11)) by

$$k_{\mathbf{b}}^{\mathrm{ad}}(x_2, y_2) := \int_{\alpha_s(y_2)}^{\beta_s(y_2)} \int_{\alpha_t(x_2)}^{\beta_t(x_2)} k(\chi_t(x_1, x_2), \chi_s(w_1, y_2))\, dx_1 dw_1, \tag{3.16}$$

where χ_t, χ_s denote a unitary transformation of the two-dimensional coordinate systems to the surface panels.

An illustration of a pair of admissible clusters in this case is given in Figure 3.3.

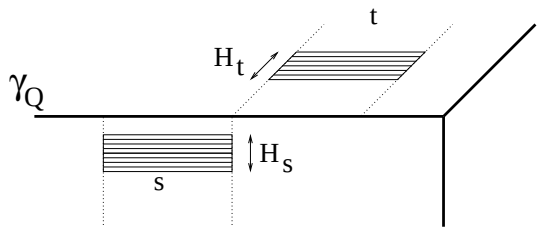


FIG. 3.3. *A pair of admissible clusters* (t, s).

The panel-clustering method for anisotropic elements uses an approximation of $k_{\mathbf{b}}^{\mathrm{ad}}$ by a degenerate expansion $\tilde{k}_{\mathbf{b}}^{\mathrm{ad}}$ of the form

$$\tilde{k}_{\mathbf{b}}^{\mathrm{ad}}(x_2, y_2) = \sum_{\nu=1}^{m} \Phi_{\mathbf{b},\nu}(x_2)\, \Psi_{\mathbf{b},\nu}(y_2). \tag{3.17}$$

The following assumption states that the approximation $\tilde{k}_{\mathbf{b}}^{\mathrm{ad}}$ converges exponentially towards $k_{\mathbf{b}}^{\mathrm{ad}}$.

ASSUMPTION 3.10. *Let k denote the kernel of the single layer operator and let $k_{\mathbf{b}}^{\mathrm{ad}}$ be as in (3.16). For sufficiently small η^{aniso}, there exist constants $0 < \delta < 1$ (depending on η^{aniso}) and $0 < C < \infty$ with the following property: For all anisotropically admissible blocks $\mathbf{b} = (t, s) \in \mathcal{L}_{\mathrm{far}}^{\mathrm{aniso}}$ there exist a family of degenerate expansions $\tilde{k}_{\mathbf{b}}^{\mathrm{ad}}(\cdot, \cdot)$ depending on a parameter $m \in \mathbb{N}$ of the form (3.17) with*

$$\left| k_{\mathbf{b}}^{\mathrm{ad}}(\mathbf{x}, \mathbf{y}) - \tilde{k}_{\mathbf{b}}^{\mathrm{ad}}(\mathbf{x}, \mathbf{y}) \right| \leq C m^2 \delta^m \qquad \forall (\mathbf{x}, \mathbf{y}) \in \Gamma_t \times \Gamma_s. \tag{3.18}$$

REMARK 3.11. *The proof of* (3.18) *in the case of the screen problem is given in Theorem 6.4, while the general case can be analysed along the same lines.*

REMARK 3.12. *Higher order elements can also be treated by anisotropic panel clustering provided we write the basis functions as a sum of products of the form $\psi(x_1)\phi(x_2)$, with x_1 being the coordinate in the direction orthogonal to the base line γ_Q.*

Along the edges of the surface Γ it usually happens that stretched elements from two different containers (meeting at the edge) are contained in the same cluster t. In this situation, we will consider a pair (t, s) admissible even if this is only true for the two subsets of t coming from each of the two containers.

DEFINITION 3.13 (Generalised Anisotropic Admissiblity Condition). *Let C_{cl} be a given constant (cf. §5, typically $C_{\mathrm{cl}} = 2$). A pair of clusters (t, s) is (generalised) anisotropically admissible if there exists a partition*

$$t = \dot{\bigcup_{i=1,\ldots,C_{\mathrm{cl}}}} t_i, \qquad s = \dot{\bigcup_{j=1,\ldots,C_{\mathrm{cl}}}} s_j$$

such that each pair (t_i, s_j) is anisotropically admissible as defined in Definition 3.9.

The generalised anisotropic admissiblity is in accordance with Assumption 3.10, because after the subdivision into $C_{\mathrm{cl}} \times C_{\mathrm{cl}}$ subblocks one can apply the one-dimensional expansion as in (3.17) for the arising sub-blocks.

4. Error Analysis. In this section, we will prove the consistency and stability of the generalised panel-clustering algorithm for the single layer potential equation from Example 2.1 discretised by piecewise constant elements on the stretched meshes defined in Algorithm 2.2. In [9] the consistency and stability of the standard panel clustering method for such meshes was proved. Thus the results here are a generalisation of those, with particular emphasis on the approximations on the anisotropic blocks. The error analysis will be based on the second Strang Lemma which we briefly recall. Let $\tilde{a} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ denote the bilinear form corresponding to the panel-clustering approximation (3.14). The corresponding approximate Galerkin method is to seek $\tilde{U} \in \mathcal{S}$ such that

$$\tilde{a}(\tilde{U}, V) = (g, V) \tag{4.1}$$

The following theorem follows from [5] (see in particular [9, Lemma 4.2, equation (4.24)]).

THEOREM 4.1. *Assume that there exists a continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mu(N) \to 0$ as $N \to \infty$ such that, for all $V, W \in \mathcal{S}$*

$$(i) \quad |a(V,W) - \tilde{a}(V,W)| \lesssim \mu(N) \|V\|_{H^{-1/2}(\Gamma)} \|W\|_{H^{-1/2}(\Gamma)},$$
$$(ii) \quad |a(V,W) - \tilde{a}(V,W)| \lesssim N^{-3/4} \|V\|_{L^2(\Gamma)} \|W\|_{H^{-1/2}(\Gamma)} .$$

Then, for N sufficiently large, a unique solution $\tilde{U} \in \mathcal{S}$ of (4.1) exists and satisfies

$$\left\| u - \tilde{U} \right\|_{H^{-1/2}(\Gamma)} \lesssim N^{-1/4} \inf_{Z \in \mathcal{S}} \|u - Z\|_{L^2(\Gamma)} + N^{-3/4} \|u\|_{L^2(\Gamma)} .$$

The fact that $u \in L^2(\Gamma)$ follows from [4, Theorem 3]. Now it is shown in [21, Satz 3.7] that a suitable choice of g can be made which ensures that $\inf_{Z \in \mathcal{S}} \|u - Z\|_{L^2(\Gamma)} = N^{-1/2}$ and so in this case we obtain the estimate:

$$\left\| u - \tilde{U} \right\|_{H^{-1/2}(\Gamma)} \lesssim N^{-3/4} . \tag{4.2}$$

In the remaining part of this section, we will prove that the bilinear form $\tilde{a}(\cdot,\cdot)$ satisfies the assumptions (i) and (ii) of Theorem 4.1. To reduce technicalities we assume for the following that the boundary element mesh $\mathcal{T}$ is conforming, i.e., the intersection $\overline{\tau} \cap \overline{\sigma}$ of any two non-identical panels $\tau, \sigma \in \mathcal{T}$ is either empty or a common vertex or a common edge. This can be easily achieved by performing an appropriate closure of the mesh for the L-shaped regions in Algorithm 2.2, Step 3. Furthermore, we assume that all panels are convex and the interior angles are bounded above by a constant $\alpha_{\max} < \pi$. We introduce a positive-valued continuous function ρ as follows. For each panel τ, let ρ_τ denote the diameter of the largest inscribed disc. Let $\widehat{S}$ denote the unit triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ and let $\widehat{Q} := (0,1)^2$ denote the unit square. For each $\tau \in \mathcal{T}$, if τ is a triangle, let $\chi_\tau : \widehat{S} \to \tau$ denote an affine bijection, and, if τ is a quadrilateral, let $\chi_\tau : \widehat{Q} \to \tau$ denote a bijection which is affine in each variable. Then ρ is defined by requiring:

- For all panels τ : $\rho \circ \chi_\tau$ is $\begin{cases} \text{affine} & \text{if } \tau \text{ is a triangle,} \\ \text{affine in each variable} & \text{if } \tau \text{ is a quadrilateral.} \end{cases}$
- For all panel vertices $\mathbf{x}_p$: $\rho(\mathbf{x}_p) = \max\{\rho_\tau : \mathbf{x}_p \in \overline{\tau}\}$.

We use the function ρ in the proof of the following result.

THEOREM 4.2. *Assume that we are solving the single layer potential equation in Example 2.1 on a Lipschitz polyhedral surface Γ, meshed as in Algorithm 2.2 and choose $g = 3 + \epsilon$, where $\epsilon > 0$ is any small number. Let σ be as in Remark 3.5 and let Assumption 3.10 be satisfied for some $0 < \delta < 1$. Choose the orders m of the panel-clustering according to*

$$m = \left\lceil \frac{(2+\epsilon)\ \log N}{|\log \sigma|} \right\rceil \text{for the isotropic panel clustering,} \tag{4.3}$$

$$m = \left\lceil \frac{(5/2+\epsilon)\ \log N}{|\log \delta|} \right\rceil \text{for the anisotropic panel clustering,} \tag{4.4}$$

where $\lceil x \rceil$ denotes the smallest integer z with $z \geq x$. Then the assumptions (i) and (ii) of Theorem 4.1 hold.

Proof. For a block $\mathbf{b} \in \mathcal{L}^I_{\rm far}$, let $\tilde{k}_{\mathbf{b}}$ denote the approximation as in (3.4) and let $\varepsilon_{\mathbf{b}} := k - \tilde{k}_{\mathbf{b}}$ denote the corresponding error. For a block $\mathbf{b} \in \mathcal{L}^{II}_{\rm far}$, the partially integrated kernel function is denoted by $k^{\rm ad}_{\mathbf{b}}$ (cf. 3.16) and the approximation by $\tilde{k}^{\rm ad}_{\mathbf{b}}$ (cf. (3.12)). The corresponding error is denoted by $\varepsilon^{\rm ad}_{\mathbf{b}} = k^{\rm ad}_{\mathbf{b}} - \tilde{k}^{\rm ad}_{\mathbf{b}}$. Thus

$$\begin{aligned}
|a(U,V) - \tilde{a}(U,V)| \leq & \sum_{\mathbf{b}\in\mathcal{L}^I_{\rm far}} \int_{\mathbf{b}} |\varepsilon_{\mathbf{b}}(\mathbf{x},\mathbf{y})|\,|U(\mathbf{x})|\,|V(\mathbf{y})|\,ds_{\mathbf{x}}ds_{\mathbf{y}} \\
& + \sum_{\mathbf{b}\in\mathcal{L}^{II}_{\rm far}} \int_{A^t}\int_{A^s} \left|\varepsilon^{\rm ad}_{\mathbf{b}}(x_2,y_2)\right|\,|U(x_2)|\,|V(y_2)|\,dx_2dy_2 \\
=: & \, E^I(U,V) + E^{II}(U,V)\,,
\end{aligned} \tag{4.5}$$

where for all $\mathbf{b} = (s,t) \in \mathcal{L}^{II}_{\rm far}$ and $(i,j) \in (s,t)$ we have set

$$\begin{aligned}
U(x_2) &:= U(\xi_i) \qquad \text{for all } x_2 : (x_1,x_2) \in \tau_i, \\
V(y_2) &:= V(\xi_j) \qquad \text{for all } y_2 : (y_1,y_2) \in \tau_j.
\end{aligned}$$

In the definition of E^{II} we have assumed the scenario and notation as in (3.9), i.e. $\mathbf{b} = (t,s)$, with

$$\Gamma_t = \bigcup\nolimits_{i=1}^{\#t} \tau_i\ , \quad \text{with } \ \tau_i = (\alpha^t,\beta^t)\times(A^t_i,B^t_i)\ ,\ h_i = \beta^t - \alpha^t\ ,\ \rho_i = B^t_i - A^t_i\ , \tag{4.6}$$

and the analogous definition of Γ_s.

We first turn to the estimate of E^I and proceed as in [9]. From Assumption 3.5, and the fact that $\theta = 1$, we obtain

$$\begin{aligned}
E^I(U,V) &\lesssim \sigma^m \sum_{(t,s)\in\mathcal{L}^I_{\rm far}} \int_{\Gamma_t\times\Gamma_s} \frac{1}{\operatorname{dist}(\mathcal{B}_t,\mathcal{B}_s)} |U(\mathbf{x})||V(\mathbf{y})|ds_{\mathbf{y}}ds_{\mathbf{x}} \\
&= \sigma^m \sum_{(t,s)\in\mathcal{L}^I_{\rm far}} \frac{1}{\operatorname{dist}(\mathcal{B}_t,\mathcal{B}_s)} \int_{\Gamma_t} |U(\mathbf{x})|ds_{\mathbf{x}} \int_{\Gamma_s} |V(\mathbf{y})|ds_{\mathbf{y}}.
\end{aligned}$$

Since (t,s) is an isotropically admissible pair, from (3.3), we have

$$\operatorname{dist}(\mathcal{B}_t,\mathcal{B}_s)^{-1} \ \leq\ \eta^{iso}\max(\operatorname{diam}\mathcal{B}_t,\operatorname{diam}\mathcal{B}_s)^{-1} \leq \eta^{iso}h_i^{-1}\ , \quad \text{for all } i \in t\ .$$

So, using the fact that the sets $\{\Gamma_t\times\Gamma_s : (t,s)\in\mathcal{L}^I_{\rm far}\}$ are pairwise disjoint (and also $|\tau_i| \leq h_i^2$), we have

$$\begin{aligned}
E^I(U,V) &\lesssim \eta^{iso}\sigma^m \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} h_i^{-1}\|U\|_{L^1(\tau_i)}\|V\|_{L^1(\tau_j)} \\
&\leq \eta^{iso}\sigma^m \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} h_j\ \|U\|_{L^2(\tau_i)}\|V\|_{L^2(\tau_j)}\ .
\end{aligned}$$

Now, for any $\theta \in [0,1/2]$, using $h_j \lesssim N^{-1/2}$ and $\rho_j \gtrsim N^{-g/2}$, we obtain,

$$\begin{aligned}
E^I(U,V) &\lesssim \eta^{iso}\sigma^m N^{-1/2} \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} \rho_i^{-\theta}\|\rho^\theta U\|_{L^2(\tau_i)}\rho_j^{-1/2}\|\rho^{1/2}V\|_{L^2(\tau_j)} \\
&\lesssim \eta^{iso}\sigma^m N^{-1/2}N^{g\theta/2}N^{g/4} \sum_{i\in\mathcal{I}} \|\rho^\theta U\|_{L^2(\tau_i)} \sum_{j\in\mathcal{I}} \|\rho^{1/2}V\|_{L^2(\tau_j)}.
\end{aligned}$$

Hence using the Cauchy-Schwarz inequality and then the inverse estimates in [9, Theorem 3.6], we have

$$\begin{aligned}
E^I(U,V) &\lesssim \eta^{iso}\sigma^m N^{-1/2}N^{g\theta/2}N^{g/4}N\|\rho^\theta U\|_{L^2(\Gamma)}\|\rho^{1/2}V\|_{L^2(\Gamma)} \\
&\lesssim \eta^{iso}\sigma^m N^{1/2+g\theta/2+g/4}\|U\|_{H^{-\theta}(\Gamma)}\|V\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

The required estimate for $E^I(U,V)$ follows by observing that by taking m as given in (4.3) and with $g = 3+\epsilon$,

$$\sigma^m N^{1/2+g\theta/2+g/4} \lesssim \begin{cases} N^{-3(1+\epsilon)/4} & \text{if } \theta = 0, \\ N^{-\epsilon/2} & \text{if } \theta = 1/2\ . \end{cases}$$

In order to estimate $E^{II}(U,V)$, take a typical block $\mathbf{b} = (t,s)$ in $\mathcal{L}^{II}_{\rm far}$ and assume the notation as in (4.6). Assumption 3.10 implies

$$\left|\varepsilon^{\rm ad}_{\mathbf{b}}(x_2,y_2)\right| \ \lesssim\ m^2\delta^m\ .$$

Hence

$$E^{II}(U,V) \lesssim m^2\delta^m \sum_{\mathbf{b}=(t,s)\in\mathcal{L}^{II}_{\rm far}} \int_{A^t} |U(x_2)|dx_2 \int_{A^s} |V(x_2)|dx_2\ . \tag{4.7}$$

Moreover, for any $\theta \in [0,1/2]$

$$\begin{aligned}\int_{A^t} |U(x_2)|dx_2 = \sum_{i\in t}\int_{A^t_i} |U(x_2)|dx_2 \ &\lesssim\ \sum_{i\in t} \rho_i^{1/2}\left\{\int_{A^t_i}|U(x_2)|^2dx_2\right\}^{1/2} \\ &\lesssim \sum_{i\in t}\rho_i^{1/2-\theta}h_i^{-1/2}\|\rho^\theta U\|_{L^2(\tau_i)}.\end{aligned}$$

Since $\rho_i \le h_i$ and $1/2-\theta \ge 0$, we have $\rho_i^{1/2-\theta}h_i^{-1/2} \lesssim h_i^{-\theta} \lesssim N^{g\theta/2}$, and so

$$\int_{A^t} |U(x_2)|dx_2 \ \lesssim\ N^{g\theta/2}\sum_{i\in t}\|\rho^\theta U\|_{L^2(\tau_i)}. \tag{4.8}$$

Hence combining (4.7) and (4.8) and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}E^{II}(U,V) &\lesssim m^2\delta^m N^{(g\theta/2+g/4)}\sum_{i\in\mathcal{I}}\|\rho^\theta U\|_{L^2(\tau_i)}\sum_{j\in\mathcal{I}}\|\rho^{1/2}V\|_{L^2(\tau_j)} \\ &\lesssim m^2\delta^m N^{(g\theta/2+g/4+1)}\|\rho^\theta U\|_{L^2(\Gamma)}\|\rho^{1/2}V\|_{L^2(\Gamma)}.\end{aligned}$$

Then, making use of [9, Theorem 3.6], we have

$$E^{II}(U,V) \ \lesssim\ m^2\delta^m N^{g\theta/2+g/4+1}\|U\|_{H^{-\theta}(\Gamma)}\|V\|_{H^{-1/2}(\Gamma)}. \tag{4.9}$$

Now if we choose m as in (4.4) it follows that

$$\delta^m N^{g\theta/2+g/4+1} \lesssim \begin{cases} N^{-3/4(1+\epsilon)}\ , & \theta = 0 \\ N^{-\epsilon/2}\ , & \theta = 1/2\ , \end{cases}$$

and so the required estimates for E^{II} follow. □

5. Complexity Analysis. In this section we analyse the complexity of the storage requirements and of the matrix-vector multiplication for our new version of the panel clustering algorithm. First we introduce a measure of the sparsity of the block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$.

DEFINITION 5.1 (Sparsity). *The sparsity of a block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$ based on a cluster tree $T_{\mathcal{I}}$ is characterised by the quantity*

$$C_{\rm sp} := \max\left\{\max_{s\in T_{\mathcal{I}}}\#\{t\in T_{\mathcal{I}} \mid s\times t\in T_{\mathcal{I}\times\mathcal{I}}\},\ \max_{t\in T_{\mathcal{I}}}\#\{s\in T_{\mathcal{I}} \mid s\times t\in T_{\mathcal{I}\times\mathcal{I}}\}\right\}.$$

The importance of $C_{\rm sp}$ was discussed in [12]. In the case of shape regular meshes and with cluster trees created using standard bisection algorithms, using the isotropic admissibility condition (3.1) it was shown in

[12, Lemma 4.5] that $C_{\rm sp}$ is bounded independently of N (the number of degrees of freedom in the mesh). Here we shall extend this result to the anisotropic case. Before we do this, we will state the complexity bounds for the storage and matrix-vector product of the panel clustering approximation $\tilde{a}(U,V)$ of the bilinearform $a(U,V)$ based on the measure $C_{\rm sp}$ and the depth p of the cluster tree. Both quantities will be bounded in Theorem 5.11 under quite general conditions, which are fulfilled for the screen mesh from Example 2.4.

THEOREM 5.2 (Storage and Matrix by Vector Complexity). *Let $T := T_{\mathcal{I}\times\mathcal{I}}$ be a block cluster tree with sparsity $C_{\rm sp}$ and depth $p \geq 1$. Let m be the (maximal) interpolation order used for the interpolation of the kernel function in isotropically or anisotropically admissible blocks. Then the storage requirements $N_{\rm St}(T,m)$ and the matrix by vector complexity $N_{\tilde{a}(U,V)}(T,m)$ for the panel clustering approximation (3.13) are bounded by*

$$N_{\rm St}(T,m) = \mathcal{O}(C_{\rm sp} m^2 p \#\mathcal{I}), \qquad N_{\tilde{a}(U,V)}(T,m) \sim N_{\rm St}(T,m).$$

In particular for the screen mesh from Example 2.4, we have $pm^2\#\mathcal{I} = \mathcal{O}(gN\log^3 N)$.

Proof. a) For each block $(t,s) \in \mathcal{L}_{\rm near}$ we have to store the entries $V_{ij}^{\rm near}$ for all $(i,j) \in t \times s$. The construction of the block cluster tree ensures that either t or s is a leaf of $T_{\mathcal{I}}$, i.e., $\#t \leq n_{\min}$ or $\#s \leq n_{\min}$ so that at most $n_{\min}(\#t + \#s)$ entries have to be stored.

b) For each block $\mathbf{b} = (t,s) \in \mathcal{L}_{\rm far}$ we have to store the entries $L^{\mathbf{b}}_{i,\nu}$ and $R^{\mathbf{b}}_{\nu,j}$ for all $i \in t, j \in s, \nu \in M$, where $M = \{1,\ldots,m\}$ for anisotropically admissible blocks and $M = \{1,\ldots,m^q\}$ (q from Remark 3.5) for isotropically admissible blocks. In total we can bound the number of entries to be stored for the block $\mathbf{b}$ by $m^q(\#t + \#s)$.

c) Due to (a) and (b) the storage complexity for each block $\mathbf{b} \in \mathcal{L}(T_{\mathcal{I}\times\mathcal{I}})$ is bounded by $\max\{n_{\min}, m^q\}(\#t + \#s)$. We conclude

$$\begin{aligned}
N_{\rm St}(T,m) &\leq \sum_{(t,s)\in\mathcal{L}(T_{\mathcal{I}\times\mathcal{I}})} \max\{n_{\min}, m^q\}(\#t + \#s) \\
&\leq \sum_{t\in T_{\mathcal{I}}} \sum_{(t,s)\in T_{\mathcal{I}\times\mathcal{I}}} \max\{n_{\min}, m^q\}\#t + \sum_{s\in T_{\mathcal{I}}} \sum_{(t,s)\in T_{\mathcal{I}\times\mathcal{I}}} \max\{n_{\min}, m^q\}\#s \\
&\leq 2C_{\rm sp} \max\{n_{\min}, m^q\} \sum_{t\in T_{\mathcal{I}}} \#t \\
&\overset{[12,\ \text{R.1.10}]}{\leq} 2C_{\rm sp} \max\{n_{\min}, m^q\}(p+1)\#\mathcal{I} \quad = \quad \mathcal{O}(C_{\rm sp} m^q p \#\mathcal{I}).
\end{aligned}$$

d) The matrix by vector complexity is proportional to the storage complexity because every stored entry of $V^{\rm near}$, $L^{\mathbf{b}}$ and $R^{\mathbf{b}}$ is accessed exactly once when computing a matrix-vector product. □

COROLLARY 5.3 (Complexity for the Cluster Tree Construction). *A cluster tree $T_{\mathcal{I}}$ of depth $p \geq 1$ has at most $\#T_{\mathcal{I}} = \mathcal{O}(p\#\mathcal{I})$ nodes. The complexity for the standard construction (so-called geometrically regular clustering) as introduced in Section 3.1 is $\mathcal{O}(p\#\mathcal{I})$.*

Proof. Using [12, Remark 1.10] we obtain $\#\mathcal{L}(T_{\mathcal{I}}) \leq \#\mathcal{I}$, so that the property $t = \dot{\bigcup}_{s\in\text{sons}(t)} s$ for interior nodes yields $\#T_{\mathcal{I}} = \mathcal{O}(p\#\mathcal{I})$. In the standard construction of the cluster tree $T_{\mathcal{I}}$ we compute the boxes $\tilde{\mathcal{B}}_1, \tilde{\mathcal{B}}_2$ for each node t in $\mathcal{O}(1)$ and determine for each point $(\xi_i)_{i\in t}$ the box to which it belongs, in total $\mathcal{O}(\#t)$ operations. Due to [12, Remark 1.10] the sum over all $t \in T_{\mathcal{I}}$ is $\mathcal{O}(p\#\mathcal{I})$. □

For the following bound on the sparsity measure $C_{\rm sp}$ we will now introduce some mesh-dependent quantities. The first of them is the quantity C_{ov} defined below. This quantifies the maximal number of nearby panels.

DEFINITION 5.4. *We define (cf. Figure 5.1)*

$$C_{\rm ov} := \max_{i\in\mathcal{I}} \#\left\{j \in \mathcal{I} \mid \operatorname{dist}(\xi_i, \xi_j) \leq \rho_i/2\right\},$$

where ρ_i is the diameter of the largest inscribed circle in τ_i and ξ_i, ξ_j are the incenters of τ_i, τ_j, respectively.

In many cases, and in particular for the screen mesh from Example 2.4, we have $C_{\rm ov} = 1$. In general (e.g. for highly folded surfaces) $C_{\rm ov} > 1$ might hold.

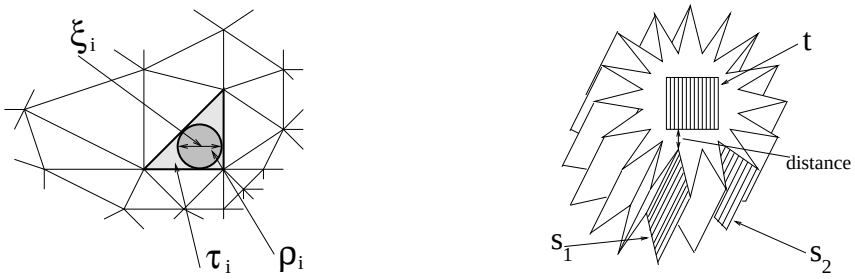


FIG. 5.1. *Left: In the inscribed circle around the incenter ξ_i there lies no other incenter $\xi_j, j \neq i$, i.e., $C_{\rm ov} = 1$. Right: The cluster t is surrounded by many clusters of the type of s_1, s_2 with very small distance to t.*

Next we want to characterise the alignment (in the sense of (3.8), (3.9) or (3.15)) of stretched panels. A situation where the constants in the complexity estimates might become large is depicted in Figure 5.1: the cluster t containing stretched panels is surrounded by a large number C of clusters $s_1, \ldots, s_C$ that also contain stretched panels. The cluster s_i are pairwise unaligned (the union of any two of those clusters would not fulfil the requirements of the anisotropic admissibility condition). In the following we want to consider the neighbourhood of a panel.

DEFINITION 5.5. *Let $C_{\rm str} > 2 + 3\eta^{\rm iso}$. We define the neighbourhood of a panel τ_i, $i \in \mathcal{I}$, by*

$$N_{str}(\tau_i) := \{j \in \mathcal{I} \mid \operatorname{dist}(\tau_j, \tau_i) < C_{\rm str}(\eta^{\rm iso})^{-1} \max\{\operatorname{diam}(\tau_i), \operatorname{diam}(\tau_j)\}.$$

A partition of the form $N_{str}(\tau_i) = \dot\cup_{\nu=1}^{C} N_{str}^{\nu}(\tau_i)$ is an alignment-classification of $N_{str}(\tau_i)$, if for all $\nu = 1, \ldots, C$ the set $N_{str}^{\nu}(\tau_i)$ is aligned, i.e. of the form (3.8), (3.9) or (3.15).

ASSUMPTION 5.6. *We assume that for any panel τ_i, $i \in \mathcal{I}$, with $h_i > C_{\rm aniso}\,\rho_i$ there exists an alignment-classification of $N_{str}(\tau_i)$ with not more than $C_{\rm cl}$ classes, where $C_{\rm cl}$ depends on the given mesh but is bounded independently of i, the grading parameter g and the number of unknowns N.*

REMARK 5.7. *For the screen mesh the number of classes is $C_{\rm cl} = \mathcal{O}(C_{\rm str}/\eta^{\rm iso})$ independent of $C_{\rm aniso}$.*

The following three auxiliary Lemmata are needed to prove the final Theorem 5.11.

LEMMA 5.8. *Let $t \in T_{\mathcal{I}}$ be a cluster of cardinality $\#t > C_{\rm ov}$. If the diameter of the minimal bounding box $\mathcal{B}_t$ is larger than the diameter of the regular box $\tilde{\mathcal{B}}_t$ (cf. Section 3.1, Figure 3.1) of the cluster tree, i.e.,*

$$\operatorname{diam}(\mathcal{B}_t) = C\operatorname{diam}(\tilde{\mathcal{B}}_t), \quad C > 1,$$

then there exists an index $i \in t$ such that

$$h_i \geq (C-1)/C\operatorname{diam}(\mathcal{B}_t) \qquad \text{and} \qquad h_i > \frac{1}{2}(C-1)\rho_i.$$

Proof. Let $\operatorname{diam}(\mathcal{B}_t) = C\operatorname{diam}(\tilde{\mathcal{B}}_t)$. Then there exists at least one panel τ_i with centroid $\xi_i \in \tilde{\mathcal{B}}_t$ and a point $\zeta \in \tau_i \cap \partial\mathcal{B}_t$ with $\operatorname{dist}(\zeta, \tilde{\mathcal{B}}_t) \geq (C-1)/2\operatorname{diam}(\tilde{\mathcal{B}}_t)$. We conclude $h_i \geq 2|\zeta - \xi_i| \geq (C-1)\operatorname{diam}(\tilde{\mathcal{B}}_t) = (C-1)/C\operatorname{diam}(\mathcal{B}_t)$. Assume that $h_i \leq (C-1)\rho_i/2$. Then $\rho_i \geq 2h_i/(C-1) \geq 2\operatorname{diam}(\mathcal{B}_t)/C = 2\operatorname{diam}(\tilde{\mathcal{B}}_t)$, so that $|\xi_i - \xi_j| \leq \rho_i/2$ holds for all $j \in t$. Due to the assumption $\#t > C_{\rm ov}$ this is not possible. ∎

LEMMA 5.9. *Let $t, s \in T_{\mathcal{I}}$ be two clusters of cardinality $\#t > C_{\rm ov}, \#s > C_{\rm ov}$ that are not isotropically admissible: $\eta^{\rm iso}\operatorname{dist}(\mathcal{B}_t, \mathcal{B}_s) < \max\{\operatorname{diam}(\mathcal{B}_t), \operatorname{diam}(\mathcal{B}_s)\}$. Let $\operatorname{diam}(\mathcal{B}_t) = C\operatorname{diam}(\tilde{\mathcal{B}}_t)$ or $\operatorname{diam}(\mathcal{B}_s) = C\operatorname{diam}(\tilde{\mathcal{B}}_s)$ for some $C \geq C_{\rm str}/(C_{\rm str} - 1 - 2\eta^{\rm iso})$ and $\operatorname{diam}(\tilde{\mathcal{B}}_t) = \operatorname{diam}(\tilde{\mathcal{B}}_s)$. Then one of the following two cases is true:*

1. *$\exists i \in t : s \subset N_{str}(\tau_i)$ and $h_i \geq \frac{1}{2}(C-1)\rho_i$, or*
2. *$\exists j \in s : t \subset N_{str}(\tau_j)$ and $h_j \geq \frac{1}{2}(C-1)\rho_j$.*

Proof. **Case a)** Let $\operatorname{diam}(\mathcal{B}_t) = C\operatorname{diam}(\tilde{\mathcal{B}}_t)$ and $\operatorname{diam}(\mathcal{B}_t) \geq \operatorname{diam}(\mathcal{B}_s)$. According to Lemma 5.8 there exists $i \in t$ such that $h_i \geq \frac{1}{2}(C-1)\rho_i$ and $h_i \geq \frac{C-1}{C}\operatorname{diam}(\mathcal{B}_t)$. We can now bound the distance of τ_i to any $\tau_j, j \in s$, by

$$\begin{aligned}\operatorname{dist}(\tau_i, \tau_j) &\leq \operatorname{diam}(\mathcal{B}_t) + \operatorname{diam}(\mathcal{B}_s) + \operatorname{dist}(\mathcal{B}_t, \mathcal{B}_s) \;\leq\; (2 + (\eta^{\rm iso})^{-1})\operatorname{diam}(\mathcal{B}_t) \\ &\leq \frac{C}{C-1}(2 + (\eta^{\rm iso})^{-1})h_i \;=\; \frac{C}{C-1}(1 + 2\eta^{\rm iso})(\eta^{\rm iso})^{-1}h_i \;\leq\; C_{\rm str}(\eta^{\rm iso})^{-1}h_i.\end{aligned}$$

According to Definition 5.5 $j \in N_{str}(\tau_i)$, i.e. $s \subset N_{str}(\tau_i)$.
Case b) Let $\operatorname{diam}(\mathcal{B}_s) = C\operatorname{diam}(\tilde{\mathcal{B}}_s)$ and $\operatorname{diam}(\mathcal{B}_s) \geq \operatorname{diam}(\mathcal{B}_t)$. Analogously to Case 1 we have $t \subset N_{str}(\tau_j)$ for some $j \in s$ and $h_j \geq \frac{1}{2}(C-1)\rho_j$. □

LEMMA 5.10. *Let $t, s_1, \ldots, s_\ell \in T_{\mathcal{I}}$ be clusters of cardinality $\#t > C_{\mathrm{ov}}, \#s_i > C_{\mathrm{ov}}$ that are not isotropically admissible and $\max\{\operatorname{diam}(\mathcal{B}_{s_i}), \operatorname{diam}(\mathcal{B}_t)\} \geq C\operatorname{diam}(\tilde{\mathcal{B}}_{s_i}) = \operatorname{diam}(\tilde{\mathcal{B}}_t)$ where $C := max\{2C_{\mathrm{aniso}}+1, C_{\mathrm{str}}/(C_{\mathrm{str}}-2-3\eta^{\mathrm{iso}})\}$. Then there exists $i \in t \cup_{i=1}^{\ell} s_i$ such that $t \cup_{i=1}^{\ell} s_i \subset N_{str}(\tau_i)$ and the panel τ_i is stretched in the sense $h_i > C_{\mathrm{aniso}}\rho_i$ required in Assumption 5.6.*

Proof. Let s_j denote the cluster with maximal diameter $\operatorname{diam}(\mathcal{B}_{s_j})$ among the $s_1, \ldots, s_\ell, t$. Due to Lemma 5.9 there is an element $i \in s_j$ with $\operatorname{diam}(\mathcal{B}_{s_j}) \leq \frac{C}{C-1}\operatorname{diam}(\tau_i)$. For any element $\nu \in s \in \{t, s_1, \ldots, s_\ell\}$ we can bound

$$\begin{aligned}\operatorname{dist}(\tau_i, \tau_\nu) &\leq \operatorname{diam}(\mathcal{B}_{s_j}) + \operatorname{dist}(\mathcal{B}_{s_j}, \mathcal{B}_t)\operatorname{diam}(\mathcal{B}_t) + \operatorname{dist}(\mathcal{B}_t, \mathcal{B}_s) + \operatorname{diam}(\mathcal{B}_s)\\ &\leq (3 + 2(\eta^{\mathrm{iso}})^{-1})\operatorname{diam}(\mathcal{B}_{s_j}) \;\leq\; (3 + 2(\eta^{\mathrm{iso}})^{-1})\frac{C}{C-1}\operatorname{diam}(\tau_i)\\ &\leq (\eta^{\mathrm{iso}})^{-1}C_{\mathrm{str}}\operatorname{diam}(\tau_i),\end{aligned}$$

i.e. $\nu \in N_{str}(\tau_i)$. Lemma 5.9 gives the bound for h_i. □

For the following theorem we use the notation $\operatorname{level}(t)$ for the distance of a cluster $t \in T_{\mathcal{I}}$ to the root. The construction of the block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$ ensures that for any block cluster (t, s) there holds $\operatorname{level}(t) = \operatorname{level}(s)$ and the corresponding boxes $\tilde{\mathcal{B}}_t, \tilde{\mathcal{B}}_s$ used for the clustering are identical up to translation.

THEOREM 5.11. *Let $T := T_{\mathcal{I}\times\mathcal{I}}$ be the block cluster tree constructed in Section 3.1 from the cluster tree $T_{\mathcal{I}}$, where $n_{\min} \geq C_{\mathrm{ov}}$, i.e. a cluster with not more than C_{ov} elements is not further subdivided. We assume that $N^{-g} \lesssim |\xi_i - \xi_j|$. Under Assumption 5.6 the following statements hold:*
(a) The depth of the tree is bounded by

$$\operatorname{depth}(T) = \mathcal{O}(g \log N).$$

(b) The sparsity constant is bounded (independently of the mesh parameters g and N) by

$$C_{\mathrm{sp}} = \mathcal{O}(1).$$

Proof. **(a)** Let $t \in T_{\mathcal{I}}$ be a non-leaf node and ℓ the level of t. We denote the box corresponding to t by $\tilde{\mathcal{B}}_t$, where due to the construction $\operatorname{diam}(\tilde{\mathcal{B}}_t) \sim 2^{-\ell/d}$. Since t is a non-leaf node, the size of t is at least $\#t > 1$. Let $i \neq j \in t$ be two different indices. Due to the assumed bound $N^{-g} \lesssim |\xi_i - \xi_j|$ we have $N^{-g} \lesssim \operatorname{diam}(\tilde{\mathcal{B}}_t) \lesssim \sqrt{d}2^{-\ell/d}$ so that $\ell \lesssim g \log N$ follows.
(b) We exploit the structure of the regular subdivision of the box $\tilde{\mathcal{B}} = [a_1, b_1) \times [a_2, b_2) \times [a_3, b_3)$ used for the clustering in Section 3.1.

FIG. 5.2. *Left: The first two layers $p = 1$ and $p = 2$ around the box $\tilde{\mathcal{B}}_t$ of t. The first layer consists of 9 boxes, the second one of 25 boxes (it includes the first). Right: A cluster t with box $\tilde{\mathcal{B}}_t$ that is much smaller than the bounding box $\mathcal{B}_t$ of Γ_t. The dashed boxes $\tilde{\mathcal{B}}_s$ correspond to non-empty clusters s.*

1. Let $t \in T_{\mathcal{I}}$ be a node with $\operatorname{level}(t) = \ell$ and $\#t > C_{\mathrm{ov}}$. We count the number of clusters s with $\operatorname{level}(t) = \ell$ and $\#s > C_{\mathrm{ov}}$ that are **not admissible** to t. Let

$$C := \max\{2C_{\mathrm{aniso}} + 1, C_{\mathrm{str}}/(C_{\mathrm{str}} - 2 - 3\eta^{\mathrm{iso}})\}.$$

Case 1) $\operatorname{diam}(\mathcal{B}_t) \le C\operatorname{diam}(\tilde{\mathcal{B}}_t)$ and $\operatorname{diam}(\mathcal{B}_s) \le C\operatorname{diam}(\tilde{\mathcal{B}}_s)$. Since (t,s) is not admissible the relation

$$\begin{aligned}\operatorname{dist}(\mathcal{B}_t,\mathcal{B}_s) &\le (\eta^{\mathrm{iso}})^{-1}\max\{\operatorname{diam}(\mathcal{B}_t),\operatorname{diam}(\mathcal{B}_s)\}\\ &\le (\eta^{\mathrm{iso}})^{-1}C\max\{\operatorname{diam}(\tilde{\mathcal{B}}_t),\operatorname{diam}(\tilde{\mathcal{B}}_s)\} \;=\; C(\eta^{\mathrm{iso}})^{-1}\operatorname{diam}(\tilde{\mathcal{B}}_t)\end{aligned}$$

is valid. On the other hand

$$\begin{aligned}\operatorname{dist}(\mathcal{B}_t,\mathcal{B}_s) &\ge \operatorname{dist}(\tilde{\mathcal{B}}_t,\tilde{\mathcal{B}}_s) - (\operatorname{diam}(\mathcal{B}_t)+\operatorname{diam}(\mathcal{B}_s))/2\\ &\ge \operatorname{dist}(\tilde{\mathcal{B}}_t,\tilde{\mathcal{B}}_s) - C\operatorname{diam}(\tilde{\mathcal{B}}_t),\end{aligned}$$

i.e., $\operatorname{dist}(\tilde{\mathcal{B}}_t,\tilde{\mathcal{B}}_s) \le C(1+(\eta^{\mathrm{iso}})^{-1})\operatorname{diam}(\tilde{\mathcal{B}}_t)$.
Now the distance of the regular boxes $\tilde{\mathcal{B}}_t,\tilde{\mathcal{B}}_s$ can be measured in layers around t (cf. Figure 5.2): There are 3^d boxes touching $\tilde{\mathcal{B}}_t$. By induction it follows that all boxes $\tilde{\mathcal{B}}_r$ with $\operatorname{dist}(\tilde{\mathcal{B}}_t,\tilde{\mathcal{B}}_r) \le p\operatorname{diam}(\tilde{\mathcal{B}}_t)/(2\sqrt{d})$ are contained in the first p layers, which are at most $(2p+1)^d$ boxes.
For $p \ge 2\sqrt{d}C(1+(\eta^{\mathrm{iso}})^{-1})$ the box $\tilde{\mathcal{B}}_s$ is included in the first p layers, i.e., s is one of the $(2p+1)^d = \mathcal{O}(1)$ clusters of the first p layers.

Case 2) Either $\operatorname{diam}(\mathcal{B}_t) > C\operatorname{diam}(\tilde{\mathcal{B}}_t)$ or $\operatorname{diam}(\mathcal{B}_s) > C\operatorname{diam}(\tilde{\mathcal{B}}_s)$. Lemma 5.10 shows that there exists an index i such that t and all clusters s (with the above property) (isotropically) inadmissible to t are contained in $N_{str}(\tau_i)$ and that τ_i is stretched in the sense $h_i > C_{\mathrm{aniso}}\rho_i$. Assumption 5.6 says that we can find an alignment classification of $N_{str}(\tau_i)$ with $C_{\mathrm{cl}} = \mathcal{O}(1)$ classes. In each class, the one-dimensional anisotropic admissibility (3.15) applies. In this situation at most $\mathcal{O}((\eta^{\mathrm{ansio}})^{-1})$ clusters s form an inadmissible pair (t,s).

2. As a consequence of 1), the number of nodes $s \in T_{\mathcal{I}}$ (with $\operatorname{level}(s) = \operatorname{level}(t)$, $\#s > C_{\mathrm{ov}}$ and $\#t > C_{\mathrm{ov}}$) not admissible to t is bounded by $\mathcal{O}(1)$.
3. Let $t' \in T_{\mathcal{I}}$ be arbitrary. If t' is the root of $T_{\mathcal{I}}$, then there is exactly one cluster on the same level, namely t'. Therefore a sparsity constant $C_{\mathrm{sp}} \ge 1$ would be sufficient. If t' is not the root, then the father cluster t of t' fulfils $\#t > n_{\min} \ge C_{\mathrm{ov}}$. Due to 2) we conclude that there are at most $\mathcal{O}(1)$ clusters $s \in T_{\mathcal{I}}$ with $s \times t \in T_{\mathcal{I}\times\mathcal{I}}$ so that there are at most $\mathcal{O}(1)$ clusters $s' \in T_{\mathcal{I}}$ with $s' \times t' \in T_{\mathcal{I}\times\mathcal{I}}$. This is the desired bound for C_{sp}.

□

The previous Theorem gives a rigorous proof that the sparsity constant is independent of N or the geometry and behaves like $\mathcal{O}(1)$ for fixed $\eta^{\mathrm{iso}},\eta^{\mathrm{aniso}}$ from the admissibility condition and fixed constants $C_{\mathrm{ov}},C_{\mathrm{str}},C_{\mathrm{aniso}},C_{\mathrm{cl}}$ describing the geometry and mesh. For the screen mesh with $\eta^{\mathrm{iso}} = \eta^{\mathrm{aniso}} = 1$ the constants $C_{\mathrm{ov}} := 1$, $C_{\mathrm{str}} := 6$, $C_{\mathrm{aniso}} := 2$ and $C_{\mathrm{cl}} := 12$ fulfil all the requirements.

6. Exact Integration.

6.1. The Partially Integrated Kernel: Simple Case. In this section we investigate the analytical properties of the partially integrated kernel k^{ad} defined in (3.11). We show that the Assumption 3.10 holds true in the case of the Laplace single layer potential kernel in Example 2.1. Thus we are concerned with the function:

$$k^{\mathrm{ad}}_{\mathbf{b}}(x,y) = \frac{1}{4\pi}\int_{\alpha^t}^{\beta^t}\int_{\alpha^s}^{\beta^s}((y_1-x_1)^2+(y-x)^2)^{-1/2}\mathrm{d}y_1\mathrm{d}x_1. \tag{6.1}$$

A suitable analytic formula is given in the following lemma.

LEMMA 6.1. *For any $x,y \in \mathbb{R}$,*

$$\begin{aligned}k^{\mathrm{ad}}_{\mathbf{b}}(x,y) = {}&\kappa(\alpha^s-\beta^t,x-y)+\kappa(\beta^s-\alpha^t,x-y)\\ &-\kappa(\alpha^s-\alpha^t,x-y)-\kappa(\beta^s-\beta^t,x-y),\end{aligned} \tag{6.2}$$

where

$$\kappa(\xi,z) := \frac{1}{4\pi}\left[\sqrt{\xi^2+z^2}-\xi\ln\left(\xi+\sqrt{\xi^2+z^2}\right)\right]. \tag{6.3}$$

Proof. The assertion follows by using

$$\frac{\partial\kappa\left(\xi,z\right)}{\partial\xi}=-\frac{\ln\left(\xi+\sqrt{z^2+\xi^2}\right)}{4\pi}\quad\text{and}\quad\frac{\partial^2\kappa\left(\xi,z\right)}{\partial\xi^2}=-\frac{1}{4\pi\sqrt{z^2+\xi^2}}$$

and some obvious substitutions. □

(Note that this lemma could be generalised to the case of higher order basis functions, in which case an additional polynomial factor appears in (6.1), cf [13]).

Our next result obtains bounds for the derivatives with respect to x of a typical term from the right-hand side of (6.2).

LEMMA 6.2. *There exists an absolute constant C such that*

$$\left|\left\{\partial^j\kappa/\partial z^j\right\}\left(\xi,z\right)\right|\;\leq\;C\;j!\;\left|\xi\right|\;\left(\frac{2}{\sqrt{\xi^2+z^2}}\right)^j,\quad\textit{for all}\quad j\geq1,\ldots\quad,$$

for all $\xi,z\in\mathbb{R}$ with $(\xi,z)\neq(0,0)$. (The factor 2 in the right-hand side can be replaced by 1 by a more refined analysis.)

Proof. First consider fixed $\xi\neq0$, and write $\kappa(\xi,z)=\left(f(\xi,z)-g(\xi,z)\right)/\left(4\pi\right)$, where

$$f\left(\xi,z\right):=\sqrt{\xi^2+z^2}\quad\text{and}\quad g(\xi,z):=\xi\ln\left(\xi+f\left(\xi,z\right)\right).$$

To bound the derivatives of f, note first that $\left\{\partial^j f/\partial z^j\right\}\left(\xi,z\right)=\xi^{1-j}\hat{f}^{(j)}\left(z/\xi\right)$, where $\hat{f}(z):=\sqrt{1+z^2}$. Then for any $z\in\mathbb{R}$, let $\mathcal{D}_z$ denote the disc centred on z with radius $\frac{\sqrt{1+z^2}}{2}$, and let $\mathcal{C}_z$ denote its boundary. Since $\hat{f}$ is analytic in $\mathcal{D}_z$, Cauchy's integral formula yields

$$\frac{1}{j!}\hat{f}^{(j)}\left(z\right)=\frac{1}{2\pi i}\oint_{\mathcal{C}_z}\frac{\hat{f}\left(t\right)}{\left(t-z\right)^{j+1}}dt\;.\tag{6.4}$$

Elementary but somewhat tedious arguments lead to the estimates

$$\frac{3}{8}\sqrt{1+z^2}\;\leq\;\left|\hat{f}\left(t\right)\right|\;\leq\;\frac{\sqrt{13}}{2}\sqrt{1+z^2}\;,\quad\text{for}\quad t\in\mathcal{C}_z\;,\tag{6.5}$$

and using this in (6.4) yields

$$\left|f^{(j)}\left(z\right)\right|\;\leq\;\frac{\sqrt{13}}{4}\;j!\;2^{j+1}\;\left\{\sqrt{\xi^2+z^2}\right\}^{1-j}.\tag{6.6}$$

To estimate the derivatives of g, let $\hat{g}\left(z\right):=\ln\left(1+\hat{f}\left(z\right)\right)$ and observe that, for $j\geq1$, $\left\{\partial^j g/\partial z^j\right\}\left(\xi,z\right)=\xi^{1-j}\hat{g}^{(j)}\left(z/\xi\right)$. Now, to estimate $\hat{g}^{(j)}$, for $j\geq1$, we shall apply (6.4) with $\hat{f}$ replaced by $\hat{g}'$. To do this note that, for $t\in\mathcal{C}_z$,

$$\left|\hat{g}'\left(t\right)\right|=\left|\left(\frac{t}{\hat{f}\left(t\right)}\right)\left(\frac{1}{1+\hat{f}\left(t\right)}\right)\right|\;\leq\;\frac{\left|t-z\right|+\left|z\right|}{\left|\hat{f}(t)\right|}\frac{1}{\left|1+\hat{f}\left(t\right)\right|}\;\leq\;\frac{4}{\left|1+\hat{f}\left(t\right)\right|}\;,$$

(where the last inequality follows from (6.5)). Since $\operatorname{Re}\hat{f}\left(t\right)>0$ for all $t\in\mathcal{C}_z$, we have $|1+\hat{f}(t)|\geq|\hat{f}(t)|$ and hence (using also the left-hand side of (6.5)),

$$\left|\hat{g}'\left(t\right)\right|\;\leq\;\frac{4}{\left|f\left(t\right)\right|}\;\leq\;\frac{32}{3\sqrt{1+z^2}}\;.$$

Using this in (6.4), with $\hat{f}$ replaced by $\hat{g}'$, we obtain, for $j\geq1$,

$$\left|\hat{g}^{(j)}\left(z\right)\right|\;\leq\;\frac{32}{3}\;\left(j-1\right)!\;2^{j-1}\;\left\{\sqrt{1+z^2}\right\}^{-j}\;.$$

For the original function g this implies

$$\left|\left\{\partial^j g/\partial z^j\right\}(\xi,z)\right| \;\le\; \frac{32}{3}\;(j-1)!\;2^{j-1}\;|\xi|\;\left\{\sqrt{\xi^2+z^2}\right\}^{-j}\;, \quad \text{for} \quad j\ge 1\;. \tag{6.7}$$

The required result for $\xi \neq 0$ now follows on collection of (6.6) and (6.7). The result for $\xi = 0$ is trivial since then $\kappa(0,z) = |z|$.
□

The bounds proved in Lemma 6.2 can now be used to bound the error in polynomial approximation of κ.

LEMMA 6.3. *Let $-\infty < a < b < \infty$ and suppose $y \notin [a,b]$. Then, for any $\xi \in \mathbb{R}$ and any $m \ge 1$, the function $\kappa(\xi, \cdot - y)$ can be interpolated on $[a,b]$ by an m-th order Chebyshev polynomial $\tilde\kappa(\xi, \cdot - y)$ with error bounded by*

$$\|\kappa(\xi,\cdot-y) - \tilde\kappa(\xi,\cdot-y)\|_{L^\infty([a,b])} \;\le\; C|\xi|\,(m+1)^2\,(1+\eta)\left(1+\frac{1}{\eta}\right)^{-m-1}\;,$$

where C is an absolute constant and $\eta = \frac{b-a}{\operatorname{dist}(y,I)}$.

Proof. We apply [1, Theorem 3.2] where the constants C_u, γ_u appearing there are estimated using Lemma 6.2. □

THEOREM 6.4. *Let $([\alpha^t,\beta^t]\times[A,B],\ [\alpha^s,\beta^s]\times[A',B'])$ the domain corresponding to an η^{aniso}-admissible block cluster with respect to (3.10). The integrated kernel $k_{\mathbf{b}}^{\text{ad}} : [A,B]\times[A',B'] \to \mathbb{R}$ (see (3.11)) can be approximated by an m-th order Chebyshev interpolation $\tilde k_{\mathbf{b}}^{\text{ad}}$ with error bounded by*

$$|k_{\mathbf{b}}^{\text{ad}} - \tilde k_{\mathbf{b}}^{\text{ad}}|_{[A,B]\times[A',B'],\infty} \;\lesssim (m+1)^2(1+\eta^{\text{aniso}})(1+\frac{1}{\eta^{\text{aniso}}})^{-m-1},$$

Proof. From Lemma 6.3 we get a bound on the interpolation error for each of the four functions on the right-hand side of (6.2). Each of the functions is of the form considered in in Lemma 6.3 where the parameter $\xi \in \{\beta^t-\alpha^s, \beta^t-\beta^s, \alpha^t-\alpha^s, \alpha^t-\beta^s\}$. We can trivially bound $|\xi| \le \operatorname{diam}\Gamma$. Hence, the assertion is a direct consequence of Lemma 6.3. □

6.2. General Polyhedral Domains. The key idea of the anisotropic panel clustering relies on the analytic integration of pairs of stretched panels. In the previous section, the relevant antiderivatives of the kernel function have been developed and analysed for the special case of the screen mesh and piecewise constant shape functions.

In this section, we will present the antiderivatives of the kernel function for general polyhedral domains. We restrict to piecewise constant elements in order not to overload this paper with technicalities. The generalisation to higher order elements is straightforward because the antiderivatives of the kernel function can be computed analytically also for this case and the resulting integrated kernel has analogous analyticity properties as the one for piecewise constant elements.

We consider pairs of panels which are trapezias of the following form. By a suitable translation and rotation of the coordinate system we may assume that one panel has the form

$$t = \left\{(x_1,x_2,0)\in\mathbb{R}^3 \mid A_t \le x_2 \le B_t \text{ and } \alpha_t(x_2) \le x_1 \le \beta_t(x_2)\right\},$$

where $0 \le A_t < B_t$ are constants and α_t, β_t are affine functions which satisfy $\alpha_t(x_2) < \beta_t(x_2)$ for all $0 \le x_2 \le B_t$ and $\alpha_t(0) = 0$. The second trapezia is described by

$$s = \left\{(w_1, w_2\cos\varphi, w_2\sin\varphi)^\intercal \mid A_s \le w_2 \le B_s \wedge \alpha_s(w_2) \le w_1 \le \beta_s(w_2)\right\}.$$

Again, $0 \le A_s < B_s$ are constants and α_s, β_s are affine functions which satisfy $\alpha_s(w_2) < \beta_2(w_2)$ for all $0 \le w_2 \le B_S$. The compatibility conditions are $\alpha_s(0) = \alpha_t(0) = 0$ and $\beta_s(0) = \beta_t(0)$. The angle $\varphi \in [0, 2\pi[$ is fixed and denotes the angle between t and s with respect to the x_1-axes. The antiderivative of the single layer kernel is given by

$$k_{\mathbf{b}}^{\text{ad}}(x_2,y_2) = \frac{1}{4\pi}\int_{\alpha_s(y_2)}^{\beta_s(y_2)}\int_{\alpha_t(x_2)}^{\beta_t(x_2)} \frac{1}{\sqrt{(x_1-w_1)^2+z^2}}\mathrm{d}x_1\mathrm{d}w_1,$$

where $z = z(x_2, y_2) = \sqrt{(x_2 - y_2\cos\varphi)^2 + (y_2\sin\varphi)^2}$. Explicit calculations yield (cf. Lemma 6.1) in

$$k_{\mathbf{b}}^{\mathrm{ad}}(x_2, y_2) = \kappa_{\mathbf{b}}(z, d)|_{d=\alpha_s(y_2)-\alpha_t(x_2)}^{\alpha_s(y_2)-\beta_t(x_2)} - \kappa_{\mathbf{b}}(z, d)|_{d=\beta_s(y_2)-\alpha_t(x_2)}^{\beta_s(y_2)-\beta_t(x_2)} \tag{6.8}$$

where $\kappa_{\mathbf{b}}$ is as in Lemma 6.1.

REMARK 6.5. *The qualitative analysis of $k_{\mathbf{b}}^{\mathrm{ad}}$ can be developed along the same lines as for the screen problem. However, the analysis is more subtle since we can neither assume $B_s < A_t$ nor $B_t < A_s$ and the integration bounds in (6.8) have to be taken into account.*

7. Numerical Experiments. In this section we consider the model problem from Example 2.4 for three different grading parameters $g \in \{1, 3, 5\}$ and an increasing number N of panels, where $N = 9n^2$. We construct the cluster tree $T_{\mathcal{I}}$ by geometrically regular clustering as was introduced in Section 3.1, with minimal size $n_{\min} = 4$. The block cluster tree $T_{\mathcal{I}\times\mathcal{I}}$ and corresponding matrix partition is based either on the isotropic admissibility alone or, additionally, on the anisotropic admissibility condition, both with parameters $\eta^{\mathrm{iso}} = \eta^{\mathrm{aniso}} = 3$. For the isotropic admissibility condition we expect the number of nearfield entries to be at least $N^{3/2-1/g}$, cf. Example 3.6.

In the first numerical test we use the isotropic admissibility condition and a blockwise rank of $k = 1$ for all farfield blocks (for a larger rank $k > 1$ one has to multiply the respective numbers in the column "Far" in Table 7.1 by k). We measure the storage requirements in kilobyte per degree of freedom, separately for the nearfield part (isotropically inadmissible leaves of the block cluster tree) and the farfield part (isotropically admissible leaves of the block cluster tree). In Table 7.1 the results are reported along with the sparsity constant C_{sp} from the (theoretical) complexity estimates. The column "Near" contains the nearfield storage requirements in kilobytes per degree of freedom and the column "Far" the respective farfield storage requirements. We observe an increase of the nearfield part and the unboundedness of the sparsity constant C_{sp} for $g > 1$ as $n \to \infty$. since the columns "Far" grow with $O(\log N)$, this shows that the far field part of the representation is growing with $O(N \log N)$.

	$g=1$			$g=3$			$g=5$		
n	Near	Far	C_{sp}	Near	Far	C_{sp}	Near	Far	C_{sp}
100	0.6	1.7	24	0.7	1.6	30	1.0	1.5	30
200	0.6	2.0	24	0.8	2.0	30	1.3	1.9	30
400	0.6	2.3	24	0.8	2.3	36	1.7	2.2	42
800	0.6	2.6	24	1.0	2.6	48	2.4	2.4	66
1600	0.6	2.9	24	1.1	2.8	72	3.3	2.7	120

TABLE 7.1
Matrix storage requirements using isotropic admissibility.

For the second numerical test we employ the combined isotropic and anisotropic admissibility conditions and measure the storage requirements in kilobyte per degree of freedom, separately for the nearfield part (isotropically and anisotropically inadmissible leaves of the block cluster tree) and the farfield part (isotropically or anisotropically admissible leaves of the block cluster tree). The results in Table 7.2 confirm the theoretical estimates that the storage requirements are $\mathcal{O}(N \log N)$ and the sparsity constant C_{sp} remains bounded, independently of g and N.

	$g=1$			$g=3$			$g=5$		
n	Near	Far	C_{sp}	Near	Far	C_{sp}	Near	Far	C_{sp}
100	0.6	1.7	24	0.6	1.7	30	0.8	1.7	30
200	0.6	2.0	24	0.6	2.0	30	0.7	2.1	30
400	0.6	2.3	24	0.6	2.4	30	0.6	2.5	30
800	0.6	2.6	24	0.6	2.7	30	0.6	2.8	30
1600	0.6	2.9	24	0.5	3.0	30	0.6	3.1	30

TABLE 7.2
Matrix storage requirements using isotropic and anisotropic admissibility.

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