

A Galerkin Method for Retarded Boundary Integral Equations with Smooth and Compactly Supported Temporal Basis Functions.

Part II: Implementation and Reference Solutions.

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Abstract

In this paper we consider the three-dimensional wave equation in unbounded domains with Dirichlet boundary conditions. We start from a retarded single layer potential ansatz for the solution of these equations which leads to the retarded potential integral equation (RPIE) on the bounded surface of the scatterer. We formulate an algorithm for the space-time Galerkin discretization with smooth and compactly supported temporal basis functions which have been introduced in [S. Sauter and A. Veit: A Galerkin Method for Retarded Boundary Integral Equations with Smooth and Compactly Supported Temporal Basis Functions, Preprint 04-2011, Universität Zürich].

For the debugging of an implementation and for systematic parameter tests it is essential to have some explicit representations and some analytic properties of the exact solutions for some special cases at hand. We will derive such explicit representations for the case that the scatterer is the unit ball. The obtained formulas are easy to implement and we will present some numerical experiments for these cases to illustrate the convergence behaviour of the proposed method.

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1 Introduction

Mathematical modeling of acoustic and electromagnetic wave propagation and its efficient and accurate numerical simulation is a key technology for numerous engineering applications as, e.g., in detection (nondestructive testing, radar), communication (optoelectronic and wireless) and medicine (sonic imaging, tomography). An adequate model problem for the development of efficient numerical methods for such types of physical applications is the three-dimensional wave equation in unbounded exterior domains. In this setting the method of integral equations is an elegant approach since it reduces the problem in the unbounded domain to an retarded potential integral equation (RPIE) on the bounded surface of the scatterer.

In the literature there exist different approaches for the numerical discretization of these retarded boundary integral equations. They include collocation schemes with some stabilization techniques (cf. [7, 10, 11, 13, 8, 26]), methods based on bandlimited interpolation and extrapolation

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(cf. [31, 33, 32, 34]), convolution quadrature (cf. [5, 19, 18, 23, 30, 3, 4, 6]) as well as methods using space-time integral equations (cf. [2, 16, 12, 17, 28]).

In [28], [29] a space-time Galerkin method for the discretization of RPIEs with smooth and compactly supported temporal basis functions has been introduced conceptually. In this paper we present an algorithmic formulation of the method. The debugging of the implementation and the investigation of the performance and sensitivity of the method with respect to various parameters require a careful implementation and some exact solutions for performing appropriate numerical experiments. It turns out that the derivation of an explicit representation of the solution for some special geometry (here: unit sphere in \mathbb{R}^3) and Dirichlet boundary conditions is by no means trivial. We start from a retarded single layer potential ansatz for the solution of this equation which results in a retarded boundary integral equation on the sphere with unknown density function ϕ . We use Laplace transformations in order to transfer these problems to univariate problems in time which we solve analytically. The obtained explicit formulas for ϕ lead to exact solutions of the full scattering problem on the sphere and they are easy to implement. We employ these reference solutions for verifying the accuracy of our new method by numerical experiments and to study its performance and convergence behaviour. Furthermore these formulas are suitable to study analytic properties of these density functions.

An easy to use MATLAB script is available at <https://www.math.uzh.ch/compmath/?exactsolutions>, which implements the formulas obtained in this article.

2 Integral formulation of the wave equation

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain with boundary Γ . We consider the homogeneous wave equation

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \times [0, T] \quad (2.1a)$$

with initial conditions

$$u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 \quad \text{in } \Omega \quad (2.1b)$$

and Dirichlet boundary conditions

$$u = g \quad \text{on } \Gamma \times [0, T] \quad (2.1c)$$

on a time interval $[0, T]$ for $T > 0$. In applications, Ω is often the unbounded exterior of a bounded domain. For such problems, the method of boundary integral equations is an elegant tool where this partial differential equation is transformed to an equation on the bounded surface Γ . We employ an ansatz as a *single layer potential* for the solution u

$$u(x, t) := S\phi(x, t) := \int_{\Gamma} \frac{\phi(y, t - \|x - y\|)}{4\pi\|x - y\|} d\Gamma_y, \quad (x, t) \in \Omega \times [0, T] \quad (2.2)$$

with unknown density function ϕ . S is also referred to as *retarded single layer potential* due to the retarded time argument $t - \|x - y\|$ which connects time and space variables.

The ansatz (2.2) satisfies the wave equation (2.1a) and the initial conditions (2.1b). Since the single layer potential can be extended continuously to the boundary Γ , the unknown density function ϕ is determined such that the boundary conditions (2.1c) are satisfied. This results in the boundary integral equation for ϕ ,

$$\int_{\Gamma} \frac{\phi(y, t - \|x - y\|)}{4\pi\|x - y\|} d\Gamma_y = g(x, t) \quad \forall (x, t) \in \Gamma \times [0, T]. \quad (2.3)$$

Existence and uniqueness results for the solution of the continuous problem are proven in [24].

3 Temporal Galerkin discretization of retarded potentials with smooth basis functions

In this section we recall the Galerkin discretization of the boundary integral equation (2.3) using smooth and compactly supported basis functions in time. For details and an analysis of the scheme we refer to [28].

A coercive space-time variational formulation of (2.3) is given by (cf. [2, 16]): Find ϕ in an appropriate Sobolev space V such that

$$\int_0^T \int_{\Gamma} \int_{\Gamma} \frac{\dot{\phi}(y, t - \|x - y\|) \zeta(x, t)}{4\pi \|x - y\|} d\Gamma_y d\Gamma_x dt = \int_0^T \int_{\Gamma} \dot{g}(x, t) \zeta(x, t) d\Gamma_x dt \quad (3.1)$$

for all $\zeta \in V$, where we denote by $\dot{\phi}$ the derivative with respect to time. The Galerkin discretization of (3.1) now consists of replacing V by a finite dimensional subspace V_{Galerkin} being spanned by L basis functions $\{b_i\}_{i=1}^L$ in time and M basis functions $\{\varphi_j\}_{j=1}^M$ in space. This leads to the discrete ansatz

$$\phi_{\text{Galerkin}}(x, t) = \sum_{i=1}^L \sum_{j=1}^M \alpha_i^j \varphi_j(x) b_i(t), \quad (x, t) \in \Gamma \times [0, T], \quad (3.2)$$

for the approximate solution, where α_i^j are the unknown coefficients. As mentioned above we will use smooth and compactly supported temporal shape functions b_i in (3.2). Their definition was addressed in [28] and is as follows. Let

$$f(t) := \begin{cases} \frac{1}{2} \operatorname{erf}(2 \operatorname{artanh} t) + \frac{1}{2} & |t| < 1, \\ 0 & t \leq -1, \\ 1 & t \geq 1 \end{cases}$$

and note that $f \in C^\infty(\mathbb{R})$.^{*} Next, we will introduce some scaling. For a function $g \in C^0([-1, 1])$ and real numbers $a < b$, we define $g_{a,b} \in C^0([a, b])$ by

$$g_{a,b}(t) := g\left(2 \frac{t-a}{b-a} - 1\right).$$

We obtain a bump function on the interval $[a, c]$ with joint $b \in (a, c)$ by

$$\rho_{a,b,c}(t) := \begin{cases} f_{a,b}(t) & a \leq t \leq b, \\ 1 - f_{b,c}(t) & b \leq t \leq c, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now consider the closed interval $[0, T]$ and l (not necessarily equidistant) timesteps

$$0 = t_0 < t_1 < \dots < t_{l-2} < t_{l-1} = T.$$

A smooth partition of unity of the interval $[0, T]$ then is defined by

$$\mu_1 := 1 - f_{t_0, t_1}, \quad \mu_l := f_{t_{l-2}, t_{l-1}}, \quad \forall 2 \leq i \leq l-1 : \mu_i := \rho_{t_{i-2}, t_{i-1}, t_i}.$$

Smooth and compactly supported basis functions b_i in time can then be obtained by multiplying these partition of unity functions with suitably scaled Legendre polynomials (cf. [28] for details). For the discretization in space we use standard piecewise polynomials basis functions φ_j . The solution of (3.1) using the discrete ansatz (3.2) leads to a linear system with $L \cdot M$ unknowns.

^{*}Note that this choice of f is by no means unique. In [9, Sec. 6.1], $C^\infty(\mathbb{R})$ bump functions are considered (in a different context) which have certain *Gevrey* regularity. They also could be used for our partition of unity.

We partition the resulting system matrix $\underline{\underline{\mathbf{A}}}$ and right-hand side $\underline{\mathbf{g}}$ as a block matrix/block vector according to

$$\underline{\underline{\mathbf{A}}} := \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \cdots & \mathbf{A}_{1,L} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \cdots & \mathbf{A}_{2,L} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{L,1} & \mathbf{A}_{L,2} & \cdots & \mathbf{A}_{L,L} \end{bmatrix}, \quad \underline{\mathbf{g}} := \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_L \end{bmatrix}, \quad (3.3)$$

where

$$\mathbf{A}_{k,i} \in \mathbb{R}^{M \times M}, \quad \mathbf{g}_k \in \mathbb{R}^M \quad \text{for } i, k \in \{1, \dots, L\}.$$

Furthermore we denote

$$\min_k := \min \text{supp } b_k, \quad \max_k := \max \text{supp } b_k$$

for $k \in \{1, \dots, L\}$. The following algorithm computes the unknown coefficients α_i^j in (3.2) and leads to a solution of the boundary integral equation (2.3).

Algorithm 1 Computation of the coefficients α_i^j in (3.2)

Input: • A triangulation $\mathcal{G} := \{\tau_i : 1 \leq i \leq \overline{M}\}$ of Γ consisting of (possibly curved) triangles τ_i .

- L : number of timesteps t_i (not necessarily equidistant).
 - Time derivative $\dot{g}(x, t)$ of right-hand side.
-

{Generation of right-hand side}

for $k = 1$ to L **do**

$$\mathbf{g}_k \leftarrow \left(\int_0^T \int_{\Gamma} \dot{g}(x, t) \varphi_l(x) b_k(t) d\Gamma_x dt \right)_{l=1}^M \in \mathbb{R}^M$$

{Blockwise generation of system matrix}

for $i = 1$ to L **do**

if $\min_i \geq \max_k$ **then**

$$\mathbf{A}_{k,i} \leftarrow \mathbf{0} \in \mathbb{R}^{M \times M} \quad (3.4)$$

else

for $j, l = 1$ to M **do**

 mindist \leftarrow dist(supp φ_j , supp φ_l)

 maxdist \leftarrow sup $_{(x,y) \in \text{supp } \varphi_l \times \text{supp } \varphi_j} \|x - y\|$

if $[\min_k - \max_i, \max_k - \min_i] \cap [\text{mindist}, \text{maxdist}] = \emptyset$ **then**

$$\mathbf{A}_{k,i}(j, l) \leftarrow 0 \in \mathbb{R} \quad (3.5)$$

else

$$\mathbf{A}_{k,i}(j, l) \leftarrow \int_0^T \int_{\Gamma} \int_{\Gamma} \frac{\varphi_j(y) \varphi_l(x)}{4\pi \|x - y\|} \dot{b}_i(t - \|x - y\|) b_k(t) d\Gamma_y d\Gamma_x dt \quad (3.6)$$

end if

end for

end if

end for

end for

{Solution of linear system}

Solve:

$$\underline{\mathbf{A}} \cdot \underline{\mathbf{x}} = \underline{\mathbf{g}} \quad \text{with } \mathbf{x} \in \mathbb{R}^{LM}$$

Output: The vector $\underline{\mathbf{x}}$ corresponds to the unknown coefficients in (3.2).

Remark 1. (Numerical quadrature)

The most time consuming part of this algorithm is the computation of the matrix entries $\mathbf{A}_{k,i}(j, l)$ by numerical quadrature. Define

$$\psi_{k,i}(r) = \int_0^T \dot{b}_i(t - r) b_k(t) dt = \int_{\min_k}^{\max_k} \dot{b}_i(t - r) b_k(t) dt \quad (3.7)$$

with $\text{supp } \psi_{k,i} = [\min_k - \max_i, \max_k - \min_i]$. Then, we can rewrite (3.6) as

$$\mathbf{A}_{k,i}(j, l) = \int_{\Gamma} \int_{\Gamma} \frac{\varphi_j(y) \varphi_l(x)}{4\pi \|x - y\|} \psi_{k,i}(\|x - y\|) d\Gamma_y d\Gamma_x. \quad (3.8)$$

In order to compute integrals of the form (3.8) the regularizing coordinate transform as explained in [27] can be applied. This transform removes the spatial singularity at $x = y$ via the determinant

of the Jacobian. The resulting integration domain is the four-dimensional unit cube. In order to approximate the transformed integrals, tensor-Gauss quadrature can be used. Note that the integrands in (3.8) are C^∞ -smooth but not analytic and therefore classical error estimates are not valid. In [28] we have developed a quadrature error analysis for this type of integrand. Instead of tensor-Gauss quadrature it might also be suitable to use other quadrature schemes like sparse grid or adaptive quadrature in order to reduce the computational complexity. In [21] we proposed a method based on sparse tensor approximation to evaluate these integrals.

For the approximation of the matrix entries (3.8) the function $\psi_{k,i}$ has to be evaluated multiple times. Since such an evaluation by a quadrature rule is costly we suggest to approximate $\psi_{k,i}$ on its support accurately by a polynomial (e.g. by interpolation), which can be evaluated efficiently.

Remark 2. (Sparsity pattern of the matrix)

The matrix $\underline{\mathbf{A}}$ in (3.3) is a blockmatrix where the lower triangular part in general is non-zero while - according to (3.4), (3.5) only very few upper off-diagonals are non-vanishing. The matrix blocks $\mathbf{A}_{k,i}$ in general are sparse - only the entries which are enlightened by the support of the relevant temporal basisfunctions are non-zero.

In Figure 3.1 the sparsity pattern of $\underline{\mathbf{A}}$ and its matrix blocks are depicted. For illustration purpose, we choose Γ to be the one-dimensional interval $[0, 2]$, subdivided into 80 equidistant subintervals, and that the time interval is given by $[0, 3]$, subdivided into 30 equidistant subintervals. As temporal basis functions we used the smooth partition of unity described above.

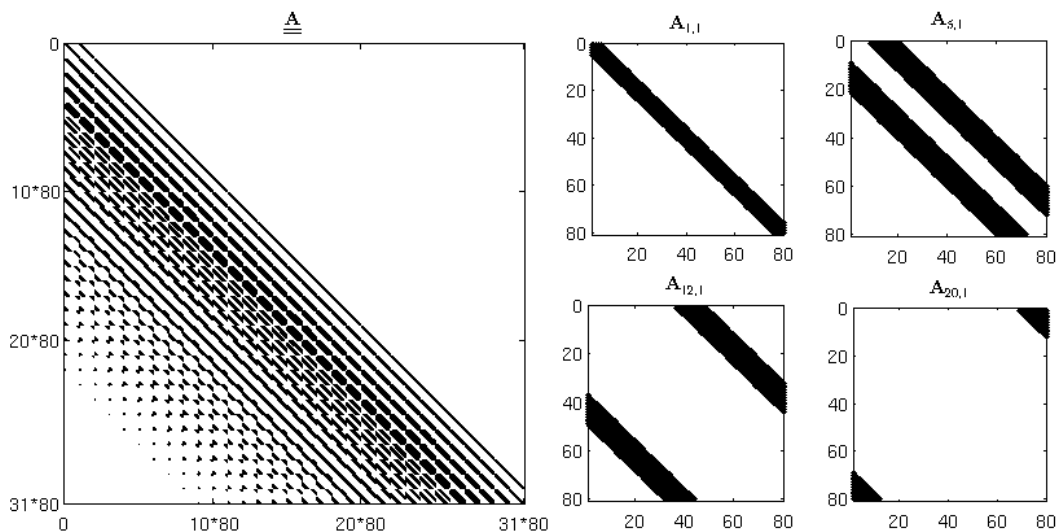


Figure 3.1: Sparsity pattern of the matrix $\underline{\mathbf{A}}$ and its blocks $\mathbf{A}_{k,i}$.

4 Exact solutions of the wave equation for $\Gamma = \mathbb{S}^2$

The systematic numerical testing of the convergence behaviour of our discretization requires the knowledge of exact solutions for some specific model problems whose derivation is far from trivial. Hence, a substantial part of this paper is devoted to the derivation of such solutions for a spherical scatterer. In Section 5 we will report on the approximation of these solutions by our method.

In this section we will derive analytic solutions of (2.3) for the special case that the boundary of the scatterer Γ is the unit sphere in \mathbb{R}^3 . Note that an equivalent formulation of the retarded

single layer potential (2.2) is given by

$$S\phi(x, t) = \int_0^t \int_{\Gamma} k(x - y, t - \tau)\phi(y, \tau)d\Gamma_y d\tau, \quad (x, t) \in \Omega \times [0, T], \quad (4.1)$$

where $k(z, t)$ is the fundamental solution of the wave equation,

$$k(z, t) = \frac{\delta(t - \|z\|)}{4\pi\|z\|},$$

$\delta(t)$ being the Dirac delta distribution. This representation is usually the starting point of discretization methods based on convolution quadrature, where only the Laplace transform of the kernel function is used. We introduce the single layer potential for the Helmholtz operator $\Delta U - s^2 U = 0$ which is given by

$$(V(s)\varphi)(x) := \int_{\Gamma} K(s, x - y)\varphi(y, \tau)d\Gamma_y, \quad x \in \mathbb{R}^3$$

where

$$K(s, z) := \frac{e^{-s\|z\|}}{4\pi\|z\|}$$

is the fundamental solution of the Helmholtz equation in three spatial dimensions. We now adopt the setting in [5]. We want to solve the boundary integral equation (2.3) in the case where Γ is the unit sphere \mathbb{S}^2 . For the right-hand side g we assume causality i.e. $g(x, t) = 0$ for $t \leq 0$ and furthermore that at least the first time derivative of g vanishes at $t = 0$. Moreover, g is supposed to be of the form

$$g(x, t) = g(t)Y_n^m$$

where Y_n^m denotes a spherical harmonic of degree n and order m . The Y_n^m are eigenfunctions of the single layer potential for the Helmholtz operator i.e.

$$V(s)Y_n^m = \lambda_n(s)Y_n^m \quad (4.2)$$

with eigenvalues $\lambda_n(s)$.

Remark 3. *The availability of eigenfunctions and eigenvalues of the frequency domain operator is crucial for the computation of exact solutions of (2.3). We refer to [22, 25] for the derivation of those in case of the single layer potential for the stationary Helmholtz equation. For the double layer potential, the adjoint double layer potential and the hypersingular operator in the frequency domain similar formulas exist (cf. [25]). In the same way as described below we can therefore obtain exact solutions also for other time-domain boundary integral equations arising in Dirichlet and Neumann problems in acoustic scattering. Details and explicit formulas for other problems can be found in [29].*

We express the eigenvalues $\lambda_n(s)$ in terms of modified Bessel functions I_{κ} and K_{κ} (see [1])

$$\lambda_n(s) = I_{n+\frac{1}{2}}(s)K_{n+\frac{1}{2}}(s).$$

Next, we will reduce equation (2.3) to a univariate problem in time. Recall the definition of the Laplace transform

$$\hat{\phi}(s) := (\mathcal{L}\phi)(s) = \int_0^{\infty} \phi(t) e^{-st} dt$$

with inverse

$$(\mathcal{L}^{-1}\hat{\phi})(s) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{\phi}(s) e^{st} ds \quad \text{for some } \sigma > 0.$$

Note that the fundamental solution of the Helmholtz equation is the Laplace transform of the fundamental solution of the wave equation. Using the representation (4.1) for S and expressing k in terms of its Laplace transform leads to the integral equation

$$\begin{aligned} g(t)Y_n^m &= \int_0^t \int_{\Gamma} k(t-\tau, \|x-y\|)\phi(y, \tau)d\Gamma_y d\tau \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_0^t e^{s\tau} \int_{\Gamma} K(s, \|x-y\|)\phi(y, t-\tau)d\Gamma_y d\tau ds \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_0^t e^{s\tau} (V(s)\phi(\cdot, t-\tau))(x) d\tau ds. \end{aligned}$$

Inserting the ansatz $\phi(x, t) = \phi(t)Y_n^m$ into (4.2) leads to the one dimensional problem: Find $\phi(t)$ s.t.

$$g(t) = \int_0^t \mathcal{L}^{-1}(\lambda_n)(\tau)\phi(t-\tau)d\tau. \quad (4.3)$$

Applying the Laplace transformation to both sides yields

$$\hat{g}(s) = \lambda_n(s)\hat{\phi}(s).$$

Rearranging terms and applying an inverse Laplace transformation finally leads to an expression for ϕ :

$$\phi(t) = \int_0^t g(\tau)\mathcal{L}^{-1}\left(\frac{1}{\lambda_n}\right)(t-\tau)d\tau. \quad (4.4)$$

Note that $\phi(t)Y_n^m$ with $\phi(t)$ as above is a solution of the full problem (2.3) in the case where $\Gamma = \mathbb{S}^2$ and $g(x, t) = g(t)Y_n^m$.

Before we proceed with the computation of (4.4), note that with the above formulas it is also possible to find an expression for the solution $\phi(x, t)$ in (2.3) for more general right-hand sides. If we choose the normalization convention for Y_n^m such that they form an orthonormal system in $L^2(\mathbb{S}^2)$: $(Y_n^m, Y_{n'}^{m'})_{L^2(\mathbb{S}^2)} = \delta_{n,n'}\delta_{m,m'}$, the following Theorem holds.

Theorem 4. *Let the right-hand side in (2.3) be causal and assume that $\partial_t g(x, 0) = 0, \forall x \in \mathbb{S}^2$. Let g be of the form*

$$g(x, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^n g_{n,m}(t)Y_n^m.$$

Then, the solution ϕ has the form

$$\phi(x, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \phi_{n,m}(t)Y_n^m,$$

where

$$\phi_{n,m} = \int_0^t g_{n,m}(\tau)\mathcal{L}^{-1}\left(\frac{1}{\lambda_n}\right)(t-\tau)d\tau.$$

Note that the expressions in Theorem 4 are considered as formal series. However, the existence and uniqueness results in [16] imply that for given right-hand side g with $\dot{g} \in H^{1/2, 1/2}(\Gamma \times [0, T]) := L^2(0, T; H^{1/2}(\Gamma)) \cap H^{1/2}(0, T; L^2(\Gamma))$ the solution ϕ exists in $H^{-1/2, -1/2}(\Gamma \times [0, T]) := L^2(0, T; H^{-1/2}(\Gamma)) + H^{-1/2}(0, T; L^2(\Gamma))$.

If only finitely many Fourier coefficients of g are non-zero, then, the expansion of ϕ and the existence in the classical pointwise sense is obvious.

For simplicity we return to the situation in (4.4) where we consider only one mode of such an expansion. In order to find an analytic expression for $\phi(t)$, it is necessary to find a representation for the inverse Laplace transform of $\frac{1}{\lambda_n(s)}$. With the formulas [15, Sec. 8.467 and 8.468] we get:

$$\lambda_n(s) = I_{n+\frac{1}{2}}(s)K_{n+\frac{1}{2}}(s) = \frac{y_n(-\frac{1}{s})y_n(\frac{1}{s}) + (-1)^{n+1}y_n^2(\frac{1}{s})e^{-2s}}{2s} \quad (4.5a)$$

where

$$y_n(s) := \sum_{k=0}^n (n, k) s^k \quad \text{and} \quad (n, k) := \frac{(n+k)!}{2^k k! (n-k)!} \quad (4.5b)$$

are the Bessel polynomials (see [20, Sec. 4.10]). This is equivalent to

$$\lambda_n(s) = (-1)^n \frac{\theta_n(s)}{2s^{2n+1}} (\theta_n(-s) - \theta_n(s) e^{-2s})$$

where θ_n are the reversed Bessel polynomials

$$\theta_n(s) := \sum_{k=0}^n (n, k) s^{n-k}.$$

After some manipulations we therefore get for the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda_n}\right) = 2\delta' + (-1)^n 2 \partial_t \mathcal{L}^{-1}\left(\frac{\tilde{\theta}_{2n-2}(s) + (-1)^n \theta_n(s)^2 e^{-2s}}{\theta_n(-s)\theta_n(s) - \theta_n(s)^2 e^{-2s}}\right) \quad (4.6)$$

where

$$\mathbb{P}_{\max(0, 2n-2)} \ni \tilde{\theta}_{2n-2}(s) = s^{2n} - (-1)^n \theta_n(-s)\theta_n(s).$$

We expand the term in the brackets in the right-hand side of (4.6) with respect to $\varepsilon = e^{-2s}$ about 0 and obtain

$$\begin{aligned} \frac{\tilde{\theta}_{2n-2}(s) + (-1)^n \theta_n(s)^2 e^{-2s}}{\theta_n(-s)\theta_n(s) - \theta_n(s)^2 e^{-2s}} &= \underbrace{\frac{\tilde{\theta}_{2n-2}(s)}{\theta_n(-s)\theta_n(s)}}_{R_n^{(1)}} \\ &+ \sum_{k=1}^{\infty} \left\{ \underbrace{(-1)^n \frac{\theta_n(s)^k}{\theta_n(-s)^k} e^{-2ks}}_{R_{n,k}^{(2)}} + \underbrace{\frac{\tilde{\theta}_{2n-2}(s)\theta_n(s)^{k-1}}{\theta_n(-s)^{k+1}} e^{-2ks}}_{R_{n,k}^{(3)}} \right\}. \end{aligned} \quad (4.7)$$

The computation of the inverse Laplace transforms of $R_n^{(1)}$, $R_{n,k}^{(2)}$ and $R_{n,k}^{(3)}$ boils down to the inversion of rational functions. This is done with the formulas in [14, Sec. 5.2]. Note that $\theta_n(s)$ is a polynomial of degree n and has exactly n complex-valued, simple zeros (cf. [20]). Let

$$\theta_n(\alpha_i) = 0 \quad \text{for } i = 1 \dots n \quad \text{where } \alpha_i = \alpha_i^{\text{re}} + i\alpha_i^{\text{im}} \quad \text{with } \alpha_i^{\text{re}}, \alpha_i^{\text{im}} \in \mathbb{R}.$$

It follows that the zeros of $\theta_n(-s)$ are $-\alpha_1, \dots, -\alpha_n$. Thus we get

$$\mathcal{L}^{-1}\left(R_n^{(1)}\right)(t) = \sum_{j=1}^n c_{n,j}^{(1)} e^{\alpha_j t} + \tilde{c}_{n,j}^{(1)} e^{-\alpha_j t}.$$

Since the solution ϕ is real, we may restrict our consideration to the real part of $\mathcal{L}^{-1}(R_n^{(1)})$. We denote the real part of $c_{n,j}^{(1)}$ by $c_{n,j}^{(1),\text{re}}$ and its imaginary part by $c_{n,j}^{(1),\text{im}}$. The notations for $\tilde{c}_{n,j}^{(1)}$ are chosen accordingly. We get

$$\begin{aligned} \mathcal{L}_{\text{re}}^{-1}\left(R_n^{(1)}\right)(t) &= \sum_{j=1}^n c_{n,j}^{(1),\text{re}} e^{\alpha_j^{\text{re}} t} \cos(\alpha_j^{\text{im}} t) - c_{n,j}^{(1),\text{im}} e^{\alpha_j^{\text{re}} t} \sin(\alpha_j^{\text{im}} t) \\ &+ \tilde{c}_{n,j}^{(1),\text{re}} e^{-\alpha_j^{\text{re}} t} \cos(-\alpha_j^{\text{im}} t) - \tilde{c}_{n,j}^{(1),\text{im}} e^{-\alpha_j^{\text{re}} t} \sin(-\alpha_j^{\text{im}} t). \end{aligned}$$

Remark 5. In Section 4.1 and 4.2 we will state explicit representations of ϕ for $n = 0, 1$. In this case the above formula simplifies considerably. We get

$$\mathcal{L}_{\text{re}}^{-1} \left(R_0^{(1)} \right) (t) = 0$$

and

$$\mathcal{L}_{\text{re}}^{-1} \left(R_1^{(1)} \right) (t) = \frac{1}{2} (e^{-t} - e^t) = -\sinh(t). \quad (4.8)$$

For larger n the arising coefficients from the inversions can be easily computed with computer algebra systems (see <https://www.math.uzh.ch/compmath/?exactsolutions> for a MATLAB implementation).

For the computation of $\mathcal{L}^{-1} \left(R_{n,k}^{(2)} \right)$ we use the time shifting property of the Laplace transformation. We employ the Heaviside step function

$$H(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t > 0, \end{cases}$$

to obtain

$$\begin{aligned} \mathcal{L}^{-1} \left(R_{n,k}^{(2)} \right) (t) &= \mathcal{L}^{-1} \left(\frac{\theta_n(s)^k}{\theta_n(-s)^k} e^{-2ks} \right) (t) = H(t-2k) \mathcal{L}^{-1} \left(\frac{\theta_n(s)^k}{\theta_n(-s)^k} \right) (t-2k) \\ &= (-1)^{nk} \delta(t-2k) H(t-2k) + \sum_{i=1}^n \sum_{j=1}^k c_{n,k,j,i}^{(2)} H(t-2k) (t-2k)^{j-1} e^{-\alpha_i(t-2k)} \end{aligned}$$

with some complex coefficients $c_{n,k,j,i}^{(2)} = c_{n,k,j,i}^{(2),\text{re}} + i c_{n,k,j,i}^{(2),\text{im}}$. For the real part of $\mathcal{L}^{-1} \left(R_{n,k}^{(2)} \right)$ we get:

$$\begin{aligned} \mathcal{L}_{\text{re}}^{-1} \left(R_{n,k}^{(2)} \right) (t) &= (-1)^{nk} \delta(t-2k) H(t-2k) \quad (4.9) \\ &+ \sum_{i=1}^n \sum_{j=1}^k c_{n,k,j,i}^{(2),\text{re}} H(t-2k) (t-2k)^{j-1} e^{-\alpha_i^{\text{re}}(t-2k)} \cos(-\alpha_i^{\text{im}}(t-2k)) \\ &- \sum_{i=1}^n \sum_{j=1}^k c_{n,k,j,i}^{(2),\text{im}} H(t-2k) (t-2k)^{j-1} e^{-\alpha_i^{\text{re}}(t-2k)} \sin(-\alpha_i^{\text{im}}(t-2k)). \end{aligned}$$

For the inverse Laplace transform of $R_{n,k}^{(3)}$ we use again the shift property and get

$$\begin{aligned} \mathcal{L}^{-1} \left(R_{n,k}^{(3)} \right) (t) &= \mathcal{L}^{-1} \left(\frac{\tilde{\theta}_{2n-1}(s) \theta_n(s)^{k-1}}{\theta_n(-s)^{k+1}} e^{-2ks} \right) (t) \\ &= H(t-2k) \mathcal{L}^{-1} \left(\frac{\tilde{\theta}_{2n-1}(s) \theta_n(s)^{k-1}}{\theta_n(-s)^{k+1}} \right) (t-2k) \\ &= H(t-2k) \left[\sum_{i=1}^n \sum_{j=1}^k c_{n,k,j,i}^{(3)} (t-2k)^j e^{-\alpha_i(t-2k)} \right]. \end{aligned}$$

The real part of $\mathcal{L}^{-1} \left(R_{n,k}^{(2)} \right)$ can therefore be written as

$$\begin{aligned} \mathcal{L}_{\text{re}}^{-1} \left(R_{n,k}^{(3)} \right) (t) &= \sum_{i=1}^n \sum_{j=1}^k c_{n,k,j,i}^{(3),\text{re}} H(t-2k) (t-2k)^j e^{-\alpha_i^{\text{re}}(t-2k)} \cos(-\alpha_i^{\text{im}}(t-2k)) \quad (4.10) \\ &- \sum_{i=1}^n \sum_{j=1}^k c_{n,k,j,i}^{(3),\text{im}} H(t-2k) (t-2k)^j e^{-\alpha_i^{\text{re}}(t-2k)} \sin(-\alpha_i^{\text{im}}(t-2k)). \end{aligned}$$

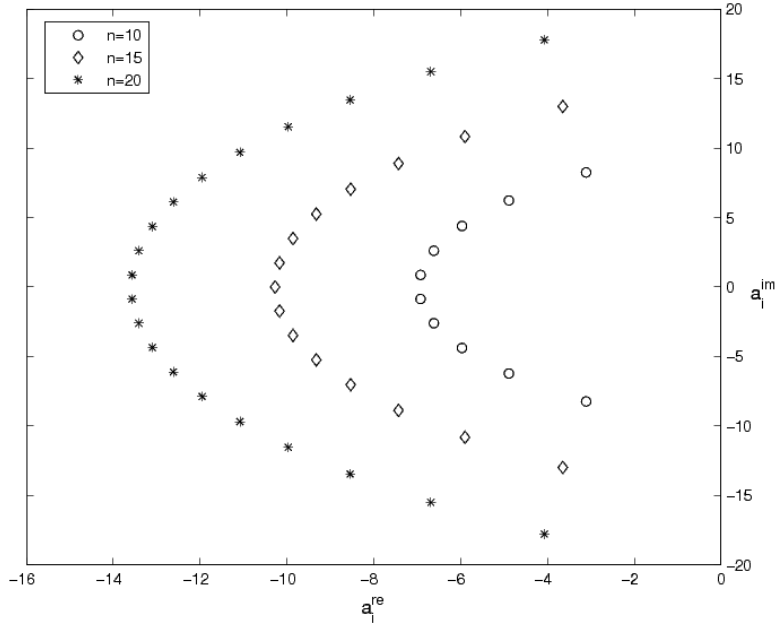


Figure 4.1: Complex zeros of $\theta_{10}(s)$, $\theta_{15}(s)$ and $\theta_{20}(s)$.

With these formulas for $\mathcal{L}_{\text{re}}^{-1}\left(R_n^{(1)}\right)$, $\mathcal{L}_{\text{re}}^{-1}\left(R_{n,k}^{(2)}\right)$ and $\mathcal{L}_{\text{re}}^{-1}\left(R_{n,k}^{(3)}\right)$ it is now possible to invert the remaining term in (4.6). Inserting this in (4.4) leads to explicit formulas for the exact solution $\phi(t)$.

Note that the complex zeros of $\theta_n(s)$ are located in left half plane of \mathbb{R}^2 , i.e., $-\alpha_i^{\text{re}} > 0$ for any i and n (cf. Figure 4.1, [20]). The representations of $\phi(t)$ that we will derive, have therefore to be handled with care when being evaluated numerically because they contain exponentially increasing functions which cancel each other and the behaviour of the final solution typically is bounded and oscillatory.

4.1 The case $n = 0$

For $n = 0$ the eigenfunctions in (4.2) are constant. We are therefore in the case where

$$g(x, t) := 2\sqrt{\pi}Y_0^0 g(t) = g(t)$$

is purely time-dependent. This case was already treated in [5] and an explicit representation of $\phi(t)$ in (4.4) was given for $t \in [0, 2[$. We generalize this to $t \geq 0$. Therefore note that the associated eigenvalue in this case is given by

$$\lambda_0(s) = \frac{1 - e^{-2s}}{2s}$$

and from the above computations we can see that

$$\mathcal{L}\left(\frac{1}{\lambda_0}\right)(t) = 2\delta'(t) + 2\partial_t\left(\sum_{k=1}^{\infty}\delta(t-2k)H(t-2k)\right).$$

Therefore the exact solution in this simple case is given by

$$\begin{aligned}
\phi(t) &= \int_0^t g(t-\tau) \left[2\delta'(\tau) + 2\partial_\tau \left(\sum_{k=1}^{\infty} \delta(\tau-2k)H(\tau-2k) \right) \right] d\tau \\
&= 2g'(t) + 2 \sum_{k=1}^{\infty} \int_0^t g(t-\tau) \partial_\tau (\delta(\tau-2k)H(\tau-2k)) d\tau \\
&= 2g'(t) + 2 \sum_{k=1}^{\infty} g'(t-2k) \\
&= 2 \sum_{k=0}^{\lfloor t/2 \rfloor} g'(t-2k)
\end{aligned} \tag{4.11}$$

due to the causality of g . Figure 4.2 shows a typical behaviour of $\phi(t)$. Note the oscillatory, non-decaying shape of the solution for larger times t . This is due to the fact that in indirect methods $\phi(t)$ is the trace difference of the solution of the exterior and the solution of the interior wave equation. The latter is determined by the many reflections inside the sphere and therefore causes the oscillations in the solution.

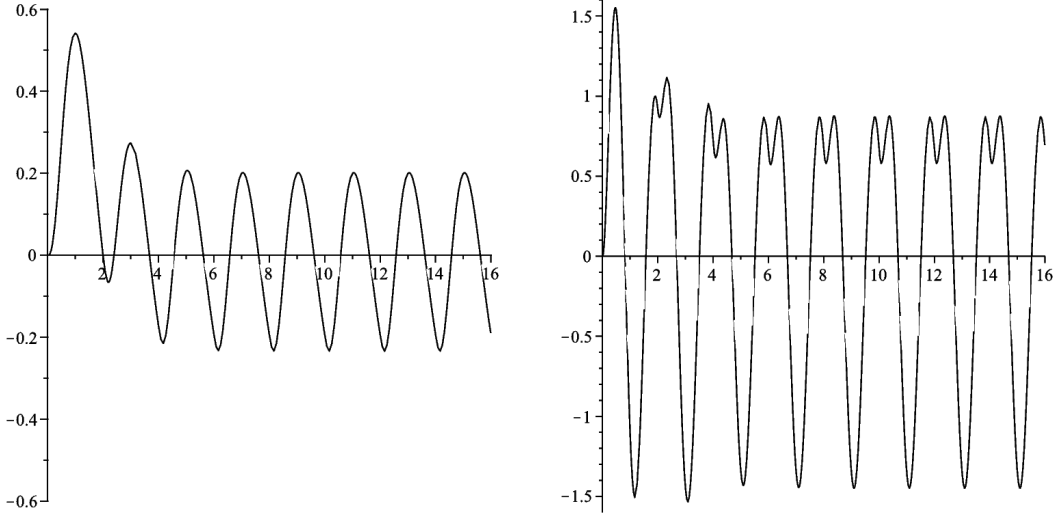


Figure 4.2: Exact solution $\phi(t)$ of (4.3) with $n = 0$ for $g(t) = t^4 e^{-2t}$ and $g(t) = \sin(2t)^2 t e^{-t}$.

A closer look at Figure 4.2 suggests that $\phi(t)$ becomes a very regular function for large times. Indeed it can be shown that $\phi(t)$ tends to a periodic function for sufficiently fast decaying right-hand sides $g(t)$. In order to see that we set

$$t = 2l + \tau \quad \tau \in [0, 2], \quad l \in \mathbb{N}_0$$

and get

$$\phi(2l + \tau) = 2 \sum_{k=0}^l g'(2k + \tau).$$

Suppose that $g(t)$ satisfies

$$g(0) = g'(0) = 0, \tag{4.12}$$

$$|g'(t)| \leq C t^{-\alpha}, \tag{4.13}$$

for $t > 0$ with $\alpha > 1$ and a positive constant C . With these assumptions, the following Lemma holds.

Lemma 6. *Let (4.12) and (4.13) be satisfied. Then the sequence of functions $\{\phi(2l + \tau)\}_{l \in \mathbb{N}_0}$ converges uniformly to a function $f(\tau) : [0, 2[\rightarrow \mathbb{R}$.*

Proof. Let $\varepsilon > 0$. Since $\alpha > 1$, we find $N \in \mathbb{N}$ such that

$$\sum_{k=l+1}^m (2k+2)^{-\alpha} < \frac{\varepsilon}{2C}$$

for all $m > l > N$. Thus,

$$\begin{aligned} |\phi(2m + \tau) - \phi(2l + \tau)| &\leq 2 \sum_{k=l+1}^m |g'(2k + \tau)| \leq 2C \sum_{k=l+1}^m (2k + \tau)^{-\alpha} \\ &\leq 2C \sum_{k=l+1}^m (2k + 2)^{-\alpha} \leq \varepsilon \end{aligned}$$

for all $m > l > N$ and therefore the uniform convergence. \square

Corollary 7. *The limit function $f(\tau)$ is continuous and satisfies*

$$f(0) = \lim_{\tau \rightarrow 2} f(\tau).$$

The solution of the scattering problem therefore tends to a periodic function for large times for every right hand side satisfying (4.12) and (4.13).

Proof. $f(\tau)$ is continuous since the uniform limit of continuous functions is continuous. Furthermore,

$$\lim_{\tau \rightarrow 2} f(\tau) = \lim_{\tau \rightarrow 2} \lim_{n \rightarrow \infty} \phi(2n + \tau) = \lim_{n \rightarrow \infty} \lim_{\tau \rightarrow 2} \phi(2n + \tau) = \lim_{n \rightarrow \infty} \phi(2n + 2) = f(0)$$

again due to the continuity of ϕ . \square

Let us suppose now that $g(t)$ is of the form

$$g(t) = v(t) e^{-\alpha t} \quad \text{with} \quad v(t) = t^2 p(t), \quad (4.14)$$

where $p \in \mathbb{P}_q$ is a polynomial of degree q . In this case we can compute the limit function $f(\tau)$ explicitly. Let the constant c_m be defined as

$$c_m := \frac{v^{(m+1)}(0) - \alpha v^{(m)}(0)}{m!}.$$

Expanding $v(t)$ and $v'(t)$ about 0 leads to

$$\begin{aligned} \phi(2l + \tau) &= 2 \sum_{k=0}^l [v'(2k + \tau) - \alpha v(2k + \tau)] e^{-\alpha\tau - 2\alpha k} \\ &= 2 \sum_{k=0}^l \left[\sum_{m=1}^q c_m (2k + \tau)^m \right] e^{-\alpha\tau - 2\alpha k} \\ &= 2 e^{-\alpha\tau} \sum_{m=1}^q \sum_{k=0}^l c_m (2k + \tau)^m e^{-2\alpha k} \\ &= 2 e^{-\alpha\tau} \sum_{m=1}^q \sum_{k=0}^l c_m \left(\sum_{j=0}^m \binom{m}{j} \tau^{m-j} (2k)^j \right) e^{-2\alpha k} \end{aligned}$$

$$= 2e^{-\alpha\tau} \sum_{m=1}^q \sum_{j=0}^m 2^j \binom{m}{j} c_m \tau^{m-j} \underbrace{\sum_{k=0}^l k^j e^{-2\alpha k}}_{=: R_{l,j,\alpha}}.$$

We are interested in $\phi(t)$ for large times t . Therefore we need an expression for $R_{l,j,\alpha}$ when l tends to infinity.

Lemma 8. *Let $j \in \mathbb{N}$ and $\alpha \in \mathbb{R}_{>0}$ be fixed. Then*

$$\sum_{k=0}^{\infty} k^j e^{-2\alpha k} = \sum_{m=0}^j \sum_{q=0}^m \frac{(-1)^{m-q} q^j \binom{j+1}{m-q} e^{2\alpha(j-m+1)}}{(e^{2\alpha} - 1)^{j+1}}.$$

Proof. Since we want to compute $\lim_{l \rightarrow \infty} R_{l,j,\alpha}$, we assume that $l \geq j$. We get

$$\begin{aligned} \left[\sum_{k=0}^l k^j e^{-2\alpha k} \right] (e^{2\alpha} - 1)^{j+1} &= \sum_{k=0}^l \sum_{q=0}^{j+1} (-1)^{j-q+1} k^j \binom{j+1}{q} e^{-2\alpha(k-q)} \\ &= \sum_{m=-(j+1)}^{-1} \sum_{q=-m}^{j+1} (-1)^{j+1-q} (q+m)^j \binom{j+1}{q} e^{-2\alpha m} \\ &\quad + \sum_{m=0}^{l-j-1} \sum_{q=0}^{j+1} (-1)^{j+1-q} (q+m)^j \binom{j+1}{q} e^{-2\alpha m} \\ &\quad + \sum_{m=l-j}^l \sum_{q=0}^{l-m} (-1)^{j+1-q} (q+m)^j \binom{j+1}{q} e^{-2\alpha m}. \end{aligned}$$

The second double sum in the last term is zero since for any polynomial p of degree less than j the equation

$$\sum_{q=0}^j (-1)^q p(q) \binom{j}{q} = 0$$

holds. Therefore

$$\begin{aligned} \left[\sum_{k=0}^l k^j e^{-2\alpha k} \right] (e^{2\alpha} - 1)^{j+1} &= \sum_{m=-(j+1)}^{-1} \sum_{q=-m}^{j+1} (-1)^{j+1-q} (q+m)^j \binom{j+1}{q} e^{-2\alpha m} \\ &\quad + \sum_{m=-j}^0 \sum_{q=0}^{-m} (-1)^{j+1-q} (q+l+m)^j \binom{j+1}{q} e^{-2\alpha(l+m)}. \end{aligned}$$

Now we can pass to the limit for $l \rightarrow \infty$ where the second double sum vanishes. After a reordering of the terms we get

$$\left[\sum_{k=0}^{\infty} k^j e^{-2\alpha k} \right] (e^{2\alpha} - 1)^{j+1} = \sum_{m=0}^j \sum_{q=0}^m (-1)^{m-q} q^j \binom{j+1}{m-q} e^{2\alpha(j-m+1)}.$$

Dividing by $(e^{2\alpha} - 1)^{j+1}$ leads to the desired result. \square

If we assume a right-hand side of the form (4.14) we get by Lemma 8 that

$$\phi(2l + \tau) \xrightarrow[l \rightarrow \infty]{} f(\tau) \quad \tau \in [0, 2[, \quad (4.15)$$

where f is given by

$$f(\tau) = 2 e^{-\alpha\tau} \sum_{m=1}^q \sum_{j=0}^m \tilde{c}_{m,j,\alpha} \tau^{m-j} \quad (4.16)$$

and

$$\tilde{c}_{m,j,\alpha} = c_m \sum_{k=0}^j \sum_{q=0}^k (-1)^{k-q} (2q)^j \binom{m}{j} \binom{j+1}{k-q} e^{2\alpha(j-k+1)} (e^{2\alpha} - 1)^{-j-1}.$$

With Lemma 8 it is also possible to show that the convergence in (4.15) is exponentially fast in l up to a polynomial factor if $g(t)$ is decaying exponentially.

Lemma 9. *Suppose that $g(t)$ is of the form*

$$g(t) = v(t) e^{-\alpha t} \quad (4.17)$$

with $\alpha > 0$, where $v(t)$ is a continuous function satisfying

$$\begin{aligned} v(0) &= v'(0) = 0, \\ |v(t)| &\leq C_1 t^{p_1}, \\ |v'(t)| &\leq C_2 t^{p_2}, \end{aligned}$$

for some $p_1, p_2 \in \mathbb{N}$ and positive constants C_1 and C_2 . For $l \geq \max\{p_1, p_2\}$ we have

$$\sup_{\tau \in [0, 2[} |f(\tau) - \phi(2l + \tau)| \leq p(l+1) e^{-2\alpha(l+1)},$$

where p is a polynomial of degree $\max\{p_1, p_2\}$ and f is as in Lemma 6.

Proof. From the proof of Lemma 8 it follows

$$\sum_{k=l+1}^{\infty} k^j e^{-2\alpha k} \leq l^j e^{-2\alpha l} \underbrace{\sum_{m=-j}^0 \sum_{i=0}^{-m} \frac{\binom{j+1}{i} e^{-2\alpha m}}{(e^{2\alpha} - 1)^{j+1}}}_{=: c_{\alpha, j}}$$

for $l \geq j$. Then we get

$$\begin{aligned} |f(\tau) - \phi(2l + \tau)| &\leq 2 \sum_{k=l+1}^{\infty} |g'(2k + \tau)| \\ &= 2 \sum_{k=l+1}^{\infty} |u'(2k + \tau) - \alpha u(2k + \tau)| e^{-\alpha\tau - 2\alpha k} \\ &\leq 2 e^{-\alpha\tau} \left(\sum_{k=l+1}^{\infty} |u'(2k + \tau)| e^{-2\alpha k} + \sum_{k=l+1}^{\infty} \alpha |u(2k + \tau)| e^{-2\alpha k} \right) \\ &\leq 2 e^{-\alpha\tau} \left(\sum_{k=l+1}^{\infty} C_2 (2k + \tau)^{p_2} e^{-2\alpha k} + \sum_{k=l+1}^{\infty} \alpha C_1 (2k + \tau)^{p_1} e^{-2\alpha k} \right) \\ &\leq C_2 2^{p_2+1} \sum_{k=l+1}^{\infty} (k+1)^{p_2} e^{-2\alpha k} + \alpha C_1 2^{p_1+1} \sum_{k=l+1}^{\infty} (k+1)^{p_1} e^{-2\alpha k} \\ &= C_2 2^{p_2+1} e^{2\alpha} \sum_{k=l+2}^{\infty} k^{p_2} e^{-2\alpha k} + \alpha C_1 2^{p_1+1} e^{2\alpha} \sum_{k=l+2}^{\infty} k^{p_1} e^{-2\alpha k} \\ &\leq [C_2 2^{p_2+1} c_{\alpha, p_2} (l+1)^{p_2} + \alpha C_1 2^{p_1+1} c_{\alpha, p_1} (l+1)^{p_1}] e^{-2\alpha(l+1)} \end{aligned}$$

for arbitrary $\tau \in [0, 2[$. □

4.2 The case $n = 1$

In the case of linear eigenfunctions in (4.2) the representation of the solution $\phi(t)$ becomes more complicated than in the previous case. For $n = 1$ the eigenvalue is given by

$$\lambda_1(s) = \frac{-\theta_1(-s)\theta_1(s) + \theta_1^2(s)e^{-2s}}{2s^3}$$

where

$$\theta_1(s) = s + 1.$$

Note that λ_1 has one real zero namely $\alpha_1 = -1$. With the above computations we get

$$\begin{aligned} & \mathcal{L}^{-1} \left(\frac{\tilde{\theta}_0(s) - \theta_1(s)^2 e^{-2s}}{\theta_1(-s)\theta_1(s) - \theta_1(s)^2 e^{-2s}} \right) (t) \\ & \stackrel{(4.7)}{=} \mathcal{L}^{-1} \left(R_1^{(1)} \right) (t) + \sum_{k=1}^{\infty} \left(-\mathcal{L}^{-1} \left(R_{1,k}^{(2)} \right) (t) + \mathcal{L}^{-1} \left(R_{1,k}^{(3)} \right) (t) \right) \\ & \stackrel{(4.8)}{=} -\sinh(t) \\ & \stackrel{(4.9)}{-} \sum_{k=1}^{\infty} (-1)^k \delta(t-2k) H(t-2k) - \sum_{k=1}^{\infty} \sum_{j=1}^k c_{1,k,j,1}^{(2),\text{re}} H(t-2k) (t-2k)^{j-1} e^{t-2k} \\ & \stackrel{(4.10)}{+} \sum_{k=1}^{\infty} \sum_{j=1}^k c_{1,k,j,1}^{(3),\text{re}} H(t-2k) (t-2k)^j e^{t-2k} \\ & = -\sinh(t) + \sum_{k=1}^{\infty} (-1)^{k+1} \delta(t-2k) H(t-2k) \\ & \quad + \sum_{k=1}^{\infty} \left[\sum_{j=1}^k \left(c_{k,j}^{(2)} + c_{k,j}^{(3)} t - c_{k,j}^{(3)} 2k \right) (t-2k)^{j-1} e^{t-2k} \right] H(t-2k), \end{aligned}$$

where

$$c_{k,j}^{(2)} := c_{1,k,j,1}^{(2),\text{re}} \quad \text{and} \quad c_{k,j}^{(3)} := c_{1,k,j,1}^{(3),\text{re}}.$$

With the formulas in [14, Sec 5.2] we obtain the following explicit expressions for these constants:

$$\begin{aligned} c_{k,j}^{(2)} &= (-1)^{k+1} \sum_{m=0}^{j-1} \frac{(1 - (-1)^{j-m})k!}{(j-1)!m!(k-j)!(j-m)!} \quad \text{and} \\ c_{k,j}^{(3)} &= (-1)^{k+1} \frac{2^{j-1}(k-1)!}{(j-1)!j!(k-j)!}, \end{aligned}$$

where we used

$$\frac{(1+s)^k}{(1-s)^k} = (-1)^k + \frac{\sum_{i=0}^{k-1} \binom{k}{i} (-1)^k (1 - (-1)^{k-i}) s^i}{(s-1)^k}$$

in order to compute $c_{k,j}^{(2)}$. With (4.6) and (4.4) we therefore get for the solution

$$\begin{aligned} \phi(t) &= \int_0^t g(t-\tau) \mathcal{L}^{-1} \left(\frac{1}{\lambda_1} \right) (\tau) d\tau \\ &= 2g'(t) - 2 \int_0^t \left(-\sinh(\tau) + \sum_{k=1}^{\infty} (-1)^{k+1} \delta(\tau-2k) H(\tau-2k) \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \sum_{j=1}^k \left(c_{k,j}^{(2)} + c_{k,j}^{(3)} \tau - c_{k,j}^{(3)} 2k \right) (\tau-2k)^{j-1} e^{\tau-2k} H(\tau-2k) \right) g'(t-\tau) d\tau \end{aligned}$$

$$\begin{aligned}
&= 2g'(t) + 2 \sum_{k=1}^{\lfloor t/2 \rfloor} (-1)^k g'(t-2k) + 2 \int_0^t \sinh(\tau) g'(t-\tau) d\tau \\
&- 2 \sum_{k=1}^{\infty} \sum_{j=1}^k \int_0^t (c_{k,j}^{(2)} + c_{k,j}^{(3)} \tau - c_{k,j}^{(3)} 2k) (\tau-2k)^{j-1} e^{\tau-2k} H(\tau-2k) g'(t-\tau) d\tau \\
&= 2 \sum_{k=0}^{\lfloor t/2 \rfloor} (-1)^k g'(t-2k) + 2 \int_0^t \sinh(\tau) g'(t-\tau) d\tau \\
&- 2 \sum_{k=1}^{\lfloor t/2 \rfloor} \sum_{j=1}^k \int_{2k}^t (c_{k,j}^{(2)} + c_{k,j}^{(3)} \tau - c_{k,j}^{(3)} 2k) (\tau-2k)^{j-1} e^{\tau-2k} g'(t-\tau) d\tau. \tag{4.18}
\end{aligned}$$

Figure 4.3 shows solutions for different right-hand sides $g(t)$. As for the case $n = 0$ we have an oscillatory behaviour for larger times t which is again due to shape of the solution of the interior wave problem. Similar properties of these solutions as before could not be observed i.e. in general $\phi(t)$ does not seem to adopt a simple periodic pattern as time evolves.

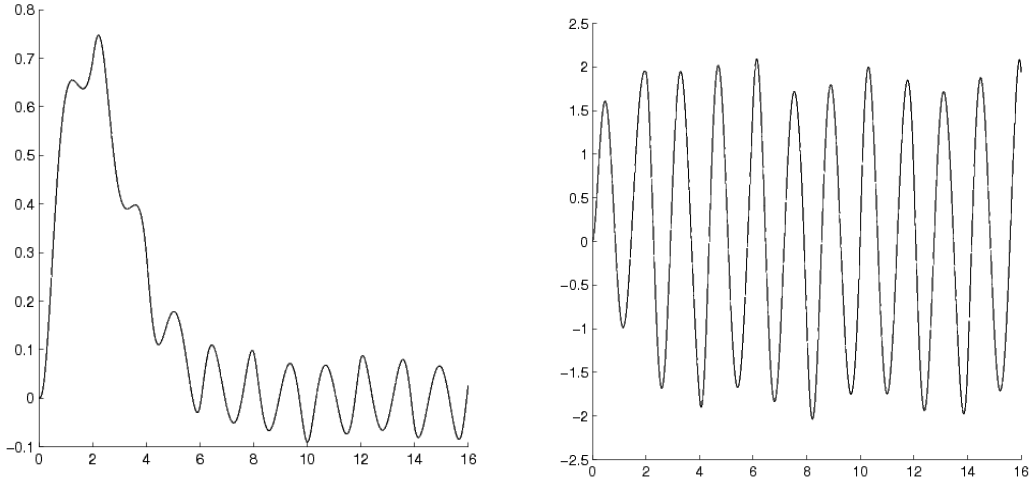


Figure 4.3: Exact solution $\phi(t)$ of (4.3) with $n = 1$ for $g(t) = t^4 e^{-2t}$ and $g(t) = \sin(2t)^2 e^{-t}$.

5 Numerical experiments

In this section we present the results of numerical experiments. We first want to verify the sharpness of Lemma 9 for different right hand sides g . Let

$$\begin{aligned}
g_1(t) &= t^4 e^{-2t}, & g_2(t) &= t^2 e^{-2t}, \\
g_3(t) &= t \sin(t) e^{-t}, & g_4(t) &= \frac{1}{4} t \sin(5t) e^{-t},
\end{aligned}$$

and denote by $\phi_j, j \in \{1, 2, 3, 4\}$ the corresponding solutions of the boundary integral equation. Let $f_j : [0, 2[\rightarrow \mathbb{R}, j \in \{1, 2, 3, 4\}$ be the limit functions corresponding to these solutions as in Lemma 6. We define the errors

$$\text{err}_j(l) := \|f_j(\cdot) - \phi_j(2l - \cdot)\|_{L^\infty([0, 2])}, \quad j \in \{1, 2, 3, 4\}$$

and illustrate the convergence in Figure 5.1 and 5.2. As predicted by Lemma 9 the solutions converge in all cases exponentially fast against the corresponding limit functions due to the

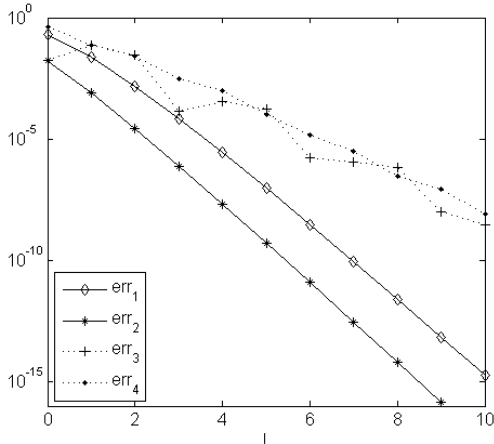


Figure 5.1: $err_j(l)$ for $j = 1, 2, 3, 4$.

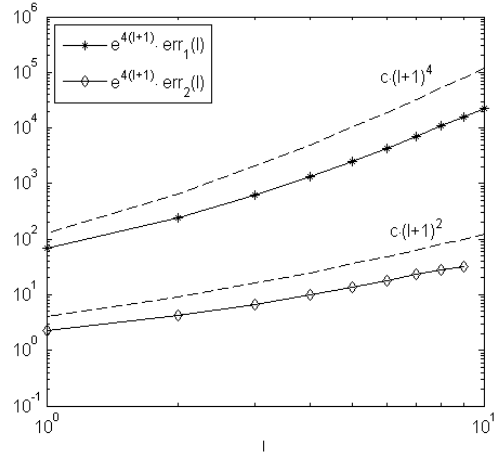


Figure 5.2: Log-log scale plot of $e^{4(l+1)} \cdot err_j(l)$ for $j = 1, 2$.

exponential decay of the right-hand sides. Since the degree of the increasing polynomial factor in g_1 is higher than in g_2 the error err_1 decays slower than err_2 by a polynomial factor (cf. Figure 5.2). The cases g_3 and g_4 indicate that more oscillatory right-hand sides (and therefore more oscillatory solutions) do not lead to a slower convergence rate if the decay behaviour of these functions is the same.

We now turn our attention on the approximation of ϕ in (4.3) by a Galerkin method using the basis functions b_i defined in Section 3 (cf. [28] for details). We compute approximations of the form

$$\phi_S = \sum_{i=1}^{N_p} \alpha_i b_i, \quad \alpha_i \in \mathbb{R}, i = 1, \dots, N_p, \quad (5.1)$$

where the number of basis function N_p depends on the number of timesteps and the degree p of the local polynomial approximation spaces used. We measure the resulting error, $\phi - \phi_S$, in the $L^2(0, T)$ norm.

Figure 5.3 shows the exact solutions of problem (4.3) for $g(t) = t^4 e^{-2t}$ and $n = 2, 3$. They were computed using the formulas derived in the last section. Figure 5.4 shows the error that results from approximating these solutions by functions of the form (5.1). In this case we computed approximations in the time interval $[0, 6]$ using equidistant time steps and local polynomial approximation spaces of degree $p = 0$, i.e., the approximations are simply linear combinations of the partition of unity functions defined in Section 3. In both cases a convergence order of N^{-1} is obtained, where N is the number of timesteps.

In order to obtain a higher convergence order we use local polynomial approximation spaces of degree $p = 1$ to approximate the solutions shown in Figure 5.5 for the right-hand side $g(t) = t^2 \sin(t)^2 e^{-t}$ and $n = 2, 3$. Figure 5.6 shows the again the error $\|\phi - \phi_S\|_{L^2(0,6)}$ for different numbers of timesteps. As expected, the convergence is quadratic in this case.

6 Conclusion

We considered retarded boundary integral formulations of the three-dimensional wave equation in unbounded domains. We formulated an algorithm for the space-time Galerkin discretization using the smooth and compactly supported temporal basis function developed in [28]. In order to test these basis functions numerically we derived explicit representations of the exact solutions

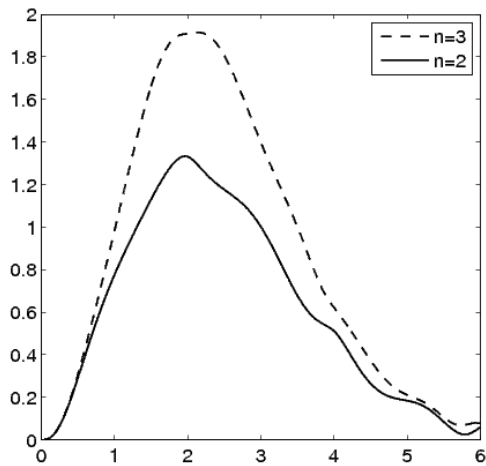


Figure 5.3: Exact solution $\phi(t)$ of (4.3) for $g(t) = t^4 e^{-2t}$ and $n = 2, 3$.

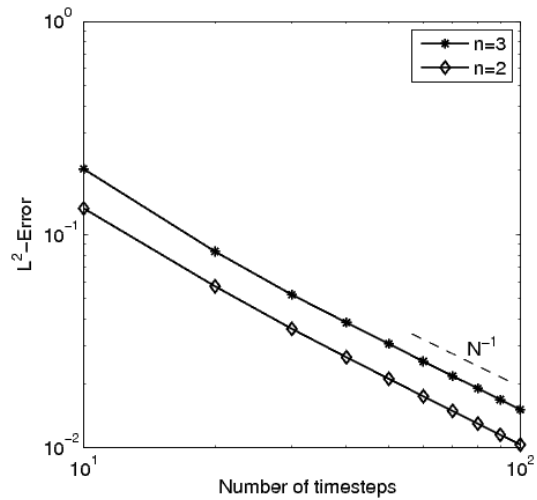


Figure 5.4: $\|\phi - \phi_S\|_{L^2(0,T)}$ for $T = 6$, $g(t) = t^4 e^{-2t}$ and $n = 2, 3$, where local polynomial approximation spaces of degree $p = 0$ were used.

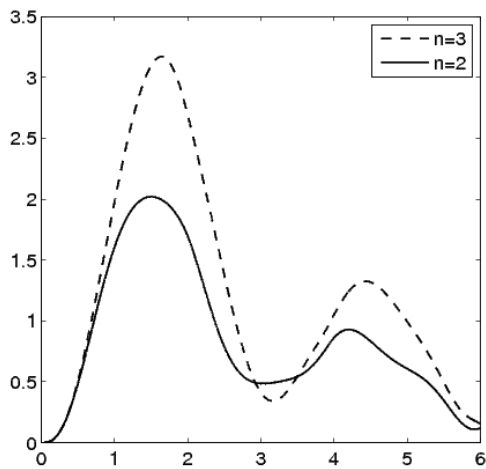


Figure 5.5: Exact solution $\phi(t)$ of (4.3) for $g(t) = t^2 \sin(t)^2 e^{-t}$ and $n = 2, 3$.

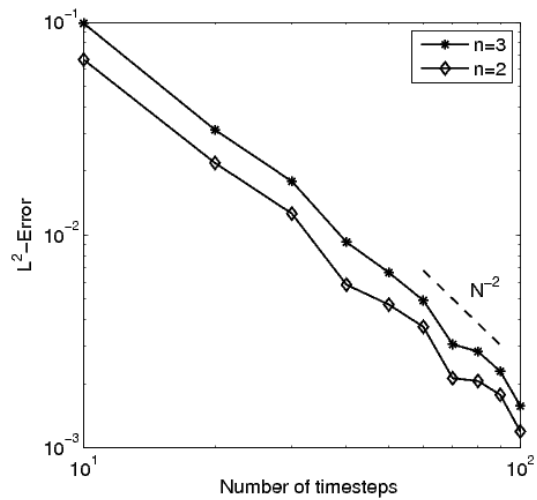


Figure 5.6: $\|\phi - \phi_S\|_{L^2(0,T)}$ for $T = 6$, $g(t) = t^2 \sin(t)^2 e^{-t}$ and $n = 2, 3$, where local polynomial approximation spaces of degree $p = 1$ were used.

of the integral equations in the case that the scatterer is the unit ball in \mathbb{R}^3 and special Dirichlet boundary conditions have to be satisfied. Furthermore we showed some analytic properties of these solutions in the case that the right-hand side is purely time-dependent.

The implementation of the obtained formulas is simple since only the right-hand side, its first derivative with respect to time and, depending on n , numerical quadrature is needed for the numerical evaluation. They can therefore serve as reference solutions in order to test numerical approximations schemes.

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