Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Boundary integral equations for second order elliptic boundary value problems

by

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Preprint no.: 55 1999

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Abstract

In this paper, we will present integral equations for general elliptic boundary value problems of second order with constant coefficients. The jump conditions, existence and uniqueness theorems are proved. In combination with black box numerical integration schemes developed recently for general boundary integral equations it becomes feasible to implement a black box software package for solving this class of problems, just by providing the coefficients of the boundary value problems. Numerical examples performed by such a black-box software package will illustrate the good convergence behaviour of the integral equation method.

1 Introduction

The integral equation method for solving partial differential equations has a long history (see [23], [24], [29], [30], [31], [15], [25], [26], [8], [21], [22], [38], [37], [2]). Since the treatment of partial differential equations via *variational principles* was established in the first part of this century, *integral equation methods* have lost significantly importance from the theoretical point of view due to the difficulties related to sharp existence and uniqueness theorems for partial differential equations in the classical setting. However, with increasing importance of *numerical techniques* for solving boundary value problems, integral equation method are becoming more and more popular as a starting point for numerically solving boundary value problems for the reasons listed below:

- The treatment of equations on complicated (time depending) 3-d domains is simpler from the viewpoint of mesh generation since only the surface of the physical body has to be (re-) meshed,

- Fast techniques for the sparse representation of the arising pseudo-local operators (panel clustering, multipole, wavelets) have been developed overcoming the drawback of full system matrices for boundary integral equations.

- The treatment of problems on *unbounded* domains is especially simple.

- Parameter dependent problems (as, e.g., the Helmholtz equation with high wave number and problems where finite element discretizations suffer from "locking") cause less difficulties as for the corresponding finite element discretizations.

- The arising large systems of linear equations are, typically, better conditioned as the direct finite element discretizations of the underlying boundary value problem.

However, in mathematical textbooks and also in engineering software packages, usually, only integral equations for the prototype operators as, e.g. the Laplace operator, the biharmonic operator, the Lamé operator, the Stokes operator are discussed and realised numerically. From the practical viewpoint, it would be interesting to develop the relevant integral equations for the general second order elliptic boundary value problems with constant coefficients

$$Lu := -\operatorname{div}(A \operatorname{grad} u) + 2\langle b, \nabla u\rangle + cu, \qquad (1)$$

since in the farfield, i.e., as $|x|$ becomes large, equations with non-constant coefficients or non-linear equations could be linearized.
In [35], [36], [14], [32], [13], [1], [33], [6], [34], black box numerical quadrature schemes have been developed for a class of integral kernels including those arising by treating (1) with integral equation methods. In the present paper, we will define potential operators for the (elliptic) operator L in (1) and derive corresponding integral equations. We will prove the relevant jump conditions by employing weak assumptions on the smoothness of the surface, either Hölder continuity or only continuity which is not elaborated in the classical references. Such an approach can be found in [12] while that exposition is

mainly restricted to the Laplace operator. A much more general approach can be found in [4], [42] while the formulae are not completely explicit.
Our motivation of this paper was to define explicit integral operators for boundary value problems with differential operator L as in (1) and prove the relevant jumping conditions, existence and uniqueness theorems. In combination with the numerical integration techniques described in [36], [14], [32], [13], [1], [33], [6] a black box software package for solving boundary values problems with differential operator L as in (1) could be derived. We have tested both, the integral equation formulations and the black box implementation (based on the program libbem described in [18], [19]) and show that the expected convergence rates are obtained also for general second order elliptic equations.
The numerical methods and formulations of the integral equations are such that is sufficient to specify the (positive definite) matrix $A \in \mathbb{R}^{d\times d}$, the vector $b \in \mathbb{C}^d$ and the coefficients $c \in \mathbb{C}$ and the program solves the corresponding boundary value problem, similarly, as it is widely realised in software packages for solving elliptic boundary value problems by, e.g., finite elements directly.
The paper is organized as follows:
After having introduced some preliminary notations in the next section we formulate the boundary value problems which we want to solve via integral equations. The key role for the transformation into integral equations plays the fundamental solution to the elliptic operator. In Section 4, the relevant fundamental solutions are provided and some properties concerning their singular behaviour at the origin and their decay behaviour for large $|x|$ are proved. By employing these fundamental solutions, the corresponding potential operators are defined in the next section. The jump relations, i.e., the behaviour of these potentials as x crosses the surface, play the essential role for the transformation of the boundary value problem into integral equations. These relations are derived in Section 6 for the principal part of the operator and the corresponding boundary integral equations are obtained. The case of general elliptic equations is considered in Section 7. Numerical results showing the good convergence behaviour of the method and the possibility of designing a black box software package where just the coefficients of the underlying elliptic operator have to be described are included in Section 8.

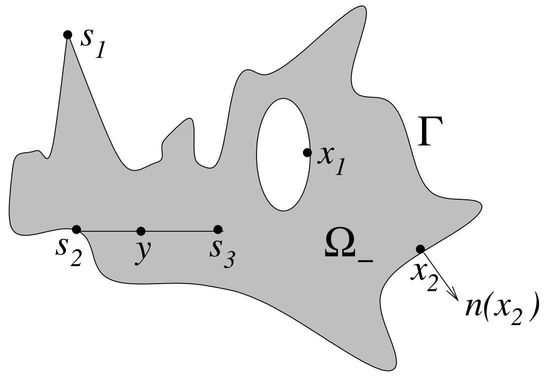

Figure 1: Domain with slit $\overline{s_2 s_3}$

2 Preliminaries and Notations

Let Ω_- be an open and bounded set in $\mathbb{R}^d$ with boundary Γ which coincides with the boundary of $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega}_-$.

Definition 1. A boundary point $u \in \Gamma$ is *regular* if there exists a neighbourhood U of u in $\mathbb{R}^d$ so that $\Gamma \cap U$ is a $(d-1)$-dimensional submanifold in $\mathbb{R}^d$. All non-regular boundary points of Ω_- are *singular boundary points.*

The set of regular boundary points of Ω_- is denoted by $\mathrm{X} := \mathrm{X}(\Omega_-)$, the singular boundary points form the set $\mathrm{S} := \mathrm{S}(\Omega_-)$.

Example 2. In Figure 1 the boundary points x_1, x_2 and y are regular points, s_1, s_2 and s_3 are singular boundary points. Because of our assumption, that Γ is the common boundary of Ω_- and Ω_+, boundary points like y, s_2 and s_3 could not occur in Γ.

Throughout the paper, we assume that Assumption 3 holds:

Assumption 3. Let $E = \{t \in \mathbb{R}^{d-1} : |t_1|, ..., |t_{d-1}| < 1\}$. There exist a neighbourhood $\tilde{E}$ of $\overline{E}$ and $C^{1+\mu}$-parametrisations $\psi_j : \tilde{E} \to \mathbb{R}^d$, $j = 1, ..., m$, with sets $\tilde{M}_j := \psi_j(\tilde{E})$ and disjoint subsets $M_j := \psi_j(E)$ which satisfies

(i) $\cup_{j=1}^m \overline{M}_j = \Gamma$,

(ii) $\cup_{j=1}^m M_j \subset \mathrm{X}$,

(iii) for all $\tilde{M}_j$, there exists a continuous unit normal vector field which coincides with the exterior normal vector field n of X on $\tilde{M}_j \cap \mathrm{X}$.

Conclusion 4. *Assumption 3 implies a vanishing* $(d-1)$*-dimensional Minkowski measure of* S*. Hence, the Gauß' divergence theorem can be applied to* Ω_- *(cf. [16], Folgerung 7.5).*

Let $A \in \mathbb{R}^{d\times d}$ denote a symmetric and positive definite matrix, $b \in \mathbb{C}^d$ and $c \in \mathbb{C}$. Let $\langle\,\cdot,\,\cdot\rangle$ denote the symmetric bilinear form on $\mathbb{C}^d$ defined by $(x, y) \mapsto \sum_{j=1}^d x_j y_j$. We define the symmetric bilinear form $\langle\,\cdot,\,\cdot\rangle_A$ on $\mathbb{C}^d$ by $\langle x, y\rangle_A := \langle A^{-1}x, y\rangle$. If x is in $\mathbb{R}^d$, we write $|x|_A$ short for $\sqrt{\langle x, x\rangle_A}$. Let us consider the elliptic operator

$$L := -\operatorname{div}(A\,\operatorname{grad}) + 2\langle b, \nabla\cdot\rangle + c\,. \tag{2}$$

In the classical sense and in the sense of distribution theory any elliptic differential operator of order 2 with constant coefficients is of the form (2) modulo multiplication by the factor -1. The *conormal vector field* l on X is given by $l(x) := An(x)$.

3 Boundary Value Problems

Let us consider the following boundary value problems:

Interior Dirichlet Problem. Find $u \in C^2(\Omega_-) \cap C(\overline{\Omega}_-)$ satisfying

$$Lu = 0 \quad \text{in } \Omega_- \qquad \text{and} \qquad u = f \quad \text{on } \Gamma \tag{3}$$

where f is a given continuous function.

Exterior Dirichlet Problem. Find $u \in C^2(\Omega_+) \cap C(\overline{\Omega}_+)$ such that

$$Lu = 0 \quad \text{in } \Omega_+ \qquad \text{and} \qquad u = f \quad \text{on } \Gamma \tag{4}$$

holds where f is a given continuous function on Γ.

Interior Natural Boundary Value Problem. Find u in $C^2(\Omega_-) \cap C(\overline{\Omega}_-)$ satisfying, for a given $g \in C(\mathrm{X})$, the equations

$$\begin{gathered} Lu = 0 \;\text{ in } \Omega_- \;\text{ and} \\ \lim_{\alpha\searrow 0}\langle l(x), \nabla u(x - \alpha l(x))\rangle = g(x) \;\text{ locally uniformly on X.} \end{gathered} \tag{5}$$

Exterior Natural Boundary Value Problem. Find u in $C^2(\Omega_+) \cap C(\overline{\Omega}_+)$ satisfying, for a given $g \in C(\mathrm{X})$, the equations

$$\begin{gathered} Lu = 0 \ \text{ in } \Omega_+ \ \text{ and} \\ \lim_{\alpha \searrow 0} \langle l(x), \nabla u(x + \alpha l(x)) \rangle = g(x) \ \text{ locally uniformly on X.} \end{gathered} \tag{6}$$

Uniqueness and Radiation Conditions. For the uniqueness of the exterior problems certain (physically motivated) radiation conditions, depending on L, have to be imposed. We give some examples:
(a) Consider $L_0 = -\operatorname{div}(A \operatorname{grad})$ (if $A = I$, this is the *Laplace operator*) and the radiation condition

$$\begin{aligned} u(x) &= O(1) \quad \text{as } |x| \to \infty \quad (d = 2)\,, \\ u(x) &\to 0 \quad \text{as } |x| \to \infty \quad (d \geq 3)\,. \end{aligned} \tag{7}$$

For the following uniqueness results we refer to [17], §6.2, and [9], §4.1.
(a1) The exterior Dirichlet problem for L_0 has at most one solution u satisfying (7).
(a2) If the interior Dirichlet problem for L_0 has a solution u, then u is uniquely determined.
These statements are proved by using the maximum-minimum principle of harmonic functions. Thus it is not necessary to require any kind of regularity of the boundary Γ .
For the exterior natural boundary value problem we impose the radiation condition

$$u(x) \to 0 \quad \text{as } |x| \to \infty\,. \tag{8}$$

(a3) If Ω_+ is connected and Γ is a C^2-submanifold, the exterior natural value problem has at most one solution satisfying (8).
(a4) If Ω_- is connected and Γ a C^2-submanifold, two solutions of the interior natural value problem can differ only by a constant.
The last two results can be proved by applying Equation (20), i.e., Green's theorem.
(b) Consider the *Helmholtz equation* $\Delta u + k^2 u = 0$, $\mathrm{Im}(k) \geq 0$ in $\mathbb{R}^3$. The *emission* or *radiation condition of Sommerfeld*

$$\left\langle \frac{x}{|x|}, \nabla u(x) \right\rangle - iku(x) = o\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \tag{9}$$

ensures, that the solution u of the Helmholtz equation represents a divergent travelling wave.
(b1) If Ω_+ is connected and Γ is the union of a finite number of disjoint, closed C^2-submanifolds, the exterior Dirichlet problem, and also the exterior Neumann problem, has at most one solution satisfying (9).
(b2) On the other hand, there exist discrete values $k \in \mathbb{R}$ such that the interior Dirichlet problem is not uniquely solvable. The same holds true for the interior Neumann problem (cf. [3], Section 3.3).
(c) Let $k > 0$ and consider the *convection-diffusion problem*

$$\begin{aligned} -\Delta u + k\partial_1 u &= g \quad \text{in } \Omega_+ \,, \\ u &= 0 \quad \text{on } \Gamma \,. \end{aligned} \tag{10}$$

in three dimensions. For $a, b \in \mathbb{R}$ define the weighted L^2-space

$$L^2_{a,b}(\Omega_+) = \{v \in L^2_{loc}(\Omega_+) : \textstyle\int_{\Omega_+} |v(x)|^2 (1+|x|)^a (1+|x|-x_1)^b \, dx < \infty\} \,.$$

Let Γ be a Lipschitz continuous boundary, $\beta > 0$, $-\beta < \alpha \leq 0$ and $g \in L^2_{\alpha+1,\beta}(\Omega_+)$. Then there exists a unique weak solution $u \in H^1_{loc}(\overline{\Omega}_+)$ of (10) satisfying the *generalized radiation condition*

$$u \in L^2_{\alpha-1,\beta}(\Omega_+) \quad \text{and} \quad \nabla u \in L^2_{\alpha,\beta}(\Omega_+) \,,$$

i.e., u is an element of an *anisotropically weighted Sobolev space* (cf. [7], Theorem 2.7).
The exterior Dirichlet problem for the convection-diffusion operator $L = -\Delta + k\partial_1$ and a given boundary function f can be transformed into (10) if a function v_0 on Ω_+ is known with $v_0|_\Gamma = f$: If u is a solution of (10) for $g = \Delta v_0 - k\partial_1 v_0$, then $v := u + v_0$ is a solution of the exterior Dirichlet problem.

4 Fundamental Solutions

The boundary value problems can be transformed into integral equations over (a subset of) Γ by the so called "*integral equation method*". The keyrole in this transformation plays a fundamental solution of the differential operator L, i.e., a distribution F satisfying the equation $LF = \delta_0$ where δ_0 is the Dirac distribution supported in the origin.

A table of fundamental solutions of the most common differential operators can be found in the second part of [27]. For the construction of fundamental solutions we refer to [28].
The definition of fundamental solutions for L involves Macdonald functions K_ν which, for example, are stated in [39], p.79, 80, [20], §5.7.

Theorem 5. *Let $\vartheta := c + \langle b, b\rangle_A = 0$. Then, $\kappa_0 : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\kappa_0(x) := \begin{cases} \dfrac{1}{2\pi\sqrt{\det A}}\, e^{\langle b,x\rangle_A} \ln \dfrac{1}{|x|_A} & \text{for} \quad d = 2 \\ \\ \dfrac{1}{(d-2)\omega_d\sqrt{\det A}}\, \dfrac{e^{\langle b,x\rangle_A}}{|x|_A^{d-2}} & \text{for} \quad d \neq 2\,, \end{cases} \tag{11}$$

where ω_d is the volume of the unit sphere in $\mathbb{R}^d$, is a fundamental solution of L. For $\vartheta \neq 0$, there exists $\lambda \in \mathbb{C}\backslash]-\infty, 0]$ with $\lambda^2 = \vartheta$. A fundamental solution κ_λ of L is given by

$$\kappa_\lambda(x) := \frac{e^{\langle b,x\rangle_A}}{(2\pi)^{d/2}\sqrt{\det A}} \left(\frac{|x|_A}{\lambda}\right)^{1-d/2} K_{d/2-1}(\lambda |x|_A), \quad x \neq 0. \tag{12}$$

For $d = 2, 3$, we obtain

$$d = 2: \qquad \kappa_\lambda(x) = \frac{e^{\langle b,x\rangle_A}}{4\sqrt{\det A}}\, i H_0^{(1)}(i\lambda|x|_A) \quad \text{if} \quad -\pi < \arg(\lambda) < \frac{\pi}{2}\,,$$

$$d = 3: \qquad \kappa_\lambda(x) = \frac{1}{4\pi\sqrt{\det A}} \frac{e^{\langle b,x\rangle_A - \lambda|x|_A}}{|x|_A}\,,$$

where $H_0^{(1)}$ is a Hankel function *(or* Bessel function of the third kind*).*

For Hankel functions we refer to [20], §5.7, and [39]. A proof of Theorem 5 can be found in [9], Kapitel 3.

Lemma 6. *The gradient of the fundamental solution κ_λ, $\lambda \in \mathbb{C}\backslash]-\infty, 0[$, has the asymptotic behaviour*

$$\nabla\kappa_\lambda(x) = -\frac{1}{\omega\sqrt{\det A}} \frac{A^{-1}x}{|x|_A^d} + o\left(\frac{1}{|x|^{d-1-\nu}}\right) \quad \text{for any } \nu \in]0, 1[.$$

If d is odd, we could replace $o(|x|^{1+\nu-d})$ by $O(|x|^{2-d})$.

The behaviour of the fundamental solution κ_λ is considered in

Lemma 7. *Let $b \in \mathbb{C}^d$ and $c \in \mathbb{C}$ with $\vartheta := c + \langle b, b\rangle_A = 0$ and $L = -\operatorname{div}(A \operatorname{grad}) + 2\langle b, \nabla\cdot\rangle + c$.*
Further, let $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{C}^d$ and $\mathbb{C}$ which converge to b and c respectivly with $\vartheta_n := c_n + \langle b_n, b_n\rangle_A \neq 0$. Let L_n denote the operator $-\operatorname{div}(A \operatorname{grad}) + 2\langle b_n, \nabla\cdot\rangle + c_n$ and let $\lambda_n \in \mathbb{C}\backslash]-\infty, 0]$ be a square root of ϑ_n.
$d = 2$: For all $x \in \mathbb{R}^2 \setminus \{0\}$: $|\kappa_{\lambda_n}(x)| \to \infty$ as $n \to \infty$.
$d \geq 3$: The singularity functions κ_{λ_n} of L_n, $n \in \mathbb{N}$, converge uniformly on compact subsets of $\mathbb{R}^d \setminus \{0\}$ to the singularity function κ_0 of L.
Moreover $(\kappa_{\lambda_n})_{n\in\mathbb{N}}$ converges to κ_0 in the sense of distribution theory, i.e., for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$, there holds

$$\kappa_{\lambda_n}(\varphi) := \int_{\mathbb{R}^d} \kappa_{\lambda_n} \varphi \, dx \to \int_{\mathbb{R}^d} \kappa_0 \varphi \, dx =: \kappa_0(\varphi) \quad as \quad n \to \infty .$$

This lemma and the next theorem are proved in [9], Kapitel 3.

Theorem 8. *Let $b \in \mathbb{R}^d$ and $c \in \mathbb{C}$. Further let $\vartheta = c + |b|_A^2$ and $\lambda \in \mathbb{C}\backslash]-\infty, 0[$ with $\lambda^2 = \vartheta$ and $\operatorname{Re}(\lambda) \geq 0$. Let κ_λ be as in Theorem 5.*

(a) Let $b = 0$. For $c \in \mathbb{C}\backslash]-\infty, 0]$, $|\kappa_\lambda|$ decreases exponentially as $|x| \to \infty$.

If $c = 0$, we have, in two dimensions, $|\kappa_\lambda(x)| = O(\ln |x|)$, in higher dimensions $|\kappa_\lambda(x)| = O(|x|^{2-d})$ as $|x|$ tends to infinity.

If $c \in]-\infty, 0[$, then $|\kappa_\lambda|$ decreases as $O(|x|^{\frac{1-d}{2}})$.

(b) Let $b \neq 0$ and $\Theta = -\left(\frac{1}{2}\frac{\operatorname{Im}(c)}{|b|_A}\right)^2$. If $\operatorname{Re}(c) > \Theta$, $|\kappa_\lambda|$ decreases exponentially as $|x| \to \infty$. If $\operatorname{Re}(c) = \Theta$, $|\kappa_\lambda(x)| = O(|x|^{\frac{1-d}{2}})|$ as $|x| \to \infty$. If $\operatorname{Re}(c) < \Theta$, $|\kappa_\lambda|$ grows exponentially in some directions (cf. Figure 2).

5 Single- and Double-Layer Potentials

Recall that the patches M_j and $\tilde{M}_j$ belong to $C^{1+\mu}$ (cf. Assumption 3).

Lemma 9. *There exists a constant $C > 0$ such that, for every $j \in \{1, ..., m\}$ and all x, $y \in \overline{M}_j$, we obtain*

$$|\langle n(y), x - y\rangle| \leq C|x - y|^{1+\mu} . \tag{13}$$

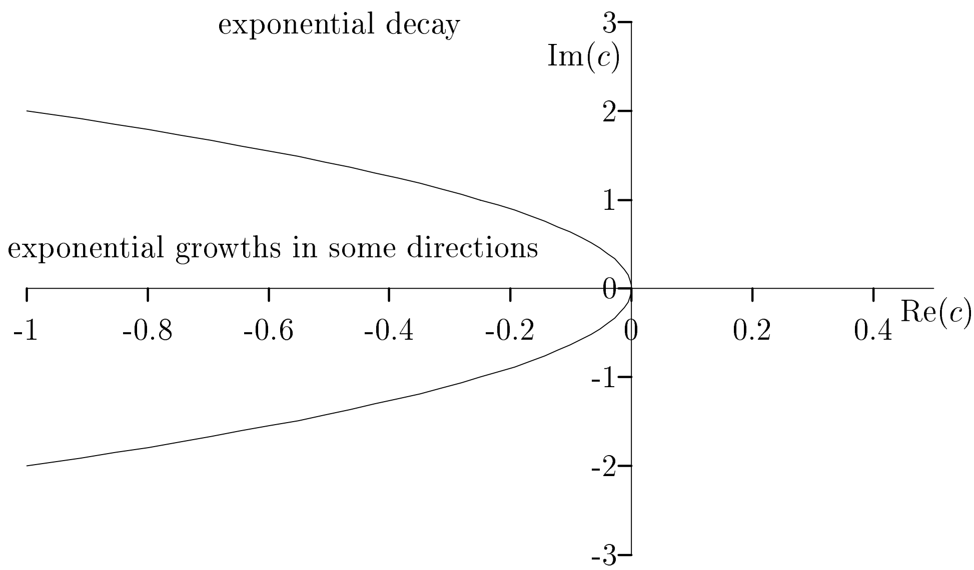

Figure 2: The decay of κ_λ for $b \in \mathbb{R}^d$, $|b|_A = 1$

The proof is analogous with the proof in [9], Satz 3.11.

Definition 10. For a given elliptic differential operator L let κ_λ be as in Theorem 5. The *single-layer potential* Φ_E and the *double-layer potential* Φ_D with density $\varphi \in L^\infty(\mathrm{X})$ are given on $\Omega_- \cup \Omega_+$ by

$$\Phi_E(x) = \int_{\mathrm{X}} \kappa_\lambda(x-y)\varphi(y)\, d\sigma(y) \tag{14}$$

and

$$\Phi_D(x) = \int_{\mathrm{X}} \frac{\partial \kappa_\lambda(x-y)}{\partial l(y)} \varphi(y)\, d\sigma(y)\,, \tag{15}$$

where σ is the surface measure of X.

Proposition 11.

(a) On $\mathbb{R}^d \setminus \Gamma$ the functions Φ_E and Φ_D are arbitrarily often differentiable under the integral sign and there holds $L\Phi_E \equiv 0$ and $L\Phi_D \equiv 0$.

(b) Φ_E can be extended continuously on $\mathbb{R}^d$ by

$$\Phi_E(x) = \int_X \kappa_\lambda(x-y)\varphi(y)\, d\sigma(y) \quad \forall x \in \mathbb{R}^d\,. \tag{16}$$

Proof. (a): Using a standard theorem about differentiation under the integral sign it is easy to verify that Φ_E and Φ_D are arbitrarily often differentiable on $\mathbb{R}^d \setminus \Gamma$. Therefore $L\kappa_\lambda \equiv 0$ on $\mathbb{R}^d \setminus \{0\}$ implies $L\Phi_E \equiv 0 \equiv L\Phi_D$ on $\mathbb{R}^d \setminus \Gamma$.
(b): The singular behaviour of $\kappa_\lambda(x)$ as $|x| \to 0$ can be characterized by

$$\kappa_\lambda(x) = \begin{cases} \dfrac{1}{2\pi\sqrt{\det A}} \ln \dfrac{1}{|x|_A} + O(1) & \text{for} \quad d = 2\,, \\[2ex] \dfrac{1}{(d-2)\omega_d\sqrt{\det A}} |x|_A^{2-d} + O(|x|^{3-d}) & \text{for} \quad d \geq 3\,. \end{cases}$$

In view of the continuity of κ_λ in $\mathbb{R}^d \setminus \{0\}$, the assertion (b) follows, for example, from [12], Lemma 8.1.5. □

6 The Principal Part

In this section, we will study the behaviour of elliptic differential operators with constant coefficients of the form $L_0 := -\operatorname{div}(A \operatorname{grad})$. According to Theorem 5 a singularity function for L_0 is given by

$$k_A(x) := \begin{cases} \dfrac{1}{2\pi\sqrt{\det A}} \ln \dfrac{1}{|x|_A} & \text{for} \quad d = 2 \\[2ex] \dfrac{1}{(d-2)\omega_d\sqrt{\det A}} \dfrac{1}{|x|_A^{d-2}} & \text{for} \quad d \neq 2\,. \end{cases} \tag{17}$$

For $x \in \mathbb{R}^d$, $y \in \mathrm{X} \setminus \{x\}$, the gradient of k_A has the representation

$$\nabla k_A(x) = -\frac{1}{\omega_d\sqrt{\det A}} \frac{A^{-1}x}{|x|_A^d} \tag{18}$$

and

$$\frac{\partial k_A(x-y)}{\partial l(y)} = \frac{1}{\omega_d\sqrt{\det A}} \frac{\langle n(y), x-y \rangle}{|x-y|_A^d} = \frac{\partial k_A(y-x)}{\partial l(y)}\,. \tag{19}$$

We are interested in the decay behaviour of the single- and double-layer potential at infinity:

Proposition 12.

(a) *For* $d \geq 3$, *we have* $\Phi_E(x) = O(|x|^{2-d})$ *and* $\nabla\Phi_E(x) = O(|x|^{1-d})$ *as* $|x| \to \infty$.

For $d = 2$, $\lim_{|x|\to\infty} \Phi_E(x) = 0$ *iff* $\int_X \varphi \, d\sigma = 0$.

In general, for all $d \geq 2$, *the relation* $\int_X \varphi \, d\sigma = 0$ *implies* $\Phi_E(x) = O(|x|^{1-d})$ *and* $\nabla\Phi_E(x) = O(|x|^{-d})$ *for* $|x| \to \infty$.

(b) *The decay of* Φ_D *at infinity is* $O(|x|^{1-d})$.

The proof is analogous with the proof of Satz 4.16 in [9].
In the sequel, we will state some auxiliary theorems.

Theorem 13. *For* $u, v \in C^2(\Omega_-) \cap C^1(\overline{\Omega}_-)$ *the equation*

$$\int_{\Omega_-} (\langle \nabla u, A\nabla v\rangle - uL_0 v)\, dx = \int_X u \frac{\partial v}{\partial l}\, d\sigma\,, \tag{20}$$

holds if both integrals exist. If, in addition, the integrals in (20) exist after interchanging the roles of u *and* v, *we have*

$$\int_{\Omega_-} (vL_0 u - uL_0 v)\, dx = \int_X \Big(u\frac{\partial v}{\partial l} - v\frac{\partial u}{\partial l}\Big)\, d\sigma\,. \tag{21}$$

Since the $(d-1)$-dimensional Minkowski measure of S is zero, formulae (20) and (21) can be proved using a version of Gauß' integral theorem stated in [16], Folgerung 7.5.
The next theorem is a generalisation of Green's representation formula of harmonic functions.

Theorem 14. *Let* $u \in C^2(\Omega_-) \cap C^1(\overline{\Omega}_-)$ *with* $L_0 u \equiv 0$ *in* Ω_-. *Then the following formula holds for each* $x \in \Omega_-$:

$$u(x) = \int_X \Big(\frac{\partial u}{\partial l}(y) k_A(x-y) - u(y)\frac{\partial k_A(x-y)}{\partial l(y)}\Big)\, d\sigma(y)\,. \tag{22}$$

By using the following Lemma 15, the proof is analogous with the proof of Green's representation formula (cf. [17], Theorem 6.5) by taking into account that, for $B \in \mathrm{GL}(\mathbb{R}^d)$ with $B^T B = A^{-1}$, we have $|Bx| = |x|_A$ and $|\det B|^{-1} = \sqrt{\det A}$ (cf. [9], Satz 3.37). Such B exists and is unique up to multiplication by orthogonal matrices from the left hand side. A possible choice is $B = F^{-1}$ where F is the Cholesky factorization of A: $FF^T = A$.

Lemma 15. *For every $B \in \mathrm{GL}(\mathbb{R}^d)$ there holds*

$$\int_{S_1} \frac{1}{|Bx|^d}\, d\sigma(x) = \frac{\omega_d}{|\det B|}\,. \tag{23}$$

Proof. We define $\varphi : \mathbb{R} \to \mathbb{R}$ by $r \mapsto r^2 e^{-r^2}$. On one hand we have

$$\begin{aligned}
&|\det B| \int_{\mathbb{R}^d} \frac{\varphi(|x|)}{|Bx|^d}\, dx = |\det B| \int_0^\infty \int_{S_1} \frac{\varphi(r) r^{d-1}}{r^d |By|^d}\, d\sigma(y)\, dr \\
&= |\det B| \int_{S_1} \frac{1}{|By|^d}\, d\sigma(y) \int_0^\infty \frac{\varphi(r)}{r}\, dr = \frac{1}{2} |\det B| \int_{S_1} \frac{1}{|By|^d}\, d\sigma(y)\,,
\end{aligned}$$

on the other hand there holds

$$\begin{aligned}
&|\det B| \int_{\mathbb{R}^d} \frac{\varphi(|x|)}{|Bx|^d}\, dx = \int_{\mathbb{R}^d} \frac{\varphi(|B^{-1}x|)}{|x|^d}\, dx = \int_{S_1} \int_0^\infty \frac{\varphi(r|B^{-1}y|)}{|ry|^d\, r^{1-d}}\, dr\, d\sigma(y) \\
&= \int_{S_1} \int_0^\infty \frac{\varphi(r|B^{-1}y|)}{r}\, dr\, d\sigma(y) = \omega_d \int_0^\infty \frac{\varphi(r)}{r}\, dr = \frac{1}{2}\, \omega_d\,.
\end{aligned}$$

□

6.1 Jump Relations

For $r > 0$ and $x \in \mathbb{R}^d$, we define $H(x,r) := S(x,r) \cap \Omega_-$, where $S(x,r)$ is the sphere of radius r centred in x, and the function $\delta_i : \mathbb{R}^d \to [0, \omega_d]$ by

$$\delta_i(x) := \lim_{r \searrow 0} \frac{r}{\sqrt{\det A}} \int_{H(x,r)} \frac{1}{|x-y|_A^d}\, d\sigma(y)\,. \tag{24}$$

The proof of Lemma 16 implies that the limit in (24) exists for all $x \in \mathbb{R}^d$. Hence, Lemma 15 ensures that $\delta_i(x) \in [0, \omega_d]$ holds (let therefor $B \in \mathrm{GL}(\mathbb{R}^d)$ such that $B^T B = A^{-1}$).
For $x \in \mathrm{X}$, there exists a neighbourhood V of x in X and a constant $L > 0$ satisfying $|\langle n(x), x - y\rangle| \le L|x-y|^{1+\mu}$ for all $y \in V$. That implies $\frac{\delta_i(x)}{\omega_d} = \frac{1}{2}$.

Lemma 16. *The function $\Xi : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\Xi(x) := \int_X \frac{\partial k_A(x-y)}{\partial l(y)}\, d\sigma(y)\,, \quad \textit{has the representation } \; \Xi(x) = -\frac{\delta_i(x)}{\omega_d}\,.$$

Proof. First let $x \in \Omega_+$. This implies $(L_0)_y k_A(x-y) \equiv 0$ in Ω_-. Equation (20) with $u \equiv 1$ and $v = k_A(x-\cdot)$ implies $\Xi(x) = 0$.
For $x \in \Omega_-$, the relation $\Xi(x) = -1$ follows directly from Theorem 14 applied to $u \equiv 1$. Lemma 15 shows that $\delta_i(x) = \omega_d$.
Now let $x \in \Gamma$. The normal vector field n on $H(x,r)$ is chosen such that $n(\xi) = \frac{1}{r}(\xi - x)$. Denote with $B(x,r)$ the open ball with radius r centred in x and define $\tilde{\Omega}_- := \Omega_- \setminus \overline{B}(x,r)$. If r is sufficiently small, we have a vanishing $(d-1)$-dimensional Minkowski measure of $S(\tilde{\Omega}_-)$ and, modulo a set of surface measure zero,

$$\mathrm{X}(\tilde{\Omega}_-) = \big(\mathrm{X} \setminus \overline{B}(x,r)\big) \cup H(x,r)\,.$$

Hence Gauß' divergence theorem (cf. [16], Folgerung 7.5.) can be applied and we obtain

$$\int_{\mathrm{X}\setminus\overline{B}(x,r)} \frac{\partial k_A(x-y)}{\partial l(y)}\, d\sigma(y) - \int_{H(x,r)} \frac{\partial k_A(x-y)}{\partial l(y)}\, d\sigma(y) = 0\,.$$

Using the dominated convergence theorem we conclude

$$\int_{\mathrm{X}} \frac{\partial k_A(x-y)}{\partial l(y)}\, d\sigma(y) = \lim_{r\searrow 0} \int_{H(x,r)} \frac{\partial k_A(x-y)}{\partial l(y)}\, d\sigma(y)$$
$$= -\lim_{r\searrow 0} \frac{r}{\omega_d\sqrt{\det A}} \int_{H(x,r)} \frac{1}{|x-y|_A^d}\, d\sigma(y) = -\frac{\delta_i(x)}{\omega_d} \in [-1,0]\,.$$

□

For $n \in \mathbb{N}$ we use the notation (cf. Figure 3)

$$K_n := \{z \in B_1 : z_d = 0 \wedge |z| < \tfrac{1}{n}\} \quad \text{and} \quad R_n := K_n \setminus K_{n+1}\,.$$

The $(d-1)$-dimensional Lebesgue measure of R_n is bounded by $\omega_{d-1} n^{-d}$.

Remark 17. There exists $\delta > 0$ such that for every $x \in \Gamma$ we find an open set U containing the ball $B(x,\delta)$ and, if $x \in \overline{M}_k$, there is a diffeomorphism $\vartheta : B_1 \to U$ with $\vartheta(0) = x$ and $\vartheta(K_1) = \tilde{M}_k \cap U =: V$. Let us denote $\vartheta^{-1}(y)$ by y' and $\vartheta(z')$ by z. There exist constants g, N, independent of x, with

$$\sup_{y\in U} \left\{ |D\vartheta(y')|_{op}\,, |D\vartheta^{-1}(y)|_{op} \right\} \leq N$$

and g is an upper bound for the Gram determinant of the parametrisation $\vartheta|_{K_1}$. Thus, we obtain for all $y, z \in U$ the equivalence of the norms

$$\frac{1}{N}|y' - z'| \leq |y - z| \leq N|y' - z'|\,.$$

The proof of Remark 17 is elementary (cf. [10]).

Theorem 18. *Let $(x_n^\pm)$ be a sequence in $\Omega_\pm$ converging to $x \in \Gamma$. Then, the jump relation (25) holds for the double-layer potential Φ_D with a density $\varphi \in L^\infty(X)$ which is continuous in x:*

$$\lim_{n\to\infty} \Phi_D(x_n^\pm) = \int_X \frac{\partial k_A(x-y)}{\partial l(y)} \varphi(y)\, d\sigma(y) \pm \frac{1}{2}\delta_\pm(x)\varphi(x)\,, \tag{25}$$

with

$$\delta_+(x) := 2\frac{\delta_i(x)}{\omega_d}\,, \quad \delta_-(x) := 2\Big(1 - \frac{\delta_i(x)}{\omega_d}\Big)\,.$$

In particular, for $x \in X$, we have

$$\lim_{n\to\infty} \Phi_D(x_n^\pm) = \int_X \frac{\partial k_A(x-y)}{\partial l(y)} \varphi(y)\, d\sigma(y) \pm \frac{1}{2}\varphi(x)\,.$$

Proof. For fixed $x \in \Gamma$ and $\xi \in \mathbb{R}^d$, we define

$$\Psi(\xi) := \int_X \frac{\partial k_A(\xi - y)}{\partial l(y)} (\varphi(y) - \varphi(x))\, d\sigma(y)\,.$$

Hence, for $\xi \notin \Gamma$, $\Phi_D(\xi) = \varphi(x)\Xi(\xi) + \Psi(\xi)$. Using Lemma 16 we obtain

$$\lim_{n\to\infty} \varphi(x)\Xi(x_n^\pm) = \int_X \frac{\partial k_A(x-y)}{\partial l(y)} \varphi(x)\, d\sigma(y) \pm \frac{1}{2}\delta_\pm(x)\varphi(x)\,.$$

Thus, it suffices to prove

$$\lim_{n\to\infty} (\Psi(x) - \Psi(x_n^\pm)) = 0\,.$$

Define

$$\zeta(\xi, x, y) := \left| \frac{\langle n(y), x - y\rangle}{|x-y|_A^d} - \frac{\langle n(y), \xi - y\rangle}{|\xi - y|_A^d} \right| |\varphi(y) - \varphi(x)|\,.$$

(1) First, for ξ close to x, it is clear that the term

$$I(\xi) := \int_{\mathrm{X}\setminus B(x,\delta)} \zeta(\xi, x, y)\, d\sigma(y)$$

tends to 0 as ξ tends to x.

(2) Let M_k be a surface piece with $x \in \overline{M}_k$. We extend φ by $\varphi(y) = \varphi(x)$ for all $y \in \tilde{M}_k \setminus M_k$. In the sequel we use the notation as in Remark 17.
To prove the theorem, it suffices to show that

$$J(\xi) := \int_V \zeta(\xi, x, y)\, d\sigma(y) \to 0 \quad \text{as } \xi \to x\,. \tag{26}$$

Let (x_n) be a sequence in $U \setminus \Gamma$ converging to x.
(a) Consider (x'_n) to be a null sequence in $B_1 \setminus K_1$ satisfying the "*angle condition*", i.e., there exists $\alpha \in]0,1[$ with $|(x'_n)_d| \geq \alpha |x'_n|$.
For sufficiently large $n \in \mathbb{N}$, there exists $m := m(n) \in \mathbb{N}$ with

$$\frac{1}{4m} \leq |x'_n| \leq \frac{1}{2m}\,.$$

If $n \to \infty$, then also $m \to \infty$. We decompose K_1 into K_m and R_1,...,R_{m-1} (cf. Figure 3). Employing the notations

$$T := \int_{K_m} \zeta(x_n, x, y)\, dy' \quad \text{and} \quad T_j := \int_{R_j} \zeta(x_n, x, y)\, dy'\,, \quad \text{where } y = y(y')\,,$$

we have

$$J(x_n) \leq \sqrt{g}\left(T + \sum_{j=1}^{m-1} T_j\right).$$

Introducing

$$\rho(\alpha) := \sup\left\{|\varphi(\eta) - \varphi(x)| :\ |\eta'| \leq \alpha\right\} \quad \text{for } \alpha > 0\,,$$

we obtain

$$T \leq C\rho(\tfrac{1}{m}) \int_{K_m} \left(\frac{1}{|y'|^{d-1-\mu}} + \left(\frac{m}{\alpha}\right)^{d-1}\right) dy'\,,$$

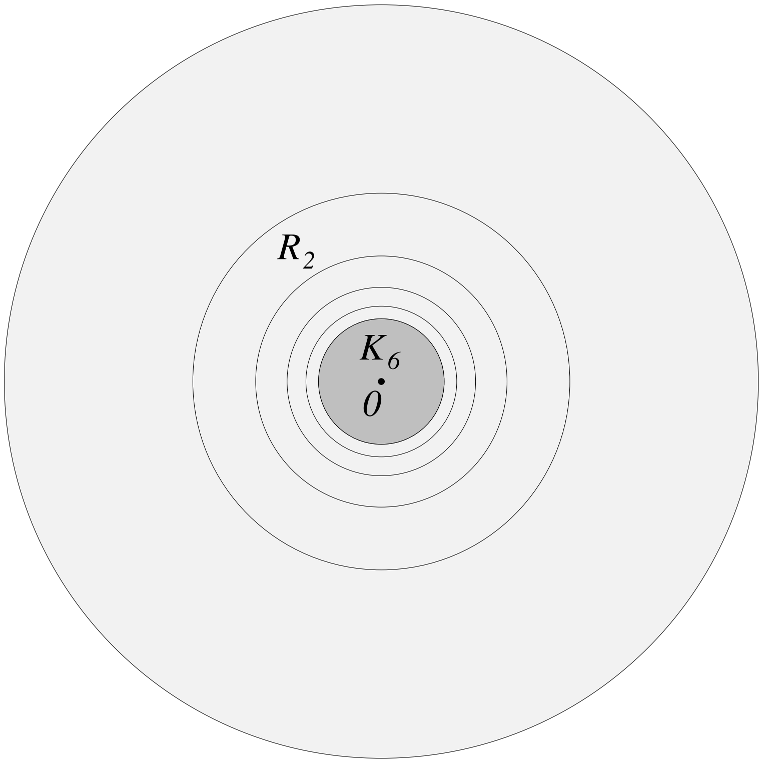

Figure 3: "Centred Decomposition" of K_1 for $m = 6$

which becomes arbitrarily small for sufficiently large n, because φ is continuous in x. Consider the integrands of T_j, $j = 1, ..., m-1$:

$$\left| \frac{\langle n(y), x-y \rangle}{|x-y|_A^d} - \frac{\langle n(y), x_n - y \rangle}{|x_n - y|_A^d} \right|$$

$$\leq \left| \frac{1}{|x-y|_A} - \frac{1}{|x_n-y|_A} \right| \sum_{k=1}^{d} \frac{|\langle n(y), x-y \rangle|}{|x-y|_A^{d-k} |x_n - y|_A^{k-1}} + \frac{|\langle n(y), x - x_n \rangle|}{|x_n - y|_A^d}$$

$$\leq \sum_{k=0}^{d} \frac{C|x_n'|}{|y'|^{d-k} |x_n' - y'|^k} =: \Upsilon_j \, .$$

The inequalities $|y'| \geq \frac{1}{j+1}$ and $|x_n' - y'| \geq \frac{1}{2(j+1)}$ for $y' \in R_j$ imply

$$\Upsilon_j \leq C \frac{(j+1)^d}{m} \, .$$

We derive

$$T_j \le C\frac{(j+1)^d}{m}|R_j|\,\rho(\tfrac{1}{j}) \le \frac{C}{m}\,\rho(\tfrac{1}{j})\,.$$

The continuity of φ at x implies that $(\rho(\frac{1}{j}))$ is a null sequence. This yields

$$\sum_{j=1}^{m-1} T_j \le \frac{C}{m}\sum_{j=1}^{m-1}\rho(\tfrac{1}{j}) \qquad \to 0 \quad \text{if } m\to\infty.$$

(b) Consider (x_n') to be a null sequence in $B_1\backslash K_1$ violating the angle condition in the following sense:
If y_n' is the orthogonal projection of x_n' on K_1 and $m := m(n)$, $\tilde m := \tilde m(n)$ are positive numbers with

$$\frac{2}{m} \le |x_n'| \le \frac{4}{m} \quad \text{ and } \quad \frac{1}{4\tilde m} \le |(x_n')_d| = |x_n' - y_n'| \le \frac{1}{2\tilde m}\,,$$

the sequence (κ_n), defined by $\kappa_n := \frac{m}{\tilde m}$, is a null sequence.
Let $n\in\mathbb{N}$ be sufficiently large such that $\tilde m > m$ holds. We define

$$\begin{aligned} K^n_{\tilde m} &:= \{z' \in K_1 : \, |y_n' - z'| < \tfrac{1}{\tilde m}\}\,,\\ S^n_j &:= \left\{\eta' \in K_1 : \, \tfrac{1}{j+1} < |\eta' - y_n'| \le \tfrac{1}{j}\right\},\\ D^n_j &:= \{\eta' \in K_1 : \, |\eta'| \ge \tfrac{1}{j+1} \wedge |y_n' - \eta'| \ge \tfrac{1}{j+1}\,,\, |\eta'| < \tfrac{1}{j} \vee |y_n' - \eta'| < \tfrac{1}{j}\}\,. \end{aligned}$$

We decompose K_1 into the subsets K_n, $K^n_{\tilde m}$, D^n_i, $i = 1,...,m-1$ and S^n_j, $j = m,...,\tilde m - 1$ (cf. Figure 4).
(i)

$$\int_{K_m} \zeta(x_n,x,y)\,dy' \le C\rho(\tfrac{1}{m})\int_{K_m}\Big(\frac{1}{|y'|^{d-1-\mu}} + m^{d-1}\Big)\,dy' \to 0 \quad \text{as } n\to\infty\,.$$

(ii) For $y' \in K^n_{\tilde m}$ we have $|y'| \ge \frac{6}{m}$ and $|x_n' - y'| \ge \frac{1}{4\tilde m}$. That implies the following estimate:

$$\int_{K^n_{\tilde m}} \zeta(x_n,x,y)\,dy' \le C\big(1 + |K^n_{\tilde m}|(\tilde m)^{d-1}\big)\rho(\tfrac{6}{m}) \to 0 \quad \text{as } n\to\infty.$$

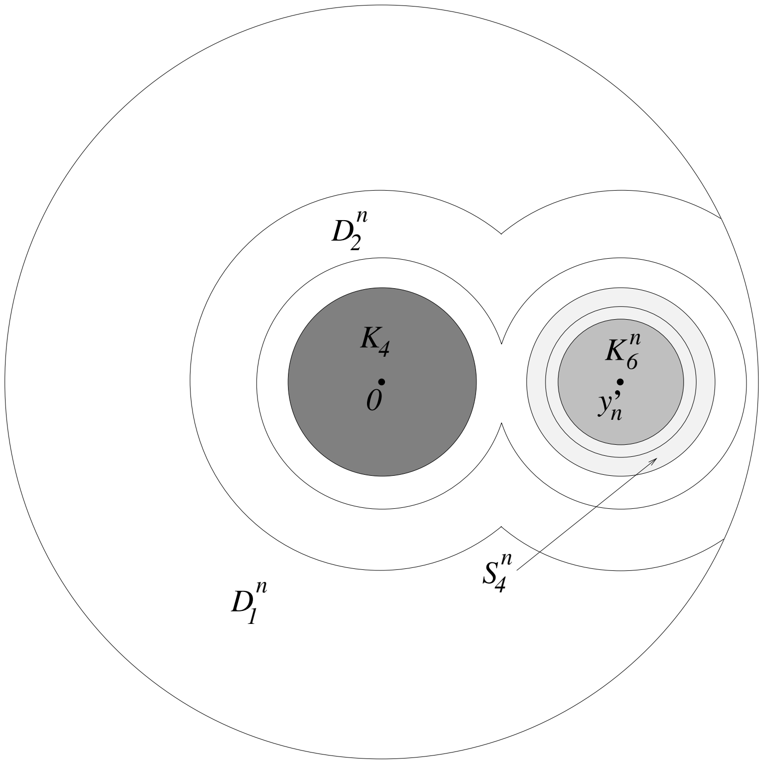

Figure 4: "Dipole-Decomposition" of K_1 with $m = 4$, $\tilde{m} = 6$

(iii) We obtain, for $y' \in D_j^n$, $j \in \{1, ..., m-1\}$, the estimates $|x_n' - y'| \geq \frac{1}{j+1}$ and $|y'| < \frac{6}{j}$. This results in

$$\left| \frac{\langle n(y), x - y \rangle}{|x - y|_A^d} - \frac{\langle n(y), x_n - y \rangle}{|x_n - y|_A^d} \right| \leq \frac{C}{m} \sum_{k=0}^{d} \frac{1}{|y'|^{d-k} |x_n' - y'|^k} \leq C \frac{j^d}{m}\,.$$

Due to $|D_j^n| \sim j^{-d}$, we conclude

$$\sum_{j=1}^{m-1} \int_{D_j^n} \zeta(x_n, x, y)\, dy' \leq \frac{C}{m} \sum_{j=1}^{m-1} \rho(\tfrac{6}{j}) \quad \to 0 \quad \text{if } n \to \infty\,.$$

(iv) Let $y' \in S_j^n$, $j \in \{m, ..., \tilde{m} - 1\}$, then $\frac{1}{2m} \leq |y'| \leq \frac{6}{m}$. We have the estimates

$$\frac{|\langle n(y), x - y \rangle|}{|x - y|_A^d} \leq C m^{d-1-\mu} \leq C j^{d-1-\mu}$$

and

$$\frac{|\langle n(y), x_n - y\rangle|}{|x_n - y|_A^d} \le \frac{|\langle n(y), x_n - y_n\rangle| + |\langle n(y), y_n - y\rangle|}{|x_n - y|_A^d} \le C\left(\frac{j^d}{\tilde{m}} + j^{d-1-\mu}\right).$$

In view of $|S_j^n| \sim j^{-d}$, we conclude

$$\begin{aligned}\sum_{j=m}^{\tilde{m}-1} \int_{S_j^n} \zeta(x_n, x, y)\, dy' \le& C \sum_{j=m}^{\tilde{m}-1} \left(\frac{j^d}{\tilde{m}} + j^{d-1-\mu}\right)|S_j^n| \rho(\tfrac{6}{m}) \\ \le& C\left(1 + \sum_{j=1}^{\infty} \frac{1}{j^{1+\mu}}\right)\rho(\tfrac{6}{m}) \qquad \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

(c) Because $\vartheta(K_1)$ may contain points not lying in Γ, we have to consider a sequence (x_n) in $\tilde{M}_k \setminus \Gamma$ converging to x. Let $y' \in K_1$. Then,

$$\frac{|\langle n(y), x - y\rangle|}{|x - y|_A^d} \le C|y'|^{1+\mu-d} \qquad \text{and} \qquad \frac{|\langle n(y), x_n - y\rangle|}{|x_n - y|_A^d} \le C|x_n' - y'|^{1+\mu-d}.$$

Let $m := m(n)$ with $\frac{1}{4m} \le |x_n'| \le \frac{1}{2m}$. We obtain

$$\int_{K_m} \zeta(x_n, x, y)\, dy' \le C\rho(\tfrac{1}{m}).$$

To estimate the integrals over the annuli R_j, $j = 1, ..., m-1$, we can proceed as in part (a).
Next, consider an arbitrary sequence (x_n) in $U \setminus \Gamma$ converging to $x \in \Gamma$:
If (x_n) has a subsequence (x_l) in $\tilde{M}_k \setminus \Gamma$, we have shown in part (c) that $\lim_{k\to\infty} J(x_l) = 0$. Without loss of generality, assume that the whole sequence lies in $U \setminus \tilde{M}_k$. We prove (26) by contradiction: Assume that $(J(x_n))$ does not converge to zero. This implies that there exists $\varepsilon > 0$ and a subsequence (x_l) of (x_n) with $|J(x_l)| > \varepsilon$ for all l. Since (x_l) converges to x, our results in (a) imply, that any infinite subsequence of (x_l) violates the angle condition. According to (b), $J(x_l)$ converges to zero yielding a contradiction. Hence, $\lim_{n\to\infty} J(x_n) = 0$. □

Lemma 19. *Let X be a C^2-submanifold, $x \in X$ and $\varphi \in L^\infty(X)$ continuous in x. Then*

$$\lim_{\alpha \searrow 0} \langle l(x), \nabla\Phi_D(x + \alpha\, l(x)) - \nabla\Phi_D(x - \alpha\, l(x))\rangle = 0.$$

For a proof see [9], Satz 4.38(c). The proof is a generalization of the proof of Lemma 8.2.17 in [12] treating the special case $L_0 = -\Delta$ in dimension $d = 2, 3$.
Let us turn to the conormal derivative of the single-layer potential:

Theorem 20. *Let $W \subset X$ be open and $\varphi \in L^\infty(X)$ continuous in W. Then the limits*

$$\lim_{h \searrow 0} \langle l(x), \nabla \Phi_E(x \pm h l(x)) \rangle = \int_X \frac{\partial k_A(x-y)}{\partial l(x)} \varphi(y)\, d\sigma(y) \mp \frac{1}{2}\varphi(x) \qquad (27)$$

exist locally uniformly on W.

Proof. Let K be a compact subset of W. Then there exists $\varepsilon > 0$ such that, for $s \in]0, \varepsilon[$ and $x \in K$, the inclusions $x - sl(x) \in \Omega_-$ and $x + sl(x) \in \Omega_+$ hold (cf. [9], Satz 3.23).
Theorem 18 implies that the double-layer potential Φ_D with density φ satisfies

$$\lim_{h \searrow 0} \Phi_D(x \pm h l(x)) = \int_{\mathrm{X}} \frac{\partial k_A(x-y)}{\partial l(y)} \varphi(y)\, d\sigma(y) \pm \frac{1}{2}\varphi(x)$$

uniformly on K. So it is sufficient to prove

$$\begin{aligned} &\lim_{h \searrow 0} \Big(\langle l(x), \nabla \Phi_E(x \pm h l(x)) \rangle + \Phi_D(x \pm h l(x)) \Big) \\ =&\frac{1}{\omega_d \sqrt{\det A}} \int_{\mathrm{X}} \frac{\langle n(y) - n(x), x - y \rangle}{|x-y|_A^d} \varphi(y)\, d\sigma(y) \end{aligned}$$

uniformly on K.
There exists $\delta > 0$ such that, for every $x \in K$, $B(x, \delta) \cap \Gamma \subset W$ and $\tilde{K} := \{\xi \in \Gamma : \operatorname{dist}(\xi, K) \leq \delta\}$ is compact. Furthermore we find for any $x \in K$ a diffeomorphism $\vartheta : B_1 \to U$, $B(x, \delta) \subset U$, with $\vartheta(0) = x$ as described in Remark 17. We denote $x_\alpha := x + \alpha l(x)$, $\alpha \in \mathbb{R}$, and $\kappa := \max\{|l(\xi)| : \xi \in \tilde{K}\}$. For $|\alpha| < \frac{\delta}{2\kappa}$ and $y \in \mathrm{X} \setminus B(x, \delta)$, using a generic constant C which does not

depend on $x \in K$ or α, we can estimate

$$\gamma(x_\alpha, x, y) := \left| \frac{\langle n(y)-n(x), x_\alpha - y\rangle}{|x_\alpha - y|_A^d} - \frac{\langle n(y)-n(x), x - y\rangle}{|x - y|_A^d} \right|$$
$$\leq \left| \frac{\langle n(y)-n(x), x_\alpha - x\rangle}{|x-y|_A^d} \right| + \left| \frac{1}{|x_\alpha - y|_A} - \frac{1}{|x-y|_A} \right| \sum_{j=1}^{d} \frac{|n(y)-n(x)||x_\alpha - y|}{|x_\alpha - y|_A^{d-j} |x-y|_A^{j-1}}$$
$$\leq C\frac{\kappa|\alpha|}{\delta^d} + |x_\alpha - x|_A \sum_{j=1}^{d} \frac{2|x_\alpha - y|}{|x_\alpha - y|^{d-j+1}|x-y|_A^j} \leq C\frac{\kappa}{\delta^d}|\alpha| \quad \to 0 \quad \text{as } \alpha \to 0\,.$$

This yields

$$\int_{\mathrm{X}\setminus B(x,\delta)} \gamma(x_\alpha, x, y)|\varphi(y)|\, d\sigma(y) \quad \to 0 \quad \text{as } \alpha \to 0$$

uniformly on K.
Now we use the notation introduced in Remark 17. Let $m \in \mathbb{N}$ with

$$\frac{1}{4m} \leq |x' - x'_\alpha| \leq \frac{1}{2m}\,.$$

Remark 17 implies $\frac{1}{cm} \leq |\alpha| \leq \frac{c}{m}$, with $c > 0$ independent from $x \in K$. We decompose K_1 into its subsets K_m and R_j, $j = 1, ..., m-1$, as in the proof of Theorem 18 (cf. Figure 3). By using Lemma 9 it follows

$$\int_{K_m} \gamma(x_\alpha, x, y)\, dy' \leq C \int_{K_m} |n(x) - n(y)| \Big(|x_\alpha - y|^{-d+1} + |x-y|^{-d+1} \Big)\, dy'$$
$$\leq C\Bigg(\left(\frac{N}{m}\right)^\mu \int_{K_m} |x_\alpha - y|^{-d+1}\, dy' + \int_{K_m} \frac{1}{|x-y|^{d-1-\mu}}\, dy' \Bigg)\,.$$

For $|\alpha|$ sufficiently small, there exists $\tilde{C} > 0$ independent of $x \in K$ such that $\operatorname{dist}(x_\alpha, \tilde{K}) \geq \tilde{C}|\alpha|$. That ensures the existence of $C > 0$, independent of $x \in K$, such that

$$\int_{K_m} \gamma(x_\alpha, x, y)\, dy' \leq C\left(\frac{1}{m^\mu} + \int_{K_m} |y'|^{-d+1+\mu}\, dy' \right),$$

i.e., $\int_{K_m} \gamma(x_\alpha, x, y)\, dy'$ tends uniformly to 0 for all $x \in K$ as $|\alpha| \to 0$.

For $y \in R_j$, $j \in \{1, ..., m-1\}$, we obtain $|y'| \geq \frac{1}{j+1}$ and $|x'_\alpha - y'| \geq \frac{1}{2(j+1)}$. Thus we can estimate

$$\gamma(x_\alpha, x, y) \leq C(j+1)^{d-\mu}|x_\alpha - x| \leq C\frac{j^{d-\mu}}{m}.$$

Finally

$$\sum_{j=1}^{m-1} \int_{R_j} \gamma(x_\alpha, x, y)\, dy' \leq \frac{C}{m}\sum_{j=1}^{m-1}\Big(j^{d-\mu}|R_j|\Big) = \frac{C}{m}\sum_{j=1}^{m-1} j^{-\mu} \quad \to 0$$

holds uniformly in $|\alpha|$ for all $x \in K$. That proves the theorem. □

Theorem 21. *Let $x \in X$ and $\varphi \in L^\infty(X)$ be continuous at x. Let $(x_n^\pm)$ be a sequence in $\Omega_\pm$ converging to x. If φ is Hölder continuous with exponent λ on a neighbourhood of x* or *if there exist c, $p > 0$ with* $\mathrm{dist}(x_n^\pm, \Gamma) \geq c|x - x_n^\pm|^p$, *we obtain*

$$\lim_{n\to\infty} \langle l(x), \nabla\Phi_E(x_n^\pm)\rangle = \int_X \frac{\partial k_A(x-y)}{\partial l(x)}\, \varphi(y)\, d\sigma(y) \mp \frac{1}{2}\varphi(x)\,. \tag{28}$$

The proof is done in detail in [10].

6.2 Integral Equations

Definition 22. The integral operators K and K^* are given formally, for every $\varphi \in L^\infty(\mathrm{X})$, by

$$K\varphi(x) = 2\int_{\mathrm{X}} \frac{\partial k_A(x-y)}{\partial l(y)}\, \varphi(y)\, d\sigma(y) + \Big(\frac{2\delta_i(x)}{\omega_d} - 1\Big)\varphi(x)\,, \quad x \in \Gamma\,,$$
$$K^*\varphi(x) = 2\int_{\mathrm{X}} \frac{\partial k_A(x-y)}{\partial l(x)}\, \varphi(y)\, d\sigma(y)\,, \quad x \in \mathrm{X}\,.$$

Theorem 23. *K is a continuous linear operator on $(L^\infty(\Gamma), |\cdot|_\infty)$ and on $(C(\Gamma), |\cdot|_\infty)$.*

Proof. (a) $\varphi \in C(\Gamma)$ implies $K\varphi \in C(\Gamma)$.
Proof of (a): With Theorem 18: $K\varphi(\xi) = 2\Phi_{D+}(\xi) - \varphi(\xi)$.
(b)

$$\sup_{\xi\in\Gamma}\left\{2\int_{\mathrm{X}}\left|\frac{\partial k_A(\xi-y)}{\partial l(y)}\right| d\sigma(y) + \left|\frac{2\delta_i(\xi)}{\omega_d} - 1\right|\right\} < \infty\,.$$

Proof of (b): (i) Since $\delta_i(\xi)$ is an element of the interval $[0, \omega_d]$ for all $\xi \in \Gamma$ we can estimate $|\frac{2\delta_i(\xi)}{\omega_d} - 1| \leq 3$.
(ii) Let δ be as in Remark 17. Recall that $\cup_{i=1}^m \overline{M}_i = \Gamma$. For $\xi \in \Gamma$, the set $\{1, ..., m\}$ can be decomposed into disjoint sets $I := \{i : \xi \in \overline{M}_i\}$,
$J := \{j : \text{dist}(\xi, M_j) \geq \delta/2\}$ and $K := \{k : 0 < \text{dist}(\xi, M_j) < \delta/2\}$. For $i \in I$, we obtain with Lemma 9

$$\int_{M_i} \left| \frac{\partial k_A(\xi - y)}{\partial l(y)} \right| d\sigma(y) \leq C \int_{M_i} \frac{1}{|\xi - y|^{d-1-\mu}} \, d\sigma(y) \, .$$

There is a constant $C' > 0$, independent of ξ and i, so that the last term of the inequality can be majorized by C'. We derive

$$\sum_{i \in I} \int_{M_i} \left| \frac{\partial k_A(\xi - y)}{\partial l(y)} \right| d\sigma(y) \leq mC' \, .$$

Furthermore:

$$\sum_{j \in J} \int_{M_j} \left| \frac{\partial k_A(\xi - y)}{\partial l(y)} \right| d\sigma(y) \leq C \sum_{j \in J} \sigma(M_j) \delta^{1-d} \leq C \sigma(\mathrm{X}) \delta^{1-d} \, .$$

Let $k \in K$ and $x \in \overline{M}_k$ with $\text{dist}(\xi, M_k) = |\xi - x|$. Let V be as in Remark 17. Then we obtain

$$\int_{M_k} \left| \frac{\partial k_A(\xi - y)}{\partial l(y)} \right| d\sigma(y) \leq \int_{M_k \setminus B(x,\delta)} \left| \frac{\partial k_A(\xi - y)}{\partial l(y)} \right| d\sigma(y) +$$
$$+ \int_V \left| \frac{\partial k_A(\xi - y)}{\partial l(y)} - \frac{\partial k_A(x - y)}{\partial l(y)} \right| d\sigma(y) + \int_V \left| \frac{\partial k_A(x - y)}{\partial l(y)} \right| d\sigma(y) \, .$$

The first intgral on the right hand side is smaller than $C\sigma(M_k)\delta^{1-d}$, the third integral is bounded by a constant independent of ξ, x and k. Let ϑ be a diffeomorphism with $\vartheta(0) = x$ as in Remark 17. Let $m \in \mathbb{N}$ with $\frac{1}{4m} \leq |\xi'| \leq \frac{1}{2m}$. The proof of

$$\int_{K_1} \left| \frac{\langle n(y), \xi - y \rangle}{|\xi - y|_A^d} - \frac{\langle n(y), x - y \rangle}{|x - y|_A^d} \right| dy' \leq C \, ,$$

where C is independent of ξ, x and k, is derived by decomposing K_1 into its subsets K_m, R_j, $j = 1, ..., m-1$, and using similar arguments as in part (a) of the proof of Theorem 18.

□

Remark 24. Although K is a compact operator on $C(\Gamma)$ if Γ is globally smooth (see Theorem 31), this is in general not true if $S \neq \emptyset$:
Consider the special case $L_0 = -\Delta$ and $d = 3$. In [40] it is shown that

$$\inf\{|K - V|_{op} : V \text{ linear and compact on } C(\Gamma)\} = \omega \tag{29}$$

for boundaries being piecewise C^2, having only convex (in the sense of [40]) corners and edges and

$$\omega := \sup\{|1 - \delta_i(x)/2\pi| : x \in \Gamma\} < 1\,,$$

i.e., boundaries, which differ slightly from those defined in section 1. Due to $\omega < 1$, equation (29) implies that Fredholm theory is applicable to the operator $I \pm K$ (cf. [12], Remark 8.2.25): the *Fredholm radius* of K is $1/\omega$ and exceeds one. For boundaries, containing "non-convex" edges, a modified version of (29) could be derived (cf. [40], §4.7, [41], §3).
In [12], §8.2.7, an analogous statement for the Laplace operator in two dimensions is proved, provided Γ is piecewise Hölder differentiable.

Proposition 25. *K^* maps $L^\infty(X)$ into $C(X)$.*

Proof. Let $\varphi \in L^\infty(\mathrm{X})$ and $x \in \mathrm{X}$. Without loss of generality we can assume that $x \in M := M_j$ for $j \in \{1, ..., m\}$. Let (x_n) be a sequence in M converging to x. We want to show $K^*\varphi(x_n) \to K^*\varphi(x)$ as $n \to \infty$. We define

$$\zeta(x_n, x, y) := \left| \frac{\langle n(x_n), x_n - y\rangle}{|x_n - y|_A^d} - \frac{\langle n(x), x - y\rangle}{|x - y|_A^d} \right|.$$

Then there holds

$$|K^*\varphi(x) - K^*\varphi(x_n)| \le C \int_{\mathrm{X}} \zeta(x_n, x, y)|\varphi(y)|\, d\sigma(y)\,. \tag{30}$$

If we restrict the domain of integration to the complement of a small neighbourhood of x in X, it is clear that the integral in (30) would tend to 0 as n tends to infinity. So let $\vartheta : B_1 \to U$ be a diffeomorphism with $\vartheta(0) = x$ as described in Remark 17. To prove that the integral in (30), with U as domain of integration instead of X, tends to 0 as $n \to \infty$, we can employ the centered decomposition (cf. Figure 3) and estimate similarlily as in the proof of Theorem 18.

□

Remark 26. Functions $u \in K^*(C(\mathrm{X}) \cap L^\infty(\mathrm{X}))$ are continuous on X while, in general, they can be singular at non-regular boundary points $x \in \mathrm{S}$.

Proof. We consider the square $Q := \Omega_-$ in $\mathbb{R}^2$ with boundary

$$\Gamma = \partial Q = \{(0,t),(1,t) : 0 \le t \le 1\} \cup \{(s,0),(s,1) : 0 \le s \le 1\}$$

and $L_0 := -\Delta$, i.e., $A = I$. Put $\varphi \equiv 1$ and define boundary points x_ε, $0 < \varepsilon \le 1$, by $x_\varepsilon = (0,\varepsilon)$. We obtain:

$$\begin{aligned} &|K^*\varphi(x_\varepsilon)| = \frac{2}{\omega_d}\left| \int_{\mathrm{X}} \frac{\langle n(x_\varepsilon), x_\varepsilon - y\rangle}{|x_\varepsilon - y|^2}\, d\sigma(y)\right| \\ \ge &\frac{2}{\omega_d}\int_0^1 \frac{t}{t^2+\varepsilon^2}\, dt = \frac{1}{\omega_d}[\ln(1+\varepsilon^2) - \ln(\varepsilon^2)] \quad \to \infty \quad \text{as} \quad \varepsilon \to 0\,, \end{aligned}$$

i.e., $K^*\varphi \notin L^\infty(\mathrm{X})$. □

Theorem 27.

(a) The double-layer potential Φ_D with density $\varphi \in C(\Gamma)$ solves the interior Dirichlet problem for the operator L_0 and a given boundary function $f \in C(\Gamma)$ iff φ solves the integral equation

$$\varphi - K\varphi = -2f \qquad \textit{on} \quad \Gamma\,. \tag{31}$$

(b) The double-layer potential Φ_D solves the exterior Dirichlet problem for a given boundary function $f \in C(\Gamma)$ iff $\varphi \in C(\Gamma)$ solves the integral equation

$$\varphi + K\varphi = 2f \qquad \textit{on} \quad \Gamma\,. \tag{32}$$

Proof. The assertion follows from Proposition 11 and Theorem 18. □

Remark 28. It follows from Lemma 16 that $\varphi \equiv 1$ lies in $\ker(I+K)$, i.e., a solution of (32) is not uniquely determined. For this reason it is preferable to work with a modified double-layer potential to solve the exterior Dirichlet problem (cf. Section 6.3).

Theorem 29.

(a) $\Phi_E|_{\overline{\Omega}_-}$ *solves the interior natural boundary value problem for a given $g \in C(X)$ iff the density $\varphi \in C(X) \cap L^\infty(X)$ is a solution of the integral equation*

$$\varphi + K^*\varphi = 2g \qquad \text{on} \quad X. \tag{33}$$

(b) $\Phi_E|_{\overline{\Omega}_+}$ *is a solution of the exterior natural boundary value problem for a given $g \in C(X)$ iff the density $\varphi \in C(X) \cap L^\infty(X)$ solves the integral equation*

$$\varphi - K^*\varphi = -2g \qquad \text{on} \quad X. \tag{34}$$

Remark 30. If we require that the solution $u = \Phi_E$ of the exterior natural boundary value problem suffices the radiation condition (8), it follows from Proposition 12 that, in the case of dimension $d \geq 3$, the radiation condition is always fulfiled, whereas in the situation of dimension $d = 2$ the radiation condition is fulfiled iff the continuous density φ satisfies the identity $\int_X \varphi \, d\sigma = 0$.

Proof of the theorem. The assertion follows from Proposition 11 and the jump relations (27). □

6.3 The Case of C^2-Boundaries

Let us consider the situation where Ω_- and Ω_+ are connected sets and the C^2-submanifold Γ is the common boundary of Ω_- and Ω_+. Here, we have better results than in the previous situation. The results stated in this subsection are proved in detail in [9], Kapitel 4, or, for the case $L_0 = -\Delta$ and dimension $d = 2, 3$, in [17], Chapter 6.

Theorem 31. *If $\Gamma = X$ is a C^2-submanifold of $\mathbb{R}^d$ we have $2\delta_i(x) = \omega_d$ for every $x \in X$. The linear operators K and K^* are compact operators on $(C(\Gamma), |\cdot|_\infty)$. Furthermore, there holds*

$$\int_\Gamma (K\varphi)\psi \, d\sigma = \int_\Gamma \varphi(K^*\psi) \, d\sigma \quad \text{for all } \varphi, \psi \in C(\Gamma).$$

Theorem 32.

(a) *The operators $I - K$ and $I - K^*$ are bijections on $C(\Gamma)$.*

(b) *The kernels of $I + K$ and $I + K^*$ are one-dimensional. To be more precise,*

$$\ker(I + K) = \operatorname{span}\{1\} \qquad \textit{and} \qquad \ker(I + K^*) = \operatorname{span}\{\rho\} ,$$

where ρ is a function in $C(\Gamma)$ satisfying $\int_\Gamma \rho \, dS \neq 0$. Furthermore,

$$(I + K^*)(C(\Gamma)) = \{\psi \in C(\Gamma) : \textstyle\int_\Gamma \psi \, d\sigma = 0\} =: C_0(\Gamma) .$$

Since the operator $I + K$ is neither injectiv nor surjectiv (according to the Fredholm alternative), we will define a modified operator $I + K'$ for the exterior Dirichlet problem.

Definition 33. Let $\xi \in \Omega_-$. The *modified double-layer potential* Φ_M on Ω_+ with continuous density φ is given by

$$\Phi_M(x) := \int_\Gamma \varphi(y) \left(\frac{\partial k_A(x - y)}{\partial l(y)} + \frac{1}{|x - \xi|_A^{d-2}} \right) d\sigma(y) \qquad \text{for} \quad x \in \Omega_+ . \quad (35)$$

The integral operator $K' : C(\Gamma) \to C(\Gamma)$ is given for all $\psi \in C(\Gamma)$ by

$$K'\psi(x) = 2 \int_\Gamma \psi(y) \left(\frac{\partial k_A(x - y)}{\partial l(y)} + \frac{1}{|x - \xi|_A^{d-2}} \right) d\sigma(y) , \qquad x \in \Gamma .$$

Conclusion 34.

(i) *On Ω_+, Φ_M is infinitly often differentiable under the integral sign and $L_0 \Phi_M \equiv 0$. The modified double-layer potential satisfies the radiation condition $\Phi_M(x) = O(|x|^{2-d})$ at infinity.*

(ii) *K' is a compact operator on $C(\Gamma)$.*

(iii) *The jump relations (25) for the double-layer potential Φ_D imply that the modified double-layer potential Φ_M could be continuously extended on $\overline{\Omega}_+$ by*

$$\Phi_M(x) = \int_\Gamma \varphi(y) \left(\frac{\partial k_A(x - y)}{\partial l(y)} + \frac{1}{|x - \xi|_A^{d-2}} \right) d\sigma(y) + \frac{1}{2}\varphi(x) , \;\; x \in \Gamma.$$

Theorem 35. *The integral operator $I + K'$ is an automorphism on $C(\Gamma)$.*

Theorem 36. *The extended modified double-layer potential Φ_M solves the exterior Dirichlet problem for a given boundary function $f \in C(\Gamma)$ iff the continuous density φ satisfies the equality*

$$\varphi + K'\varphi = 2f \qquad on \quad \Gamma\,. \tag{36}$$

Theorem 37.

(a) *For a given boundary function $f \in C(\Gamma)$ the interior Dirichlet problem has a unique solution. This solution is the double-layer potential Φ_D with density φ, fulfilling (31).*

(b) *For a given boundary function $f \in C(\Gamma)$ the exterior Dirichlet problem with the radiation condition (7) has a unique solution. This solution is the modified double-layer potential Φ_M with density φ, fulfilling (36).*

Theorem 38.

(a) *The interior natural boundary value problem is solvable iff the integral over Γ of the given boundary function $g \in C(\Gamma)$ vanishes, i.e., $g \in C_0(\Gamma)$. The solution is then given by the single-layer potential Φ_E with density φ, satisfying (33).*

(b) *For a given boundary function $g \in C(\Gamma)$ and $d \geq 3$, the exterior natural boundary value problem with radiation condition (8) is uniquely sovable.*

   In $\mathbb{R}^2$, the exterior natural boundary value problem with radiation condition (8) has a unique solution iff the given boundary function g lies in $C_0(\Gamma)$.

   The solution is in both cases given by the single-layer potential Φ_E with density φ, satisfying (34).

In the case of C^2-boundaries all integral equations mentioned before are of the form

$$(\lambda I + T)u = f\,,$$

where $\lambda \neq 0$ and T is a compact integral operator. Such kind of equations can be solved very efficiently by applying the multi-grid method of second kind (cf. [11], Chapter 16, [12], Chapter 5 and §9.3.1). Note that, due to the compactness of Γ, classical Neumann series converge as well while the convergence speed is much slower as for multi-grid methods.

Theorem 39. *The solutions to the Dirichlet und natural boundary value problems (with radiation conditions (7) and (8) for exterior problems) depend continuously on the given boundary data in $C(\Gamma)$ in the maximum norm.*

7 The General Situation

In this section, we will state the integral equations for Dirichlet and natural boundary value problems corresponding to a general elliptic partial differential operator L of order two with constant coefficients. We have to compute the jump relations of the double-layer potential and the conormal derivative of the single-layer potential. Therefor consider the fundamental solution k_A of the principal part $L_0 = -\operatorname{div}(A \operatorname{grad})$ of the differential operator L (defined in (17)), the double-layer potential Φ_D^0 and the single-layer potential Φ_E^0 of L_0 with density φ (as defined in Chapter 4).

Proposition 40. *Consider a density function $\varphi \in L^\infty(X)$. Then $\Phi_D - \Phi_D^0$ can be continuously extended on $\mathbb{R}^d$ by*

$$(\Phi_D - \Phi_D^0)(x) = \int_X \Big(\frac{\partial \kappa_\lambda(x-y)}{\partial l(y)} - \frac{\partial k_A(x-y)}{\partial l(y)}\Big)\varphi(y)\, d\sigma(y)\,, \quad x \in \mathbb{R}^d\,.$$

Proof. Let R be sufficiently large so that B_R contains Γ, define $G := \overline{B}_R$ and the function $f : G \times \mathrm{X} \setminus \{(\xi,\xi) : \xi \in \mathrm{X}\} \to \mathbb{C}$ by

$$f(x,y) := \frac{\partial}{\partial l(y)}(\kappa_\lambda - k_A)(x-y)\,.$$

The continuity of $\Phi_D - \Phi_D^0$ follows from Lemma 6 and [12], Lemma 8.1.5. $\square$

Theorem 41. *If $(x_n^\pm)$ is a sequence in $\Omega_\pm$ which converges to $x_0 \in \Gamma$, the jump relation (37) holds for the double-layer potential Φ_D with a density $\varphi \in L^\infty(X)$ being continuous at x_0:*

$$\lim_{n\to\infty} \Phi_D(x_n^\pm) = \int_X \frac{\partial \kappa_\lambda(x_0 - y)}{\partial l(y)}\varphi(y)\, d\sigma(y) \pm \frac{1}{2}\delta_\pm(x_0)\varphi(x_0)\,, \qquad (37)$$

where $\delta_\pm$ are as in Theorem 18. In particular, for $x_0 \in X$, we have

$$\lim_{n\to\infty} \Phi_D(x_n^\pm) = \int_X \frac{\partial \kappa_\lambda(x_0 - y)}{\partial l(y)}\varphi(y)\, d\sigma(y) \pm \frac{1}{2}\varphi(x_0)\,.$$

Proof. There holds

$$\Phi_D(\xi) = (\Phi_D(\xi) - \Phi_D^0(\xi)) + \Phi_D^0(\xi)\,.$$

Hence, Theorem 41 follows from Theorem 18 and Proposition 40. □

Lemma 42. *Let $\varphi \in L^\infty(X)$. Then the function $\Phi_E - \Phi_E^0$ is continuously differentiable on $\mathbb{R}^d$. For all $x \in \mathbb{R}^d$ holds*

$$\nabla(\Phi_E - \Phi_E^0)(x) = \int_X (\nabla\kappa_\lambda(x-y) - \nabla k_A(x-y))\varphi(y)\,d\sigma(y)\,.$$

Proof. According to Lemma 6 there exists a compact neighbourhood G of Γ, $\nu \in]0,1[$ and $C > 0$ such that on $(G \times \Gamma) \setminus \{(\xi,\xi) : \xi \in \Gamma\}$ the inequality

$$|\nabla(\kappa_\lambda - k_A)(x-y)| \le C|x-y|^{1+\nu-d}$$

holds. Hence, Lemma 42 follows from [12], Lemma 8.1.5. □

Theorem 43. *Let $W \subset X$ be open and $\varphi \in L^\infty(X)$ continuous in W. Then the limits*

$$\lim_{h\searrow 0}\langle l(x), \nabla\Phi_E(x \pm hl(x))\rangle = \int_X \frac{\partial\kappa_\lambda(x-y)}{\partial l(x)}\,\varphi(y)\,d\sigma(y) \mp \frac{1}{2}\varphi(x) \qquad (38)$$

exist locally uniformly on W.

Proof. There exists $c > 0$ such that $\frac{1}{c} \le |l(x)| \le c$ holds on X. Hence, Lemma 42 implies

$$\lim_{h\searrow 0}\Big\langle l(x), \Big(\nabla(\Phi_E - \Phi_E^0)(x \pm hl(x)) - \int_{\mathrm{X}} \nabla(\kappa_\lambda - k_A)(x-y)\varphi(y)\,d\sigma(y)\Big)\Big\rangle = 0$$

uniformly on X. Now the theorem follows from Theorem 20 and

$$\begin{aligned}&\lim_{h\searrow 0}\langle l(x), \nabla\Phi_E(x \pm hl(x))\rangle \\ &= \lim_{h\searrow 0}\left(\langle l(x), \nabla(\Phi_E - \Phi_E^0)(x \pm hl(x))\rangle + \langle l(x), \nabla\Phi_E^0(x \pm hl(x))\rangle\right).\end{aligned}$$

□

Because of (37) and (38), we get the same integral equations for the Dirichlet and natural boundary value problems for L as stated in the Theorems 27 and 29 (of course in Definition 22 k_A has to be replaced by κ_λ).

8 Numerical Examples

In this section, we will report on the numerical realization of the integral equation formulations presented in the previous sections for solving elliptic boundary value problems. As stated in the introduction the motivation for this paper was to develop explicit integral equation formulations so that only the coefficients of the underlying operator L as in (1) have to be specified. The essential bottle neck in the realization of the boundary element method (as the most flexible discretization method for boundary integral equations being applicable to a large class of problems of engineering interest) is the computation of the entries of the arising stiffness matrix. These entries are defined as surface integrals with singular, nearly singular, and regular integrands. In [35], [36], [14], [13], [1], [33], [6] efficient, black-box quadrature methods have been developed for these integrals in the sense that

- it is sufficient to provide the coefficients of the differential operator L and the numerical integrator automatically computes a sufficiently accurate result,
- the numerical quadrature method converges *exponentially* with respect to the order (independent on the distance from the singularity).

The class of integral kernels includes those which arise by applying the method of integral equations to the operator L. Along with the boundary integral equations derived in this paper it is now possible to *solve* boundary value problems via integral equation methods just by prescribing the coefficients of the operator L.

The numerical implementation was based on the boundary element library `LIBBEM` written by C. Lage (cf. [18], [19]) where the black-box numerical integrator was incorporated by S. Erichsen (cf. [6], [5]).

As a model problem serves the natural boundary value problem in the (classical) form: Seek $u \in C^2(\Omega_+) \cap C\left(\overline{\Omega_+}\right)$ satisfying

$$\begin{array}{ll} Lu = 0 & \text{in } \Omega_+ := \mathbb{R}^3 \backslash \overline{B_1} \\ \frac{\partial u_+}{\partial l} = g & \text{uniformly on } \Gamma := \partial B_1 \end{array} \qquad (39)$$

with $L = -\nabla (A\nabla \cdot)$ and $g = -\langle x, A^{-1}x\rangle^{-3/2}$. The positive definite matrix A is chosen as

$$A = \begin{bmatrix} \frac{23}{24} & \frac{7}{24} & -\frac{1}{4\sqrt{2}} \\ \frac{7}{24} & \frac{23}{24} & -\frac{1}{4\sqrt{2}} \\ -\frac{1}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} & \frac{5}{4} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} \frac{7}{6} & -\frac{1}{3} & \frac{1}{6\sqrt{2}} \\ -\frac{1}{3} & \frac{7}{6} & \frac{1}{6\sqrt{2}} \\ \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{5}{6} \end{bmatrix}$$

and has eigenvalues $2/3, 1, 3/2$. The exact solution is given by

$$u(x) = \left\langle x, A^{-1}x \right\rangle^{-1/2}.$$

We represent the solution as a single-layer potential with density φ: $u = \Phi_E$ leading to the integral equation:

$$\varphi(x) - 2\int_\Gamma \varphi(y) \frac{k_A(x-y)}{\partial l(x)} d\sigma(y) = -2g(x), \qquad x \in \Gamma. \tag{40}$$

In order to employ the Galerkin method for its discretization we start with defining finite element spaces on the surface Γ. Let Γ_0 denoting the double pyramid with vertices $(\pm 1, 0, 0)^\intercal$, $(0, \pm 1, 0)^\intercal$, $(0, 0, \pm 1)^\intercal$. The surface triangles define the initial mesh $\mathcal{T}_0$. Finer triangulations are defined recursively. Assume T_{i-1} is already generated. Then, refine each triangle $\tau = \Delta(A_1, A_2, A_3)$ (with edge midpoints M_{12}, M_{13}, M_{23}) by projecting M_{ij} onto the true surface (resulting in $\tilde{M}_{ij}$) and defining the four sons of τ by

$$\begin{aligned} \tau_1 &= \Delta\left(A_1, \tilde{M}_{12}, \tilde{M}_{13}\right), \qquad \tau_2 = \Delta\left(A_2, \tilde{M}_{23}, \tilde{M}_{12}\right), \\ \tau_3 &= \Delta\left(A_3, \tilde{M}_{13}, \tilde{M}_{23}\right), \qquad \tau_4 = \Delta\left(\tilde{M}_{12}, \tilde{M}_{13}, \tilde{M}_{23}\right). \end{aligned}$$

This results in the triangulation T_i and the approximate surface $\Gamma_i = \bigcup T_i$. The set of vertices in T_i is denoted by Θ_i.
The space of continuous, piecewise linear finite elements on Γ_i is given by

$$S_i := \{v \in C(\Gamma_i) \mid \forall \tau \in T_i : v\,|_\tau \text{ is affine}\}.$$

The computations have been performed on levels $i = 0, 1, 2, 3, 4$ where the dimension of S_i is listed below:

i	0	1	2	3	4
$\sharp T_i$	8	32	128	512	2048
$\dim S_i$	6	18	66	258	1026

i	$-\Delta$		L	
	e_i	e_i/e_{i+1}	e_i	e_i/e_{i+1}
0	2.167162e-01	2.6878009	2.333187e-01	2.5749276
1	8.062956e-02	3.9331032	9.061175e-02	4.3091592
2	2.050024e-02	4.1212427	2.102771e-02	4.0420282
3	4.974286e-03	4.0502795	5.202267e-03	4.0012206
4	1.228134e-03		1.300170e-03	

Table 1: $L = -\operatorname{div}(A \text{ grad})$

The Galerkin formulation to (40) is given by seeking $\varphi_i \in S_i$ such that

$$(\varphi_i, v)_{0,\Gamma} - 2 \int_{\Gamma\times\Gamma} v(x)\, \varphi_i(y) \frac{k_A(x-y)}{\partial l(x)} d\Gamma_y d\Gamma_x = -2\,(g, v)_{0,\Gamma}, \qquad \forall v \in S_i,$$

where $(\cdot,\cdot)_{0,\Gamma}$ denotes the L_2 scalar product on Γ. The approximate Galerkin solution to (39) on level i is denoted by $u_i = \Phi_{E,i}$, where $\Phi_{E,i}$ is the single-layer potential with density φ_i.
The relative error in the discrete ℓ_2-norm is defined by

$$e_i := \sqrt{\frac{\sum_{x\in\Theta_i} |u_i(x) - u(x)|^2}{\sum_{x\in\Theta_i} |u(x)|^2}}.$$

Due to the regularity of our problem and the stability of the Galerkin method we expect that the Galerkin method converges quasi-optimally, i.e., quadratically. The table below clearly shows this convergence behaviour and, additionally, that the error has about the same magnitude as the error for the Laplace problem.

References

[1] Atkinson, K., ‘Solving Integral Equations on Surfaces in Space’, in G. Hämmerlin and K. Hoffmann, editors, *Constructive Methods for the Practical Treatment of Integral Equations*, 20-43, Birkhäuser: ISNM, 1985.

[2] Calderon, A. and Zygmund, A., ‘Singular Integral Operators and Differential Equations’, *Amer. J. Mat.*, **79**, 901-921 (1957).

[3] Colton, D. and Kress, R., *Integral Equation Methods in Scattering Theory*, John Wiley & Sons, 1983.

[4] Costabel, M. and Wendland, W., 'Strong Ellipticity of Boundary Integral Operators', *J. Reine Angew. Math.*, **372**, 34-63 (1986).

[5] Erichsen, S., *Entwurf und Implementierung von Black-Box-Quadraturverfahren für die Randelementmethode in* $\mathbb{R}^3$, PhD thesis (in preparation), Lehrstuhl Prakt. Math., Universität Kiel.

[6] Erichsen, S. and Sauter, S., 'Efficient automatic quadrature in 3-d Galerkin BEM', *Comp. Meth. Appl. Mech. Eng.*, **157**, 215-224 (1998).

[7] Farwig, R., 'A variational approach in weighted Sobolev spaces to the operator $-\Delta + \partial/\partial_{x_1}$ in exterior domains of $\mathbb{R}^3$', *Mathematische Zeitschrift*, **210**, 449-464 (1992).

[8] Giraud, G., 'Sur une classe generale d'equation a integrales principales', *C.R. Acad. Sc. Paris*, **202**, 2124-2126 (1936).

[9] Gnewuch, M., *Die Integralgleichungsmethode für elliptische Randwertprobleme*, Diploma thesis, Lehrstuhl Prakt. Math., Universität Kiel, 1998.

[10] Gnewuch, M. and Sauter, S., 'Boundary Integral Equations for Second Order Elliptic Boundary Value Problems' (Extended Version), Preprint, Max-Planck-Institut für Mathematik, Leipzig, 1999 (http://www.mis.mpg.de/cgi-bin/preprints.pl).

[11] Hackbusch, W., *Multi-Grid Methods and Applications*, Springer-Verlag Berlin Heidelberg New York, 1985.

[12] Hackbusch, W., *Integral Equations: Theory and Numerical Treatment*, Birkhäuser, Basel, 1995.

[13] Hackbusch, W. and Sauter, S., 'On the Efficient Use of the Galerkin Method to Solve Fredholm Integral Equations', *Appl. of Math.*, **38(4-5)**, 301-322 (1993).

[14] Hackbusch, W. and Sauter, S., 'On Numerical Cubatures of Nearly Singular Surface Integrals arising in BEM Collocation', *Computing*, **52**, 139-159 (1994).

[15] Hadamard, J., *Lectures on Cauchy's Problem*, Yale Univ. Press, 1923.

[16] König, H., 'Ein einfacher Beweis des Integralsatzes von Gauß', *Jahresbericht der DMV*, **66**, 119-138 (1964).

[17] Kress, R., *Linear Integral Equations*, Springer, Berlin Heidelberg, 1989.

[18] Lage, C., *Software Development for Boundary Element Methods: Analysis and Design of Efficient Techniques (in German)*, PhD thesis, Lehrstuhl Prakt. Math., Universität Kiel, 1995.

[19] Lage, C., 'Concept Oriented Design of Numerical Software', Technical Report 98-07, ETHZ, Zürich, 1998.

[20] Lebedew, N., *Special Functions and their Applications*, Dover Publications, Inc., New York, 1972.

[21] Mikhlin, S., *Integral Equations*, Pergamon Press, London, 1957.

[22] Muskhelishvili, N., *Singular Integral Equations*, Noordhoff, Groningen, 1953.

[23] Neumann, C., *Untersuchungen über das logarithmische und Newtonsche Potential*, Teubner, Leipzig, 1877.

[24] Noether, F., 'Über eine Klasse Singulärer Integralgleichungen', *Math. Ann.*, **82**, 42-63 (1921).

[25] Nyström, E., 'Über die praktische Auflösung von linearen Integralgleichungen mit Anwendungen auf Randwertaufgaben der Potentialtheorie', *Soc. Sci. Fenn. Comment. Phys.-Math.*, **4**, 1-52 (1928).

[26] Nyström, E., 'Über die praktische Auflösung von Integralgleichungen mit Anwendungen auf Randwertaufgaben', *Acta Math.*, **54**, 185-204 (1930).

[27] Ortner, N., 'Regularisierte Faltung von Distributionen', *Zeit. für angew. Math. und Phys.*, **31** 133-173 (1980).

[28] Ortner, N., 'Construction of Fundamental Solutions', in C. Brebbia, editor, *Topics in Boundary Element Research*, Springer Verlag.

[29] Plemelj, J., 'Ein Ergänzungssatz zur Cauchyschen Integraldarstellung analytischer Funktionen, Randwerte betreffend', *Monatshefte f. Math. u. Phys.*, **19**, 205-210 (1908).

[30] Plemelj, J., *Potentialtheoretische Untersuchungen*, Preisschriften der Fürstlichen Jablonowskischen Gesellschaft, Teubner, Leipzig, 1911.

[31] Radon, J., 'Über die Randwertaufgaben beim logarithmischen Potential', *Sitzungsberichte der Akademie der Wissenschaften Wien*, **128**, 1123-1167 (1919).

[32] Sauter, S., *Über die effiziente Verwendung des Galerkinverfahrens zur Lösung Fredholmscher Integralgleichungen*, PhD thesis, Lehrstuhl Prakt. Math., Universität Kiel, 1992.

[33] Sauter, S., 'Cubature Techniques for 3-d Galerkin BEM', in W. Hackbusch and G. Wittum, editors, *BEM: Implementation and Analysis of Advanced Algorithms*, Vieweg Verlag, 1996.

[34] Sauter, S. and Lage, C., 'Transformation of hypersingular integrals and black-box cubature (extended version)', Technical Report 97-17, Universität Kiel.

[35] Schwab, C. and Wendland, W., 'Kernel Properties and Representations of Boundary Integral Operators', *Math. Nachr.*, **156**, 187-218 (1992).

[36] Schwab, C. and Wendland, W., 'On Numerical Cubatures of Singular Surface Integrals in Boundary Element Methods', *Numer. Math.*, **62**, 343-369 (1992).

[37] Seeley, R., 'Singular Integrals on Compact Manifolds', *Amer. J. Math.*, **81**, 658-690 (1958).

[38] Tricomi, F., *Integral Equations*, Interscience Publishers, Ltd., New York-London, 1957.

[39] Watson, G., *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 2.Edition, 1958.

[40] Wendland, W., *Lösung der ersten und zweiten Randwertaufgaben des Innen- und Außengebietes für die Potentialgleichung im R_3 durch Randbelegungen*, PhD thesis, Fakultät für Allgemeine Ingenieurwissenschaften, TU Berlin, 1965.

[41] Wendland, W., 'Die Behandlung von Randwertaufgaben im R_3 mit Hilfe von Einfach- und Doppelschichtpotentialen', *Numerische Mathematik*, **11**, 380-404 (1968).

[42] Wendland, W., 'Strongly elliptic boundary integral equations', in A. Iserles and M. Powell, editors, *The State of the Art in Numerical Analysis*, 511-561, Clarendon Press, Oxford, 1987.