Composite Finite Elements for the Approximation of PDEs on Domains with complicated Micro-Structures

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Abstract

Usually, the minimal dimension of a finite element space is closely related to the geometry of the physical object of interest. This means that sometimes the resolution of small micro-structures in the domain requires an inadequately fine finite element grid from the viewpoint of the desired accuracy.

This fact limits also the application of multi-grid methods to practical situations because the condition that the coarsest grid should resolve the physical object often leads to a huge number of unknowns on the coarsest level.

We present here a strategy for coarsening finite element spaces independently of the shape of the object. This technique can be used to resolve complicated domains with only few degrees of freedom and to apply multi-grid methods efficiently to PDEs on domains with complex boundary.

In this paper we will prove the approximation property of these generalized FE spaces.

Mathematics Subject Classification (1991): 65D05, 65N12, 65N15, 65N30, 65N50, 65N55

1 Introduction

In this paper, we will introduce so-called *Composite Finite Elements* on two-dimensional domains. However, we state that generalizations to more spatial variables are obvious. We have in mind that these domains may have boundaries with complicated micro-structures. Consequently, every reasonable finite element grid (quasi-uniform, satisfying the minimal angle condition) which has to resolve the boundary will have a huge number of elements. Finite element spaces corresponding to such grids and also

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finer grids usually satisfy an asymptotic approximation property. We will define subspaces of these finite element spaces corresponding to “coarser” FE grids which also satisfy the asymptotic approximation property. The minimal number of unknowns will not be limited by the shape of the domain.

This new class of finite elements is called *Composite Finite Elements* for the following reason. According to the definition of [4, Chapter 2.3], finite elements are triples consisting of the element domain, the space of shape functions, and the set of nodal functionals. Usually, the element domains are smooth images of a reference element and the shape functions are smooth at least in the interior of the element domain. For composite finite elements, however, the element domain K is the union of many small standard elements. The shape functions on K are *composed* locally of piecewise polynomials on the small elements along with suitable global constraints on K which leads to the name *composite* finite elements. [1]

The ideas are closely related to Shortley-Weller discretizations in the context of finite difference approximations as described in [13], [7], [10] implemented in a hierarchical way using the Galerkin product (see [5]).

Another approach for coarsening finite element spaces can be found in [2] and [3]. There, the authors define a hierarchical basis on non-nested grids and prove grid-independent convergence rates for the corresponding BPX method. In contrast to the method presented in our paper the coarsening strategy of the mentioned authors can be applied to arbitrarily unstructured grids, while our approach uses the logically regular grid. Consequently, it turns out that, a priori, we know that the coarsest grid will consist of extremely few degrees of freedom (typically smaller than 10) independent of the shape of the domain. The coarsening approach in [2] is heuristic and, hence, it is beforehand not known what the number of unknowns at the coarsest level will be, when the algorithm terminates.

A further related method is presented in [11]. In that paper, the physical domain is embedded in a domain of easy shape which is refined by standard methods. The FE spaces are given by the restriction of the functions on the artificial larger domain to the physical domain. It was shown that subspace correction methods can be applied successfully to this method.

Knowing the approximation property and stability behaviour, it is well known that the Galerkin FEM has quasi-optimal convergence behaviour. Thus, if one is interested in a relatively crude approximation of the solution, we are now able to use composite finite element spaces of low dimension independent of the shape of the domain and obtain the corresponding accuracy.

Following the theory of [6], the convergence of multi-grid methods can be split in the proof of the approximation and the smoothing property. The approximation property for multi-grid methods follows from the approximation quality of the finite element spaces and assumptions on the differential equation on the continuous but

[1]After submitting the paper we noticed that, in the context of approximating curved boundaries, a similar finite element was introduced in [12].

not on the discrete level (see [6, Section 6.3.1]).

This paper is organized as follows. In the next chapter, we will introduce strategies to coarsen triangulations of domains independently of the shape of the domain. Then, in Chapter 3 we will define finite element spaces on these grids by introducing suitable interpolation operators. In Chapter 4, we will prove the approximation property of these FE spaces in the case that the domain is the whole plane. Chapter 5 addresses the approximation quality of composite finite element spaces on bounded domains Ω using the previous results. Finally, in the Appendix we prove a stability theorem for the interpolation process involved in the definition of the FE space. This stability result plays the crucial role for the estimates in the H^1-norm of Chapters 3 and 5.

The paper is the first in a sequence of two. A second paper discusses the efficient construction of the generalized FE spaces, the complexity of the method and will include numerical experiments.

2 The Construction of Generalized FE Grids

Composite Finite Elements will be defined in Chapter 3 in an abstract way. There, some geometric assumptions will be imposed on the hierarchy of grids. In order to make these assumptions more transparent we will first present an example of a grid generator and a coarsening algorithm which generates an admissible hierarchy of grids. It turns out that this algorithm carries over to the 3-d case in a straightforward manner (see [10]).

We will present a strategy of generating FE grids on a complicated domain $\Omega \subset \mathbf{R}^2$ which can easily be coarsened to grids which will be related to FE spaces having only very few degrees of freedom. Before presenting the detailed description of the method, we will outline the principal underlying idea. An illustration of the process described below is given in Figure 1. We consider an infinite (virtual) sequence of uniform square grid triangulations $\{\tilde{\tau}_\ell\}_{0 \leq \ell < \infty}$ covering the whole plane $\mathbf{R}^2$. These grids are thought to be nested in the sense that each triangle $\tilde{\Delta} \in \tilde{\tau}_\ell$ has a father on a coarser level and four sons on the finer level, which arise by connecting the midpoints of the edges of $\tilde{\Delta}$. Let us assume that the grid $\tilde{\tau}_{\ell_{\max}}$ is fine enough in the sense that small displacements of grid points in $\tilde{\tau}_{\ell_{\max}}$, which may not destroy the logical connectivity, result in a grid $\tau^{\infty}_{\ell_{\max}}$ having the following property. There is a (finite) subset $\tau_{\ell_{\max}} \subset \tau^{\infty}_{\ell_{\max}}$ which is a proper triangulation of Ω. "Proper" is meant in the sense that standard refinement procedures as, e.g., projecting the midpoint of edges onto the physical boundary, can be applied successfully. We emphasize that $\tau_{\ell_{\max}}$ may not necessarily be the finest grid in the disrcetization process, but can be viewed as the coarsest grid, where standard refinement procedures (including adaptivity) can be applied. A fully adaptive version of the coarsening was presented in [8].

Since we have a one-to-one correspondence of $\tau^{\infty}_{\ell_{\max}}$ and the virtual grid $\tilde{\tau}_{\ell_{\max}}$, coarsening can be performed easily by the following procedure. Let Δ be a triangle of $\tau_{\ell_{\max}}$ and $\tilde{\Delta}$ the corresponding triangle of $\tilde{\tau}_{\ell_{\max}}$. The father of $\tilde{\Delta}$, $\tilde{\Delta}_f \in \tilde{\tau}_{\ell_{\max}-1}$ with vertices $\left\{\tilde{X}_i\right\}_{1\leq i\leq 3}$, is well defined. The vertices $\{X_i\}_{1\leq i\leq 3}$ denote grid points corresponding to $\left\{\tilde{X}_i\right\}_{1\leq i\leq 3}$ arising by adapting the virtual grid to the physical domain. The triangle with vertices $\left\{\tilde{X}_i\right\}_{1\leq i\leq 3}$ is contained in the coarser triangulation $\tau_{\ell_{\max}-1}$. This process can be iterated ending with a coarsest grid τ_0 which consists only of very few triangles. This grid will not have much to do with the domain Ω. However, we will not define standard finite element spaces on these non-fitting grids, but they are only used to connect degrees of freedom with each other. The corresponding finite element space will consist only of functions which are defined on the physical domain. To avoid confusion, we state that the virtual grids $\tilde{\tau}_\ell$ and grids τ_ℓ^∞ are never used in actual computations, because, due to the regularity of them, the positions and connectivity of the triangles are known beforehand.

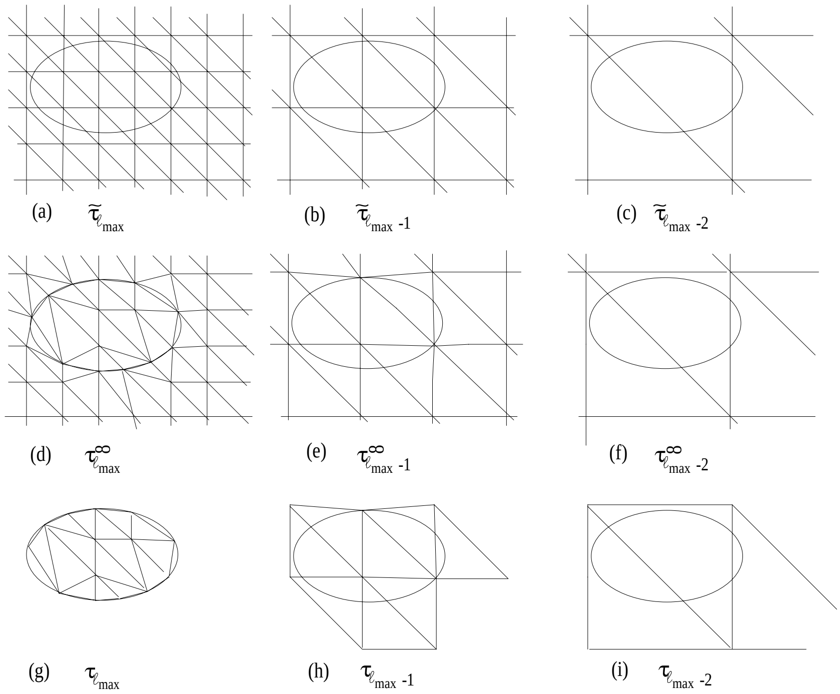

Figure 1: In the first line, the virtual grid $\tilde{\tau}_{\ell_{\max}}$ and coarser grids $\tilde{\tau}_\ell$ are depicted. The grid $\tau^\infty_{\ell_{\max}}$ arise by moving grid points of $\tilde{\tau}_{\ell_{\max}}$ being close to the boundary onto the boundary. Coarser grids as, e.g., $\tau_{\ell_{\max}-1}$ arise by collecting the fathers of triangles in $\tau^\infty_{\ell_{\max}}$, using the logical connection to the uniform reference grid. The triangulation $\tau_{\ell_{\max}}$ which is used for computations consists of triangles which lie "inside" the domain. Coarser triangulations consist of the fathers of triangles on finer levels and cannot be regarded as an approximation of the domain.

2.1 The Hierarchy of Virtual Reference Grids

In this subsection, we will give the precise definition of the sequence of reference grids. In order to indicate that a quantity belongs to the reference grid, we will use a tilde, e.g., $\tilde{\tau}$ for the reference grid and $\tilde{x}$ for a grid point of $\tilde{\tau}$. The corresponding quantities on the true triangulation are denoted by τ, x, etc.

The set $\tilde{\Theta}_\ell$ of vertices is the square grid of size $\tilde{h}_\ell$ given by $\tilde{\Theta}_\ell = \tilde{h}_\ell \mathbf{Z}^2$. We choose an infinite sequence $\left\{\tilde{h}_\ell\right\}_{0\le\ell<\infty}$ of step sizes with $\tilde{h}_\ell = 2\tilde{h}_{\ell+1}$. Consequently, we obtain that the vertex sets form a hierarchy $\left\{\tilde{\Theta}_\ell\right\}_{0\le\ell<\infty}$ satisfying $\Theta_\ell \subset \Theta_{\ell+1}$. The corresponding hierarchy of triangulations $\{\tilde{\tau}_\ell\}_{0\le\ell<\infty}$ is given by the following procedure. Put lines along the co-ordinate axes through the grid points of Θ_ℓ resulting in a Cartesian square grid and insert diagonals through the pairs of points $\tilde{h}_\ell \begin{pmatrix} m \\ 0 \end{pmatrix}$ and $\tilde{h}_\ell \begin{pmatrix} m-1 \\ 1 \end{pmatrix}$, $m \in \mathbf{Z}$. The arising triangles define the grid $\tilde{\tau}_\ell$ (cf. Figure 1(a)-(c)). The triangulations $\tilde{\tau}_\ell$ are nested in a natural way. For any triangle $\tilde{\Delta} \in \tilde{\tau}_\ell$, there exist four sons $\left\{\tilde{\Delta}'_j\right\}_{1\le j\le 4} \in \tilde{\tau}_{\ell+1}$, satisfying $\bigcup_{j=1}^4 \tilde{\Delta}'_j = \tilde{\Delta}$. The triangle $\tilde{\Delta}$ is the father of each $\tilde{\Delta}'_j$, and hence, each triangle in $\tilde{\tau}_\ell$ has a father in $\tilde{\tau}_{\ell-1}$ provided $\ell > 0$.

2.2 Construction of the Fine Grid

Let us assume that the boundary of the domain Ω has to be resolved with a step width $\tilde{h}_{\ell_{\max}}$ and micro-structures being smaller can be neglected. Then, an intermediate grid $\tau^\infty_{\ell_{\max}}$ is defined by moving grid points $\tilde{x} \in \tilde{\Theta}_{\ell_{\max}}$ of the reference grid $\tilde{\tau}_{\ell_{\max}}$ which are close to the boundary, i.e., satisfying $\operatorname{dist}(\tilde{x}, \partial\Omega) \ll h_{\ell_{\max}}$, together with the corresponding edges onto the physical boundary $\partial\Omega$. This procedure defines a one-to-one mapping $\Phi : \tilde{\Theta}_{\ell_{\max}} \to \Theta^\infty_{\ell_{\max}}$. The triangles of $\tau^\infty_{\ell_{\max}}$ are given by the condition:

A triangle with vertices A, B, C belongs to $\tau^\infty_{\ell_{\max}}$, if and only if the triangle with vertices $\Phi^{-1}(A), \Phi^{-1}(B), \Phi^{-1}(C)$ belongs to $\tilde{\tau}_{\ell_{\max}}$.

Thus, any triangle $\tilde{\Delta} \in \tau_{\ell_{\max}}$ is linked to one and only one triangle $\Delta \in \tau^\infty_{\ell_{\max}}$. The corresponding mapping is denoted by $\Phi^\star : \tilde{\tau}_{\ell_{\max}} \to \tau^\infty_{\ell_{\max}}$. Since no confusion is possible, we skip the superscript $\star$.

The following procedure **adapt** illustrates, how the reference grid might be adapted to the domain Ω. The procedure **adapt** is called by

adapt$\left(\tilde{\Theta}_{\ell_{\max}}, \tilde{\tau}_{\ell_{\max}}, \Phi, \Theta^\infty_{\ell_{\max}}, \tau^\infty_{\ell_{\max}}\right)$;

and is defined by

procedure adapt$\left(\tilde{\Theta}, \tilde{\tau}, \Phi, \Theta, \tau\right)$;

Comment This routines generates the adapted triangulation τ and the corresponding set of nodal points Θ.

begin

$\Theta := \tilde{\Theta}; \tau := \tilde{\tau}; \Phi := Identity,$

for each triangle $\tilde{\Delta}$ of $\tilde{\tau}$ **do begin**

$\Delta := \Phi\left(\tilde{\Delta}\right)$;

if $\Delta \cap \partial\Omega \neq \emptyset$ **then begin**

for $i = 1$ **to** 3 **do begin**

Let $e := \overline{x_\mu, x_\lambda}$ be the ith edge of Δ;

if $e \cap \partial\Omega \neq \emptyset$ **then begin**

$A_\eta := \arg \min\limits_{x \in \partial\Omega \cap e} \|x - x_\eta\|$ for $\eta \in \{\mu, \lambda\}$;

Comment Θ and Φ are updated in the following step;

if $\|x_\mu - A_\mu\| \leq \|x_\lambda - A_\lambda\|$ **then** $x_\mu := A_\mu$ **else** $x_\lambda := A_\lambda$;

Comment τ is updated in the following step;

$\tau := \Phi\left(\tilde{\tau}\right)$;

end end end end end.

The result of the procedure **adapt** applied to the triangulation $\tilde{\tau}_{\ell_{\max}}$ is depicted in Figure 1(d).

Note that the algorithm **adapt** is not regarded as a subroutine in an implementation, but as a formal description of the explanations above. In order to obtain the finite grid $\tau_{\ell_{\max}}$ which represents a proper triangulations of the domain Ω, we neglect all triangles, lying essentially outside of the domain.

$$\tau_{\ell_{\max}} = \left\{\Delta \in \tau^\infty_{\ell_{\max}} \mid \text{all vertices of } \Delta \text{ lie in } \bar{\Omega}\right\}.$$

In view of this definition, it is clear, how to modify the procedure **adapt** such that only a finite number of triangles appear. One should consider only those elements of $\tilde{\tau}_{\ell_{\max}}$ which intersects the boundary and construct the corresponding elements of $\tau^\infty_{\ell_{\max}}$ and, then, extending the triangulation over the whole interior of the domain. We skip the algorithmic details, since they will be discussed in a second part of the paper.

2.3 Coarsening of the Fine Grid

Since the grid $\tau^\infty_{\ell_{\max}}$ is linked to the reference grid $\tilde{\tau}_{\ell_{\max}}$ by the mapping Φ, we can use the logical regularity of the reference grid to construct coarser grids τ^∞_ℓ, for $\ell < \ell_{\max}$. We define the mapping Φ_ℓ acting on triangles $\tilde{\Delta} \in \tilde{\tau}_\ell$ by the following conditions. Let $\left\{\tilde{X}_i\right\}_{1\leq i\leq 3}$ denote the vertices of $\tilde{\Delta}$ and $X_i = \Phi\left(\tilde{X}_i\right)$. The triangle with vertices $\{X_i\}_{1\leq i\leq 3}$ is denoted by Δ and we put $\Delta = \Phi_\ell\left(\tilde{\Delta}\right)$. Since no confusion is possible, we skip the index ℓ and simply write Φ. The adapted triangulation τ^∞_ℓ are given by (cf. Figure 1(d)-(f))

$$\tau^\infty_\ell := \Phi\left(\tilde{\tau}_\ell\right) := \left\{\Delta \mid \Phi^{-1}\left(\Delta\right) \in \tilde{\tau}_\ell\right\}.$$

Obviously, the grids τ^∞_ℓ consist of infinitely many triangles and, hence, cannot be used for practical computations. The coarser finite grids τ_ℓ and the corresponding sets of grid points Θ_ℓ, for $\ell < \ell_{\max}$ are defined recursively by

$$\begin{aligned}&\tau_{\ell_{\max}} \text{ is defined as above,}\\ &\Theta_{\ell_{\max}} \text{ consists of all vertices of } \tau_{\ell_{\max}}.\end{aligned}$$

Assume that $\tau_{\ell+1}$ and $\Theta_{\ell+1}$ are given. Then, τ_ℓ is defined by

$$\begin{aligned}\tau_\ell \quad : \quad &= \left\{\Delta \in \tau_\ell^\infty \mid \exists \Delta' \in \tau_{\ell+1} : \Phi^{-1}(\Delta) \text{ is the father of } \Phi^{-1}(\Delta')\right\}\\ &\cup \left\{\Delta \in \tau_\ell^\infty \mid \exists x \in \Theta_{\ell+1} : x \in \mathring{\Delta}\right\}\end{aligned} \tag{1}$$

and Θ_ℓ is the set of all vertices of τ_ℓ.

We will not go further into algorithmic details as, e.g., the application of relaxation strategies to the grids in order to avoid too large angles in triangles, edge swapping, the generation of coarse grid triangulations without generating the full fine grid, etc., but refer to the announced second part of this paper. The main issue of this paper lies in the definition of suitable finite element spaces for such grids and to prove the approximation property. This is done in a more abstract setting, thus, the construction presented in procedure **adapt** can be regarded as an illustration how the abstract assumptions which are made in the following chapters can be satisfied.

3 Composite Finite Element Spaces on $\Omega = \mathbf{R}^2$

In this chapter, we will introduce so-called *Composite Finite Element Spaces* on coarsened finite element grids. We will present the adaption of the uniform, virtual reference grid $\tilde{\tau}_{\ell_{\max}}$ to the true triangulation $\tau_{\ell_{\max}}$ in a more general setting in order to treat adaptation strategies, possibly different from that described in procedure **adapt**, within the same framework. All finite element functions will be defined on the grid $\tau_{\ell_{\max}}$. We recall that in applications $\tau_{\ell_{\max}}$ usually will not be the finest grid but can be viewed as the coarsest grid where standard refinement strategies apply. On the coarser grids τ_ℓ, for $0 \leq \ell < \ell_{\max}$, we will use the nodal points to define grid functions in a purely algebraic way. Then, these vectors are interpolated by using standard finite element interpolation on τ_ℓ in order to define the corresponding grid function on a finer level. Finally, we will get a grid function on $\tau_{\ell_{\max}}$, which will be interpreted as a finite element function by standard prolongation.

The reason for separating the investigation of the case $\Omega = \mathbf{R}^2$ from the case of a bounded domain is to avoid as much as possible technicalities in the presentation of the principal ideas.

We consider here the approximation of functions $u \in H^2 := H^2(\mathbf{R}^2)$ by piecewise linear functions. For this purpose, let $\mathbf{R}^2$ be partitioned into a hierarchy of uniform reference triangulations $\{\tilde{\tau}_\ell\}_{0 \leq \ell < \infty}$ as explained in the previous chapter. We do not restrict ourself to the case that the grid $\tau_{\ell_{\max}}$ has to be generated by the procedure

adapt, but assume in an abstract way that $\Phi : \tilde{\Theta}_{\ell_{\max}} \to \Theta^{\infty}_{\ell_{\max}}$ and $\Phi^{\star} : \tilde{\tau}_\ell \to \tau_\ell^{\infty}$ transfer the reference grid onto the true triangulation. The correspondence of Φ and $\Phi^{\star}$ is the same as explained in the previous chapter. Since no confusion is possible, we skip the superscript $\star$. Since the domain $\Omega = \mathbf{R}^2$, it is not necessary to restrict τ_ℓ^{∞} to a finite triangulation τ_ℓ. Here, we identify τ_ℓ^{∞} with τ_ℓ and skip the superscript ∞.

The triangulations $\{\tau_\ell\}_{0 \leq \ell < \ell_{\max}}$ are not physically nested. However, we will define a logical hierarchy using the physical hierarchy of the reference grid. For this, we have to introduce some notations.

3.1 Notations

Let $H^s(\Omega)$ denote the usual Sobolev spaces as, e.g., defined in the book of Adams (see [1]), equipped with the scalar product $(\cdot,\cdot)_{s,\Omega}$ and norm $\|\cdot\|_{s,\Omega} = \sqrt{(\cdot,\cdot)_{s,\Omega}}$. The seminorm containing only the derivatives of highest order is denoted by $|\cdot|_{s,\Omega}$.

We have to distinguish between a set of triangles and the domain defined by the union of these triangles. For any set of triangles ω, we define $\operatorname{dom}\omega$ by

$$\operatorname{dom}\omega := \bigcup_{\Delta \in \omega} \Delta.$$

Since no confusion is possible, we write $\|v\|^2_{t,\omega}$ instead of $\|v\|^2_{t,\operatorname{dom}\omega}$. On level $\ell + k$, each reference triangle $\tilde{\Delta} \in \tilde{\tau}_\ell$ has 4^k sons characterized by the conditions

$$\begin{aligned} son_\ell^{\ell+k}\left(\tilde{\Delta}\right) &\subset \tau_{\ell+k} \\ \operatorname{dom} son_\ell^{\ell+k}\left(\tilde{\Delta}\right) &= \tilde{\Delta}. \end{aligned}$$

Similarly, we define the sons of a triangle $\Delta \in \tau_\ell$ on level $\ell + k$ as the set

$$son_\ell^{\ell+k}(\Delta) := \Phi\left(son_\ell^{\ell+k}\left(\Phi^{-1}(\Delta)\right)\right)$$

and as an abbreviation

$$\sigma(\Delta) = son_\ell^{\ell_{\max}}(\Delta). \tag{2}$$

The sons of a triangle Δ are **not** nested in the sense that $\Delta = \operatorname{dom}\left(son_\ell^{\ell+k}(\Delta)\right)$ is true in general. A hierarchical structuring is given by $\sigma(\Delta)$ of (2). For all triangles $\Delta \in \tau_\ell$, we obtain

$$\operatorname{dom}\sigma\left(son_\ell^{\ell+k}(\Delta)\right) = \operatorname{dom}\sigma(\Delta)$$

and

$$\operatorname{dom}\sigma(\Delta') \subset \operatorname{dom}\sigma(\Delta), \qquad \forall \Delta' \in son_\ell^{\ell+k}(\Delta).$$

This situation is illustrated in Figure 2.

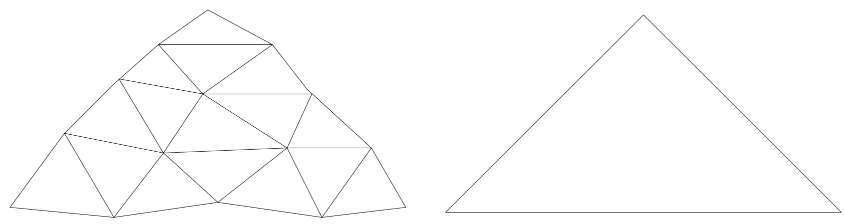

Figure 2: The left picture shows the domain $\operatorname{dom}(\sigma(\Delta))$ of a triangle $\Delta \in \tau_{\ell_{\max}-2}$, while the right one shows Δ.

The father $\mathcal{F}^{\ell}_{\ell+k}(\Delta')$ of a triangle $\Delta' \in \tau_{\ell+k}$ on coarser levels τ_ℓ is defined correspondingly by

$$\mathcal{F}^{\ell}_{\ell+k}(\Delta') = \Delta \Leftrightarrow \Delta' \in son_{\ell}^{\ell+k}(\Delta). \tag{3}$$

Furthermore, we have to associate sets of triangles with the corresponding vertices. For any set of triangles $\omega \subset \tau_\ell$, we define $\mathbf{V}$ by

$$\mathbf{V}(\omega) = \Theta_\ell \cap \bar{\omega}. \tag{4}$$

3.2 Construction of Composite Finite Element Spaces

In order to define the finite element spaces on τ_ℓ, we first have to introduce grid functions which are mappings $\gamma_\ell : \Theta_\ell \to \mathbf{C}$. The space of grid functions on level ℓ is denoted by $\mathbf{C}^{\Theta_\ell}$.

We introduce prolongation operators $P_\ell^{\ell+1} : \mathbf{C}^{\Theta_\ell} \to \mathbf{C}^{\Theta_{\ell+1}}$ by

$$\left(P_\ell^{\ell+1}\gamma_\ell\right)(x) = \left(I_\ell^{int}\gamma_\ell\right)(x), \quad \forall x \in \Theta_{\ell+1},$$

where the interpolation $I_\ell^{int} : \mathbf{C}^{\Theta_\ell} \to \mathcal{C}^0(\mathbf{R}^2)$ is defined by the conditions

$$\begin{aligned} &I_\ell^{int}\gamma_\ell \text{ is affine on each } \Delta \in \tau_\ell, \\ &\left(I_\ell^{int}\gamma_\ell\right)(x) = \gamma_\ell(x) \quad \forall x \in \Theta_\ell. \end{aligned} \tag{5}$$

The prolongation operator P_ℓ, which associates to each grid function $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$ a grid functions on level $\ell_{\max}$, finally is defined by

$$P_\ell := P_{\ell_{\max}-1}^{\ell_{\max}} P_{\ell_{\max}-2}^{\ell_{\max}-1} \cdots P_\ell^{\ell+1}.$$

The interpolation of $P_\ell\gamma_\ell$ at level $\ell_{\max}$ describes the following finite element space

$$S_\ell := \left\{v \in H^1\left(\mathbf{R}^2\right) \mid \exists \gamma_\ell \in \mathbf{C}^{\Theta_\ell} : v = I_{\ell_{\max}}^{int} P_\ell \gamma_\ell\right\}.$$

We will illustrate this definition by characterizing the basis functions of S_ℓ. For simplicity we choose $\ell = \ell_{\max} - 1$. Let γ_ℓ^μ denote the unit vector on τ_ℓ, i.e.

$$\gamma_\ell^\mu(x_\nu) := \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

for all nodal points $x_\nu \in \Theta_\ell$. The affine interpolant of γ_ℓ on the grid τ_ℓ is the standard hat function $\varphi_\mu(x)$ on the grid τ_ℓ. This function $\varphi_\mu(x)$ is now used to define the values of the prolonged unit vector $P_\ell^{\ell+1}\gamma_\ell^\mu$, i.e. ,

$$\left(P_\ell^{\ell+1}\gamma_\ell^\mu\right)(x) = \varphi_\mu(x), \qquad \forall x \in \Theta_{\ell+1}.$$

Finally, the linear interpolant of $P_\ell^{\ell+1}\gamma_\ell^\mu$ is the basis function of S_ℓ corresponding to the nodal point x_μ. The situation is illustrated in Figure 3.

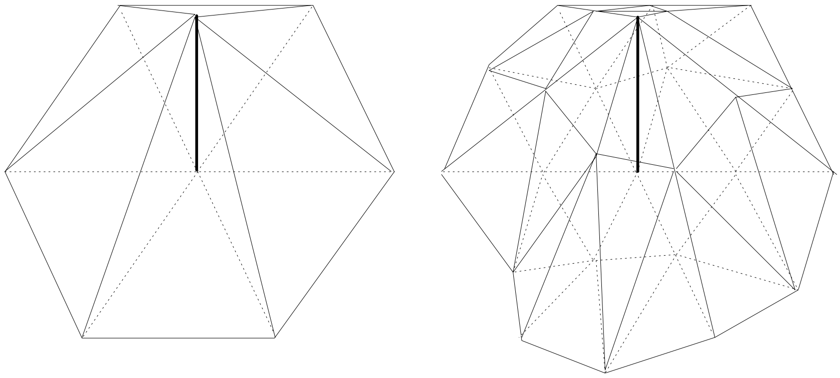

Figure 3: Basisfunction of S_ℓ generated by interpolating the standard basis function in the nodal points of the finer level.

Remark 1 *If the mapping $\Phi : \tilde{\Theta}_{\ell_{\max}} \to \Theta_{\ell_{\max}}$ is the identity, then the space S_ℓ is the standard finite element space on the grid τ_ℓ.*

In any case, the spaces S_ℓ are nested in the sense that $S_j \subset S_k$ for $k > j$.

3.3 Localization of the Interpolation Process

By the linearity of P_ℓ, it follows that, for all $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$, we can write

$$(P_\ell\gamma_\ell)(x) = \sum_{y\in\Theta_\ell} c_y(x)\,\gamma_\ell(y) \tag{6}$$

with some coefficients $c_y(x)$ which are independent of γ_ℓ. The mapping P_ℓ has to be local in the sense that, for the computation of a value $(P_\ell\gamma_\ell)(x)$, only values $\gamma_\ell(y)$ are needed which correspond to grid points y lying close to x. In order to give the formal definition of this, we need the following

Definition 2 *Let $\omega \subset \tau_{\ell_{\max}}$. The set of triangles on level ℓ, which influence the computation of $\{(P_\ell\gamma_\ell)(x)\}_{x\in\mathbf{V}(\omega)}$ in (6) is given by*

$$\mathfrak{I}_\ell(\omega) = \left\{\Delta' \in \tau_\ell \mid \exists y_1, y_2 \in \mathbf{V}(\Delta'), y_1 \neq y_2, x \in \mathbf{V}(\omega) : c_{y_j}(x) \neq 0\right\}. \tag{7}$$

This means that the computation of $P_\ell \gamma_\ell$ in the vertices of the sons of a triangle $\Delta \in \tau_\ell$ on the finest level requires the values of γ_ℓ in the vertices of the influence set $\mathfrak{I}_\ell(\sigma(\Delta))$ which is a subset of τ_ℓ. The definition of $\mathfrak{I}_\ell(\sigma(\Delta))$ is illustrated in Figure 4.

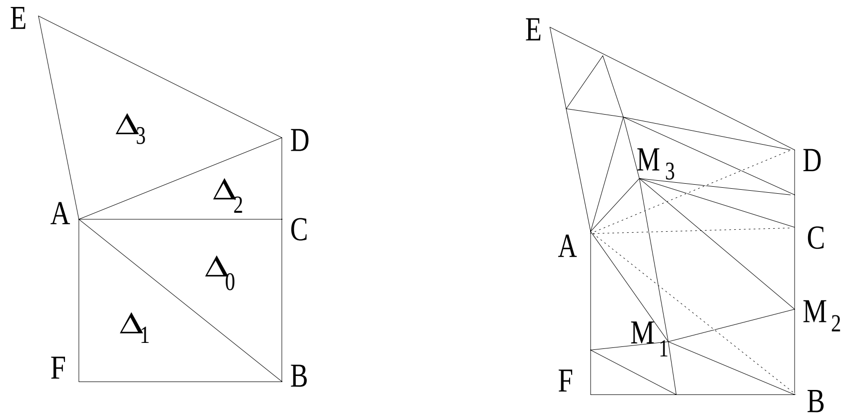

Figure 4: The set $\mathbf{V}(\sigma(\Delta_0))$ consists of the points $\{A, B, C, M_1, M_2, M_3\}$. Since M_1 lies in Δ_1 and M_3 in Δ_3, the computation of the prolongation for points in $\mathbf{V}(\sigma(\Delta_0))$ uses the points $\{A, B, C, D, E, F\}$. Thus, $\mathfrak{I}_\ell(\sigma(\Delta))$ is given by the union of $\{\Delta_j\}, 0 \leq j \leq 3$.

Using this definition, the representation (6) can be localized as

$$P_\ell \gamma_\ell(x) = \sum_{y \in \mathbf{V}(\mathfrak{I}_\ell(\omega))} c_y(x) \gamma_\ell(y), \qquad \forall x \in \mathbf{V}(\omega). \tag{8}$$

We require that the prolongation is local in the following sense.

Assumption 3 *(a) We require that, for all $\Delta \in \tau_\ell$, there are only finitely many triangles $\Delta' \in \tau_\ell$ such that $\mathfrak{I}_\ell(\sigma(\Delta'))$ intersects Δ, i.e.,*

$$\sup_{0 \leq \ell \leq \ell_{\max}} \sup_{\Delta \in \tau_\ell} \# \{\Delta' \in \tau_\ell \mid \Delta \cap \mathfrak{I}_\ell(\sigma(\Delta')) \neq \emptyset\} \leq C_{local}. \tag{9}$$

(b) Furthermore, the number of triangles in $\mathfrak{I}_\ell(\sigma(\Delta'))$ have to be bounded, i.e.,

$$\sup_{0 \leq \ell \leq \ell_{\max}} \sup_{\Delta \in \tau_\ell} \# \mathfrak{I}_\ell(\sigma(\Delta)) \leq C_{\mathfrak{I}}. \tag{10}$$

Obviously, Assumptions (a) and (b) are implicit assumptions on the mapping Φ. If, e.g., Φ is the identity, we obtain $\mathfrak{I}_\ell(\sigma(\Delta)) = \Delta$, $C_{\mathfrak{I}} = 1$ and $C_{local} \leq C(\alpha_0)$, where α_0 denotes the smallest angle of the triangulation τ_ℓ.

Remark 4 *Let $v = I_\ell^{int} P_\ell \gamma_\ell$ and $\Delta \in \tau_\ell$. Then the restriction $v\,|_{\operatorname{dom}\sigma(\Delta)}$ is uniquely determined by the values $\gamma_\ell(x)$ for $x \in \mathbf{V}(\Im_\ell(\sigma(\Delta)))$. For example, $\gamma_\ell(x) = 0$ for all $x \in \mathbf{V}(\Im_\ell(\sigma(\Delta)))$ implies that $v\,|_{\operatorname{dom}\sigma(\Delta)} \equiv 0$.*

The following assumption controls the regularity of the grid and the distortion of triangles by Φ.

Assumption 5 **(a)** *Each triangle $\Delta = \Phi\left(\tilde{\Delta}\right) \in \tau_\ell$ has the same orientation as $\tilde{\Delta} \in \tilde{\tau}_\ell$,*

(b) $h_\ell := \sup_{\Delta \in \tau_\ell} \operatorname{diam}\{\Delta\}$,

(c) $h_\ell \leq C \operatorname{diam}\{\Delta\}, \quad \forall \Delta \in \tau_\ell$, *i.e., τ_ℓ is quasi-uniform, while $\tilde{\tau}_\ell$ is uniform,*

(d) $\sup\{\operatorname{diam} S \mid S$ *is a ball contained in* $\Delta\} \geq C h_\ell, \qquad \forall \Delta \in \tau_\ell$,

(e) $h_\ell \geq (1 + C_{ref})\, h_{\ell+1}$, *with* $1/2 < C_{ref} \leq 1$

while all constants above are positive and independent of Δ and ℓ.

(f) *Let $\Delta \in \tau_\ell$ and $\ell \leq m < \ell_{\max}$. We introduce a parameter which controls the distortion of $\operatorname{dom} son_m^{m+1}(\Delta')$ relative to a triangle $\Delta' \in \Im_m(\sigma(\Delta))$ by*

$$d_m(\Delta) := \max_{\Delta' \in \Im_m(\sigma(\Delta))} \frac{\max\limits_{x \in \operatorname{dom} son_m^{m+1}(\Delta')} \operatorname{dist}(x, \Delta')}{\operatorname{diam} \Delta'}. \tag{11}$$

We assume that Φ is such that for all $\Delta \in \tau_\ell$

$$\sum_{m=\ell}^{\ell_{\max}-1} d_m(\Delta) \leq C \tag{12}$$

is satisfied with a constant C independent of $\ell, \ell_{\max}$, and Δ.

Assumption 5(f) can be interpreted in the following way. Let $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$ denote a grid function. The computation of $\gamma_{\ell_{\max}} := P_\ell \gamma_\ell$ can be split by introducing local intermediate grid function γ_{m+1} for $\ell \leq m \leq \ell_{\max} - 1$ by the recursion

$$\gamma_{m+1}(x) = \sum_{y \in \mathbf{V}(\Im_m(\sigma(\Delta)))} c_y(x)\, \gamma_m(y), \qquad \forall x \in \mathbf{V}(\Im_{m+1}(\sigma(\Delta))).$$

Condition (12) controls the distortion of the triangles of $\Im_m(\sigma(\Delta))$ compared with its sons on the finer level. Later, this will be used in order to prove stability of the interpolation process P_ℓ. Some relations to typical refinement strategies and implications are concerned in the following

Lemma 6 *(a) If the grid $\tau_{\ell_{\max}}$ was constructed by the procedure **adapt,** then, Assumption 5(f) is satisfied.*

(b) Let $\Delta \in \tau_\ell$ be a triangle with an edge $e = \overline{X_1 X_2}$ corresponding to a boundary piece e_Γ of class $\mathcal{C}^2$. Let the midpoint of e be projected onto e_Γ by a refinement procedure resulting in $x \in e_\Gamma$. Then, we obtain

$$\operatorname{dist}(x, \Delta) \leq C h_\ell^2. \tag{13}$$

This assumption implies (12), too.

(c) If Assumption 5(f), is satisfied, we get

$$\frac{|\sigma(\Delta)|}{|\Delta|} \leq C$$

while for any set of triangles ω, $|\omega|$ denotes the area measure of $\operatorname{dom} \omega$.

Proof. By the procedure **adapt** each grid point $\tilde{\Theta}_{\ell_{\max}}$ is moved at most by a distance of $O(h_{\ell_{\max}})$. Let $\tilde{\Delta} \in \tilde{\tau}_\ell$ and $\tilde{e}_j$ an edge of $\tilde{\Delta}$. Let M_j denote the midpoint of this edge. Then, we know that

$$\operatorname{dist}\left(M_j, \tilde{\Delta}\right) = 0.$$

In view of (11), we have to estimate

$$\max_{x \in \operatorname{dom} son_\ell^{\ell+1}(\Delta)} \operatorname{dist}(x, \Delta) = \max_{1 \leq j \leq 3} \operatorname{dist}(\Phi(M_j), \Delta) = \max_{1 \leq j \leq 3} \operatorname{dist}(\Phi(M_j), \Phi(\tilde{e}_j)) \leq 2C h_{\ell_{\max}}$$

and in view of the shape regularity of the triangles, i.e., Assumption 5(a)-(e), we get for any $\Delta \in \tau_\ell$

$$d_m(\Delta) = \max_{\Delta' \in \Im_m(\sigma(\Delta))} \frac{\max_{x \in \operatorname{dom} son_m^{m+1}(\Delta')} \operatorname{dist}(x, \Delta')}{\operatorname{diam} \Delta'} \leq C\left(1 + C_{ref}\right)^{m - \ell_{\max}}.$$

This implies that

$$\sum_{m=\ell}^{\ell_{\max}-1} d_m(\Delta) \leq \sum_{m=0}^{\ell_{\max}-1} C\left(1 + C_{ref}\right)^{m-\ell_{\max}} \leq \frac{C}{C_{ref}}.$$

Estimate (13) is well known and proven by introducing a local coordinate system with origin in the point $\hat{x}$ of e_Γ having maximal distance from e and expanding e_Γ as a Taylor series about $\hat{x}$. Here, we skip the details. It follows that in this case

$$d_m(\Delta) \leq C\left(1 + C_{ref}\right)^{-m}, \qquad \forall \Delta \in \tau_\ell$$

holds and, hence,

$$\sum_{m=\ell}^{\ell_{\max}} d_m(\Delta) \leq \sum_{\ell=0}^{\ell_{\max}} C\left(1 + C_{ref}\right)^{-\ell} \leq \frac{\hat{C}}{C_{ref}}.$$

In order to prove statement (c), we proceed as follows. Let $\omega_\ell := \Delta \in \tau_\ell$ and $\omega_{\ell+j} := \operatorname{dom} son_\ell^{\ell+j}(\Delta)$. In view of the coarsening process we know that $\omega_{\ell+j}$ is a polygon having a boundary which consists of at most $3 \cdot 2^j$ straight lines. Let δ_ℓ be defined by

$$\delta_{\ell+j} := \max_{x \in \omega_{\ell+j}} \operatorname{dist}(x, \omega_{\ell+j-1}).$$

Therefore, we can estimate

$$|\omega_{\ell+j+1}| \leq |\omega_{\ell+j}| + 3 \cdot 2^j \frac{h_{\ell+j}\delta_{\ell+j}}{2}.$$

Let $\delta\ell := \ell_{\max} - \ell$. Inductively, we obtain

$$\begin{aligned} |\omega_{\ell_{\max}}| &\leq |\omega_\ell| + \frac{3}{2}\sum_{j=0}^{\delta\ell-1} 2^j h_{\ell+j}\delta_{\ell+j} \leq |\Delta| + \frac{3}{2}\sum_{j=0}^{\delta\ell-1} 2^j h_{\ell+j}^2 \frac{\delta_{\ell+j}}{h_{\ell+j}} \\ &\leq |\Delta| + \frac{3}{2}h_\ell^2 \sum_{j=0}^{\delta\ell-1}\left(\frac{2}{(1+C_{ref})^2}\right)^j \frac{\delta_{\ell+j}}{h_{\ell+j}} \leq |\Delta| + \frac{3}{2}h_\ell^2 \sum_{j=0}^{\delta\ell} \frac{\delta_{\ell+j}}{h_{\ell+j}}, \end{aligned}$$

since $C_{ref} > \frac{1}{2}$ implies that $\frac{2}{(1+C_{ref})^2} < 1$. Due to the assumption on the shape regularity of the triangles, we obtain

$$|\omega_{\ell_{\max}}| \leq |\Delta| \left(1 + C\sum_{j=0}^{\delta\ell} d_{\ell+j}(\Delta)\right) \leq C\,|\Delta|.$$

■

Remark 7 *In Lemma 6 (a) and (b), it was shown that for two typical refinements strategies, Assumption 5(f) is satisfied. In view of (12), it is clear that it is allowed to do finitely many times (independent of $\ell_{\max}$) any reasonable adaptation process, while the sum (12) will still be bounded. This would include , e.g., edge swapping (see [3]) in the coarsening process or movement of coarse grid points during the coarsening process.*

Remark 8 *For the refinement strategies presented in Lemma 6, we have not used the fact that condition (12) is local. This would be important, if in different regions of the triangulations, the quantities $d_m(\Delta)$ have a different decreasing behaviour with respect to m. Then, using $d_m := \sup_{\Delta\in\tau_m} d_m(\Delta)$ instead of $d_m(\Delta)$ could possibly violate condition (12). For example, swapping of edges could be allowed more often imposing the local condition, provided it takes place in different parts of the triangulation.*

4 Approximation of Functions $u \in H^2(\mathbf{R}^2)$

In this chapter, we will develop the analysis of finite element approximation for functions $v \in H^2(\mathbf{R}^2)$. Throughout this chapter, we will use the notation $H^t := H^t(\Omega)$. In Chapter 5 the case of a bounded domain will be discussed. Here, we will develop an estimate of the approximation error in the form that, for all $v \in H^2$ and $t \in \{0,1\}$, there exists a function $v_\ell \in S_\ell$ such that

$$\|v - v_\ell\|_{t,\mathbf{R}^2} \leq C h_\ell^{2-t} |v|_{2,\mathbf{R}^2}.$$

The error analysis is split into the following steps. For a function $v \in H^2$, we define the restriction operator $R_\ell : \mathcal{C}^0 \to \mathbf{C}^{\Theta_\ell}$ by

$$(R_\ell v)(x) := v(x), \qquad \forall x \in \Theta_\ell. \tag{14}$$

The interpolation operator on the grid τ_ℓ was denoted by I_ℓ^{int}. We recall the definition of the nodal values $\mathbf{V}(son_\ell^m \Delta)$ corresponding to the sons of Δ on level m (see (4)). Let v_m be given by

$$v_m := I_{\ell_{\max}}^{int} P_m R_m v$$

Using the triangle inequality, we obtain

$$\begin{aligned} \|v - v_\ell\|_{0,\sigma(\Delta)}^2 &\leq 2\left(\|v - v_{\ell_{\max}}\|_{0,\sigma(\Delta)}^2 + \sum_{\Delta' \in \sigma(\Delta)} \|v_{\ell_{\max}} - v_\ell\|_{0,\Delta'}^2\right) \\ &\leq 2\left(\|v - v_{\ell_{\max}}\|_{0,\sigma(\Delta)}^2 + \sum_{\Delta' \in \sigma(\Delta)} \max_{x \in \mathbf{V}(\sigma(\Delta))} |(v - v_\ell)(x)|^2 \|1\|_{0,\Delta'}^2\right) \end{aligned}$$

For the first term on the right side above standard error estimates apply. We will show that the pointwise errors, appearing in the second term of the right hand side above, can be estimated by $Ch_\ell \|v\|_{\Im_\ell(\sigma(\Delta))}$ and hence the approximation property in $\mathcal{L}^2$ follows. The stability of the interpolation process in H^1 plays the key role for the H^1-estimate. We will show that

$$|v_{\ell_{\max}}|_1 \leq C |v_\ell|_1$$

is satisfied under moderate assumptions on the refinement (resp. coarsening) process. In combination with the $\mathcal{L}^2$-estimate and the inverse inequality the approximation property in H^1 follows.

In this light, we will assume throughout this and the following chapters that Assumptions 3 and 5 are satisfied.

We begin to estimate the approximation quality of S_ℓ in $\mathcal{L}^2$.

Lemma 9 *Let $u \in H^2$ and $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$ be the interpolating grid function of u:*

$$\gamma_\ell(x) = u(x), \qquad \forall x \in \Theta_\ell.$$

Let $\gamma_{\ell_{\max}} = P_\ell \gamma_\ell$ be the corresponding grid function on the finest level.

Then, for all $\Delta \in \tau_\ell$, the pointwise estimate,

$$\left|\gamma_{\ell_{\max}}(x) - u(x)\right| \leq C h_\ell \left|u\right|_{2,\Im_\ell(\sigma(\Delta))}, \qquad \forall x \in \mathbf{V}(\sigma(\Delta)) \tag{15}$$

is satisfied.

Proof. We define the intermediate grid function $\gamma_{m+1} \in \mathbf{C}^{\Theta_{m+1}}$ arising by $(m+1-\ell)$-times interpolating γ_ℓ:

$$\gamma_{m+1} := P_m^{m+1} P_{m-1}^m \cdots P_\ell^{\ell+1} \gamma_\ell.$$

To compute the value $\gamma_{m+1}(x)$ for a nodal point $x \in \Theta_{m+1}$, one has to determine a triangle $\Delta^m \in \tau_m$ with $x \in \overline{\Delta^m}$. The vertices of Δ^m are denoted by $\{y_j\}_{1 \leq j \leq 3}$. Then

$$\gamma_{m+1}(x) = \sum_{j=1}^{3} \alpha_j(x) \gamma_m(y_j)$$

with some coefficients $\alpha_j(x)$ satisfying

$$\begin{aligned} \alpha_j(x) &\geq 0, \qquad 1 \leq j \leq 3, \\ \sum_{j=1}^{3} \alpha_j(x) &= 1. \end{aligned} \tag{16}$$

Hence, we obtain

$$\begin{aligned} \left|\gamma_{m+1}(x) - u(x)\right| &= \left|\sum_{j=1}^{3} \alpha_j(x) \gamma_m(y_j) - u(x)\right| \\ &\leq \left|\sum_{j=1}^{3} \alpha_j(x) \left(\gamma_m(y_j) - u(y_j)\right)\right| + \left|\sum_{j=1}^{3} \alpha_j(x) u(y_j) - u(x)\right| \\ &\leq \left(\sum_{j=1}^{3} \alpha_j(x)\right) \max_{1 \leq j \leq 3} \left|\gamma_m(y_j) - u(y_j)\right| + \left|\sum_{j=1}^{3} \alpha_j(x) u(y_j) - u(x)\right|. \end{aligned}$$

The linear interpolant u_{int} of u on Δ^m in the vertices $\{y_j\}$ coincides with $\sum_{j=1}^{3} \alpha_j(x) u(y_j)$. Using standard interpolation results, we get (see [4, Theorem 3.1.5])

$$\left|\sum_{j=1}^{3} \alpha_j(x) u(y_j) - u(x)\right| \leq \max_{x \in \Delta^m} \left|u_{int}(x) - u(x)\right| \leq C h_m \left|u\right|_{2,\Delta^m}. \tag{17}$$

Together, we obtain

$$\left|\gamma_{m+1}(x) - u(x)\right| \leq \max_{1 \leq j \leq 3} \left|\gamma_m(y_j) - u(y_j)\right| + C h_m \left|u\right|_{2,\Delta^m}.$$

Let $y_k := \arg\max_{1\leq j\leq 3} |\gamma_m(y_j) - u(y_j)|$ and $y_k \in \overline{\Delta^{m-1}} \in \tau_{m-1}$. The vertices of Δ^{m-1} are denoted by $\{z_j\}_{1\leq j\leq 3}$. Using the same technique as before we get

$$|\gamma_{m+1}(x) - u(x)| \leq \max_{1\leq j\leq 3} |\gamma_{m-1}(z_j) - u(z_j)| + Ch_{m-1} |u|_{2,\Delta^{m-1}} + Ch_m |u|_{2,\Delta^m}.$$

Since $\gamma_\ell(x) = u(x)$ for all $x \in \Theta_\ell$, we get inductively

$$|\gamma_{\ell_{\max}}(x) - u(x)| \leq C \sum_{m=\ell}^{\ell_{\max}-1} h_m |u|_{2,\Delta^m}.$$

It follows that

$$|\gamma_{\ell_{\max}}(x) - u(x)| \leq C |u|_{2,\Im_\ell(\sigma(\Delta))} \sum_{m=\ell}^{\ell_{\max}} h_m \leq h_\ell \frac{C}{C_{ref}} |u|_{2,\Im_\ell(\sigma(\Delta))}, \qquad \forall x \in \mathbf{V}(\sigma(\Delta))$$

with C_{ref} defined in Assumption 5. ■

Using this Lemma, we easily obtain the $\mathcal{L}^2$-estimate of the approximation of a H^2-function by interpolation.

Theorem 10 *Let $v \in H^2$. Then, there exists a function $v_\ell \in S_\ell$ such that*

$$\|v - v_\ell\|_{0,\mathbf{R}^2} \leq CC_{local} h_\ell^2 |v|_{2,\mathbf{R}^2}$$

is satisfied.

Proof. Let $v \in H^2$ and γ_ℓ denote the interpolating grid function:

$$\gamma_\ell(x) = v(x), \qquad \forall x \in \Theta_\ell. \tag{18}$$

In order to define the corresponding finite element function, we first have to prolong γ_ℓ onto the finest grid level:

$$\gamma_{\ell_{\max}} := P_\ell \gamma_\ell \tag{19}$$

and then to interpolate: $v_\ell := I^{int}_{\ell_{\max}} \gamma_{\ell_{\max}}$. The global norm can be decomposed into local norms defined over the patches $\operatorname{dom} \sigma(\Delta)$:

$$\|v - v_\ell\|^2_{0,\mathbf{R}^2} = \sum_{\Delta\in\tau_\ell} \|v - v_\ell\|^2_{0,\sigma(\Delta)}.$$

In the following we will use the convention that

$$\sum_{j\in\Delta} \ldots := \sum_{j:\operatorname{supp}\varphi_j\cap\Delta\neq\emptyset} \ldots.$$

This means that $\{j \in \Delta\}$ denotes the indices of the vertices of Δ. We obtain

$$\begin{aligned}
\|v - v_\ell\|^2_{0,\sigma(\Delta)} &= \sum_{\Delta'\in\sigma(\Delta)} \|v - v_\ell\|^2_{0,\Delta'} \\
&\leq 2 \sum_{\Delta'\in\sigma(\Delta)} \left\| v - \sum_{j\in\Delta'} v(x_j)\varphi_j(x) \right\|^2_{0,\Delta'} + 2 \sum_{\Delta'\in\sigma(\Delta)} \left\| \sum_{j\in\Delta'} (v(x_j) - v_\ell(x_j))\varphi_j(x) \right\|^2_{0,\Delta'}.
\end{aligned}$$

The function $\sum_{j\in\Delta'} v(x_j)\varphi_j(x)$ denotes the linear interpolant of v on Δ'. Therefore we know (see [7, Theorem 8.4.4]) that

$$\left\| v - \sum_{j\in\Delta'} v(x_j)\varphi_j(x) \right\|_{0,\Delta'} \leq C h_{\ell_{\max}}^2 |v|_{2,\Delta'}$$

is fulfilled. Using the fact that $v_\ell(x) = \gamma_{\ell_{\max}}(x)$ for all nodal points on the finest level (see (19)) and the pointwise estimate of the Lemma above, we conclude with

$$\begin{aligned}
\|v - v_\ell\|_{0,\sigma(\Delta)}^2 &\leq 2 \sum_{\Delta'\in\sigma(\Delta)} C^2 h_{\ell_{\max}}^4 |v|_{2,\Delta'}^2 + 2 \sum_{\Delta'\in\sigma(\Delta)} \max_{j\in\Delta'} |\gamma_{\ell_{\max}}(x_j) - v(x_j)|^2 \|1\|_{0,\Delta'}^2 \\
&\leq 2C^2 h_{\ell_{\max}}^4 |v|_{2,\sigma(\Delta)}^2 + 2 \sum_{\Delta'\in\sigma(\Delta)} C^2 h_\ell^2 |v|_{2,\Im_\ell(\sigma(\Delta))}^2 \|1\|_{0,\Delta'}^2 \\
&= 2C^2 h_{\ell_{\max}}^4 |v|_{2,\sigma(\Delta)}^2 + 2C^2 h_\ell^2 |v|_{2,\Im_\ell(\sigma(\Delta))}^2 \|1\|_{0,\sigma(\Delta)}^2 \\
&\leq \hat{C}^2 h_\ell^4 |v|_{2,\Im_\ell(\sigma(\Delta))}^2 .
\end{aligned}$$

For the last estimate we have used Lemma 6 (c). The global estimate follows from

$$\begin{aligned}
\|v - v_\ell\|_{0,\mathbf{R}^2}^2 &= \sum_{\Delta\in\tau_\ell} \|v - v_\ell\|_{0,\sigma(\Delta)}^2 \leq C^2 h_\ell^4 \sum_{\Delta\in\tau_\ell} |v|_{2,\Im_\ell(\sigma(\Delta))}^2 \\
&\leq C^2 C_{local}^2 h_\ell^4 |v|_{2,\mathbf{R}^2}^2 .
\end{aligned}$$

■

The estimate of the error in the H^1 seminorm is more involved. The reason is the following. Let $\gamma_{\ell+1} = P_\ell^{\ell+1}\gamma_\ell$. Let $x \in \Theta_{\ell+1}$ and $x \in \overline{\Delta} \in \tau_\ell$. Then, we obtain

$$\gamma_{\ell+1}(x) = \sum_{y\in \text{vertex of } \Delta} \alpha_y(x)\gamma_\ell(y),$$

and in view of (16), we obtain

$$|\gamma_{\ell+1}(x)| \leq \max_{y\in\text{vertex of }\Delta} \gamma_\ell(y).$$

Thus, the prolongation operator P_ℓ is stable in the maximum norm with constant 1. For the gradients of the interpolation $\gamma_{\ell+1}$ this is not true. For $j \in \{0,1\}$, let $v_{\ell+j} = I_{\ell+j}^{int}\gamma_{\ell+j}$ and $\Delta' \in son_\ell^{\ell+1}(\Delta)$. $\mathcal{N}(\Delta)$ denotes the set of neighbouring triangles of Δ. We will prove the representation

$$\nabla v_{\ell+j}\,|_{\Delta'} = \sum_{\hat\Delta\in\mathcal{N}(\Delta)\cup\Delta} \epsilon_{\Delta',\hat\Delta} \nabla v_\ell\,|_{\hat\Delta},$$

where the singular values of the 2×2 matrices $\epsilon_{\Delta',\hat\Delta}$ are smaller than one and $\sum_{\hat\Delta\in\mathcal{N}(\Delta)\cup\Delta} \epsilon_{\Delta',\hat\Delta} = I$. Unfortunately, an estimate of the form

$$\|\nabla v_{\ell+j}\,|_{\Delta'}\| \leq \max_{\hat\Delta\in\mathcal{N}(\Delta)\cup\Delta} \|\nabla v_\ell\,|_{\hat\Delta}\|$$

is **not** true for all grid functions $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$. Under reasonable assumption we can still prove stability of P_ℓ in the maximum norm, but, since it is rather technical, we postpone the proof to the Appendix. The assertion is stated in the following

Lemma 11 *For any ℓ and any grid function $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$, the estimate*

$$\left|I_m^{int} P_{m-1}^m P_{m-2}^{m-1} \cdots P_\ell^{\ell+1} \gamma_\ell\right|_{1,\mathbf{R}^2} \leq C \left|I_\ell^{int} \gamma_\ell\right|_{1,\mathbf{R}^2}, \qquad \forall \ell \leq m \leq \ell_{\max}$$

is satisfied with a constant independent of ℓ and γ_ℓ, i.e., the interpolation process P_m is stable in H^1.

Using this Lemma, the proof of the approximation property of S_ℓ is straightforward.

Theorem 12 *Let $v \in H^2$. Then, there is a function $p \in S_\ell$ such that*

$$|v - p|_{1,\mathbf{R}^2} \leq C h_\ell |v|_{2,\mathbf{R}^2}.$$

Proof. For a function $v \in H^2$, we set $v_m := I_{\ell_{\max}}^{int} P_m R_m v$ (cf. (14)). We will show that the interpolant $p = v_\ell$ has the asserted approximation property. We know that

$$\begin{aligned} |v - v_\ell|_{1,\mathbf{R}^2} &\leq |v - v_{\ell_{\max}}|_{1,\mathbf{R}^2} + |v_\ell - v_{\ell_{\max}}|_{1,\mathbf{R}^2} \\ &\leq |v - v_{\ell_{\max}}|_{1,\mathbf{R}^2} + \sum_{m=\ell}^{\ell_{\max}-1} |v_m - v_{m+1}|_{1,\mathbf{R}^2}. \end{aligned} \tag{20}$$

Since $v_{\ell_{\max}}$ is the interpolant of v on the grid $\tau_{\ell_{\max}}$, we can apply the standard finite element estimate (see e.g. [7, Theorem 8.4.4]) and obtain

$$|v - v_{\ell_{\max}}|_{1,\mathbf{R}^2} \leq C h_{\ell_{\max}} |v|_{2,\mathbf{R}^2}.$$

We know that $\delta_{m+1} := v_m - v_{m+1}$ belongs to S_{m+1}. Let $\gamma_{m+1} \in \mathbf{C}^{\Theta_{m+1}}$ be the corresponding grid function:

$$\gamma_{m+1}(x) = (v_m - v_{m+1})(x), \qquad \forall x \in \Theta_{m+1}$$

and $\delta_{m+1}^{int} := I_{m+1}^{int} \gamma_{m+1}$ the interpolant on the grid τ_{m+1}. Using Lemma 11, we obtain

$$|\delta_{m+1}|_{1,\mathbf{R}^2} \leq C \left|\delta_{m+1}^{int}\right|_{1,\mathbf{R}^2}. \tag{21}$$

We know that

$$\delta_{m+1}^{int}(x) = 0, \qquad \forall x \in \Theta_m.$$

Similarly as in the proof of Lemma 9, we will show that for each triangle $\Delta \in \tau_m$ the estimate

$$\left|\delta_{m+1}^{int}(x)\right| \leq C h_m |v|_{2,\Im_\ell(\sigma(\Delta))}, \qquad \forall x \in \mathbf{V}\left(son_m^{m+1}(\Delta)\right) \tag{22}$$

holds. For this, let $x \in \mathbf{V}\left(son_m^{m+1}(\Delta)\right)$. Then

$$\left|\delta_{m+1}^{int}(x)\right| = |v_m(x) - v_{m+1}(x)| = |v_m(x) - v(x)|$$

and (22) follows from Lemma 9. Since the triangulation τ_m was assumed to be quasi-uniform, we obtain for each $\Delta \in \tau_m$:

$$\begin{aligned}
\max_{\Delta' \in son_m^{m+1}(\Delta)} \left\| \nabla \delta_{m+1}^{int}|_{\Delta'} \right\| &\leq C h_{m+1}^{-1} \left\| \delta_{m+1}^{int} \right\|_{\mathcal{L}^\infty\left(son_m^{m+1}(\Delta)\right)} \\
&\leq C \frac{h_m}{h_{m+1}} |v|_{2,\mathfrak{I}_\ell(\sigma(\Delta))} \leq \hat{C} |v|_{2,\mathfrak{I}_\ell(\sigma(\Delta))} .
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\delta_{m+1}^{int}\right|_{1,\mathbf{R}^2}^2 &= \sum_{\Delta \in \tau_m} \sum_{\Delta' \in son_m^{m+1}(\Delta)} \left|\delta_{m+1}^{int}\right|_{1,\Delta'}^2 \\
&= \sum_{\Delta \in \tau_m} \sum_{\Delta' \in son_m^{m+1}(\Delta)} \left\| \nabla \delta_{m+1}^{int}|_{\Delta'} \right\|^2 |\Delta'| \\
&\leq \sum_{\Delta \in \tau_m} C |v|_{2,\mathfrak{I}_\ell(\sigma(\Delta))}^2 \left| \operatorname{dom} son_m^{m+1}(\Delta) \right| \\
&\leq C C_{local}^2 h_m^2 |v|_{2,\mathbf{R}^2}^2 . \qquad (23)
\end{aligned}$$

For the last estimate, we have used Lemma 6 (c). Combining (20), (21), and (23), we get

$$\begin{aligned}
|v - v_\ell|_{1,\mathbf{R}^2} &\leq |v - v_{\ell_{\max}}|_{1,\mathbf{R}^2} + \sum_{m=\ell}^{\ell_{\max}-1} |v_m - v_{m+1}|_{1,\mathbf{R}^2} \\
&\leq C h_{\max} |v|_{2,\mathbf{R}^2} + C C_{local} |v|_{2,\mathbf{R}^2} \sum_{m=\ell}^{\ell_{\max}-1} h_m \\
&\leq C \frac{C_{local}}{C_{ref}} h_\ell |v|_{2,\mathbf{R}^2} ,
\end{aligned}$$

yielding the proof. ■

5 Composite Finite Element Spaces on Bounded Domains

In this chapter, we will define Composite Finite Element Spaces S_ℓ on bounded domains. We will prove that, for any function $u \in H^2(\Omega)$, there is a function u_ℓ such that

$$\|u - u_\ell\|_{t,\Omega} \leq C h_\ell^{2-t} \|u\|_{2,\Omega}$$

is satisfied for $t \in \{0, 1\}$. The definition of the spaces will rely on a proper restriction of the adapted grids $\{\tau_\ell\}$ which contains infinitely many triangles to the domain Ω.

Let $\tilde{\tau}_\ell$ denote the reference square grid triangulation as explained in Chapter 2. We recall that the mapping Φ, defined in Chapter 3, adapts the grid points and reference grid $\tilde{\tau}_{\ell_{\max}}$ onto the intermediate grid points $\Theta_{\ell_{\max}}^\infty$ and triangulation $\tau_{\ell_{\max}}^\infty$. The triangulation $\tau_{\ell_{\max}}$ was defined by restricting $\tau_{\ell_{\max}}^\infty$ to the finite domain Ω. The coarser triangualtion τ_ℓ were constructed by using the logical structure of the reference triangulation (see (1)). The domains corresponding to the triangulation τ_ℓ are given by

$$\Omega_\ell := \operatorname{dom} \tau_\ell.$$

We assume here for simplicity that $\Omega = \Omega_{\ell_{\max}}$. Since we assumed that $\tau_{\ell_{\max}}$ is sufficiently close to Ω, we can treat the general case, namely, that $\Omega \neq \Omega_{\ell_{\max}}$ with the standard theory of finite elements on domains with curved boundary.

Since the extremal points of the polygon Ω_ℓ are a subset of Θ_ℓ, condition (1) guarantees that

$$\Omega_0 \supseteq \Omega_1 \supseteq \ldots \supseteq \Omega_{\ell_{\max}} = \Omega. \tag{24}$$

The finite element space is again defined by a suitable prolongation of grid functions. In the case of bounded domains, the space of grid functions consists of all mappings $\gamma_\ell : \Theta_\ell \to \mathbf{C}$, i.e., $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$, where Θ_ℓ is now a finite set. The prolongation operator $P_\ell^{\ell+1} : \mathbf{C}^{\Theta_\ell} \to \mathbf{C}^{\Theta_{\ell+1}}$ is given by

$$P_\ell^{\ell+1} \gamma_\ell (x) := I_\ell^{int} \gamma_\ell (x), \qquad \forall\, x \in \Theta_{\ell+1}, \tag{25}$$

where I_ℓ^{int} is the standard finite element interpolation on the (finite) grid τ_ℓ. Due to condition (1), it is guaranteed that for all $x \in \Theta_{\ell+1}$, there exists a triangle $\Delta \in \tau_\ell$ such that $x \in \bar{\Delta}$. Hence, the interpolation process (25) is well defined. Again, we set $P_\ell := P_{\ell_{\max}-1}^{\ell_{\max}} P_{\ell_{\max}-2}^{\ell_{\max}-1} \cdots P_\ell^{\ell+1}$. The space of composite finite elements on bounded domains is defined by

$$S_\ell := \left\{ v : \mathbf{R}^2 \to \mathbf{C} \mid \exists \gamma_\ell \in \mathbf{C}^{\Theta_\ell} :\; v = I_\ell^{int} P_\ell \gamma_\ell \right\}.$$

Remark 13 *The dimension of the space S_ℓ is given by*

$$\dim S_\ell = \# \Theta_\ell.$$

In the following, we will show that for every function in $H^2(\Omega)$, there exists a function in S_ℓ which satisfies the asymptotic approximation property. This can easily be done by an extension argument and applying the theorems of Chapter 4. The following theorem concerns the existence of an extension operator for functions $u \in H^2(\Omega)$.

Theorem 14 *Let Ω be a domain with Lipschitz boundary. Then, there exists an extension operator $\mathcal{E}$ and a constant C independent of ℓ with the property that for all $0 \leq \ell \leq \ell_{\max}$ and $u \in H^2(\Omega)$:*

$$\begin{aligned} u_{ext} &: = \mathcal{E}u : \Omega_0 \to \mathbf{C}, \\ u_{ext} &|_{\Omega} = u|_{\Omega}, \\ \|u_{ext}\|_{2,\Omega_\ell} &\leq C\|u\|_{2,\Omega}. \end{aligned}$$

Proof. The proof of this theorem is given in the book of Stein [14, p.181, Theorem 5]. ■

The extension theorem is used to construct a function in S_ℓ having the required approximation property.

Theorem 15 *Let Ω be a domain with Lipschitz boundary and $u \in H^2(\Omega)$. Then, there exists a function $u_\ell \in S_\ell$ such that*

$$\|u - u_\ell\|_{t,\Omega} \leq Ch_\ell^{2-t}\|u\|_{2,\Omega}$$

is satisfied for $t \in \{0, 1\}$.

Proof. Let $u \in H^2(\Omega)$ and the extension u_{ext} defined as explained above. Since the inclusion (24) holds, we can define a grid function $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$ by

$$\gamma_\ell(x) = u_{ext}(x), \qquad \forall x \in \Theta_\ell$$

and $u_\ell := I_{\ell_{\max}}^{int} P_\ell \gamma_\ell$ the corresponding finite element function. All estimates in the case of $\Omega = \mathbf{R}^2$ which have been derived in the previous chapter were local in the sense that the error on patch $\sigma(\Delta)$ was bounded by the H^2-seminorm in a local neighbourhood of Δ. If we replace $\sigma(\Delta)$ by $\sigma(\Delta) \cap \Omega_{\ell_{\max}}$, the theorems of Chapter 4 directly apply yielding

$$\|u_{ext} - u_\ell\|_{t,\Omega} \leq Ch_\ell^{2-t}\|u_{ext}\|_{2,\Omega_\ell}$$

for $t \in \{0, 1\}$. Using the fact that $u_{ext}|_\Omega = u|_\Omega$ and the continuity of the extension operator, we get

$$\|u - u_\ell\|_{t,\Omega} \leq Ch_\ell^{2-t}\|u\|_{2,\Omega}.$$

■

6 Final Remarks

In this paper, we have developed Composite Finite Elements in two dimensions. However, the modification of procedure **adapt** to the case of uniform tetrahedral partitionings of $\mathbf{R}^3$ is obvious, where the analysis of the approximation behaviour can be carried over directly.

On bounded domains, we have considered the approximation of functions in H^2 which corresponds to the case of elliptic boundary value problems of second order with Neuman boundary conditions. Dirichlet boundary conditions can be treated by modifying the bilinear form using a penalty term. The details can be found in [8]. The construction of composite finite element spaces satisfying Dirichlet boundary conditions requires a slight modification of the prolongation operators, to ensure that the trial spaces are conforming subspaces of $H_0^1(\Omega) \cap H^2(\Omega)$. The values of prolonged grid functions at the boundary have to be set to zero. Similiar modifications are necessary, if interfaces or changing boundary conditions are present. The analysis of the approximation property has to be modified in these cases and will be presented in a forthcoming paper.

After having computed the stiffness matrix $\mathbf{A}_{\ell_{\max}}$ on the finest grid $\tau_{\ell_{\max}}$, it is easy to derive coarser discretizations by means of the Galerkin product

$$\mathbf{A}_\ell = \left(P_\ell^{\ell+1}\right)^\star \mathbf{A}_{\ell+1} P_\ell^{\ell+1}, \tag{26}$$

where $\left(P_\ell^{\ell+1}\right)^\star$ denotes the adjoint of $P_\ell^{\ell+1}$ with respect to a properly weighted Euclidean scalar product. Since the prolongations were assumed to be local, the complexity of computing the sequence of matrices $\{\mathbf{A}_\ell\}_{0\le\ell\le\ell_{\max}}$, which is needed, e.g., in a multi-grid process, is $O\left(h_{\ell_{\max}}^{-2}\right)$ arithmetical operations. However, the formula (26) is not the only way to compute $\mathbf{A}_\ell$. We state that it is possible to compute the matrix $\mathbf{A}_\ell$ by a complexity of $O\left(h_\ell^{-2} + M_{\ell_{\max}}\right)$, where $M_{\ell_{\max}}$ denotes the numbers of grid points of $\Theta_{\ell_{\max}}$ which have been moved by adapting the reference grid $\tilde{\tau}_{\ell_{\max}}$ to the physical domain. Typically $M_{\ell_{\max}} = O\left(h_{\ell_{\max}}^{-1}\right)$ is satisfied. The algorithmic details, together with a discussion of the complexity, is presented in the announced second part of this paper.

A On the Stability Condition of the Prolongation Operator in H^1

For the proof of the approximation property we have assumed that

$$\left|I_{\ell_{\max}}^{int} P_\ell \gamma_\ell\right|_{1,\mathbf{R}^2} \le C \left|I_\ell^{int} \gamma_\ell\right|_{1,\mathbf{R}^2}, \qquad \forall \gamma_\ell \in \mathbf{C}^{\Theta_\ell} \tag{27}$$

is satisfied. We will proof this condition under Assumption 5 of Chapter 2. Since some technicalities will arise in this chapter, we will outline the principal ideas. Firstly, we will investigate, how piecewise linear functions on a grid τ_ℓ are distorted by the interpolation process defined by P_ℓ. Then, in a second step, we will estimate the growth of the gradients $\nabla I_{\ell_{\max}}^{int} P_\ell \gamma_\ell$ relative to the gradients of $I_\ell^{int}\gamma_\ell$ dependent on the distortion of the nodal points relative to the reference grid. Finally, in a third step, we will use Assumption 5 to obtain an estimate of the form (27).

We have to introduce some notations, namely, the neighbours of a triangle $\Delta \in \tau_\ell$ by

$$\mathcal{N}(\Delta) := \{\Delta' \in \tau_\ell \mid \Delta' \neq \Delta \text{ and } \Delta' \text{ has a common edge with } \Delta\}.$$

We recall the definition of the father of a triangle on coarser levels (see (3)). Let a grid function $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$ be given and $v_\ell(x) := (I_\ell^{int}\gamma_\ell)(x)$ denote the linear interpolant on τ_ℓ. We further define

$$v_{\ell+j}(x) := \left(I_{\ell+j}^{int} P_{\ell+j-1}^{\ell+j} P_{\ell+j-2}^{\ell+j-1} \cdots P_\ell^{\ell+1}\gamma_\ell\right)(x) \tag{28}$$

The gradient of $v_{\ell+j}$ can be expressed by the gradients of $v_{\ell+j-1}$. The details are in the following

Lemma 16 *Let $\Delta' \in son_{\ell+j-1}^{\ell+j}(\Delta)$. Then, the gradient of $v_{\ell+j}$ can be written as*

$$\nabla v_{\ell+j}\mid_{\Delta'} = \nabla v_{\ell+j-1}\mid_{\Delta} + \sum_{\hat{\Delta}\in\mathcal{N}(\Delta)} \epsilon_{\Delta',\hat{\Delta}}\left(\nabla v_{\ell+j-1}\mid_{\hat{\Delta}} - \nabla v_\ell\mid_{\Delta}\right), \tag{29}$$

where $\epsilon_{\Delta',\hat{\Delta}}$ are 2×2 matrices of rank smaller than or equal to one. If Δ' and $\hat{\Delta}$ have disjoint interior, then $\epsilon_{\Delta',\hat{\Delta}} = 0$. The largest singular value $\rho\left(\epsilon_{\Delta',\hat{\Delta}}\right)$ can be estimated as

$$\rho\left(\epsilon_{\Delta',\Delta}\right) \leq C\frac{\max\limits_{x\in son_\ell^{\ell+1}(\Delta)} \operatorname{dist}(x,\Delta)}{\operatorname{diam}\Delta}.$$

Proof. The proof of the Lemma is elementary but technical and can be found in [9, Appendix]. ■

In the following, we will use the Lemma above to estimate the gradients of prolonged grid functions. We recall the definition of the influence set $\mathfrak{I}_\ell$ (see (7)) and representation formula (8). For given $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$, let $v_{\ell+j}$ be defined by (28) and $\Delta' \in \tau_{\ell+j}$. According to the representation formula (29), the gradients $\nabla v_{\ell+j}\mid_{\Delta'}$ can be expressed as a linear combination of the gradients of $\nabla v_{\ell+j-1}$ on the father triangle $\Delta = \mathcal{F}_{\ell+j}^{\ell+j-1}(\Delta')$ and the neighbouring triangles by (29). For a triangle $\Delta' \in \tau_{\ell+j}$, we define those neighbours $\hat{\Delta} \in \mathcal{N}(\Delta)$ which satisfy $\epsilon_{\Delta',\hat{\Delta}}^\ell \neq 0$:

$$\mathring{\mathcal{N}}(\Delta') = \left\{\hat{\Delta} \in \mathcal{N}(\Delta) \mid \epsilon_{\Delta',\hat{\Delta}}^{m-1} \neq 0\right\}.$$

The triangles which are used to compute $\nabla v_{\ell+j}$ are given by

$$\mathcal{C}(\Delta') = \Delta \cup \mathring{\mathcal{N}}(\Delta').$$

Hence, (29) can be rewritten as

$$\nabla v_{\ell+1}\mid_{\Delta'} = \nabla v_\ell\mid_\Delta + \sum_{\hat{\Delta}\in\mathring{\mathcal{N}}(\Delta')} \epsilon_{\Delta',\hat{\Delta}}^\ell\left(\nabla v_\ell\mid_{\hat{\Delta}} - \nabla v_\ell\mid_\Delta\right). \tag{30}$$

This representation will be used to estimate $\nabla v_{\ell+1}\mid_{\Delta'}$. The details are in the following

Theorem 17 *We use the notation of Lemma 16. For $\Delta \in \tau_\ell$ and $\ell \le m < \ell_{\max}$, let $d_m(\Delta)$ be defined by (11). The function v_ℓ was given by (28). For $\Delta' \in \sigma(\Delta)$, the gradients of v_ℓ on the finest level can be estimated as*

$$\left\|\nabla v_{\ell_{\max}}|_{\Delta'}\right\| \le \prod_{m=\ell}^{\ell_{\max}-1}\left(1+6d_m(\Delta)\right)\max_{\hat{\Delta}\in\mathfrak{I}_\ell(\sigma(\Delta))}\left\|\nabla v_\ell|_{\hat{\Delta}}\right\|. \tag{31}$$

Proof. Let $\Delta \in \tau_\ell$ and $\Delta' \in son_\ell^{\ell+1}(\Delta)$. Using (29), we obtain

$$\begin{aligned}\left\|\nabla v_\ell|_{\Delta'}\right\| &\le \left(1+\sum_{\hat{\Delta}\in\overset{\circ}{\mathcal{N}}(\Delta')}\rho\left(\epsilon^\ell_{\Delta',\hat{\Delta}}\right)\right)\left\|\nabla v_{\ell-1}|_{\Delta}\right\| + \sum_{\hat{\Delta}\in\overset{\circ}{\mathcal{N}}(\Delta')}\rho\left(\epsilon_{\Delta',\hat{\Delta}}\right)\left\|\nabla v_{\ell-1}|_{\hat{\Delta}}\right\| \\ &\le \left(1+6\max_{\hat{\Delta}\in\overset{\circ}{\mathcal{N}}(\Delta')}\rho\left(\epsilon^\ell_{\Delta',\hat{\Delta}}\right)\right)\max_{\hat{\Delta}\in\mathcal{C}(\Delta')}\left\|\nabla v_{\ell-1}|_{\hat{\Delta}}\right\|.\end{aligned}$$

Now, let $\Delta' \in \sigma(\Delta)$. Using Lemma 16, we get by induction

$$\begin{aligned}\left\|\nabla v_{\ell_{\max}}|_{\Delta'}\right\| &\le \left(1+6d_{\ell_{\max}-1}(\Delta)\right)\max_{\hat{\Delta}\in\mathcal{C}(\Delta')}\left\|\nabla v_{\ell_{\max}-1}|_{\hat{\Delta}}\right\| \\ &\le \left(1+6d_{\ell_{\max}-1}(\Delta)\right)\left(1+6d_{\ell_{\max}-2}(\Delta)\right)\max_{\hat{\Delta}\in\mathcal{C}(\Delta')}\max_{\hat{\hat{\Delta}}\in\mathcal{C}(\hat{\Delta})}\left\|\nabla v_{\ell_{\max}-1}|_{\hat{\hat{\Delta}}}\right\| \\ &\le \prod_{r=\ell}^{\ell_{\max}-1}\left(1+6d_r(\Delta)\right)\max_{\hat{\Delta}\in\mathfrak{I}_\ell(\sigma(\Delta))}\left\|\nabla v_{\ell_{\max}-1}|_{\hat{\Delta}}\right\|,\end{aligned}$$

since the iterated maxima appearing in the induction, namely

$$\max_{\hat{\Delta}\in\mathcal{C}(\hat{\Delta})}\max_{\hat{\hat{\Delta}}\in\mathcal{C}(\hat{\Delta})}\cdots,$$

are by definition the maximum over a subset of the influence set $\mathfrak{I}_\ell(\sigma(\Delta))$. ■

In view of (31), we will assume an estimate of the form

$$\left\|\nabla v_{\ell+j}|_{\Delta'}\right\| \le C_{\ell,j}\max_{\Delta'\in\mathfrak{I}_\ell(\sigma(\Delta))}\left\|\nabla v_\ell|_{\Delta'}\right\| \tag{32}$$

with $\Delta' \in son_\ell^{\ell+j}(\Delta)$ to estimate the H^1-seminorm of $v_{\ell+j}$. The details are in the following

Lemma 18 *Let us assume that (32) is true. Then,*

$$\left|v_{\ell+j}\right|_{1,son_\ell^{\ell+j}(\Delta)} \le \hat{C}C_{\ell,j}\left|v_\ell\right|_{1,\mathfrak{I}_\ell(\sigma(\Delta))}$$

is satisfied.

Proof. Let $\Delta \in \tau_\ell$ and consider the triangles of $son_\ell^{\ell+j}(\Delta)$. Then, we obtain

$$\begin{aligned}
|v_{\ell+j}|^2_{1,son_\ell^{\ell+j}(\Delta)} &= \sum_{\Delta' \in son_\ell^{\ell+j}(\Delta)} |\nabla v_{\ell+j}|^2_{1,\Delta'} = \sum_{\Delta' \in son_\ell^{\ell+j}(\Delta)} \left\| \nabla v_{\ell+j} \mid_{\Delta'} \right\|^2 |\Delta'| \\
&\leq C^2_{\ell,j} \left(\max_{\hat{\Delta} \in \mathfrak{I}_\ell(\sigma(\Delta))} \left\| \nabla v_\ell \mid_{\hat{\Delta}} \right\| \right)^2 \left| \operatorname{dom} son_\ell^{\ell+j}(\Delta) \right| \\
&\leq C^2_{\ell,j} \left| \operatorname{dom} son_\ell^{\ell+j}(\Delta) \right| \sum_{\hat{\Delta} \in \mathfrak{I}_\ell(\sigma(\Delta))} \left\| \nabla v_\ell \mid_{\hat{\Delta}} \right\|^2 \\
&= C^2_{\ell,j} \left| \operatorname{dom} son_\ell^{\ell+j}(\Delta) \right| \sum_{\hat{\Delta} \in \mathfrak{I}_\ell(\sigma(\Delta))} \frac{1}{\left|\hat{\Delta}\right|} |v_\ell|^2_{1,\hat{\Delta}}.
\end{aligned}$$

Due to the quasi-uniformity of the grid, we know that

$$\frac{1}{\left|\hat{\Delta}\right|} \leq C \frac{1}{|\Delta|}, \qquad \forall \hat{\Delta} \in \mathfrak{I}_\ell(\sigma(\Delta)).$$

In Lemma 6, it was shown that Assumption 5 implies that $\frac{\left|\operatorname{dom} son_\ell^{\ell+j}(\Delta)\right|}{|\Delta|} \leq C$. Consequently, we obtain

$$|v_{\ell+j}|_{1,son_\ell^{\ell+j}(\Delta)} \leq C C_{\ell,j} |v_\ell|_{1,\mathfrak{I}_\ell(\sigma(\Delta))}.$$

■

An immediate consequence of this Lemma is the global estimate.

Theorem 19 *Let $\gamma_\ell \in \mathbf{C}^{\Theta_\ell}$ be given and v_ℓ, $v_{\ell+j}$ be defined by (28). Then,*

$$|v_{\ell+j}|_{1,\mathbf{R}^2} \leq C C_{local} C_{\ell,j} |v_\ell|_{1,\mathbf{R}^2}. \tag{33}$$

Proof. This follows directly from Lemma 18 with the constant C_{local} defined by (9). ■

Obviously, a sufficient condition for an estimate of the form

$$|v_{\ell+j}|_{1,\mathbf{R}^2} \leq C |v_\ell|_{1,\mathbf{R}^2}$$

with a constant C independent of ℓ and j is that $C_{\ell,j}$ does not depend on ℓ and j. We will show that Assumption 5 implies that $C_{\ell,j} < C$. Condition (5) of Assumption 5 reads

$$\sum_{\ell=0}^{\ell_{\max}} d_\ell(\Delta) \leq C. \tag{34}$$

Let $\Delta \in \tau_\ell$. Hence,

$$\begin{aligned}\prod_{\ell=0}^{\ell_{\max}-1} \left(1 + 6 d_\ell\left(\Delta\right)\right) &\leq \exp\left\{\sum_{\ell=0}^{\ell_{\max}-1} \log\left(1 + 6\hat{C} d_\ell\right)\right\} \\ &\leq \exp\left\{\sum_{\ell=0}^{\ell_{\max}-1} 6\hat{C} d_\ell\right\} \leq e^{6\hat{C}C}.\end{aligned}$$

Condition (34) was guaranteed for the refinement strategies presented in Lemma 6 and Remark 7, and thus, result in the stability estimate of Theorem 19 and finally, in the required approximation property as has been worked out in the previous chapters.

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