Composite Finite Elements for problems with complicated boundary. Part III: Essential Boundary Conditions

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Abstract

In this paper, we define a new class of conforming finite elements for the approximation of functions in $H_0^1(\Omega)$. In contrast to standard finite elements, the approximation property can be proved without any restrictions on the (minimal) dimension of these so-called composite finite element spaces also for very complicated domains. Therefore, this class of finite elements can be used for coarse-level discretizations of PDEs on complicated domains.

1 Introduction

In [8], a new class of finite elements for the discretization of partial differential equations on complicated domains has been introduced. Such kinds of problems typically arise in environmental modelling, porous media, modelling of complicated technical engines, etc. In principle, these problems can be treated with standard finite elements as well. However, the usual requirement, namely, that the finite element grid has to resolve the boundary makes a coarse-scale discretization impossible. Every reasonable discretization will contain a huge number of unknowns being directly linked to the number of geometric details of the physical domain. On the other hand, the efficiency of many numerical solvers, e.g., multi-grid methods, extrapolation, and wavelets are based on a multi-scale discretization of the domain containing low-dimensional levels as well.

The *composite finite elements* introduced in [8] allow coarse-level discretizations of partial differential equations, where the minimal number of unknowns is independent of the number and size of geometric details. For functions in $H^k(\Omega)$, the approximation property is proved in an analogue generality as established for standard finite elements (see [8], [6], [10]).

Here, we will introduce composite finite elements for problems with Dirichlet boundary conditions. To be more concrete, the aim of this paper is to set up a family of finite elements which satisfy the approximation property for functions in

$H^k(\Omega) \cap H_0^1(\Omega)$, where the domain Ω, possibly, has a complicated boundary. The minimal dimension of this finite element space will be independent of the size and number of geometric details.

2 Composite Finite Elements

The definition of composite finite elements is based on a sequence of grids. In contrast to standard finite element grids, only the *finest* grid has to resolve the boundary. On the finest grid, the composite finite element space coincides with the usual finite element space. All lower dimensional spaces are *subspaces* of the fine grid space. The definition of these subspaces is based on the principle that values at *coarse grid points* are prolonged to values at the nodal points of the *finest grid*. The nodal interpolation of these fine grid values defines a function of the coarse grid space.

In the next section, we will define composite finite element grids.

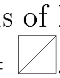
2.1 Composite Finite Element Grids

To explain the principle ideas we avoid at this point the most general definition of composite finite element grids and will explain more general situations at the end of this chapter. In this light, we assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain with boundary $\Gamma := \partial\Omega$. The definition of composite finite element grids consists of three steps.

(1) In the first step, a hierarchy of *reference* grids will be defined. The domain Ω has to be contained in the domain covered by the grids but it is *not* required that the boundary Γ is resolved. To indicate that a quantity belongs to the reference grid we will use a $\tilde{\ }$.

(2) In the second step, the finest grid will be adapted to the boundary by moving fine grid points, lying close to Γ, onto the boundary. The arising grids and quantities will be indicated by a superscript $^\infty$.

(3) Finally, we remove all triangles lying outside the domain from the grids.

We come now to the formal definition (see Figure 1). As reference grids, we use a sequence of uniform triangulations of $\mathbb{R}^2$. To be more concrete, consider the following partition of the unit square $T =$ . The translates and dilations are denoted by $T_{x,h} := h(x+T)$. For $\ell \in \mathbb{N}_0$, let $\tilde{h}_\ell := 2^{-\ell}$. The grid $\tilde{\tau}_\ell$ is given by the partition of $\mathbb{R}^2$ into $T_{x,\tilde{h}_\ell}$ where x are the Cartesian grid points $\mathbb{Z}^2$. The set of vertices of $\tilde{\tau}_\ell$ is denoted by $\tilde{\Theta}_\ell$. Throughout this paper triangles are always considered as *open* sets.

These grids are logically and physically nested, i.e., for each $\tilde{K} \in \tilde{\tau}_\ell$, there exist $4^{(\ell'-\ell)}$ members $\tilde{K}' \in \tilde{\tau}_{\ell'}$, $\ell' \geq \ell$, satisfying

$$\tilde{K}' \subset \tilde{K}.$$

This property motivates the definition of a parent/child relation:

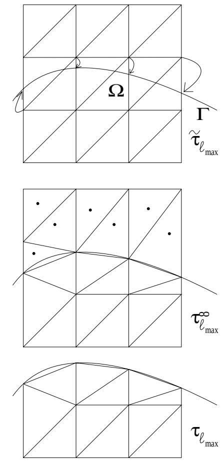


Figure 1: Grid generation: We start with the uniform reference grid on the finest level $\tilde{\tau}_{\ell_{\max}}$. Grid points lying close to the boundary are moved onto Γ. This results in the intermediate grid $\tau^{\infty}_{\ell_{\max}}$. Rejecting all elements lying (essentially) outside the domain (indicated with a black dot) results in the composite finite element grid $\tau_{\ell_{\max}}$.

Definition 1 *For $\tilde{K} \in \tilde{\tau}_\ell$, the set of children of $\tilde{K}$ on level $\ell' \geq \ell$ is denoted by $\sigma_\ell^{\ell'}\left(\tilde{K}\right)$ and defined by*

$$\sigma_\ell^{\ell'}\left(\tilde{K}\right) := \left\{\tilde{K}' \in \tilde{\tau}_{\ell'} \mid \tilde{K}' \subset \tilde{K}\right\}.$$

On the other hand, the parent of an element $\tilde{K}' \in \tilde{\tau}_{\ell'}$ on a coarser level $\ell \leq \ell'$ is denoted by $f_{\ell'}^{\ell}\left(\tilde{K}'\right)$ and defined by

$$f_{\ell'}^{\ell}\left(\tilde{K}'\right) = \tilde{K} \Leftrightarrow \tilde{K}' \in \sigma_\ell^{\ell'}\left(\tilde{K}\right).$$

These reference grids $\tilde{\tau}_\ell$ will now be adapted to the domain in the following way. Grid points of the finest grid lying close to the boundary Γ are moved onto Γ. For an element $\tilde{K} \in \tilde{\tau}_\ell$, the set of edges is denoted by $\mathbf{E}(\tilde{K})$. We formulate the adaption algorithm in a pseudo-computer language. For this, let $tol > 0$ be a user-specified tolerance reflecting the size of the geometric details of Ω which have to be resolved by the grid. Choose $\ell_{\max}$ such that $\tilde{h}_{\ell_{\max}} \approx tol$ holds. Then, the adaption is performed by the **procedure adapt**:

procedure adapt;
begin
    for all $\tilde{K} \in \tilde{\tau}_{\ell_{\max}}$ **do for all** $\tilde{e} \in \mathbf{E}\left(\tilde{K}\right)$ **do**
        if $\tilde{e} \cap \partial\Omega \neq \emptyset$ **then**
            replace one of the endpoints of $\tilde{e}$ by an appropriate boundary point;
end;

At this point we are not very precise which endpoint of $\tilde{e}$ is preferably moved onto the boundary and which boundary point has to be picked for replacing the endpoint. The reason is that this algorithm was already presented in [7] and [10] (cf. Figure 2). Here, the main concern is the definition of the finite element spaces on these grids for

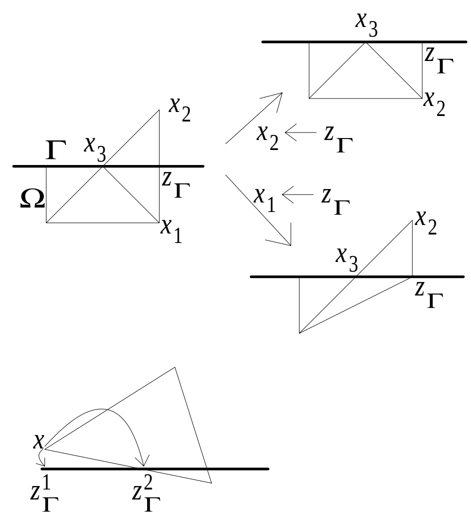


Figure 2: First line: Replacement of x_2 by z_Γ leads to more favorable interior angles of the arising triangles than replacement of x_1 by z_Γ. Second line: Replacement by the closest boundary point (z_1^Γ) might result in better triangles than replacement by the intersection point $z_\Gamma^2 \in e \cap \Gamma$.

problems with Dirichlet boundary conditions.

We emphasize that the **procedure adapt** also changes the shape of coarse grid triangles. If, e.g., a point $\tilde{x} \in \tilde{\Theta}_{\ell_{\max}} \cap \tilde{\Theta}_\ell$ is replaced by a point x^∞, then, $\tilde{x}$ has to be interchanged by x^∞ on the coarse grid as well. The resulting grids are denoted by τ_ℓ^∞, $0 \leq \ell \leq \ell_{\max}$. The movement of grid points defines a mapping $\Phi_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ by the following conditions

1. If a grid point $\tilde{x} \in \tilde{\Theta}_\ell$ is replaced by x^∞ we put $\Phi_\ell(x^\infty) = \tilde{x}$; if $\tilde{x}$ remains unchanged we put $\Phi_\ell(\tilde{x}) = \tilde{x}$.

2. For all $K^\infty \in \tau_\ell^\infty$, the restriction $\Phi_\ell|_{K^\infty}$ is affine-linear.

For the proof of the approximation property, we have to make appropriate assumptions on the compatibility of $\tilde{\tau}_\ell$ and τ_ℓ^∞. This can conveniently be expressed by assumptions on the mappings Φ_ℓ. However, for the definition of composite finite elements, these assumptions are not essential and therefore postponed to Chapter 3. Although the grid τ_ℓ^∞ is no longer physically nested, we adopt the parent/child-relation from the reference grids:

$$\sigma_\ell^j(K^\infty) := \Phi_j^{-1}\sigma_\ell^j\Phi_\ell(K^\infty) := \left\{\Phi_j^{-1}\left(\tilde{K}'\right) : \tilde{K}' \in \sigma_\ell^j(\Phi_\ell(K^\infty))\right\}.$$

The composite finite element grids are now defined by rejecting all elements lying outside the domain Ω :

$$\tau_\ell := \{K \in \tau_\ell^\infty \mid K \cap \Omega \neq \emptyset\}. \tag{1}$$

The parent/child relation of the grid τ_ℓ^∞ is adopted to τ_ℓ by

$$\sigma_\ell^j(K) \leftarrow \sigma_\ell^j(K) \cap \tau_j.$$

The domain Ω was assumed to be polygonal, hence, the following assumption can be interpreted as an assumption on the size of $\tilde{h}_{\ell_{\max}}$ and on the mapping Φ_ℓ which describes the movement of grid points. Throughout this paper, we assume that

$$\Omega_{\ell_{\max}} = \Omega$$

holds. This completes the definition of composite finite element grids. In the next section, we will define composite finite element spaces on these grids.

2.2 Composite Finite Element Spaces

First, we recall the definition of (standard) finite element spaces. Let τ_ℓ be a conforming triangulation, i.e., two elements K, K' either coincide or the intersection $\overline{K} \cap \overline{K'}$ is either empty, a common edge, or a common point. Then, the (standard) finite element space is given by

$$S_\ell := \{u \in C(\Omega_\ell) \mid \forall K \in \tau_\ell : u\,|_K \text{ is affine-linear}\}.$$

The nodal basis corresponding to a grid point $y \in \Theta_\ell$ is denoted by φ_y^ℓ :

$$\begin{aligned} \varphi_y^\ell &\in S_\ell, \\ \varphi_y^\ell(x) &= \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \in \Theta_\ell \setminus \{y\}. \end{cases} \end{aligned}$$

The composite finite element space can be regarded as an adaption of the space S_ℓ to the boundary. On the finest grid, the assumption $\Omega_{\ell_{\max}} = \Omega$ motivates the following definition

$$S_{\ell_{\max}}^{CFE} := S_{\ell_{\max}} \cap H_0^1(\Omega).$$

All coarse-level spaces S_ℓ^{CFE} will be subspaces of the fine grid space $S_{\ell_{\max}}^{CFE}$. The definition of appropriate prolongation operators $I_{\ell_{\max},\ell} : S_\ell \to S_{\ell_{\max}}^{CFE}$ will play the key role for the definition of S_ℓ^{CFE}:

Definition 2 *Let a suitable operator $I_{\ell_{\max},\ell} : S_\ell \to S_{\ell_{\max}}^{CFE}$ be given. Then, the composite finite element space for the approximation of functions with zero traces is defined by*

$$S_{\ell_{\max}}^{CFE} := S_{\ell_{\max}} \cap H_0^1(\Omega), \qquad \text{for } \ell = \ell_{\max}$$

and, for $\ell < \ell_{\max}$, by

$$S_\ell^{CFE} := \mathit{Range}(I_{\ell_{\max},\ell}) = \{u \in C(\Omega) \mid \exists u_\ell \in S_\ell : u = I_{\ell_{\max},\ell}[u_\ell]\}. \tag{2}$$

The rest of this section is concerned with the definition of an appropriate prolongation operator $I_{\ell_{\max},\ell}$. It turns out that the definition of $I_{\ell_{\max},\ell}$ can be reduced to the definition of single-step prolongations $I_{\ell+1,\ell}: S_\ell \to S_{\ell+1}$. For $i > \ell$, we define the multi-step (or iterated) prolongation $I_{i,\ell}$ by the composition:

$$I_{i,\ell} = I_{i,i-1} I_{i-1,i-2} \cdots I_{\ell+1,\ell}. \tag{3}$$

Formally, we define $I_{\ell,\ell}$ as the identity operator. We distinguish two cases. In the interior of the grid τ_ℓ, the operator $I_{\ell+1,\ell}$ will be nothing but the standard nodal interpolation (see (7)). However, near the boundary, $I_{\ell+1,\ell}$ has to be modified such that zero boundary conditions are satisfied (at least on the finest grid $\tau_{\ell_{\max}}$). To measure the distance of an element $K \in \tau_\ell$ from the boundary we define a special distance function by counting the minimal numbers of elements necessary to connect the boundary and K. The formal definition is given below and illustrated in Figure 3.

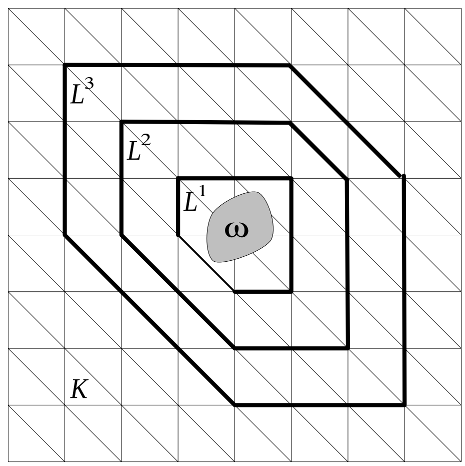


Figure 3: Layers around a set ω. The elementwise distance of K and ω is $\operatorname{dist}_\tau(K,\omega)$.

Definition 3 *For a set T of triangles, we define the domain covered by the triangles of T by*

$$\operatorname{dom} T = int \overline{\bigcup_{K \in T} K},$$

where $int(M)$ denotes the interior of a set M. Let $\omega \subset \mathbb{R}^2$. For $i \in \mathbb{N}_+$, we define layers $L_T^i(\omega) \subseteq T$ around ω via the recursion:

$$\begin{aligned} L_T^1(\omega) \quad &: \quad = \left\{ K \in T \mid \overline{K} \cap \overline{\omega} \neq \emptyset \right\}, \\ L_T^{i+1}(\omega) \quad &: \quad = L_T^1\left(\operatorname{dom} L_T^i(\omega)\right). \end{aligned}$$

The elementwise distance of two subset $\omega_1, \omega_2 \subseteq \operatorname{dom} T$ *is defined by*

$$\operatorname{dist}_T(\omega_1, \omega_2) := \min\left\{ i : \overline{\omega_2} \cap \overline{\operatorname{dom} L_T^i(\omega_1)} \neq \emptyset \right\}.$$

We emphasize that, in practical implementations, it is a much simpler task to compute the elementwise distance of two sets compared to the computation (or approximation) of the Euclidean distance of the sets. Furthermore, it turns out that, for the local analysis of the approximation error, it is essential to work with the elementwise distance to include adaptive refinement.

For the definition of $I_{\ell+1,\ell}$, we have to specify first appropriate subsets of τ_ℓ (see Figure 4).

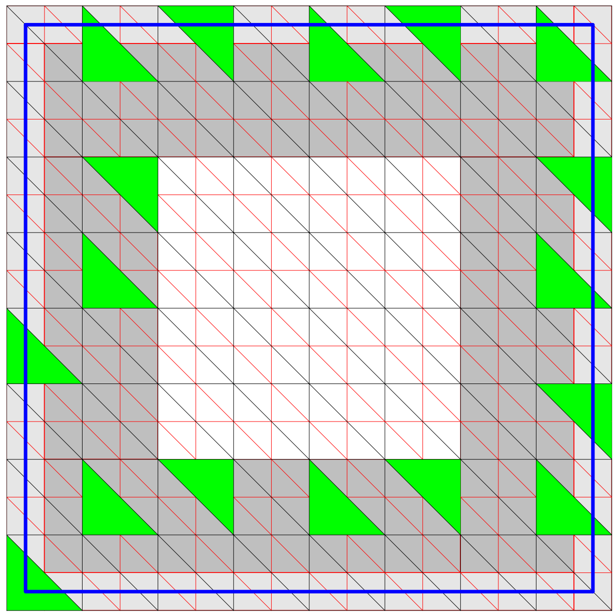


Figure 4: Ω is the square indicated by the blue boundary. The triangles on level $\ell = \ell_{\max} - 2$ are depicted with black, the triangles on level $\ell + 1$ with red lines. The light shaded region shows the near-boundary grid $\tau_{\ell+1}^\Gamma$. The triangles of τ_ℓ in the light and dark shaded region form the grid τ_ℓ^P being close to $\tau_{\ell+1}^\Gamma$. The green shaded triangles form a maximal independent set τ_ℓ^{ext} of τ_ℓ^P.

Definition 4 *Let* τ_ℓ *be a composite finite element grid as defined in the previous section (see (1)). The "near-boundary grid"* τ_ℓ^Γ *and the "interior grid"* $\mathring{\tau}_\ell$ *are defined by*

$$\begin{array}{lll} \tau_\ell^\Gamma = \tau_\ell \cap L_\ell^{b_\Gamma}(\Gamma), & \Omega_\ell^\Gamma := \operatorname{dom} \tau_\ell^\Gamma, & \Theta_\ell^\Gamma := \Theta_\ell \cap \overline{\Omega_\ell^\Gamma}, \\ \mathring{\tau}_\ell := \tau_\ell \backslash \tau_\ell^\Gamma, & \mathring{\Omega}_\ell := \operatorname{dom} \mathring{\tau}_\ell, & \mathring{\Theta}_\ell := \Theta_\ell \backslash \Theta_\ell^\Gamma \end{array} \tag{4}$$

with

$$b_\Gamma = 1. \tag{5}$$

The grid τ_ℓ^P *contains those coarse grid triangles which lie "close" to the near-boundary grid of the finer level:*

$$\tau_\ell^P := \left\{ K \in \tau_\ell^\Gamma \mid \operatorname{dist}_{\tau_{\ell+1}^\infty} \left(K, \Omega_{\ell+1}^\Gamma \right) \leq 1 \right\} \tag{6}$$

and τ_ℓ^{ext} *is a maximal independent subset of* τ_ℓ^P *:* [1]

$$\tau_\ell^{ext} := \operatorname{argmax} \left\{ \# \tau : \tau \subset \tau_\ell^P \mid \forall K, K' \in \tau_\ell^P : \textit{either } K = K' \textit{ or } \overline{K} \cap \overline{K'} = \emptyset \right\}.$$

Let, in the following, $u_\ell \in S_\ell$ be a standard finite element function. We will define the operator $I_{\ell+1,\ell}$ by explaining how the function $u_{\ell+1,\ell} = I_{\ell+1,\ell}[u_\ell]$ is computed.

First, the nodal values $\{u_{\ell+1,\ell}(y)\}_{y \in \Theta_{\ell+1}}$ will be assigned. Then, the nodal interpolation of these values defines the function $u_{\ell+1,\ell}$:

$$u_{\ell+1,\ell}(x) = \sum_{y \in \Theta_\ell} u_{\ell+1,\ell}(y) \varphi_y^{\ell+1}(x), \qquad x \in \Omega_{\ell+1}. \tag{7}$$

For $y \in \overset{\circ}{\Theta}_{\ell+1}$, we simply evaluate the function u_ℓ at the fine grid point y:

$$u_{\ell+1,\ell}(y) = u_\ell(y). \tag{8}$$

The computation of the values at the remaining grid points $y \in \Theta_{\ell+1}^\Gamma$ is more involved and split into the following steps:

1. To each grid point $y \in \Theta_{\ell+1}^\Gamma$ we assign a coarse grid triangle $K \in \tau_\ell$ lying "close" to y.

2. To each grid point $y \in \Theta_{\ell+1}^\Gamma$, we assign a point $y_\Gamma \in \Gamma$ having minimal element-wise distance from y.

3. Let u_K^{ext} be the analytic (natural) continuation of $u\,|_K$ onto $\mathbb{R}^2$. Then, we put

$$u_{\ell+1,\ell}(y) = u_K^{ext}(y) - u_K^{ext}(y_\Gamma). \tag{9}$$

4. For $\ell + 1 = \ell_{\max}$, we set $u_{\ell+1,\ell}(y) = 0$ for all $y \in \Theta_{\ell_{\max}} \cap \Gamma$.

The choice of the coarse grid triangle lying close to y (cf. Step 1) is the essential step in this algorithm. The formal definition is given below (see Figure 5).

[1] If there are several elements $x \in M$ maximizing a functional $F : M \to \mathbb{R}$, then, $\operatorname{argmax}\{F(x) : x \in M\}$ is one arbitrary but fixed maximizer. The function argmin is defined analogously.

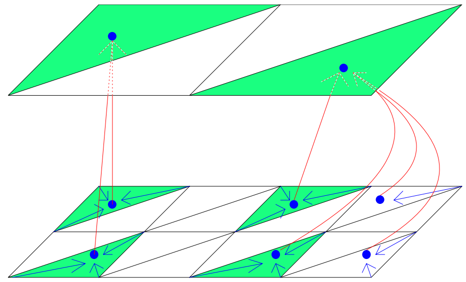


Figure 5: Prolongation from τ_ℓ to $\tau_{\ell+1}$. A detail of the near-boundary grids is depicted. The green triangles correspond to the grids τ_ℓ^{ext}, $\tau_{\ell+1}^{ext}$. The blue arrows represent the function $\pi_{\ell+1}^{\ell+1}$ while the red arrows illustrate the function $\kappa_{\ell+1}^{\ell}$. The function $\pi_{\ell+1}^{\ell}$ is the composite mapping $\kappa_{\ell+1}^{\ell} \circ \pi_{\ell+1}^{\ell+1}$.

Definition 5 *The auxiliary function $\kappa_{\ell+1}^{\ell} : \tau_{\ell+1}^{\Gamma} \to \tau_\ell^{ext}$ defined by*

$$\kappa_{\ell+1}^{\ell}\left(K'\right) = \operatorname*{argmin}_{K \in \tau_\ell^{ext}} \operatorname{dist}_{\tau_{\ell+1}^{\infty}}\left(K', K\right), \qquad \forall K' \in \tau_{\ell+1}^{\Gamma}.$$

The function which maps a grid point $x \in \Theta_{\ell+1}^{\Gamma}$ onto a coarse grid element (cf. Step 1) is denoted by $\pi_{\ell+1}^{\ell} : \Theta_{\ell+1}^{\Gamma} \to \tau_\ell^{ext}$ and defined by the following procedure. For later purpose, an auxiliary function $\pi_{\ell+1}^{\ell+1} : \Theta_{\ell+1}^{\Gamma} \to \tau_{\ell+1}^{\Gamma}$ is defined, too. Let $\mathbf{V}\left(K\right)$ denote the set of vertices of K.

procedure define_ π;
begin
  for all $y \in \Theta_{\ell+1}^{\Gamma}$ ***do*** $\pi_{\ell+1}^{\ell}\left(y\right) := undefined$;
  for all $K' \in \tau_{\ell+1}^{ext}$ ***do for all*** $y \in \mathbf{V}\left(K'\right)$ ***do begin***
    $\pi_{\ell+1}^{\ell}\left(y\right) := \kappa_{\ell+1}^{\ell}\left(K'\right); \pi_{\ell+1}^{\ell+1}\left(y\right) := K'$***;end;***
  for all $K' \in \tau_{\ell+1}^{\Gamma} \backslash \tau_{\ell+1}^{ext}$ ***do for all*** $y \in \mathbf{V}\left(K'\right)$ ***do***
    if $\pi_{\ell+1}^{\ell}\left(y\right) = undefined$ ***then begin***
      $\pi_{\ell+1}^{\ell}\left(y\right) := \kappa_{\ell+1}^{\ell}\left(K'\right); \pi_{\ell+1}^{\ell+1}\left(y\right) := K'$***;end***
end;

It remains to define the function needed in Step 2 which assigns a closest boundary point to a grid point $y \in \Theta_{\ell+1}^{\Gamma}$.

Definition 6 *The function γ^ℓ maps a subset $\omega \subset \Omega_\ell$ to a closest boundary point with respect to the elementwise distance. Let $i = \operatorname{dist}_{\tau_\ell^\infty}\left(\Gamma, \omega\right)$. Then $\gamma^\ell\left(\omega\right) \in \Gamma$ is a boundary point satisfying*

$$\begin{array}{ll} \operatorname{dist}_{\tau_\ell^\infty}\left(\gamma^\ell\left(\omega\right), \omega\right) = i & \text{if } i > 1, \\ \operatorname{dist}\left(\gamma^\ell\left(\omega\right), \omega\right) = \operatorname{dist}\left(\Gamma, \omega\right) & \text{if } i = 1. \end{array}$$

For $K \in \tau_\ell$, we write γ_K short for $\gamma^\ell(K)$ and, for $y \in \Theta_\ell$, γ_y instead of $\gamma^\ell(y)$ if no confusion is possible.

Now, we have all ingredients for the formal definition of the single-step prolongation operator $I_{\ell+1,\ell}$ (cf. Figure 6). We use the following notations. Let $u_\ell \in S_\ell$. For

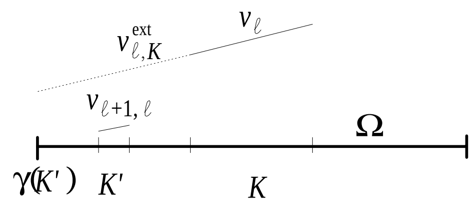


Figure 6: Prolongation operator: The function v_ℓ on the coarse-level element K is extended linearly on K' and the constant boundary value $v^{ext}_{\ell,K}(\gamma_{K'})$ is subtracted.

$K \in \tau_\ell$, the analytic extension of the affine-linear function $u_\ell|_K$ is denoted by u^{ext}_K. For $y \in \Theta^\Gamma_{\ell+1}$, we put $K_y = \pi^\ell_{\ell+1}(y)$ and $K'_y = \pi^{\ell+1}_{\ell+1}(y)$. The prolonged function is denoted by $u_{\ell+1,\ell} = I_{\ell+1,\ell}[u_\ell]$ where $I_{\ell+1,\ell}$ is defined in the following definition.

Definition 7 *For $y \in \Theta_{\ell+1}$, we set*

$$y_\Gamma := \begin{cases} \gamma_y & \text{if } \ell+1 = \ell_{\max} \text{ and } \overline{K'_y} \cap \Gamma \neq \emptyset, \\ \gamma_{K'_y} & \text{otherwise} \end{cases} \tag{10}$$

and

$$u_{\ell+1,\ell}(y) = \begin{cases} u_\ell(y) & \text{if } y \in \mathring{\Theta}_{\ell+1}, \\ u^{ext}_{K_y}(y) - u^{ext}_{K_y}(y_\Gamma) & \text{otherwise.} \end{cases} \tag{11}$$

The nodal interpolation of $\{u_{\ell+1,\ell}(y)\}_{y\in\Theta_{\ell+1}}$ on $\tau_{\ell+1}$ defines the continuous function $u_{\ell+1,\ell}$:

$$u_{\ell+1,\ell}(x) = \sum_{y\in\Theta_{\ell+1}} u_{\ell+1,\ell}(y)\,\varphi^{\ell+1}_y(x), \qquad x \in \Omega_{\ell+1}.$$

Remark 1 *We use the same notation as in the previous definition. Let $y \in \Theta^\Gamma_{\ell+1}$. We assume $\ell+1 = \ell_{\max}$ and $\overline{K'_y} \cap \Gamma \neq \emptyset$. Since $u_\ell|_{K_y}$ is affine-linear, the gradient on K_y is constant and denoted by g. Then, the value $u_{\ell+1,\ell}(y)$ can be rewritten as*

$$u_{\ell+1,\ell}(y) = \langle g, y - \gamma_y\rangle. \tag{12}$$

The function value $u_{\ell+1,\ell}(y)$ depends linearly on the values of $u_\ell(z)_{z\in\mathbf{V}(K_y)}$. An algebraic manipulation yields the formula

$$u_{\ell+1,\ell}(y) = \sum_{z\in\mathbf{V}(K_y)} \alpha_z(y)\,u_\ell(z)$$

with

$$\alpha_z(y) = \frac{\det[y - \gamma_y, z_+ - z_-]}{\det[z - z_-, z_+ - z_-]},$$

where z_-, z, z_+ denote the vertices of K_y. Having computed the three values $\alpha_z(y)$ for each grid point $y \in \Theta_{\ell+1}$, the evaluation of $I_{\ell+1,\ell}$ requires only 5 operations per grid point.

Remark 2 *The definition of the composite finite element space implies that the spaces S_ℓ^{CFE} are nested:*

$$S_\ell^{CFE} \subset S_{\ell'}^{CFE}$$

for all $\ell' \geq \ell$. Furthermore, the inclusion $S_\ell^{CFE} \subset H_0^1(\Omega)$ holds (cf. (12)).

The shape of a typical basis function is illustrated in Figure 7.

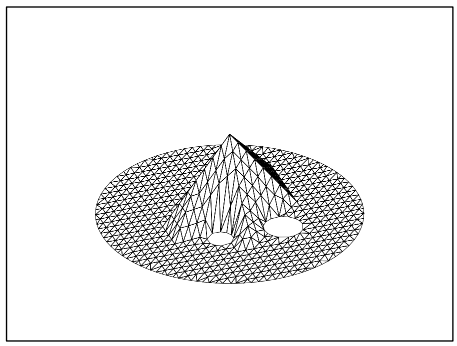


Figure 7: Basis function for problems with Dirichlet boundary conditions

In the rest of this chapter, we will explain how to extend the definitions above to more general situations.

(a) Higher approximation order.

Instead of linear elements, composite finite elements can be defined by using higher polynomial orders as well. Then, the definition is related to the standard finite element space of order p :

$$S_\ell := \{v \in C(\Omega_\ell) \mid \forall K \in \tau_\ell : v|_K \in P_p\},$$

where P_p is the space of polynomials in two variables of total degree p. The set of *grid points* Θ_ℓ has to be redefined as the set of unisolvent *nodal points*, i.e., the interpolation problem of seeking $u \in S_\ell$ such that

$$u(x) = f(x), \qquad x \in \Theta_\ell$$

has a unique solution for any continuous function $f \in C\left(\overline{\Omega_\ell}\right)$. Using these notations, the definitions above define the composite finite element space of order p with the exception that, in Definition 7, the function u_K^{ext} is no longer affine-linear but the analytic (natural) extension of the polynomial $u_\ell \mid_K$. Obviously, Remark 1 applies only to the case of linear composite finite elements.

(b) Non-uniform reference grids and adaptive refinement

We have never used the fact that the reference grids $\tilde{\tau}_\ell$ are uniform. Instead, one could start with an arbitrary (conforming) finite element grid $\tilde{\tau}_0$ satisfying $\Omega \subset \operatorname{dom} \tilde{\tau}_0$. This grid can now be refined step by step by any nested refinement strategy including adaptivity. Since the arising hierarchy of grids will still be used as reference grids we assume that these grids as well as the triangles are again physically and logically nested. This requires that during the refinement of the *reference grids* no grid points will be moved onto the boundary. Afterwards, the adaption is again performed by moving grid points of the *finest* reference grid which lie close to the boundary onto the boundary.

We recommend that, although adaptivity is allowed in principle, the elements of a coarse reference grid $\tilde{\tau}_\ell$ lying close to the boundary (to be more precise: satisfying $K \in L_\ell^{b_\Gamma+1}(\Gamma)$) will be refined instead of staying unrefined.

(c) Quadrilateral elements

The definition of composite finite elements is not restricted to triangular grids. Quadrilateral elements can be used as well. In order to avoid too many technicalities, we consider only parallelograms. Assume that τ_ℓ is a conforming finite element grid containing parallelograms and triangles. The space of polynomials in two variables of degree p in *each* variable is denoted by Q_p. Then, S_ℓ is defined by

$$S_\ell := \left\{ v \in C(\Omega_\ell) \mid \forall K \in \tau_\ell : v \in \left\{ \begin{array}{ll} P_p & \text{if } K \text{ is a triangle} \\ Q_p & \text{if } K \text{ is a parallelogram} \end{array} \right\} \right\}.$$

Then, by using the same modifications as explained in (a), the composite finite element space is defined by Definition 7 and (3), (2). We recommend that, close to the boundary, triangular grids should be used. Movement of grid points of parallelograms would result in quadrilaterals which are not necessarily parallelograms. Furthermore, by numerical experiments we found that the adaption of quadrilateral reference grids to the boundary is much more involved as the adaption of triangular meshes.

(d) Three-dimensional case

The definition of composite finite elements is independent of the space dimension. After having adapted a three-dimensional reference grid to the boundary the definition of the corresponding composite finite element spaces stays the same. Similarly as in the two-dimensional case, we recommend to use tetrahedral grids close to the boundary.

(e) Reduction of unknowns for linear elements

Since, for problems with complicated boundaries, the regularity of the solution of a partial differential equation usually is low one should use low order elements in the neighborhood of the boundary. This motivates a modification of the definition of linear composite finite elements which has, furthermore, the following numerical advantages. The grids τ_ℓ defined by (1) typically overlap the boundary. On the other hand, it appears to be somewhat strange that, for the approximation of functions having zero boundary traces, we are using degrees of freedom belonging to nodal points *outside* the domain. In addition, the following stability problem might arise. Let $z \in \Theta_\ell \backslash \overline{\Omega}$ a grid point lying outside the domain. Consider the basis function φ_z^ℓ of S_ℓ. The corresponding composite finite element function $I_{\ell_{\max},\ell}\left[\varphi_z^\ell\right]$ will be identically zero or have very small function values. If we represent the solution of the partial differential equation in the form

$$\sum_{y \in \Theta_\ell} u(y)\, I_{\ell_{\max},\ell}\left[\varphi_y^\ell\right](x)$$

the matrix line of the arising linear system which corresponds to the nodal point z will vanish or contain only very small entries. This might cause instabilities for the solver of the linear system.

As mentioned above, for linear elements, the mesh τ_ℓ can be reduced to a mesh lying *inside* of Ω. A simple definition would be

$$\tau_\ell \leftarrow \tau_\ell \backslash L_\ell^1(\Gamma)$$

which works in practical all situations. However, it might happen that there exist elements $K' \in \tau_{\ell+1}$ having too large distance from the coarse grid domain Ω_ℓ such that extrapolation from Ω_ℓ onto K' (cf. 9) will not be sufficiently accurate. This, typically, arises if K' is an isolated element surrounded by boundary pieces (cf. Figure 8). Since we know that functions in $H_0^1(\Omega)$ have zero traces one could hope that

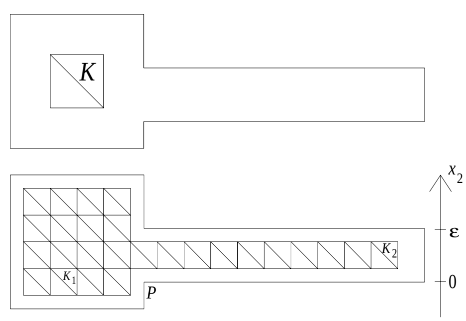


Figure 8: Coarse grid and refined grid lying completely in Ω. The distance of the fine grid element K_2 from the nearest coarse grid element K is much larger than $O(h_{K_2})$ and, hence, extrapolation from K is too inaccurate. For linear elements, the zero function is a sufficiently good approximation on K_2.

the zero function on K' will be a good approximation for a function $u \in H^k(\Omega) \cap$

$H_0^1(\Omega)$. However, one can construct examples where the boundary pieces have too large distance to K' such that the zero function does not satisfy the approximation property. In these cases, one has to enrich the coarse grid space. In this light we define, for a fine grid element $K' \in \tau_{\ell+1}$, a subset of τ_ℓ which covers K' and contains a minimal number of unknowns:

$$U_{\ell+1}^{\ell}(K') := \operatorname{argmin}\{\#\tau : \tau \subset \tau_\ell \mid K' \subset \operatorname{dom}\tau\}.$$

In order to decide whether zero is a good approximation or not we define the function $zero(K')$ as follows.

Definition 8 *Let T be either a triangle or a tetrahedron. Then, we set*

$$h_T := \operatorname{diam} T, \qquad h_T^{\max} = \max_{e \in \mathbf{E}(T)} |e|, \qquad h_T^{\min} = \min_{e \in \mathbf{E}(T)} |e|,$$
$$\rho_T := \max\{\operatorname{diam} B : B \text{ is a ball contained in } T\},$$

where $|e|$ denotes the length of an edge.

Let $K' \in \tau_{\ell+1}$. If, for all $y \in \mathbf{V}(K')$, there exists a triangle (tetrahedron in 3-d) T_y satisfying

1. $T_y \subset \operatorname{dom} L_{\ell+1}^{n_T}(K')$ *with*
$$n_T = 2, \tag{13}$$
2. $\dfrac{h_{T_y}}{\rho_{T_y}} + \dfrac{h_{T_y}^{\max}}{h_{T_y}^{\min}} \leq C_{reg}$,
3. $y \in \overline{T_y}$,
4. $\mathbf{V}(T_y) \subset \Gamma$,

   then we set $zero(K') = admissible$, otherwise, we set $zero(K') = non-admissible$.

The definition of the reduced grids for linear elements takes the form:

procedure generate_Dirichlet_grid$\left(\{\tau_\ell\}_{\ell=0}^{\ell_{\max}}\right)$;
begin
  for $\ell := 0$ **to** $\ell_{\max}-1$ **do** $\tau_\ell := \tau_\ell \backslash L_\ell^1(\Gamma)$;
  for $\ell := \ell_{\max}-1$ **downto** 0 **do for all** $K' \in \tau_{\ell+1}$ **do**
    if $zero(K') = non-admissible$ **and** $\operatorname{dist}_{\tau_{\ell+1}^{\infty}}(K', \Omega_\ell) > 1$ **then**

$$\tau_\ell := \tau_\ell \cup U_{\ell+1}^{\ell}(K');$$

end;

A typical sequence of reduced grids is depicted in Figure 9. By the reduction of the

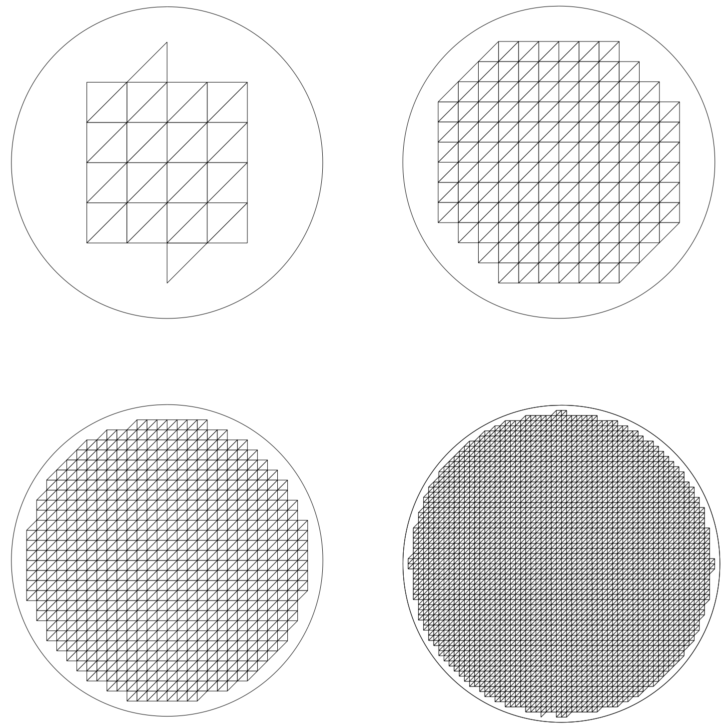


Figure 9: Sequence of reduced grids for linear elements.

grids, the condition $\overset{\circ}{\Theta}_{\ell+1} \subset \overline{\Omega_\ell}$ is no longer guaranteed. In other words, there might exist points $y \in \overset{\circ}{\Theta}_{\ell+1}$ where the prolongation cannot be defined via interpolation (see (8)). This fact leads to the following modification of the sets $\overset{\circ}{\Theta}_{\ell+1}$ and $\Theta^{\Gamma}_{\ell+1}$:

$$\begin{aligned} \overset{\circ}{\Theta}{}^{new}_{\ell+1} &:= \overset{\circ}{\Theta}_{\ell+1} \cap \overline{\Omega_\ell}, \\ \Theta^{\Gamma,new}_{\ell+1} &:= \Theta_{\ell+1} \setminus \overset{\circ}{\Theta}{}^{new}_{\ell+1} . \end{aligned}$$

Since no ambiguity is possible, we write $\overset{\circ}{\Theta}_{\ell+1}$, $\Theta^{\Gamma}_{\ell+1}$ short for $\overset{\circ}{\Theta}{}^{new}_{\ell+1}$, $\Theta^{\Gamma,new}_{\ell+1}$.

Since we now have a new category of elements, the prolongation operator has to be modified, too. The function $\kappa^{\ell}_{\ell+1}$ will be extended to a function $\kappa^{\ell}_{\ell+1} : \Theta^{\Gamma}_{\ell} \to \tau^{ext}_{\ell} \cup \{\emptyset\}$ by

$$\kappa^{\ell}_{\ell+1}(K') = \begin{cases} \emptyset & \text{if } \operatorname{dist}_{\tau^{\infty}_{\ell+1}}(K', K) > 1, \\ \underset{K \in \tau^{ext}_{\ell}}{\operatorname{argmin}} \operatorname{dist}_{\tau^{\infty}_{\ell+1}}(K', K) & \text{otherwise.} \end{cases}$$

According to **procedure define_π** the function $\pi^{\ell}_{\ell+1}$ is then extended to a function $\pi^{\ell}_{\ell+1} : \Theta^{\Gamma}_{\ell} \to \tau^{ext}_{\ell} \cup \{\emptyset\}$. The definition of $u_{\ell+1,\ell}$ (see 11) is now extended by a further category:

$$u_{\ell+1,\ell}(y) = \begin{cases} u_\ell(y) & \text{if } y \in \overset{\circ}{\Theta}_{\ell+1}, \\ 0 & \text{if } \pi_\ell^{\ell+1}(y) = \emptyset, \\ u_{K_y}^{ext}(y) - u_{K_y}^{ext}(y_\Gamma) & \text{otherwise.} \end{cases} \tag{14}$$

where y_Γ is defined by (10). We emphasize that the proposed modification for linear elements has several advantages compared to the original definition and should be used instead.

(f) Relaxing the condition $\Omega_{\ell_{\max}} = \Omega$

We assumed that, after adapting the reference grid to Γ and removing elements lying outside the domain, $\Omega_{\ell_{\max}} = \Omega$ holds. This condition can be relaxed to the condition that the boundary of $\Omega_{\ell_{\max}}$ can be mapped onto Γ by a Lipschitz-continuous mapping $g : \partial\Omega_{\ell_{\max}} \to \Gamma$ and the restriction of g to the edges of $\tau_{\ell_{\max}}$ lying on $\partial\Omega_{\ell_{\max}}$ is smooth. In other words, the grid $\tau_{\ell_{\max}}$ is not regarded as the finest grid in the discretization process but the coarsest grid where standard finite elements discretizations can be applied.

3 Approximation property

In the previous section, we have defined composite finite elements of (polynomial) order p. Next, we will show that these elements satisfy the so-called approximation property for functions in Sobolev spaces.

Definition 9 *Let $k \geq p+1 \geq 2$ and S_ℓ^{CFE} the composite finite element space of (polynomial) order p. S_ℓ^{CFE} satisfies the approximation property if the following two estimates are valid.*

(a) There exists $r \in \mathbb{N}$ such that, for all $u \in H^k(\Omega) \cap H_0^1(\Omega)$, there exists a function $u_\ell \in S_\ell^{CFE}$ satisfying the local approximation property

$$|u - u_\ell|_{m,\sigma(K)} \leq C h_K^{p+1-m} \|u\|_{p+1,L_\ell^r(K)}, \qquad \forall K \in \tau_\ell, m = 0,1 \tag{15}$$

where $\sigma(K) := \sigma_\ell^{\ell_{\max}}(K)$ denotes the children of K on the finest level.

(b) For $m = 0,1$ and $m+1 \leq k \leq p+1$, the global estimate

$$|u - u_\ell|_{m,\Omega} \leq C h_\ell^{k-m} \|u\|_{k,\Omega}$$

is satisfied.

In the following, we will prove the approximation property for composite finite elements. In order to illustrate the principle ideas we take the uniform triangular reference grid and a polygonal domain in $\mathbb{R}^d$ with $d = 2$ as a basis and consider the composite finite element space based on linear elements: $p = 1$. We have chosen $b_\Gamma = 1$ and $n_T = 2$ in (5) and (13) because we found by numerical experiments that this choice leads to the best multigrid convergence results. For the theory, however, it turns out that the choice $b_\Gamma = n_T = 5$ leads to less technicalities in the estimates. Therefore, we assume for the following convergence analysis that the parameters b_Γ and n_T are chosen according to

$$b_\Gamma = n_T = 5$$

while the proof for the other case will be included in [3].

Furthermore, we assume that $\tau_\ell^\infty = \tilde{\tau}_\ell$ holds, i.e., the adaption procedure does not move any grid points of the reference grid. In other words, we first consider the situation that there exists a subset $\tau_{\ell_{\max}} \subset \tilde{\tau}_{\ell_{\max}}$ which covers Ω. Later, we will explain how to extend the results to the general case. In view of (15), we assume that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ holds. The approximation of u will be based on an extension of u onto $\mathbb{R}^2$. Since we assumed that Ω is a polygonal domain we know that there is a continuous extension operator $E : H^2(\Omega) \to H^2(\mathbb{R}^2)$ denoted by $E[u]$, satisfying

$$\begin{aligned} &E[u]\,|_\Omega = u, \\ &\|E[u]\|_{2,\mathbb{R}^2} \leq C_E \|u\|_{2,\Omega}. \end{aligned} \tag{16}$$

In the error estimates derived below the constant C_E will appear. In order to get a robust method with respect to the size and the number of geometric details it is important to elaborate the dependence of C_E on the geometry. This was done in [9] and [10, Section 5.1]. Roughly speaking it was proved that the continuity constant C_E neither depends on the size nor on the number of geometric details as long as the distance of, e.g., holes is comparable to or larger than the minimum of their diameters.

In the following, the extension of a function u is again denoted by u if no confusion is possible.

The approximation of a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is constructed as follows. Sobolev's imbedding theorem guarantees that u is continuous and, hence, the function $u_{\ell,\ell} \in S_\ell$ is uniquely determined by the condition

$$u_{\ell,\ell}(x) = E[u](x), \qquad \forall x \in \Theta_\ell.$$

Transporting this function on the finest grid by $u_{\ell_{\max},\ell} := I_{\ell_{\max},\ell}[u_{\ell,\ell}]$ defines a function in the composite finite element space $u_{\ell_{\max},\ell} \in S_\ell^{CFE}$. We will prove that this function satisfies the approximation property. We will need the following intermediate functions $u_{i,j} \in S_i$, $i \geq j$, defined by

$$\begin{aligned} &u_{i,i}(x) = E[u](x), \quad \forall x \in \Theta_i, \\ &u_{i,j} = I_{i,j}[u_{j,j}], \end{aligned} \tag{17}$$

This leads to the following splitting ($m = 0, 1$)

$$|u - u_{\ell_{\max},\ell}|_{m,\sigma(K)} \leq |u - u_{\ell_{\max},\ell_{\max}}|_{m,\sigma(K)} + \sum_{i=\ell+1}^{\ell_{\max}} |u_{\ell_{\max},i} - u_{\ell_{\max},i-1}|_{m,\sigma(K)}. \tag{18}$$

To estimate the seminorms above we introduce the following mesh dependent norms. For $u \in S_\ell$ and $K \in \tau_\ell$, we define

$$\begin{aligned}
|||u|||_{0,K} &:= \|u\|_{0,K}\,, \\
|||u|||_{1,K} &:= \begin{cases} |u|_{1,K} & \text{if } \overline{K} \cap \overline{\Omega_\ell^\Gamma} = \emptyset, \\ \sqrt{|u|_{1,K}^2 + h_K^{d-2}\,\|u\|_{0,\infty,K}^2} & \text{otherwise,} \end{cases} \\
|||u|||_{1,\infty,K} &:= \begin{cases} |u|_{1,\infty,K} & \text{if } \overline{K} \cap \overline{\Omega_\ell^\Gamma} = \emptyset, \\ \max\left\{|u|_{1,\infty,K}\,, h_K^{-1}\,\|u\|_{0,\infty,K}\right\} & \text{otherwise.} \end{cases}
\end{aligned} \tag{19}$$

For a subset of elements $\tau_\ell' \subset \tau_\ell$, the global norm is defined by

$$|||u|||_{m,\tau_\ell'}^2 := \sum_{K \in \tau_\ell'} |||u|||_{m,K}^2\,.$$

The estimate of the terms in the telescope sum (18) will be split in a stability and a consistency part. In this light, we define a local stability constant for the multi-step prolongation. For a triangle $K \in \tau_\ell$, the aim is to prove the local error estimate on $\sigma(K)$. For a function $I_{\ell_{\max},\ell}[u_{\ell,\ell}] =: u_{\ell_{\max},\ell} \in S_\ell^{CFE}$, the values of $u_{\ell_{\max},\ell}\,|_{\operatorname{dom}\sigma(K)}$ depend on the values of $u_{\ell,\ell}$ in a neighborhood of K. To be more precise we define the influence set for an element $K \in \tau_\ell$.

Definition 10 *Let $K \in \tau_\ell$ and $\sigma(K)$ denote the children of K on the finest level. The influence sets describing which elements on level $\ell \le j \le \ell_{\max}$ are used for the evaluation of the prolongation $I_{\ell_{\max},\ell}$ on the finest level are defined via the following recursion.*

On the finest level, we set

$$\mathfrak{I}_{\ell_{\max},\ell}(K) := \sigma(K)\,, \qquad \mathfrak{Y}_{\ell_{\max},\ell}(K) := \Theta_{\ell_{\max}} \cap \overline{\operatorname{dom}\sigma(K)}$$

and, for $j = \ell_{\max}-1, \ell_{\max}-2, \ldots, \ell$, we define

$$\begin{aligned}
\mathfrak{I}_{j,\ell}(K) :=\ & \{K' \in \tau_j \mid K' \cap \operatorname{dom}\mathfrak{I}_{j+1,\ell}(K) \neq \emptyset\} \\
& \cup \left\{K' = \pi_{j+1}^{j}(x) : x \in \mathfrak{Y}_{j+1,\ell}(K)\right\}, \\
\mathfrak{Y}_{j,\ell}(K) :=\ & \Theta_j \cap \overline{\operatorname{dom}\mathfrak{I}_{j,\ell}(K)}.
\end{aligned}$$

In order to avoid too many technicalities in the proof of the approximation property we assume that $\operatorname{dom}\mathfrak{I}_{j,\ell}(K)$ is connected. This property can easily be ensured by enriching $\mathfrak{I}_{\ell_{\max}}$ or/and τ_ℓ by appropriate elements $K \in \tau_\ell$. The following assumption can then be interpreted as an assumption on the domain. For all $x, y \in \operatorname{dom}\mathfrak{I}_{j,\ell}(K)$, there exists a path s_{xy} with endpoints x, y satisfying

$$\begin{aligned}
s_{xy} &\subset \operatorname{dom}\mathfrak{I}_{j,\ell}(K)\,, \\
|s_{xy}| &\le C h_\ell
\end{aligned} \tag{20}$$

with $|s_{xy}|$ denoting the length of s_{xy}. This assumption can be problematic if, e.g. slit-domains are considered. Then, the condition that $\operatorname{dom} \mathfrak{I}_{j,\ell}(K)$ is connected can be in conflict with condition (20). In [10, Bemerkung 97], it is explained how to modify the prolongation to cover such cases, too.

Using this definition the local stability of the multi-step prolongation can be expressed as follows.

Definition 11 *Let $m \in \{0,1\}$ and $K \in \tau_\ell$. For $j \geq \ell$, we define the local stability constant $\Lambda_{j,\ell}^{(m)}(K)$ as the smallest constant satisfying*

$$|u_{\ell_{\max},j}|_{m,\sigma(K)} \leq C \, |||u_{j,j}|||_{m,\mathfrak{I}_{j,\ell}(K)}, \qquad \forall u_{j,j} \in S_j.$$

By using this definition, the terms in the telescope sum (18) can be estimated by

$$|u_{\ell_{\max},i} - u_{\ell_{\max},i-1}|_{m,\sigma(K)} \leq \Lambda_{i,\ell}^{(m)}(K) \, |||u_{i,i} - u_{i,i-1}|||_{m,\mathfrak{I}_{i,\ell}(K)}. \tag{21}$$

In other words, one has to prove the local stability of the iterated prolongation $I_{\ell_{\max},i}$ and the approximation property for the single-step prolongation $I_{i,i-1}$. We begin with estimating the error of $e_{i,i-1} = u_{i,i} - u_{i,i-1}$. For $x \in K' \in \mathfrak{I}_{i,\ell}(K)$, the error has the representation

$$e_{i,i-1}(x) = \sum_{y \in \mathbf{V}(K')} (u - u_{i,i-1})(y) \, \varphi_y^i(x).$$

Using the inverse inequality we obtain

$$|||e_{i,i-1}|||_{m,K'} \leq C h_i^{d/2-m} \max_{y \in \mathbf{V}(K')} |(u - u_{i,i-1})(y)|. \tag{22}$$

The estimate of the error in the vertices $\mathbf{V}(K')$ is given in the following Lemma. We recall that $p = 1$ holds.

Lemma 12 *Let $K' \in \tau_i$ and $y \in \mathbf{V}(K')$. Then, the estimate*

$$|u(y) - u_{i,i-1}(y)| \leq C h_i^{p+1-d/2} |u|_{p+1,L_i^6(K')}$$

is satisfied.

Proof. We consider the two cases $y \in \overset{\circ}{\Theta}_i$ and $y \in \Theta_i^\Gamma$ separately.

(a) $y \in \overset{\circ}{\Theta}_i$. Then, $u_{i,i-1}(y) = u_{i-1,i-1}(y)$. Let $F_{K'} = f_i^{i-1}(K')$ denote the father of K'. We assumed that $\tau_\ell \subset \tilde{\tau}_\ell$ and, therefore, $y \in \overline{F_{K'}}$ holds. By using the pointwise error estimate for standard finite elements (cf. [2, Theorem 3.1.5]) we obtain

$$|u(y) - u_{i,i-1}(y)| \leq |u - u_{i-1,i-1}|_{0,\infty,F_{K'}} \leq C h_{i-1}^{p+1-d/2} |u|_{2,F_{K'}}.$$

A simple consequence of the definition of the uniform reference grids is

$$\begin{aligned} h_{i-1} &= 2h_i \\ F_{K'} &\subset L_i^1(K') \end{aligned}$$

resulting in

$$|u(y) - u_{i,i-1}(y)| \leq \tilde{C} h_i^{p+1-d/2} |u|_{2,L_i^1(K')}.$$

(b) $y \in \Theta_i^\Gamma$. The functions $\pi_{\ell+1}^\ell$, $\pi_{\ell+1}^{\ell+1}$ are defined in Definition 5. We put $K'' := \pi_i^i(y)$ and $K^{ext} := \pi_i^{i-1}(y)$. The analytic extension of the restriction $u_{i-1,i-1}|_{K^{ext}}$ is denoted by u_{i-1}^{ext}. Then, $u(y) = u_{i-1}^{ext}(y) - u_{i-1}^{ext}(y_\Gamma)$ holds, where $y_\Gamma \in \Gamma$ is defined by (10). Since $K^{ext} \in \tau_\ell^{ext}$ and τ_ℓ^{ext} was defined as a maximal independent subset of τ_ℓ^P (see (6)) we know that

$$\operatorname{dist}_{\tau_i^\infty}\left(K', K^{ext}\right) \leq 3$$

and, hence,

$$K^{ext} \subset L_i^5(K').$$

From (4), it follows $y_\Gamma \subset L_i^5(K')$, too. This results in

$$y \cup K^{ext} \cup y_\Gamma \subset \operatorname{dom} L_i^5(K'). \tag{23}$$

Taking into account $u(y_\Gamma) = 0$ we derive to the error estimate

$$|e_{i,i-1}(y)| \leq \left|u(y) - u_{i-1}^{ext}(y)\right| + \left|u(y_\Gamma) - u_{i-1}^{ext}(y_\Gamma)\right| \leq 2\left|u - u_{i-1}^{ext}\right|_{0,\infty,L_i^5(K')}.$$

Since the triangles of τ_i are uniform, we know that

$$\operatorname{diam}\left(\operatorname{dom} L_i^5(K')\right) \leq 11 h_i \tag{24}$$

holds. As in the proof of [1, Theorem 7.1]

$$\left|u - u_{i-1}^{ext}\right|_{0,\infty,L_i^5(K')} \leq C h_i^{p+1-d/2} |u|_{2,L_i^5(K')}$$

follows. ■

Using this Lemma we can prove the approximation property for the single-step prolongation $I_{i,i-1}$.

Lemma 13 *Let $K \in \tau_\ell$ and $u_{i,i}$, $u_{i,i-1}$ be defined by (17). Then, the estimates*

$$\begin{aligned} |||e_{i,i-1}|||_{m,\mathfrak{I}_{i,\ell}(K)} &\leq C h_i^{p+1-m} |u|_{2,L_\ell^{n_\mathfrak{I}}(K)} \\ |||e_{i,i-1}|||_{m,\Omega} &\leq C h_i^{p+1-m} |u|_{2,\Omega} \end{aligned} \tag{25}$$

are satisfied with $n_\mathfrak{I} = O(1)$.

Proof. Inserting the error estimate of the previous Lemma into the splitting (22) results in

$$|||e_{i,i-1}|||_{m,K'} \leq Ch_i^{p+1-m} |u|_{2,L_i^6(K')}.$$

To obtain an estimate on $\operatorname{dom} \mathfrak{I}_{i,\ell}$ we sum over all $K' \in \mathfrak{I}_{i,\ell}(K)$:

$$\begin{aligned}
|||e_{i,i-1}|||^2_{m,\mathfrak{I}_{i,\ell}(K)} &= \sum_{K' \in \mathfrak{I}_{i,\ell}(K)} |||e_{i,i-1}|||^2_{m,K'} \leq Ch_i^{2(p+1-m)} \sum_{K' \in \mathfrak{I}_{i,\ell}(K)} |u|^2_{2,L_i^6(K')} \\
&= Ch_i^{2(p+1-m)} \sum_{K' \in L_i^6(\mathfrak{I}_{i,\ell}(K))} |u|^2_{2,K'} \sum_{\substack{K'' \in \mathfrak{I}_{i,\ell}(K) \\ K' \in L_i^6(K'')}} 1.
\end{aligned}$$

Since all triangles are uniform the number of elements $K'' \in \mathfrak{I}_{i,\ell}(K)$ satisfying $K' \in L_i^6(K')$ is bounded by a constant $C_\# = O(1)$. Hence, we get

$$|||e_{i,i-1}|||^2_{m,\mathfrak{I}_{i,\ell}(K)} \leq CC_\# h_i^{2(p+1-m)} |u|^2_{2,L_i^6(\mathfrak{I}_{i,\ell}(K))}.$$

In [10, Lemma 75], it was shown that $L_i^6(\mathfrak{I}_{i,\ell}(K)) \subset L_\ell^{n_\mathfrak{I}}(K)$ with a constant $n_\mathfrak{I} = O(1)$ which leads to the asserted local estimate.

For the global estimate, we sum over all $K' \in \tau_i$:

$$\begin{aligned}
|||e_{i,i-1}|||^2_{m,\Omega} &\leq \sum_{K' \in \tau_i} |||e_{i,i-1}|||^2_{m,K'} \leq \sum_{K \in \tau_{i-1}} |||e_{i,i-1}|||^2_{m,\mathfrak{I}_{i,i-1}(K)} \\
&\leq Ch_i^{2(p+1-m)} \sum_{K \in \tau_{i-1}} |u|^2_{2,L_i^6(\mathfrak{I}_{i,i-1}(K))} \\
&= Ch_i^{2(p+1-m)} \sum_{K \in L_i^6(\tau_i)} |u|^2_{2,K} \sum_{\substack{K' \in \tau_{i-1} \\ K \in L_i^6(\mathfrak{I}_{i,i-1}(K'))}} 1.
\end{aligned}$$

Again, it is easy but technical to prove that the number of $K' \in \tau_{i-1}$ satisfying $K \in L_i^6(\mathfrak{I}_{i,i-1}(K'))$ is bounded by a constant $C'_\# = O(1)$ (see [10, Lemma 15 and Lemma 75]). This leads to

$$|||e_{i,i-1}|||^2_{m,\Omega} \leq CC'_\# h_i^{2(p+1-m)} \sum_{K \in L_i^6(\tau_i)} |u|^2_{2,K}.$$

Finally, we need the continuity of the extension operator E (see (16)) to get

$$|||e_{i,i-1}|||_{m,\Omega} \leq C_E \sqrt{CC'_\#} h_i^{p+1-m} \|u\|_{2,\Omega}.$$

■

The stability of the iterated prolongation $I_{\ell_{\max},i}$ is concerned in the next step.

Lemma 14 *Let $K \in \tau_\ell$. There exists a constant $\Lambda < \infty$ independent of $\ell, \ell_{\max}$ and K such that, for $m = 0, 1$, and all $0 \leq \ell \leq i \leq \ell_{\max}$ the prolongation $I_{i,\ell}$ is bounded by Λ :*

$$\Lambda_{i,\ell}^{(m)} \leq \Lambda, \tag{26}$$

where $\Lambda_{i,\ell}^{(m)}$ denotes the local stability constant of Definition 11.

Since the proof of this Lemma is rather technical, we postpone it to the end of this chapter. We proceed with the proof of the approximation property.

Theorem 15 *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_{\ell_{\max},\ell}$ defined by (17). For $m = \{0,1\}$, the local and global approximation properties are valid:*

$$\begin{aligned} |u - u_\ell|_{m,\sigma(K)} &\le C h_\ell^{p+1-m} |u|_{p+1,L_\ell^{n_\jmath}(K)}, \qquad \forall K \in \tau_\ell, \\ |u - u_\ell|_{m,\Omega} &\le C h_\ell^{p+1-m} |u|_{p+1,\Omega} \end{aligned}$$

with $n_\jmath$ of Lemma 13.

Proof. We begin with the local estimate. We insert (26) and (25) into (21). This results in

$$|u_{\ell_{\max},i} - u_{\ell_{\max},i-1}|_{m,\sigma(K)} \le C\Lambda h_i^{p+1-m} |u|_{2,L_\ell^{n_\jmath}(K)}.$$

The function $u_{\ell_{\max},\ell_{\max}}$ is the standard nodal interpolation on the finest grid and, hence, the standard error estimate applies (see, e.g., [5, Lemma 8.4.3]):

$$|u - u_{\ell_{\max},\ell_{\max}}|_{m,\sigma(K)} \le C h_{\ell_{\max}}^{p+1-m} |u|_{m,\sigma(K)}.$$

Plugging these estimates in (18) results in the local estimate

$$\begin{aligned} |u - u_{\ell_{\max},\ell}|_{m,\sigma(K)} &\le C h_{\ell_{\max}}^{p+1-m} |u|_{m,\sigma(K)} + C\Lambda |u|_{2,L_\ell^{n_\jmath}(K)} \sum_{i=\ell+1}^{\ell_{\max}} h_i^{p+1-m} \\ &= C\Lambda |u|_{2,L_\ell^{n_\jmath}(K)} \left(2^{-(p+1-m)\ell_{\max}} + \sum_{i=\ell+1}^{\ell_{\max}} 2^{-(p+1-m)i} \right) \le C h_\ell^{p+1-m} |u|_{2,L_\ell^{n_\jmath}(K)}. \end{aligned}$$

We come now to the global estimate and have to sum over all elements $K \in \tau_\ell$:

$$\begin{aligned} |u - u_{\ell_{\max},\ell}|_{m,\Omega}^2 &\le \sum_{K\in\tau_\ell} |u - u_{\ell_{\max},\ell}|_{m,\sigma(K)}^2 \le C h_\ell^{2(p+1-m)} \sum_{K\in\tau_\ell} |u|_{2,L_\ell^{n_\jmath}(K)}^2 \\ &= C h_\ell^{2(p+1-m)} \sum_{K\in\tau_\ell} |u|_{2,K}^2 \sum_{\substack{K'\in\tau_\ell \\ K\in L_\ell^{n_\jmath}(K')}} 1. \end{aligned}$$

In [10, Lemma 15], it was proved that the number of elements $K' \in \tau_\ell$ satisfying $K \in L_\ell^{n_\jmath}(K')$ is bounded by the constant $C_\#$ appearing in the proof of Lemma 13. The results in

$$|u - u_{\ell_{\max},\ell}|_{m,\Omega} \le C C_\# h_\ell^{p+1-m} |u|_{2,\Omega_\ell}.$$

Using the continuity of the extension operator proves the assertion. ■

It remains to prove the stability of the prolongation $I_{i,\ell}$.

Proof of Lemma 14.

For the following, we fix an element $\hat{K} \in \tau_\ell$ and choose i, j such that $\ell \le j < i \le \ell_{\max}$ is fulfilled. Let $u_{j,j} \in S_j$ be an arbitrary finite element function and

$u_{i,j} := I_{i,j}[u_{j,j}] \in S_i$ the prolonged version. The aim is to estimate $|u_{i,j}|_{m,\mathfrak{I}_{i,\ell}(\hat{K})}$ by $|u_{j,j}|_{m,\mathfrak{I}_{j,\ell}(\hat{K})}$. In this light, we consider a triangle $K' \in \mathfrak{I}_{i,\ell}\left(\hat{K}\right)$. For $x \in K'$, the derivatives $\partial^\alpha[u_{i,j}]$, $\alpha \in \mathbb{N}_0^2$, can be written in the form

$$\partial^\alpha [u_{i,j}](x) = \sum_{y \in \mathbf{V}(K')} u_{i,j}(y)\, \partial^\alpha \left[\varphi_y^i\right](x). \tag{27}$$

The values $u_{i,j}(y)$ are computed from $u_{i-1,j}$ according to (11). We distinguish the following cases:

1. $K' \in \tau_i^{ext}$ and $K := \kappa_i^{i-1}(K')$. Then, for all $y \in \mathbf{V}(K')$, $u_{i,j}(y)$ was defined by

   $$u_{i,j}(y) = \langle g, y - y_\Gamma \rangle, \tag{28}$$

   where the gradient $g = \nabla u_{i-1,j}\,|_K$ is constant and the boundary point y_Γ is defined by (10).

2. $K' \in \overset{\circ}{\tau}_i$ and $\overline{K'} \cap \overline{\Omega_i^\Gamma} = \emptyset$. Then, for all $y \in \mathbf{V}(K')$, $u_{i,j}(y)$ was defined by

   $$u_{i,j}(y) = u_{i-1,j}(y).$$

3. In all other cases, $u_{i,j}(y)$ is computed by different strategies 1,2 for different nodal points $y \in \mathbf{V}(K')$. In that case we say that $u_{i,j}\,|_{K'}$ was *averaged.*

For the computation of $u_{i,j}\,|_{\Omega_i^\Gamma}$, only the values of $u_{i-1,j}\,|_{\operatorname{dom}\tau_{i-1}^{ext}}$ are needed. Hence, for the stability of the *iterated* prolongation $I_{i,j}$ only cases 1,2 are relevant. For Case 3 one has to prove stability only for the single-step prolongation. The closest boundary point y_Γ on the finest level was defined differently as on coarser levels. Again, for the stability of the iterated prolongation, only the definition for the coarser levels are relevant. We employ formula (27) and consider the above cases separately. Let $m = 0, 1$ and $\alpha \in \mathbb{N}_0^2$ satisfying $|\alpha| = \alpha_1 + \alpha_2 = m$.

1) $K' \in \tau_i^{ext}$ and $K := \kappa_i^{i-1}(K')$. We first consider the case that $y_\Gamma = \gamma_{K'}$ holds (cf. (10)). On K', the restriction $u_{i,j}\,|_{K'}$ has the representation (28). Hence, the $W^{1,q}$-seminorm of $u_{i,j}\,|_{K'}$ coincides with the $W^{1,q}$-seminorm of $u_{i-1,j}\,|_K$:

$$|u_{i,j}|_{1,\infty,K'} = |u_{i-1,j}|_{1,\infty,K}. \tag{29}$$

For the L^∞-Norm, we obtain

$$\|u_{i,j}\|_{0,\infty,K'} \le \|g\|\,\|y - y_\Gamma\| \le \sqrt{2}\,|u_{i-1,j}|_{1,\infty,K}\,\|y - y_\Gamma\|.$$

From the definition of the near-boundary grid τ_i^Γ, it follows that $y \in L_i^5(y_\Gamma)$ and, due to the uniformity of the triangulation, we get

$$\|y - y_\Gamma\| \le 5h_i.$$

Using the inverse inequality results in

$$\|u_{i,j}\|_{0,\infty,K'} \leq 5\sqrt{2}h_i \, |u_{i-1,j}|_{1,\infty,K} \leq C \, \|u_{i-1,j}\|_{0,\infty,K} \, . \tag{30}$$

Multiplying (29) and (30) with the area of K' results in

$$|u_{i,j}|_{m,K'} \leq C \, |u_{i-1,j}|_{m,K} \, . \tag{31}$$

We will also need an estimate of the modified H^1-seminorm $\|\|u_{i,j}\|\|_{1,K'}$ (see (19)). Since $\overline{K'} \cap \overline{\Omega_i^\Gamma} \neq \emptyset$ we have to estimate the additional term $Z := h_i^{d/2-1} \|u_{i,j}\|_{0,\infty,K'}$. We get

$$
\begin{aligned}
Z &\leq C h_i^{d/2-1} \|u_{i-1,j}\|_{0,\infty,K} && (32)\\
\|\|u_{i,j}\|\|_{1,K'} &= \sqrt{|u_{i,j}|_{1,K'}^2 + h_i^{d-2} \|u_{i,j}\|_{0,\infty,K'}^2} && (33)\\
&\leq \sqrt{C \, |u_{i-1,j}|_{1,\infty,K}^2 + C h_i^{d-2} \|u_{i-1,j}\|_{0,\infty,K}^2} \leq C \, \|\|u_{i-1,j}\|\|_{1,K} \, .
\end{aligned}
$$

It remains to consider the case that $i = \ell_{\max}$ and $\overline{K'} \cap \Gamma \neq \emptyset$. Then, $y_\Gamma = \gamma_y$ holds. Estimate (29) is no longer true. Instead, we get

$$\partial^\alpha u_{i,j}(x) = \sum_{y \in \mathbf{V}(K')} \langle g, y - y_\Gamma \rangle \, \partial^\alpha \varphi_y^i(x) \, , \qquad \forall x \in K'.$$

Using the inverse inequality results in

$$|u_{i,j}|_{m,\infty,K'} \leq C \, \|g\| \left(\max_{y \in \mathbf{V}(K')} \|y - y_\Gamma\| \right) h_i^{-m}.$$

As in (30) we conclude that

$$|u_{i,j}|_{m,\infty,K'} \leq C \, |u_{i-1,j}|_{1,\infty,K} \, h_i^{1-m} \leq C \, |u_{i-1,j}|_{m,\infty,K} \, h_{i-1}^{m-1} h_i^{1-m} \leq C \, |u_{i-1,j}|_{m,\infty,K} \tag{34}$$

holds. Estimates (30), (31), and (33) are derived as before. Summarizing we have shown that, for $i < \ell_{\max}$,

$$|u_{i,j}|_{1,K'} = |u_{i-1,j}|_{1,K}$$

holds and in the general case

$$
\begin{aligned}
|u_{i,j}|_{m,K'} &\leq C \, |u_{i-1,j}|_{m,K} \\
\|\|u_{i,j}\|\|_{m,K'} &\leq C \, \|\|u_{i-1,j}\|\|_{m,K}
\end{aligned}
$$

is satisfied.

2.) $K' \in \overset{\circ}{\tau}_i$ and $\overline{K'} \cap \overline{\Omega_i^\Gamma} = \emptyset$. Since the triangulations are physically nested, it follows that

$$u_{i,j}(x) = u_{i-1,j}(x) \, , \qquad \forall x \in K'$$

holds. Hence, all norms and seminorms of $u_{i,j}$ and $u_{i-1,j}$ coincide on K'.

3.) We now come to the case $K' \in \tau_i \backslash \tau_i^{ext}$ and $\overline{K'} \cap \overline{\Omega_i^{\Gamma}} \neq \emptyset$. First, we consider the case that $K' \in \tau_i^{\Gamma}$. Then, all values $u_{i,j}(y)$, $y \in \mathbf{V}(K')$ are computed according to

$$u_{i,j}(y) = \left\langle g_{K_y}, y - y_\Gamma \right\rangle \tag{35}$$

with $K_y := \pi_i^{i-1}(y)$, $g_{K_y} := \nabla u_{i-1,j} \mid_{K_y}$, and y_Γ defined by (10). Hence, we obtain, similarly as for (30) and (34), the estimate

$$\begin{aligned} |u_{i,j}|_{m,K'} &\leq C \max_{y \in \mathbf{V}(K')} \left\| g_{K_y} \right\| \|y - y_\Gamma\| \, h_i^{-m} \leq C h_i^{1-m} \max_{y \in \mathbf{V}(K')} |u_{i-1,j}|_{1,\infty,K_y} \\ &\leq C \max_{y \in \mathbf{V}(K')} |u_{i-1,j}|_{m,\infty,K_y}. \end{aligned}$$

The estimate of the modified H^1-seminorm is derived in the same way as (33).

It remains to consider the case that $K' \in \overset{\circ}{\tau}_i$ but $\overline{K'} \cap \overline{\Omega_i^{\Gamma}} \neq \emptyset$. Let $\Theta_{K'}^{ext} := \Theta_i^{\Gamma} \cap \overline{K'}$ and $\Theta_{K'}^{int} := \overset{\circ}{\Theta}_i \cap \overline{K'}$. From $K' \in \overset{\circ}{\tau}_i$, it follows that $K' \cap \Gamma = \emptyset$ holds. Hence, the values $\{u_{i,j}(y)\}_{y \in \Theta_{K'}^{ext}}$ are computed by

$$u_{i,j}(y) = u_{K_y}^{ext}(y) - u_{K_y}^{ext}(y_\Gamma)$$

with K_y defined as in (35) and $y_\Gamma := \gamma_{K'}$. The function $u_{K_y}^{ext}$ denotes the analytic (affine-linear) extension of $u_{i-1,j} \mid_{K_y}$. The father of K' is denoted by $f_{K'}$. Let $\hat{y} \in \Theta_{K'}^{int}$ be fixed and $z_y \in \overline{K_y}$ be a point having minimal distance to $\hat{y}$. Then, for $x \in K'$, we get

$$\begin{aligned} \partial^\alpha [u_{i,j}](x) &= \sum_{y \in \Theta_{K'}^{int}} u_{i-1,j}(y) \, \partial^\alpha \varphi_y^i(x) + \sum_{y \in \Theta_{K'}^{ext}} \left(u_{K_y}^{ext}(y) - u_{K_y}^{ext}(y_\Gamma) \right) \partial^\alpha \varphi_y^i(x) \\ &= \sum_{y \in \Theta_{K'}^{int}} \left(u_{i-1,j}(y) - u_{i-1,j}(\hat{y}) \right) \partial^\alpha \varphi_y^i(x) + \delta_{0,m} u_{i-1,j}(\hat{y}) \\ &\quad + \sum_{y \in \Theta_{K'}^{ext}} \left(u_{K_y}^{ext}(y) - u_{i-1,j}(\hat{y}) \right) \partial^\alpha \varphi_y^i(x) - \sum_{y \in \Theta_{K'}^{ext}} u_{K_y}^{ext}(y_\Gamma) \, \partial^\alpha \varphi_y^i(x) \\ &= \sum_{y \in \Theta_{K'}^{int}} \left\langle g_{f_{K'}}, y - \hat{y} \right\rangle \partial^\alpha \varphi_y^i(x) + \delta_{0,m} u_{i-1,j}(\hat{y}) \\ &\quad + \sum_{y \in \Theta_{K'}^{ext}} \left(u_{K_y}^{ext}(y) - u_{i-1,j}(z_y) \right) \partial^\alpha \varphi_y^i(x) - \sum_{y \in \Theta_{K'}^{ext}} u_{K_y}^{ext}(y_\Gamma) \, \partial^\alpha \varphi_y^i(x) \\ &\quad + \sum_{y \in \Theta_{K'}^{ext}} \left(u_{i-1,j}(z_y) - u_{i-1,j}(\hat{y}) \right) \partial^\alpha \varphi_y^i(x) =: S_1 + S_2 + S_3 + S_4 + S_5 \end{aligned}$$

These five terms S_i will be estimated step by step. We begin with the first sum:

$$\begin{aligned} |S_1|_{0,\infty,K'} &\leq h_i \, |u_{i-1,j}|_{1,\infty,f_{K'}} \sum_{y \in \Theta_{K'}^{int}} \left| \partial^\alpha \varphi_y^i(x) \right|_{0,\infty,K'} \\ &\leq C h_i \, |u_{i-1,j}|_{1,\infty,f_{K'}} \, h_i^{-m} \leq C h_i^{1-m} h_{i-1}^{m-1} \, |u_{i-1,j}|_{m,\infty,f_{K'}} \\ &\leq C \, |u_{i-1,j}|_{m,\infty,f_{K'}}. \end{aligned}$$

For the second term, we get

$$|S_2|_{0,\infty,K'} \leq \delta_{0,m} |u_{i-1,j}|_{m,\infty,f_{K'}}.$$

To estimate third one we observe that $u^{ext}_{K_y}(z_y) = u_{i-1,j}(z_y)$ holds. Furthermore, we have to estimate the distance $\|y - z_y\|$. From the definition of the influence sets, it follows that $f_{K'}$ and all K_y are contained in $\mathfrak{I}_{i-1,i-1}(f_{K'})$. Hence, in view of (20), we conclude that there exists a path $s_{z_y,y} \subset \operatorname{dom}\mathfrak{I}_{i-1,i-1}\left(\hat{K}\right)$ with length $\left|s_{z_y,y}\right| \leq Ch_{i-1}$ connecting z_y and y implying

$$\|y - z_y\| \leq Ch_{i-1} \leq \tilde{C}h_i. \tag{36}$$

Now, we can estimate S_3 by

$$\begin{aligned} |S_3|_{0,\infty,K'} &= \left|\sum_{y\in\Theta^{ext}_{K'}} \left(u^{ext}_{K_y}(y) - u^{ext}_{K_y}(z_y)\right)\partial^\alpha\varphi^i_y(x)\right|_{0,\infty,K'} = \left|\sum_{y\in\Theta^{ext}_{K'}} \left\langle g_{K_y}, y - z_y\right\rangle \partial^\alpha\varphi^i_y(x)\right|_{0,\infty,K'} \\ &\leq h_i \max_{y\in\mathbf{V}(K')} |u_{i-1,j}|_{1,\infty,K_y} \sum_{y\in\Theta^{int}_{K'}} \left|\partial^\alpha\varphi^i_y(x)\right|_{0,\infty,K'} \leq C \max_{y\in\mathbf{V}(K')} |u_{i-1,j}|_{m,\infty,K_y}. \end{aligned}$$

We proceed with the fourth term. We observe that

$$u^{ext}_{K_y}(y_\Gamma) = u^{ext}_{K_y}(z_y) + u^{ext}_{K_y}(y_\Gamma) - u^{ext}_{K_y}(z_y) = u_{i-1,j}(z_y) + \left\langle g_{K_y}, y_\Gamma - z_y\right\rangle$$

holds. Using (23), (24), and 36 in combination with the triangle inequality the difference $\|z_y - y_\Gamma\|$ can be estimated from above by Ch_i. This leads to

$$\begin{aligned} |S_4|_{0,\infty,K'} &= \left|\sum_{y\in\Theta^{ext}_{K'}} \left(u_{i-1,j}(z_y) + \left\langle g_{K_y}, y_\Gamma - z_y\right\rangle\right)\partial^\alpha\varphi^i_y(x)\right|_{0,\infty,K'} \\ &\leq C \max_{y\in\mathbf{V}(K')} \left(|u_{i-1,j}|_{0,\infty,K_y} h_i^{-m} + |u_{i-1,j}|_{1,\infty,K_y} h_i^{1-m}\right) \\ &\leq C \max_{y\in\mathbf{V}(K')} \left(|u_{i-1,j}|_{0,\infty,K_y} h_i^{-m} + |u_{i-1,j}|_{m,\infty,K_y}\right). \end{aligned}$$

This is the stability in L^∞, i.e., $m = 0$:

$$|S_4|_{0,\infty,K'} \leq C \max_{y\in\mathbf{V}(K')} |u_{i-1,j}|_{0,\infty,K_y}$$

and the stability in the modified $W^{1,\infty}$-seminorm ($m = 1$)

$$|S_4|_{0,\infty,K'} \leq C \max_{y\in\mathbf{V}(K')} |||u_{i-1,j}|||_{1,\infty,K_y}.$$

The estimate of the fifth term is obtained as follows. Let $s_{z_y,\hat{y}} \subset \operatorname{dom}\mathfrak{I}_{i-1,i-1}(f_{K'})$ be a path connecting z_y and $\hat{y}$ with length $\left|s_{z_y,\hat{y}}\right| \leq Ch_i$. The restriction $u_{i-1,j}\,|_{s_{z_y,\hat{y}}}$ is Lipschitz-continuous leading to

$$|S_5|_{0,\infty,K'} \leq C\,|u_{i-1,j}|_{1,\infty,s_{z_y,\hat{y}}} \left|s_{z_y,\hat{y}}\right| h_i^{-m} \leq C\,|u_{i-1,j}|_{1,\infty,m,\mathfrak{I}_{i-1,\ell}(\hat{K})} h_i^{1-m}.$$

Together, we have shown that

$$|u_{i,j}|_{m,K'} \leq C\, |||u_{i-1,j}|||_{m,\Im_{i-1,\ell}(\hat{K})}$$

is valid. The estimate of the additional term appearing in the definition of modified H^1-seminorm can be performed in the same way as the estimate of Z in (32). This results in

$$|||u_{i,j}|||_{m,K'} \leq C\, |||u_{i-1,j}|||_{m,\Im_{i-1,\ell}(\hat{K})}.$$

Now, the proof of the single-step prolongation is complete and we come to the stability of the iterated prolongation $I_{i,j}$.

First, let $\ell \leq j < i < \ell_{\max}$. We consider an element $K' \in \Im_{i,\ell}\left(\hat{K}\right) \cap \tau_i^{ext}$. We build a sequence via the recursion:

$$\begin{aligned} &K_i = K', \\ &K_r = \kappa_{r+1}^r\left(K_{r+1}\right) \qquad r = i-1, i-2, \ldots, j. \end{aligned}$$

In view of $K_r \in \tau_r^{ext}$, we know that $u_{r,i}\,|_{K_r}$ is *not* averaged for all r resulting in

$$u_{i,j}(x) = u_{K_j}^{ext}(x) - u_{K_j}^{ext}(\gamma_{K'}), \qquad \forall x \in K'.$$

The proof of stability is therefore the same as for the single-step prolongation:

$$|||u_{i,j}|||_{m,\infty,K'} \leq C\, |||u_{j,j}|||_{m,\infty,\Im_{j,\ell}(\hat{K})}. \tag{37}$$

Now, we consider the case of $K' \in \Im_{i,\ell}\left(\hat{K}\right)$ satisfying $\overline{K'} \cap \overline{\Omega_i^\Gamma} \neq \emptyset$. The stability of the single-step prolongation yields

$$|||u_{i,j}|||_{m,\infty,K'} \leq \max\left\{ |||u_{i-1,j}|||_{m,\infty,f_{K'}}, \max_{y\in\mathbf{V}(K')} |||u_{i-1,j}|||_{m,\infty,K_y} \right\}.$$

In [10, Lemma 71], it is proved that $f_{K'}$ and all K_y, $y \in \mathbf{V}(K')$, belong to τ_{i-1}^{ext}. Hence, we can apply (37) and obtain the stability of the iterated prolongation for K'.

It remains to consider those elements $K' \in \tau_i$ satisfying $\overline{K'} \cap \overline{\Omega_i^\Gamma} = \emptyset$. We build a sequence of triangles by the following procedure:

$K_i := K'; r := i;$
while $\overline{K_r} \cap \overline{\Omega_r^\Gamma} = \emptyset$ **and** $r > j$ **do begin**
  $r := r-1; K_r := f_{r+1}^r\left(K_{r+1}\right);$
end;

The triangulations are physically nested, therefore, $u_{i,j}|_{K'} = u_{r,j}|_{K_r}$ holds and all seminorms coincide. Hence, if $r = j$ there is nothing to prove since all norms coincide. For the following, we assume $r > j$ implying $\overline{K_r} \cap \overline{\Omega_r^\Gamma} \neq \emptyset$. Applying the previous results proves the stability for all cases provided $i < \ell_{\max}$. For $i = \ell_{\max}$, follows from this results and the stability of the single-step prolongation $I_{\ell_{\max},\ell_{\max}-1}$. This completes the proof of the stability of the iterated prolongation $I_{i,j}$. ■

From the proof of the stability, the following corollary follows directly.

Corollary 16 *There exists a constant For $j \geq \ell$ and all $K \in \tau_\ell$, the multi-step prolongation is also stable in $W^{1,\infty}$:*

$$\left| u_{\ell_{\max},j} \right|_{1,\infty,\sigma(K)} \leq C \left\| u_{j,j} \right\|_{1,\infty,\Im_{i,\ell}(K)} .$$

In [10, Section 5.2.2], a proof of the approximation property for the general situations described at the end of the previous section is given. One has to impose technical assumption on the mapping Φ_ℓ which can be satisfied by using an appropriate algorithm for adapting the reference grid $\tilde{\tau}_{\ell_{\max}}$ to the boundary. Roughly speaking, the composite finite elements grids τ_ℓ have to be shape-regular and the diameters of the triangles of τ_ℓ must be comparable to the diameters of the corresponding triangles of the reference grids. In [10, Annahme 27] six conditions are formulated which ensures that the approximation property is valid also for the general case.

The general proof can be obtained by transferring all Lemmata and auxiliary statements of this section to the general case.

In the next section, we explain how to use composite finite elements for the discretization of partial differential equations.

4 Discretization with composite finite elements and numerical results

We have introduced composite finite elements for the discretization of partial differential equations on complicated domains. As a model problem, we consider the problem of finding $u \in H_0^1(\Omega)$ such that, for given functional $F \in H^{-1}(\Omega)$,

$$\int_\Omega \left(\langle \nabla u, \nabla v \rangle + uv \right) dz = F(v), \qquad \forall v \in H_0^1(\Omega)$$

is satisfied. We focus our attention on domains containing small geometric details as holes, etc. The Galerkin discretization based on composite finite elements takes the form: Find $u_\ell \in S_\ell^{CFE}$ such that

$$\int_\Omega \left(\langle \nabla u_\ell, \nabla v_\ell \rangle + u_\ell v_\ell \right) dz = F(v_\ell), \qquad \forall v_\ell \in S_\ell^{CFE}(\Omega) \tag{38}$$

is fulfilled. For the reformulation of this problem as a system of linear equations we introduce the space of grid functions $\mathbb{R}^{\Theta_\ell}$ containing all mappings $\beta : \Theta_\ell \to \mathbb{R}$. Any function u_ℓ can be represented by

$$u_\ell(x) = \sum_{y \in \Theta_\ell} \beta(y) I_{\ell_{\max},\ell}\left[\varphi_y^\ell\right](x) =: \sum_{y \in \Theta_\ell} \beta(y) b_y^\ell(x).$$

Hence, equation (38) is equivalent to finding $\beta_\ell \in \mathbb{R}^{\Theta_\ell}$ such that

$$\sum_{y \in \Theta_\ell} K_\ell(x,y) \beta_\ell(y) = f_\ell(x), \qquad \forall x \in \mathbb{R}^{\Theta_\ell},$$

where $K_\ell \in \mathbb{R}^{\Theta_\ell \times \Theta_\ell}$ and $f_\ell \in \mathbb{R}^{\Theta_\ell}$ are defined by

$$\begin{aligned} K_\ell(x,y) &= \int_\Omega \left(\left\langle \nabla b_y^\ell, \nabla b_x^\ell \right\rangle + b_y^\ell b_x^\ell\right) dz, && x,y \in \Theta_\ell, \\ f_\ell(x) &= F\left(b_x^\ell\right), && x \in \Theta_\ell. \end{aligned}$$

The matrix $K_{\ell_{\max}}$ and right-hand side $f_{\ell_{\max}}$ on the finest grid can be assembled as for standard finite elements since $S_{\ell_{\max}}^{CFE} = S_{\ell_{\max}} \cap H_0^1(\Omega)$ holds. The coarser matrix and right-hand side can be obtained by coarsening the fine-grid system. Let us introduce the (discrete) prolongation and restriction operators $P_{\ell+1,\ell} \in \mathbb{R}^{\Theta_{\ell+1} \times \Theta_\ell}$ and $R_{\ell,\ell+1} \in \mathbb{R}^{\Theta_\ell \times \Theta_{\ell+1}}$ by

$$\begin{aligned} P_{\ell+1,\ell}(x,y) &= I_{\ell+1,\ell}\left[\varphi_y^\ell\right](x), && \forall x \in \Theta_{\ell+1}, \forall y \in \Theta_\ell, \\ \langle R_{\ell,\ell+1}[v], w\rangle_\ell &= \langle v, P_{\ell+1,\ell}[w]\rangle_{\ell+1}, && \forall v \in \mathbb{R}^{\Theta_{\ell+1}}, \forall w \in \mathbb{R}^{\Theta_\ell}, \end{aligned} \tag{39}$$

where $\langle \cdot, \cdot \rangle_\ell : \mathbb{R}^\ell \times \mathbb{R}^\ell$ denotes the Euclidean scalar product. Then, the coarser system is defined by

$$\begin{aligned} K_\ell &= R_{\ell,\ell+1} K_{\ell+1} P_{\ell+1,\ell} \\ f_\ell &= R_{\ell,\ell+1} f_{\ell+1}. \end{aligned} \tag{40}$$

We state that the complexity of assembling the whole sequence of linear systems $\{K_\ell\}_{\ell=0}^{\ell_{\max}}$, $\{f_\ell\}_{\ell=0}^{\ell_{\max}}$ is proportional to the work needed for the generation of the fine grid system, i.e., proportional to the number of fine grid points $\#\Theta_{\ell_{\max}}$. However, if one is interested only in the generation of a coarse grid system it is possible to localize the prolongation and restriction to the near-boundary region and reduce the arithmetic work for assembling K_ℓ from $O(N_{\ell_{\max}})$ to $O(N_\ell) + O\left(N_{\ell_{\max}}^{\frac{d-1}{d}}\right)$. The implementation and complexity analysis are worked out in detail in [7].

For numerical tests, we have chosen the unit disc and considered the approximation of the function $u(x) = e^{-10\|x\|^2} - e^{-10}$. We have used the uniform triangulation described in the beginning of Section 2.1 as reference grids. This problem is well suited for testing the approximation quality of composite finite elements since the composite finite element grids on the coarse levels cannot be regarded as an approximation of the domain. We have employed composite finite elements based on linear elements ($p = 1$) and, hence, restricted the grids to the interior of the domain as described in Section 2.2 (e). A sequence of grids is depicted in Figure 9. We first verified the approximation property. We have proven that there exists a function $u_\ell \in S_\ell^{CFE}$ such that

$$|u - u_\ell|_{m,\Omega} \leq C h_\ell^{2-m} \|u\|_{2,\Omega}, \qquad m = 0, 1$$

holds. The following table reports the observed convergence rates.

Level	dim	$\|e_\ell\|_0$	$\frac{\|e_{\ell-1}\|_0}{\|e_\ell\|_0}$	$\|e_\ell\|_1$	$\frac{\|e_{\ell-1}\|_1}{\|e_\ell\|_1}$
1	9	1.22e-1		8.83e-1	
2	45	7.51e-2	1.6	8.27e-1	1.1
3	193	2.07e-2	3.6	4.48e-1	1.8
4	777	5.43e-3	3.8	2.28e-1	2.0
5	3101	1.37e-3	4.00	1.15e-1	2.0
6	12365	3.45e-4	4.00	5.74e-2	2.0
7	49473	8.62e-5	4.00	2.87e-2	2.0

As a further application of composite finite elements we have tested the efficiency of multi-grid methods (see [4]) based on composite finite elements. The coarse-level matrices are characterized by K_ℓ from (40). The intergrid transfer is performed by the prolongation and restriction operators defined in (39). We have used the symmetric Gauß-Seidel method as a smoother and the W-cycle multi-grid method with two pre- and two post-smoothing steps. The iteration was stopped as soon as the ℓ^2-norm of the residual was smaller than 10^{-8}.

Level	dim	# iterations
1	9	direct solver
2	45	8
3	193	7
4	777	6
5	3101	5
6	12365	7
7	49473	4

Obviously, the numbers of iterations are very low and independent of the refinement level. Finally, we report also the computing times of the various steps of the algorithm. The CPU-time (in milli-seconds) for the time for assembling the fine-grid system is denoted by $t\left[K_{\ell_{\max}}\right]$, the time needed for the generation of all coarser systems by $t\left[\{K_\ell\}_{\ell=0}^{\ell_{\max}-1}\right]$, and the time needed for the multi-grid solver is denoted by $t\,[\mathrm{mg}]$. The time of adapting the finest grid to the boundary and rejecting elements from all grids lying (essentially) outside of the domain is denoted by $t\,[\mathrm{grids}]$.

level	dim	$t\,[\mathrm{grids}]$	$t\left[K_{\ell_{\max}}\right]$	$t\left[\{K_\ell\}_{\ell=0}^{\ell_{\max}-1}\right]$	$t\,[\mathrm{mg}]$
3	9		1		33
4	45		16	17	417
5	193		100	33	1466
6	777		400	133	5117
7	3101		1517	533	16883
8	12365		5950	2050	95000
9	49473	455283	34233	8000	220550

It clearly can be seen that the CPU-time for all quantities grows linearly with respect to the number of unknowns. The most time consuming step is the generation of the composite finite element grids followed by the multi-grid solver.

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