A Posteriori Estimation of Dimension Reduction Errors

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1 Introduction

The method of dimension reduction is a popular approach frequently used by engineers for the approximate solution of the problems posed in *thin* domains. The term "thin" means that the size of the original physical domain along one coordinate direction is much smaller than along the others; this allows to make some simplifying assumptions on the behaviour of the exact solution and to replace the original, for instance, three-dimensional problem by a two-dimensional one. It is, however, clear that the solution of the new, "reduced" problem will, in general, differ from the solution to the original high-dimensional problem. Thus, the dimension reduction method unavoidably produces the error that can be referred to as the dimension reduction or the *modelling error*. The essential part of the model verification is, hence, a reliable *a posteriori* control of the dimension reduction error.

Despite the practical importance of the topic, only a few a posteriori estimators for the dimension reduction error have been introduced so far. In [10] and [2] (see also [1]) residual-type estimators were proposed and proved reliable and efficient under the assumptions that the right-hand side of the given equation is zero and the original domain is a plate with plane parallel faces. In [3] and [8] implicit estimators based on the solution of local Neumann problems were developed; the estimators were intended for hierarchical modelling and involved the solution of local three-dimensional problems.

In this work we propose a reliable and efficient a posteriori estimator for the dimension reduction error in the energy norm, having no specific assumptions on the right-hand side of the given equation and considering a general geometry of the given domain. We show that, for the zero-order dimension reduction method considered here, the estimator of Babuška and Schwab (see [1], [2]) can be obtained as a particular case of our estimator when the right-hand side of the equation is zero and the original domain is a plate with plane parallel faces. We demonstrate the optimal convergence of the estimator as the plate thickness tends to zero (although, it is worth noting that the proposed estimator preserves its reliability for any positive thickness). Finally, we observe how accurately the estimator indicates the local error distribution, thus, allowing for a local improvement of the model.

2 Problem setting

We consider a three-dimensional Lipschitz domain

$$\Omega := \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \widehat{\Omega}\,,\ d_\ominus(x_1, x_2) < x_3 < d_\oplus(x_1, x_2)\}\,,$$

where $\widehat{\Omega} \subset \mathbb{R}^2$ is its projection on the (x_1, x_2)-plane ($\widehat{\Omega}$ has the Lipschitz boundary $\widehat{\Gamma}$) and $d_\ominus$ and $d_\oplus$ are Lipschitz continuous functions: $\widehat{\Omega} \to \mathbb{R}$. The lower and upper faces of Ω are denoted by

$$\Gamma_\ominus := \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \widehat{\Omega}\,,\ x_3 = d_\ominus(x_1, x_2)\}$$

and

$$\Gamma_\oplus := \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \widehat{\Omega}\,,\ x_3 = d_\oplus(x_1, x_2)\}\,,$$

the lateral boundary by

$$\Gamma_0 := \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in \widehat{\Gamma}\,,\ d_\ominus(x_1, x_2) < x_3 < d_\oplus(x_1, x_2)\}$$

(see Figure 1).

Remark. We consider $d_\ominus$ and $d_\oplus$ as explicit functions of (x_1, x_2)-coordinates only for the sake of simplicity. The generalization of the theory to the case of an arbitrary Lipschitzian domain Ω presents no difficulty from the conceptional point of view.

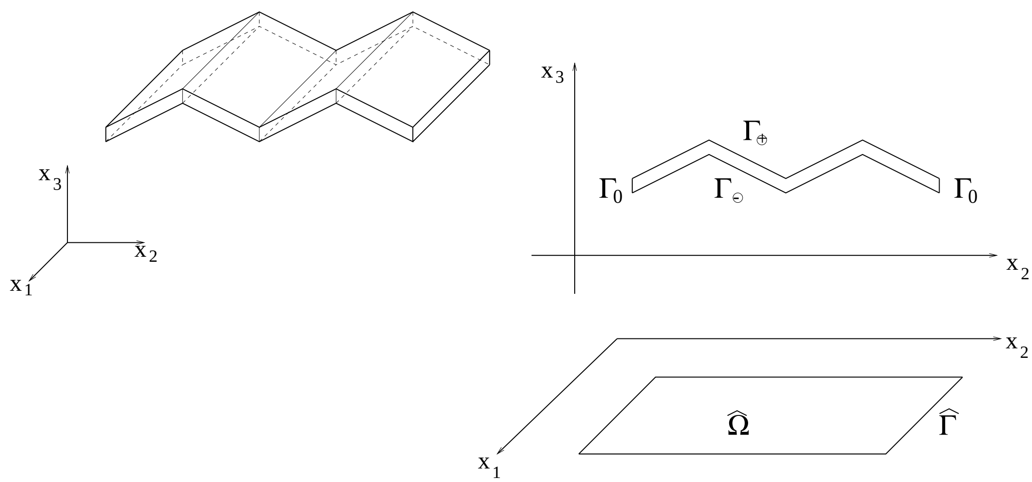

Fig. 1. Sketch of the domain geometry

The assumption that the given domain Ω is "thin" can now be written as

$$\operatorname{diam} \widehat{\Omega} \gg \max_{\overline{\widehat{\Omega}}} d\,(x_1, x_2)\,, \tag{1}$$

where $d = d_\oplus - d_\ominus$ is the domain thickness, $d\,(x_1, x_2) \geq d_* > 0\ \ \forall (x_1, x_2) \in \overline{\widehat{\Omega}}$. Although the assumption is of purely qualitative nature, it serves as a basis for the derivation of the corresponding two-dimensional reduced model. We also have to notice that Figure 1 depicts a simplified case; in the geometrical definitions we do not assume the domain thickness $d\,(x_1, x_2)$ to be a constant.

In the domain Ω we consider a model elliptic problem

$$-\mathrm{Div}\,(\mathbf{A}\nabla u) = f \quad \text{in } \Omega\,, \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_0\,, \tag{3}$$

$$\mathbf{A}\nabla u \cdot \boldsymbol{\nu}_\ominus = \widehat{F}_\ominus \quad \text{on } \Gamma_\ominus\,, \tag{4}$$

$$\mathbf{A}\nabla u \cdot \boldsymbol{\nu}_\oplus = \widehat{F}_\oplus \quad \text{on } \Gamma_\oplus\,, \tag{5}$$

where $f \in L_2(\Omega)$, $\widehat{F}_\ominus$, $\widehat{F}_\oplus \in L_2(\widehat{\Omega})$, $\boldsymbol{\nu}_\ominus$ and $\boldsymbol{\nu}_\oplus$ are outward normal vectors at $\Gamma_\ominus$ and $\Gamma_\oplus$ respectively. The matrix $\mathbf{A} = (a_{ij}(x))_{i,j=\overline{1,3}}$ with the components from $L_\infty(\Omega)$ is symmetric and uniformly positive definite, i.e. there exist constants $0 < c < C < \infty$ such that

$$c|\xi|^2 \le \mathbf{A}(x)\xi \cdot \xi \le C|\xi|^2 \quad \forall \xi \in \mathbb{R}^3\,,\ \text{a. e. in } \Omega\,.$$

From now on we will frequently use the notation $\widehat{x} = (x_1, x_2)$, $x = (\widehat{x}, x_3)$, and all functions depending only on (x_1, x_2) will be marked by $\widehat{\ }$; in addition, we will distinguish between 3- and 2-dimensional divergence operator:

$$\mathrm{Div}\,\boldsymbol{\tau} = \boldsymbol{\tau}_{1,1} + \boldsymbol{\tau}_{2,2} + \boldsymbol{\tau}_{3,3}\,,\ \mathrm{div}\,\widehat{\boldsymbol{\tau}} = \widehat{\boldsymbol{\tau}}_{1,1} + \widehat{\boldsymbol{\tau}}_{2,2}\,.$$

The weak form of the problem (2)–(5) reads
Problem (P): Find $u \in V_0 := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\}$ such that

$$\int_\Omega \mathbf{A}\nabla u \cdot \nabla w\, dx = \int_\Omega f\, w\, dx + \int_{\Gamma_\ominus} \widehat{F}_\ominus\, w\, ds + \int_{\Gamma_\oplus} \widehat{F}_\oplus\, w\, ds \qquad \forall w \in V_0\,. \tag{6}$$

3 The reduced problem

The assumption (1) allows one to suppose that

$$\text{the exact solution } u \approx \mathit{const} \text{ in the } x_3\text{-direction.} \tag{7}$$

This gives rise to the so-called *zero-order reduced model* for the original problem (6). The model is very popular due to its simplicity and purely two-dimensional formulation. The discussion on the hierarchy of the reduced models of different orders can be found in, e.g., [9], [2].

Then, introducing the subspace

$$\widehat{V}_0 := \{v \in V_0 \mid \exists\, \widehat{v} \in H^1_0(\widehat{\Omega}) \text{ such that } v(x) = \widehat{v}(\widehat{x}) \text{ for a.e. } x = (\widehat{x}, x_3) \in \Omega\}$$

and the operation $(\widetilde{\ })$ of averaging in the x_3-direction

$$\forall g \in L_1(\Omega)\ : \quad \widetilde{g}(\widehat{x}) := \frac{1}{d\,(\widehat{x})} \int\limits_{d_\ominus(\widehat{x})}^{d_\oplus(\widehat{x})} g(\widehat{x}, x_3)\, dx_3 \ \text{ for a.e. } \widehat{x} \in \widehat{\Omega}\,,$$

we can deduce from (6) the *reduced problem* (see [7]) that reads
Problem $(\widehat{P})$: Find $\widehat{u} \in \widehat{V}_0$ such that

$$\int_{\widehat{\Omega}} d\,(\widehat{x}) \widetilde{\mathbf{A}}_p(\widehat{x}) \nabla \widehat{u} \cdot \nabla \widehat{w}\, d\widehat{x} = \int_{\widehat{\Omega}} d\,(\widehat{x}) \widehat{f}(\widehat{x}) \widehat{w}\, d\widehat{x} \qquad \forall \widehat{w} \in \widehat{V}_0\,, \tag{8}$$

where $\widehat{f} = \widetilde{f} + \frac{\widehat{F}_\ominus\sqrt{1+|\nabla d_\ominus|^2}+\widehat{F}_\oplus\sqrt{1+|\nabla d_\oplus|^2}}{d}$ and $\widetilde{\mathbf{A}}_p(\widehat{x}) = (\widetilde{a}_{ij}(\widehat{x}))_{i,j=\overline{1,2}}$ is the averaged "plane" part ($\mathbf{A}_p(x) = (a_{ij}(x))_{i,j=\overline{1,2}}$) of the matrix $\mathbf{A}$.

It is clear that problem (8) is a two-dimensional elliptic problem with the homogeneous Dirichlet boundary condition:

$$-\mathrm{div}\,(d(\widehat{x})\,\widetilde{\mathbf{A}}_p(\widehat{x})\nabla\widehat{u}) = d(\widehat{x})\,\widehat{f}(\widehat{x}) \quad \text{in} \;\; \widehat{\Omega} \tag{9}$$

$$\widehat{u} = 0 \quad \text{on} \;\; \widehat{\Gamma}\,. \tag{10}$$

4 A posteriori estimation of the modelling error

In order to control the dimension reduction error $e := u - \widehat{u}$, we apply the functional-type a posteriori error estimate derived in [6] (see also [4] and [5]) to the original three-dimensional problem (6):
For all $\gamma > 0$, $\delta > 0$ and $y^* \in H_*(\Omega, \mathrm{Div})$ there holds

$$|||u-\widehat{u}|||^2 \le (1+\gamma)\,M_1^2 + \left(1+\frac{1}{\gamma}\right)(1+\delta)C_\Omega^2\,M_2^2 + \left(1+\frac{1}{\gamma}\right)\left(1+\frac{1}{\delta}\right)C_\Gamma^2(1+C_\Omega^2)\,M_3^2\,, \tag{11}$$

where $|||\cdot|||$ is the energy norm, $|||v||| := \left(\int_\Omega \mathbf{A}(x)\nabla v\cdot\nabla v\,dx\right)^{1/2}$ $\forall v \in V_0$, C_Ω is the constant from Friedrichs' inequality ($C_\Omega^{-2} = \inf\limits_{w\in V_0\setminus\{0\}} \frac{|||w|||^2}{\|w\|^2_{L_2(\Omega)}}$), C_Γ is the constant from the trace inequality ($C_\Gamma^2 = \sup\limits_{w\in V_0\setminus\{0\}} \frac{\|w\|^2_{L_2(\Gamma_\oplus)}+\|w\|^2_{L_2(\Gamma_\ominus)}}{|||w|||^2+\|w\|^2_{L_2(\Omega)}}$) and the functionals M_1^2, M_2^2, M_3^2 are defined as follows:

$$M_1^2 := \int_\Omega (\nabla\widehat{u} - \mathbf{A}^{-1}y^*)\cdot(\mathbf{A}\nabla\widehat{u} - y^*)\,dx\,,$$
$$M_2^2 := \|\mathrm{Div}\,y^* + f\|^2_{L_2(\Omega)}\,,$$
$$M_3^2 := \|\widehat{F}_\ominus - y^*\boldsymbol{\nu}_\ominus\|^2_{L_2(\Gamma_\ominus)} + \|\widehat{F}_\oplus - y^*\boldsymbol{\nu}_\oplus\|^2_{L_2(\Gamma_\oplus)}\,.$$

We emphasize that the estimate is valid for any positive numbers γ and δ and for any vector-function y^* from the space $H_*(\Omega, \mathrm{Div})$ defined as

$$H_*(\Omega,\mathrm{Div}) := \{y^* \in L_2(\Omega,\mathbb{R}^3)\,|\,\mathrm{Div}\,y^* \in L_2(\Omega)\,,\; y^*\cdot\boldsymbol{\nu}_\ominus \in L_2(\Gamma_\ominus)\,,\; y^*\cdot\boldsymbol{\nu}_\oplus \in L_2(\Gamma_\oplus)\}\,.$$

While the best possible option would be to take as y^* the exact flux $\mathbf{A}\nabla u$ (then M_2 and M_3 would vanish and M_1 would give us the energy norm of the exact error e), we have to restrict ourselves to choosing some computable quantity, i.e. not containing the unknown exact solution u. We approximate the flux by

$$y^* = \widetilde{\mathbf{A}}_p\nabla\widehat{u} + \boldsymbol{\tau}^*\,, \tag{12}$$

where $\boldsymbol{\tau}^* = \{0\,,\;0\,,\;\psi(x)\}^T$, ψ is the auxiliary function from $L_2(\Omega)$ such that $\psi_{,3} \in L_2(\Omega)$, $\psi \in L_2(\Gamma_\ominus)$ and $\psi \in L_2(\Gamma_\oplus)$. Using (9), it is easy to verify that y^* from (12) belongs to $H_*(\Omega, \mathrm{Div})$. A discussion about other choices of y^* can be found in [7].

Substituting (12) into the functionals M_1^2, M_2^2, M_3^2, we obtain (see the details in [7])

$$M_1^2 = \int_\Omega (b_{33}\psi^2 + 2(\mathbf{b}_3 \cdot \widetilde{\mathbf{A}}_p \nabla\widehat{u})\psi)\,dx + \int_{\widehat{\Omega}} d(\widehat{x})\,(\widetilde{\mathbf{B}}_p\widetilde{\mathbf{A}}_p - \mathbf{I})\nabla\widehat{u} \cdot \widetilde{\mathbf{A}}_p\nabla\widehat{u}\,d\widehat{x}\,, \tag{13}$$

$$M_2^2 = \| f - \widetilde{f} - \frac{\widehat{F}_\ominus\sqrt{1+|\nabla d_\ominus|^2} + \widehat{F}_\oplus\sqrt{1+|\nabla d_\oplus|^2}}{d} + \psi_{,3} - \frac{\nabla d}{d}\cdot\widetilde{\mathbf{A}}_p\nabla\widehat{u}\|^2_{L_2(\Omega)}\,, \tag{14}$$

$$M_3^2 = \|\widehat{F}_\ominus - \widetilde{\mathbf{A}}_p\nabla\widehat{u}\cdot\boldsymbol{\nu}_\ominus - \psi\nu_{\ominus 3}\|^2_{L_2(\Gamma_\ominus)} + \|\widehat{F}_\oplus - \widetilde{\mathbf{A}}_p\nabla\widehat{u}\cdot\boldsymbol{\nu}_\oplus - \psi\nu_{\oplus 3}\|^2_{L_2(\Gamma_\oplus)}\,, \tag{15}$$

where $\widetilde{\mathbf{B}}_p$ is the averaged “plane” part of the matrix $\mathbf{B} := \mathbf{A}^{-1}$ (i.e., if $\mathbf{B}(x) = (b_{ij}(x))_{i,j=\overline{1,3}}$, then $\mathbf{B}_p(x) = (b_{ij}(x))_{i,j=\overline{1,2}}$), the vector $\mathbf{b}_3 := \{b_{31}\,,\ b_{32}\}^T$ and $\mathbf{I}$ is the 2×2 identity-matrix.

Now we still have the freedom of choosing the auxiliary function ψ that in the case of the Poisson equation should, obviously, approximate the derivative $u_{,3}$ of the exact solution in the x_3-direction. The simplest choice is to take such a ψ that the term M_3 (i.e. the residual on the Neumann boundary condition) would be identically zero. This can be immediately achieved by letting $\psi(x) = \widehat{\alpha}(\widehat{x})\,x_3 + \widehat{\beta}(\widehat{x})$ with the coefficient functions $\widehat{\alpha}\,,\ \widehat{\beta} \in L_2(\widehat{\Omega})$ uniquely determined by the requirement $M_3 = 0$. Other options for the function ψ are considered in [7]. Then, minimizing the right-hand side of (11) with respect to the scalar parameters $\gamma > 0$ and $\delta > 0$, we arrive at the estimate

$$|||u - \widehat{u}||| \le M := M_1 + C_\Omega\, M_2\,, \tag{16}$$

where M_1 and M_2 are defined by (13) and (14). The error majorant M has been derived for quite general geometry of Ω and coefficient matrix $\mathbf{A}(x)$. However, to make the estimate more transparent, we consider two particular cases.

4.1 Plate of constant thickness

We assume that

$$d_\oplus = d_\ominus + d_0 \quad (d_0 = const\ > 0) \tag{17}$$

and, in addition, that

$$\mathbf{A} = \mathbf{A}(\widehat{x}) \ \ \text{(this immediately implies } \mathbf{B} = \mathbf{B}(\widehat{x}))\,, \tag{18}$$

$$a_{31} = a_{32} = 0 \ \ \text{(this yields } \mathbf{B}_p = \mathbf{A}_p^{-1}\,,\ b_{33} = a_{33}^{-1}\,,\ b_{31} = b_{32} = 0)\,. \tag{19}$$

With these assumptions the terms M_1 and M_2 in estimate (16) become simpler:

$$M_1 = \left(\int_\Omega a_{33}^{-1}\psi^2\,dx\right)^{1/2}, \qquad M_2 = \|f - \widetilde{f}\|_{L_2(\Omega)}\,. \tag{20}$$

One may notice that the integral in the first term M_1 of the error majorant M can be rewritten as

$$\int_\Omega a_{33}^{-1}\psi^2\,dx = d_0\cdot\int_{\widehat{\Omega}} a_{33}^{-1}(\widehat{\alpha}^2\frac{d_\oplus^2 + d_\oplus d_\ominus + d_\ominus^2}{3} + \widehat{\alpha}\widehat{\beta}(d_\oplus + d_\ominus) + \widehat{\beta}^2)\,d\widehat{x}\,,$$

which means that the term M_1 is of order $\mathcal{O}(d_0^{1/2})$ when the plate thickness d_0 tends to zero. If $f \in L_\infty(\Omega)$, the second term M_2 is obviously of the same order $\mathcal{O}(d_0^{1/2})$, i.e.

the whole estimator M converges to zero with the rate $\mathcal{O}(d_0^{1/2})$ as $d_0 \to 0$. This is the optimal convergence rate for the modelling error e in the energy norm, as was shown in [9] for the simpler case of a plate with plane parallel faces and $f = 0$. It is worth noting that, if $f \in C^1(\Omega)$, the second term in M is of higher order $\mathcal{O}(d_0^{3/2})$ as compared to the first term.

4.2 Plate with plane parallel faces

If, in addition to (18), (19), we strengthen the assumption (17) replacing it by

$$d_\oplus = \frac{d_0}{2}\,,\; d_\ominus = -\frac{d_0}{2} \quad (d_0 = const\; > 0)\,, \tag{21}$$

the auxiliary function ψ will take the simple form $\psi = \frac{\widehat{F}_\oplus + \widehat{F}_\ominus}{d_0}\, x_3 + \frac{\widehat{F}_\oplus - \widehat{F}_\ominus}{2}$ and the error estimate (16) will read

$$|||u - \widehat{u}||| \le \sqrt{\frac{d_0}{3}} \left(\int_{\widehat{\Omega}} a_{33}^{-1} (\widehat{F}_\oplus^2 + \widehat{F}_\ominus^2 - \widehat{F}_\oplus \widehat{F}_\ominus)\, d\widehat{x} \right)^{1/2} + C_\Omega \, \| f - \widetilde{f} \|_{L_2(\Omega)}\,. \tag{22}$$

If we set here $f = 0$, $a_{33} = 1$ and $\widehat{F}_\oplus = \widehat{F}_\ominus = \widehat{F}$, we obtain

$$|||u - \widehat{u}||| \le \sqrt{\frac{d_0}{3}}\, \|\widehat{F}\|_{L_2(\widehat{\Omega})}\,, \tag{23}$$

which is exactly the estimator of Babuška and Schwab (see [1]) for the zero-order reduced model. Thus, the latter estimator can be obtained as a particular case of the error majorant (16) if one makes the assumptions (18), (19), (21) and sets $f = 0$. This is a particularly interesting fact, since we advocate the estimation approach that is completely different from the one utilized in [1] (see the details in [7] and [6]).

5 Numerical example

In order to analyse the performance of the proposed error estimator, we consider a simple two-dimensional test problem in the "sine-shape" domain (see Figure 2 (left)) whose upper and lower faces are given by

$$d_{\oplus,\ominus}(x) = \sin(k\pi x) \pm \frac{d_0}{2}\,, \quad k = 1, 2, \dots\,,$$

where $d_0 > 0$ is the domain thickness. In this example, $\widehat{\Omega} = (0,1)$ and $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x \in \widehat{\Omega}\,,\; d_\ominus(x) < y < d_\oplus(x)\}$. The considered problem is

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega\,, \\ u &= 0 && \text{at } x = 0 \text{ and } x = 1\,, \\ \nabla u \cdot \boldsymbol{\nu}_{\oplus,\ominus} &= \widehat{F}_{\oplus,\ominus} && \text{at } y = d_{\oplus,\ominus}\,, \end{aligned}$$

and the right-hand sides of the equation and of the boundary condition are computed using the exact solution

$$u(x, y) = \sin(\pi x) \cdot y^m \quad (m = 1, 2, \ldots).$$

The reduced problem (8) is, in this case, a one-dimensional Dirichlet problem that, of course, can be solved very accurately (in the present work, we address the estimation of the modelling error only, assuming that the discretization error stemming from the solution of the reduced problem is negligible).

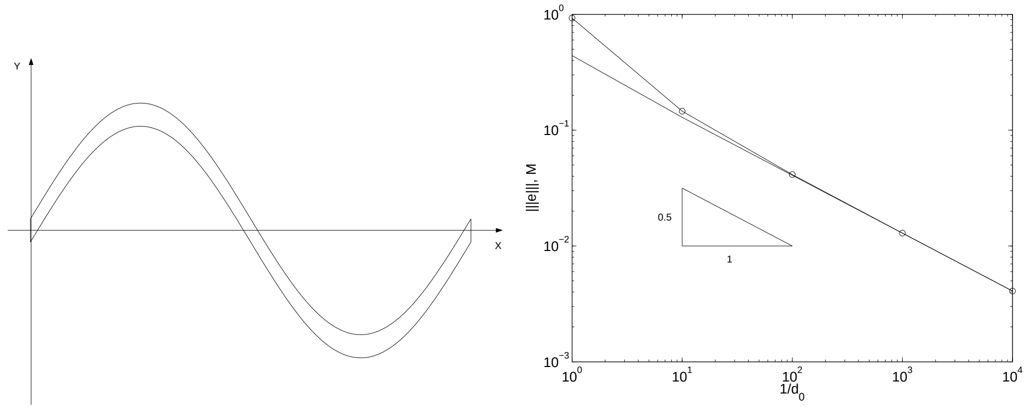

Fig. 2. (left) The domain geometry, (right) Convergence rate of the exact error and of the error majorant, $k = 2$, $m = 2$, the majorant is indicated by "$\circ$"

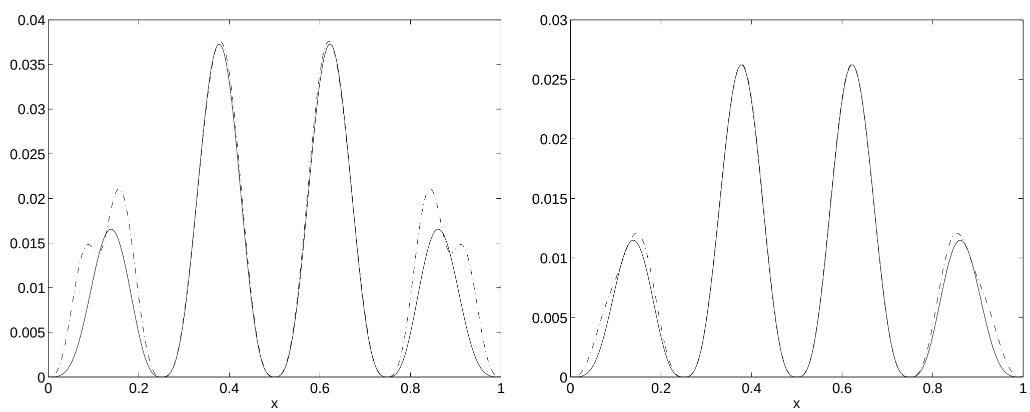

Fig. 3. Local error distribution provided by the exact error (solid line) and by the M_1-term of the majorant (dash-dot line), $k = 4$, $m = 4$: (left) $d_0 = 0.1$, (right) $d_0 = 0.05$

Figure 2 (right) shows the convergence rates of the exact modelling-error in the energy norm ($|||e|||$) and of the error majorant M as the domain thickness d_0 tends to zero. It is clear that both the exact error and the majorant converge to zero with the theoretically predicted, optimal rate $\mathcal{O}(d_0^{1/2})$, and, moreover, the *effectivity index* $\frac{M}{|||e|||}$ demonstrates the asymptotics $\frac{M}{|||e|||} = 1 + \mathcal{O}(d_0)$. It is also important to note that the presented error estimator provides a reliable upper bound for the exact error at any positive values of the domain thickness d_0, i.e. also in the cases when the domain is not “thin” at all.

Finally, the local error distribution provided by the exact error and by the first, M_1-term of the majorant are depicted in Figure 3. The figure shows that already for rather large value of the domain thickness $d_0 = 0.1$ the majorant delivers a sufficiently accurate information on the location of the regions of the biggest modelling error, while for $d_0 = 0.05$ the exact and the estimated error distributions are practically coincident.

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