Time Reversed Absorbing Conditions: Signal reconstruction and Application to inverse problems

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Content

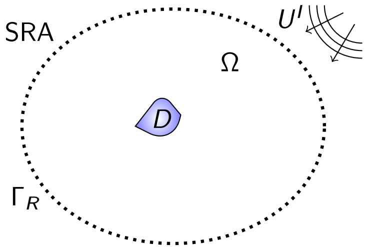
Outline

Forward problem

$\mathcal{L}$ is a hyperbolic equation: Maxwell, elasticity, wave equation, ...
An impinging wave U^I illuminates an unknwon inclusion D.

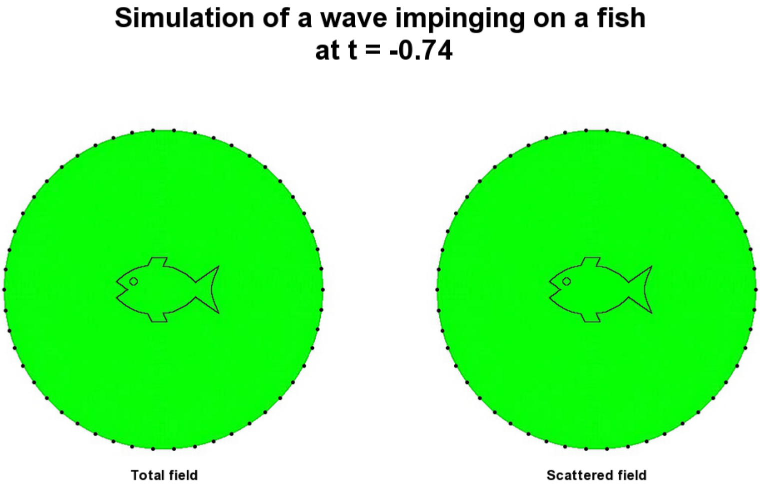
The total field U^T satisfies:

$$\begin{cases} \mathcal{L}(U^T) = 0 \text{ in } \mathbb{R}^d \\ U^S \text{ has finite energy at infinity} \\ \text{Homogeneous Initial Conditions} \end{cases}$$



The scattered field $U^S := U^T - U^I$ is recorded from $t = 0$ to $t = T_f$ on a Source-Receiver Array (SRA) located on a surface Γ_R that encloses a domain Ω.

Goal: Reconstruct the scattered field from the recorded data on Γ_R.



Numerical simulations were made using `Freefem++` (F. Hecht).

Next

Recreate the past from the SRA

Based on Time reversibility of hyperbolic equations.
Example: the wave equation

$$\mathcal{L}(U) = 0 \longrightarrow \rho\, u_{tt} - \mathrm{div}\left(\mu \nabla u\right) = 0$$

If $u(t,x)$ is a solution, $u(-t,x)$ is a solution as well since:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(-t,x)}{\partial t^2}$$

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TR approach: Find a BVP whose solution is the time reversed scattered field $U_R^S(t,\cdot) := U^S(T_f - t,\cdot)$. Thus, U_R^S satisfies:

$$\begin{cases} \mathcal{L}_0(U_R^S) = 0 & \text{in } (0,T_f) \times \Omega \backslash D \\ U_R^S(t,\cdot) = U^S(T_f - t,\cdot) & \text{on } (0,T_f) \times \partial\Omega \end{cases}$$

Problem: No boundary condition on ∂D.
This boundary value problem is underdetermined.

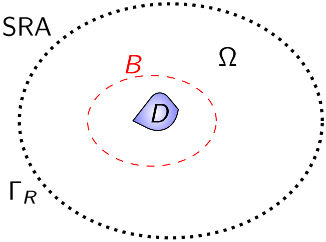
TRAC: time reversal and absorbing boundary conditions

Recall, U_R^S satisfies:

$$\begin{array}{rcll} \mathcal{L}_0(U_R^S) & = & 0 & \text{in } (0,T_f)\times\Omega\backslash D \\ U_R^S(t,\cdot) & = & U^S(T_f-t,\cdot) & \text{on } (0,T_f)\times\partial\Omega \end{array}$$

In order to remove the underdetermination, we introduce an artificial domain B enclosing D and solve the reversed problem in $\Omega\backslash B$.

Which BC on the artificial boundary ∂B ?



Recall, U_R^S satisfies:

$$\begin{aligned} \mathcal{L}_0(U_R^S) &= 0 && \text{in } (0, T_f) \times \Omega\backslash D \\ U_R^S(t,\cdot) &= U^S(T_f - t, \cdot) && \text{on } (0, T_f) \times \partial\Omega \end{aligned}$$

In order to remove the underdetermination, we introduce an artificial domain B enclosing D and solve the reversed problem in $\Omega\backslash B$.

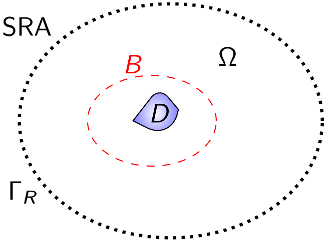
Which BC on the artificial boundary ∂B ?

The forward scattered field U^S satisfies a radiation condition at ∞, that we can approximate by an absorbing boundary condition (ABC):

$$\mathrm{ABC}(U^S) = 0 \text{ on } \partial B\,,$$

We time reverse it:

$$\mathrm{TRAC}(U_R^S) = 0 \text{ on } \partial B\,.$$



(Ref. for ABCs: Engquist & Majda 1977, Bayliss & Turkel 1980, Grote & Keller 1995)

Recreate the past via *TRAC*

Example: the wave equation

$$\mathcal{L}(U) \longrightarrow \rho\, u_{tt} - \mathrm{div}\,(\mu \nabla u)$$

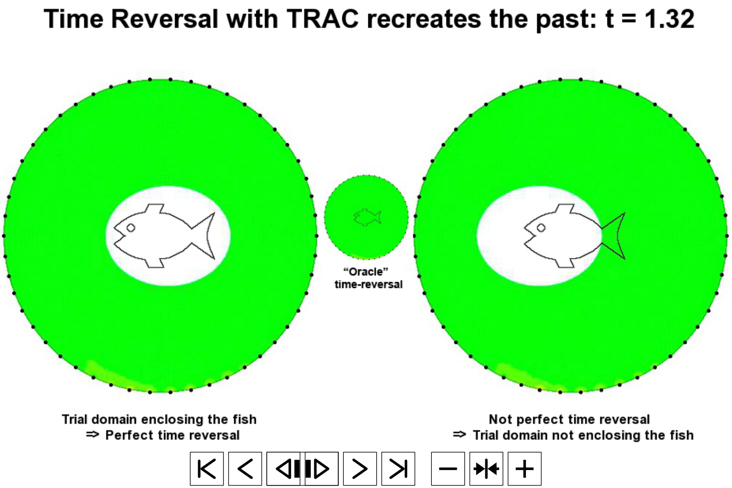
The *TRAC* problem reads

$$\begin{cases} \rho_0 \dfrac{\partial^2 u_R^S}{\partial t^2} - \mathrm{div}\,\left(\mu_0 \nabla u_R^S\right) = 0 & \text{in } \Omega \backslash B \\ u_R^S(t,\cdot) = u^S(T_f - t,\cdot) & \text{on } \Gamma_R \\ \mathrm{TRAC}(u_R^S) := \dfrac{\partial u_R^S}{\partial t} + c_0 \dfrac{\partial u_R^S}{\partial n} - c_0\, \kappa\, \dfrac{u_R^S}{2} = 0 & \text{on } \partial B \\ \text{zero Cauchy Data} & \end{cases} \quad (1)$$

where $c_0 = \sqrt{\mu_0/\rho_0}$ and κ is the curvature of ∂B.

By solving (1), we are able to recreate the past, namely reconstruct u^S in domain $\Omega \backslash B$.

Time reversal with *TRAC* (20% noise)



Left: B encloses the inclusion D; *TRAC* recreates the past

Right: B does not enclose D; past is not correctly recreated ▸ Next

Inverse problem via *TRAC*

TRAC has at least two applications in inverse problems:

1. The first application is the reduction of the size of the computational domain by redefining the reference surface on which the receivers appear to be located. This is reminiscent of the redatuming method, see Berryhill, 1979.
2. The second application is to identify an unknown inclusion D from boundary measurements.This is achieved by using a trial and error procedure on the trial domain B.

Outline

TRAC + AI methods

Illustration of the redatuming:

combination [1] of the *TRAC* method with an inverse problem technique, called Adaptive Inversion method

- the *TRAC* method (Time-Reversed Absorbing Condition method) [2] allows to reduce the computational domain and regularize the data
- the AI method (Adaptive Inversion method) [3] solve an inverse problem using mesh and basis adaptivity

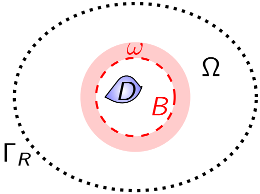
[1] M. de Buhan and M. Kray, *A new approach to solve the inverse scattering problem for waves: combining the TRAC and the Adaptive Inversion methods*, Inverse Problems, 29(8), 085009, 2013.

[2] F. Assous, M. Kray, F. Nataf, and E. Turkel, *Time Reversed Absorbing Condition: Application to inverse problems*, Inverse Problems, 27(6), 065003, 2011.

[3] M. de Buhan and A. Osses, *Logarithmic stability in determination of a 3D viscoelastic coefficient and a numerical example*, Inverse Problems, 26(9), 095006, 2010.

The interest of the *TRAC* method is threefold:

1. reduction of the computational domain by moving virtually the line of receivers from Γ_R to B



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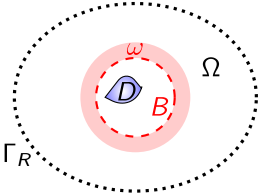

2. noise robustness: data regularization on the new virtual receivers' line (propagation of the wave by time reversal)

Redatuming with the *TRAC* method

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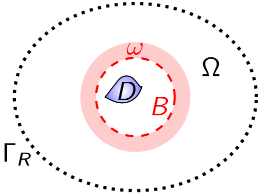
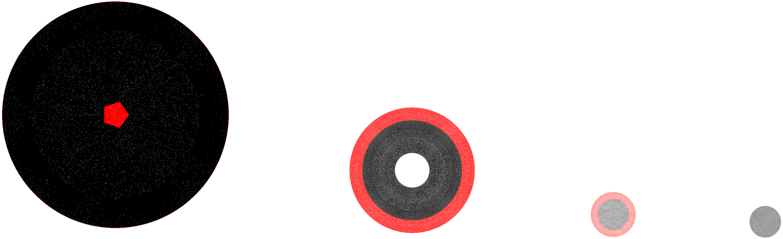


2. noise robustness: data regularization on the new virtual receivers' line (propagation of the wave by time reversal)

3. whole wave equation vs. paraxial approximation (redatuming in geophysics)

Size reduction

Reduction of the size of the computational domain: ratio 40



(a) (b) (c) (d)

(a) mesh for the forward problem, $R = 10\lambda$, 138678 vertices containing the source and inclusion D

(b) mesh of $\Omega \setminus B$ for the *TRAC* problem, $R = 5\lambda$, 31383 vertices

(c) mesh of $B \cup \omega$ for the inverse problem, $R = 2\lambda$, 2448 vertices

(d) mesh of B, $R = 1.4\lambda$, 3573 vertices

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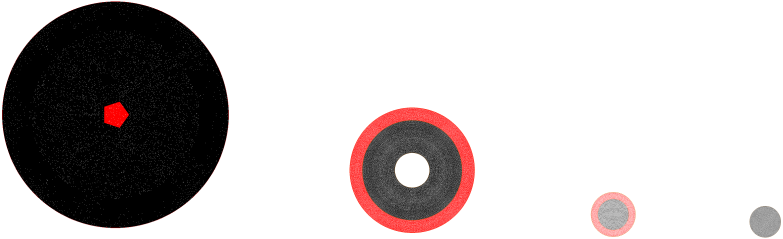

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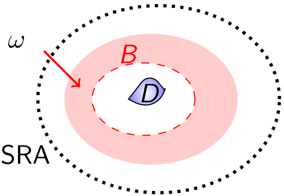
Reduction of the computational time: ratio 10

u_R^T time-reversed total field (exact)
v_R^T *TRAC* reconstruction

relative L^2-error:

$$E(v_R) = \frac{\|u_R^T - v_R^T\|_{L^2(\omega)}}{\|u_R^T\|_{L^2(\omega)}}$$

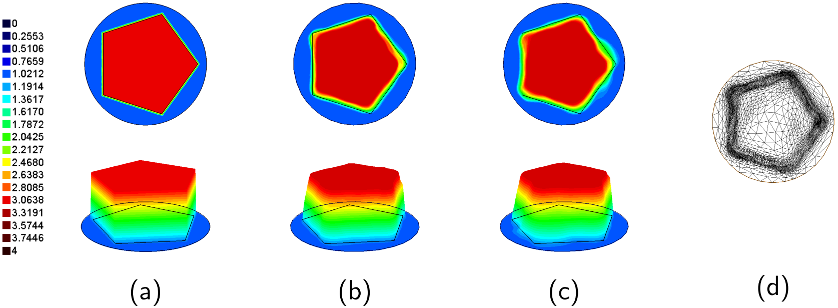
$\Gamma_R :=$SRA



Noise/Test	1	2	3	4	5	6	Mean value
0%	6.28%	3.14%	1.93%	3.92%	7.37%	5.18%	4.64%
5%	6.39%	3.37%	2.31%	4.15%	7.50%	5.31%	4.84%
10%	6.72%	4.03%	3.21%	4.68%	7.84%	5.78%	5.38%
20%	8.04%	5.98%	5.32%	6.28%	8.87%	7.21%	6.95%
Mean value	6.86%	4.13%	3.19%	4.76%	7.90%	5.87%	5.45%

Remark: penetrable inclusions such as $c_D = 3$ and $c_0 = 1$.

Reconstruction of a pentagon:



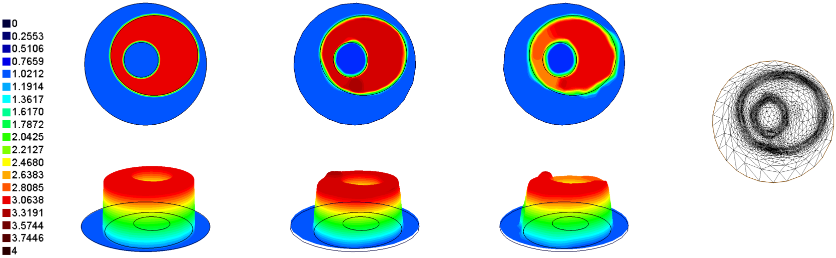
(a) Exact propagation speed in B

(b) Reconstruction with AI from exact data on ω

(c) Reconstruction with AI from TRAC *data on* ω *(with* 20% *of noise originally)*

(d) Final mesh through adaptative process

Reconstruction of a holed ellipse:



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(a) Exact propagation speed in B

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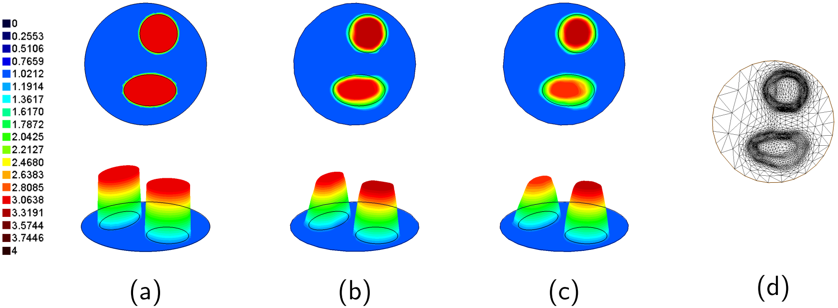
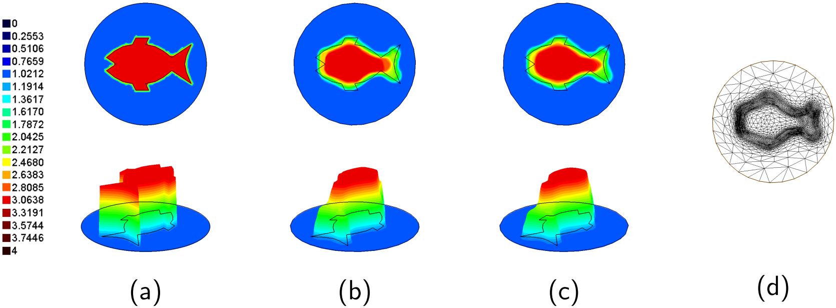
Reconstruction of two distinct ellipses:



(a) Exact propagation speed in B
(b) Reconstruction with AI from exact data on ω
(c) Reconstruction with AI from TRAC *data on* ω *(with* 20% *of noise originally)*
(d) Final mesh through adaptative process

Reconstruction of a fish:

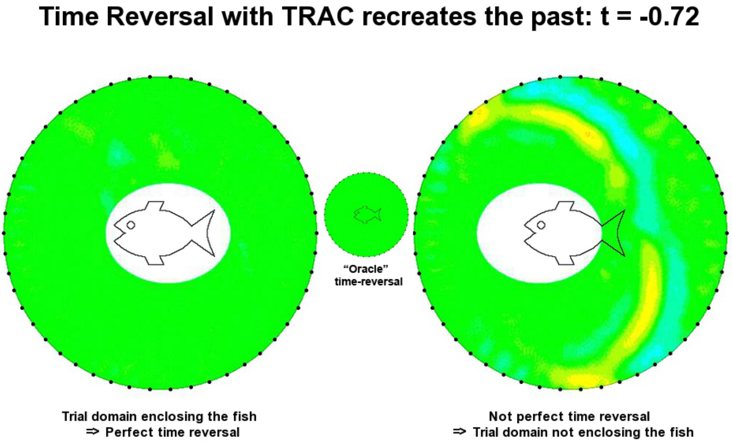


(a) Exact propagation speed in B
(b) Reconstruction with AI from exact data on ω
(c) Reconstruction with AI from TRAC *data on ω (with* 20% *of noise originally)*
(d) Final mesh through adaptative process

Outline

Principle to identify the inclusion:

- If B encloses the object D, the solution u_R^S to the time reversed BVP coincides with the time reversed of the "forward" solution u^S ("oracle").
- Conversely, if there is a difference between u_R^S and the reverse of the "forward" solution u^S, we know that domain B does not contain the inclusion.

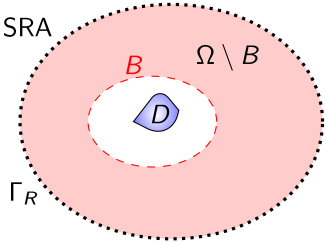

Time Reversal with TRAC recreates the past: t = -0.72
"Oracle"
time-reversal
Trial domain enclosing the fish
⇒ Perfect time reversal
Not perfect time reversal
⇒ Trial domain not enclosing the fish

Criterion of objet discrimination

In full aperture : final time criterion

$$J_{FT}(B) := \frac{\|v_R^S(T_f,\cdot)\|_{L^\infty(\Omega\setminus B)}}{\sup_{t\in[0,T_f]}\|u^I(t,\cdot)\|_{L^\infty(\Omega)}}$$

v_R^S computed time-reverse scattered field
u^I known forward incident wave



Only Final time solutions are displayed

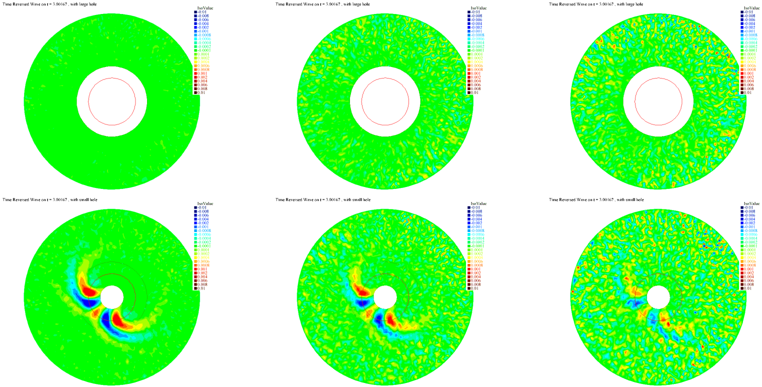


Figure: $Coeff = 10\%$, $Coeff = 30\%$, $Coeff = 50\%$

Aim: Objects identification and **counting**.

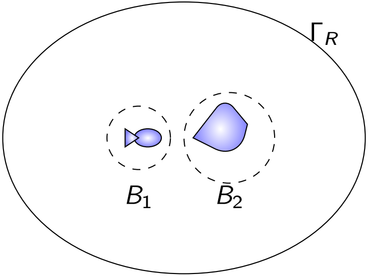
TRAC problem:

$$\begin{cases} \dfrac{\partial^2 u_R}{\partial t^2} - c^2 \Delta u_R = 0 & \text{in } (0,T) \times \Omega \setminus (B_1 \cup B_2) \\ u_R(t,\cdot) = u(T-t,\cdot) & \text{on } (0,T) \times \Gamma_R \\ \text{TRAC}_1(u_R) = 0 & \text{on } (0,T) \times B_1 \\ \text{TRAC}_2(u_R) = 0 & \text{on } (0,T) \times B_2 \\ \text{Homogeneous initial conditions} & \end{cases}$$

with

$$\text{TRAC}_k(u) := \left(-\frac{\partial}{\partial t} + c \frac{\partial}{\partial r_k} + \frac{c}{2 r_k} \right) u$$

and r_k the radial coordinate with origin at the center of B_k



Improvement of the *TRAC* method for multiscattering:

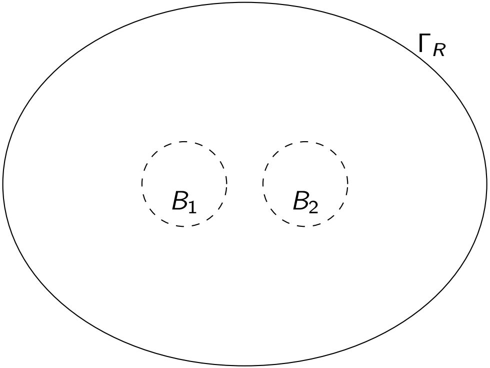
New TRAC problem:

$$\begin{cases} \dfrac{\partial^2 u_R}{\partial t^2} - c^2 \Delta u_R = 0 & \text{in } (0,T) \times \Omega \setminus (B_1 \cup B_2) \\ u_R(t,\cdot) = u(T-t,\cdot) & \text{on } (0,T) \times \Gamma_R \\ \text{TRAC}_1(u_R) = g_2 & \text{on } (0,T) \times B_1 \\ \text{TRAC}_2(u_R) = g_1 & \text{on } (0,T) \times B_2 \\ \text{Homogeneous initial conditions} & \end{cases}$$

$$g_\ell := \text{TRAC}_k(u_{\ell,R}) \qquad k \neq \ell$$

Problem: Find $u_{1,R}$ on B_2 (respectively $u_{2,R}$ on B_1)

Improvement of the *TRAC* method for multiscattering:
based on uniqueness of the decomposition of the field
see M.J. Grote & I. Sim, J. Comp. Phys. (2011)



Aim: Find u_1 and u_2 from the recorded data u on the boundary and the TRAC.

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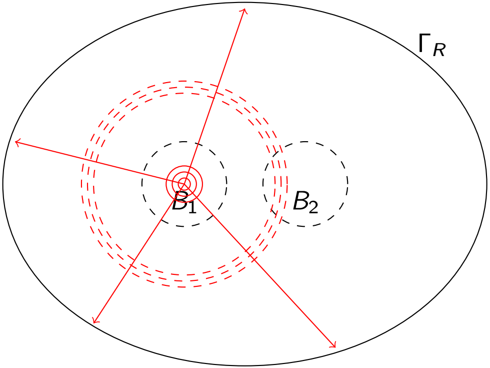

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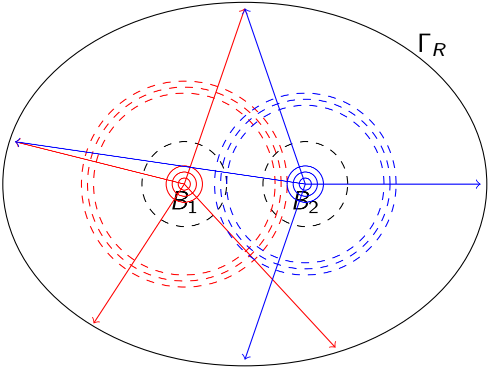

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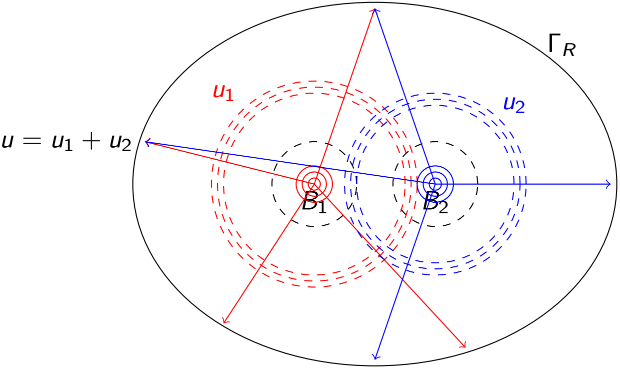
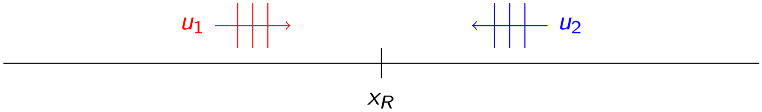


Aim: Find u_1 and u_2 from the recorded data u on the boundary and the TRAC.

TRAC and multiscattering

Analogous case in 1D

$$u(t,x) := u_1(x - ct) + u_2(x + ct)$$



On x_R, absorbing boundary condition

$$\underbrace{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}}_{\text{data}} = \underbrace{\frac{\partial u_1}{\partial t} + c\frac{\partial u_1}{\partial x}}_{=0} + \underbrace{\frac{\partial u_2}{\partial t} + c\frac{\partial u_2}{\partial x}}_{2\partial \mathbf{u_2}/\partial t}$$

By integrating in time on $[0, t]$ (initialization at 0), we get u_2, then u_1

Numerical results: accuracy at 10^{-14}

TRAC and multiscattering

2D case for θ-independent source functions

$$\begin{aligned} u(t,\vec{x}) &:= u_1(r_1 - ct) + u_2(r_2 - ct) \\ &= \frac{1}{\sqrt{r_1}} f_1(r_1 - ct) + \frac{1}{\sqrt{r_2}} f_2(r_2 - ct) \end{aligned}$$

Functions f_i are constant along the characteristics $r_i - ct$

Filter operators:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial r_i}\right)(\sqrt{r_i}u) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial r_i}\right)\left(\sqrt{\frac{r_i}{r_j}} f_j\right), \qquad \text{for } i \neq j$$

After a change of variable: first order ODE in time to get f_j
with homogeneous initial condition

Numerical result: error $\sim 1\%$

Next ?

- 2D case for θ-dependent functions (in progress)
- more than 2 sources
- 3D:

$$\begin{aligned} u(t,\vec{x}) &:= u_1(r_1 - ct, \theta_1, \phi_1) + u_2(r_2 - ct, \theta_2, \phi_2) \\ &= \frac{1}{r_1} f_1(r_1 - ct, \theta_1, \phi_1) + \frac{1}{r_2} f_2(r_2 - ct, \theta_2, \phi_2) \end{aligned}$$

Outline

Conclusion

Conclusion about the *TRAC* method

- stability and energy estimate
- time and frequency domain method
- redatuming without paraxial approximation
- for solid or penetrable object of any shape
- robust with respect to noise on the recorded data
- full and partial aperture of the SRA
- application to discrimination between one and two nearby inclusions

Actual work & prospects

- improvement of the TRAC for multiple scatterers (ABC for multiple scattering, Grote et al., 2007, 2011, joint work with Marcus Grote, Unibas)
- comparison with experimental data (joint work with Frédéric Nataf and Franck Assous)
- combination of the *TRAC* method and an inverse problem technique (joint work with Maya de Buhan, Paris 5)
- other equations: Maxwell, elasticity (ANR Medimax)

Thanks !

Publications :

- F. Assous, M. Kray, F. Nataf and E. Turkel, *Time Reversed Absorbing Conditions*, CR. Acad. Sci., (2010)
- F. Assous, M. Kray, F. Nataf and E. Turkel, *Time Reversed Absorbing Condition : Application to Inverse Problems*, Inverse problems (2011)
- F. Assous, M. Kray and F. Nataf, *Time Reversed Absorbing Condition in the Partial Aperture Case*, Wave Motion (2012)
- F. Assous, M. Kray and F. Nataf, *Retournement Temporel avec Conditions Absorbantes (TRAC) et Super-résolution : Reconstruction du passé et applications aux problèmes inverses*, I2M (2012)
- M. de Buhan and M. Kray, *A new approach to solve the inverse scattering problem for waves : combining the* TRAC *and the Adaptive Inversion methods*, Inverse Problems (2013)

all papers can be found on HAL.