Finite Elements for linear hyperbolic PDEs in polygonal domains

Fabian Müller Christoph Schwab

Seminar for Applied Mathematics, ETH Zurich

Linear second-order hyperbolic PDE

$$
\begin{aligned}
\partial_t^2 \mathbf{u}(\mathbf{x},t) - \mathcal{A}[\mathbf{u}](\mathbf{x},t) &= \mathbf{f}(\mathbf{x},t), \quad (\mathbf{x},t) \in Q,\\
\mathbf{u}(\cdot, t=0) &\equiv \mathbf{u}_0,\\
\partial_t \mathbf{u}(\cdot, t=0) &\equiv \mathbf{u}_1,\\
\mathbf{u}(\mathbf{x},t) &= \mathbf{0}, \quad (\mathbf{x},t) \in \Gamma_D \times I,\\
\mathcal{N}[\mathbf{u}](\mathbf{x},t) &= \mathbf{0}, \quad (\mathbf{x},t) \in \Gamma_N \times I,
\end{aligned}
$$

with...

$\mathbf{u}(\mathbf{x},t) \in \mathbb{C}^m$,

Polygonal domain $G \subseteq \mathbb{R}^2$,

Time interval $T_{\max} > 0$, $I := (0, T_{\max})$, define $Q := G \times I$.

Data $\mathbf{f}(\mathbf{x},t) \in L^2(I, L^2(G)^m)$, $\mathbf{u}_0 \in H^1(G)^m$, $\mathbf{u}_1 \in L^2(G)^m$.

Linear, elliptic second-order differential operator $\mathcal{A}$.

Operator $\mathcal{A}$ and its FEM-discretization

$$\mathcal{A}[\mathbf{u}](\mathbf{x}) := \sum_{k,l=1}^{2} \mathbf{A}_{kl}(\mathbf{x})\,\partial_k\partial_l\mathbf{u}, \quad \mathbf{A}_{kl} \in C^\infty(\bar{G}, \mathbb{C}^{m\times m}).$$

Symmetry: $\mathbf{A}_{kl} = \left(\overline{\mathbf{A}_{kl}}\right)^T$, $\mathbf{A}_{kl} = \mathbf{A}_{lk}$.
Ellipticity: $\exists\, c > 0$, s.t. for all tuples $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ with $\boldsymbol{\eta}_k \in \mathbb{R}^m$,

$$\sum_{k,l} \langle \mathbf{A}_{kl}\boldsymbol{\eta}_k, \boldsymbol{\eta}_l\rangle_{2,\mathbb{R}^m} \geq c \sum_k \|\boldsymbol{\eta}_k\|_2^2\ .$$

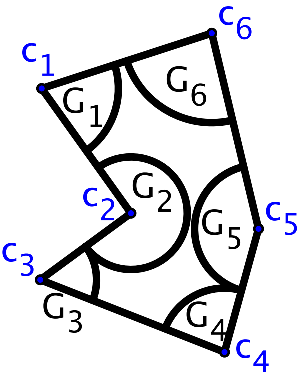
Weak form: Choose $V \subseteq H^1(G)^m$ s.t. for all $t \in I$:

$$\partial_t^2(\mathbf{u}(\cdot,t),\mathbf{v}) + a(\mathbf{u}(\cdot,t),\mathbf{v}) = (\mathbf{f}(\cdot,t),\mathbf{v}) \quad \forall \mathbf{v} \in V.$$

Semi-Discretization: conforming FEM on spatial Mesh $\mathcal{M}$:
$V_h := \mathcal{S}^{p,1}(\mathcal{M}, G)$ + Basis choice $\Rightarrow \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{A}\mathbf{u}(t) = \mathbf{l}(t)$

Domain

- Polygon $G \subseteq \mathbb{R}^2$, open, bounded, with straight sides.
- Corners $\mathbf{c}_i \in \partial G$, $i = 1, \ldots, M$.
- Interior opening angles $\phi_i \in (0, 2\pi]$.
- Border $\partial G = \Gamma_N \cup \Gamma_D$.
- Edge $\mathbf{e}_j$: $\mathbf{e}_j \in \Gamma_N$ (x-)or $\mathbf{e}_j \in \Gamma_N$.
- Local domains $G_i := G \cap B_{R_i}(\mathbf{c}_i)$.
- In G_i: Polar coordinates centered at $\mathbf{c}_i$ denoted by (r_i, ϑ_i).



Problem: Corner singularities in $\mathbf{x}$

- Convergence rate w. r. to #d.o.f.: For each $t \in I$,

$$|\mathbf{u}(\cdot,t) - \mathbf{u}_{\mathrm{FE},p}(\cdot,t)|_{H^1(G)} = O\left(N^{-\rho}\right), \quad N \to \infty,$$

with $\rho = \frac{\min(p+1,s)-1}{2}$, $N := \dim(V_h)$, $\mathbf{u} \in C^2(\bar{I}, H^s(G))^m$.

- Towards $\mathbf{c}_i$: $\mathbf{u}(\mathbf{x},t) \simeq c_i(t)\, r_i^{\lambda^{(i)}}\, \mathbf{\Phi}^{(i)}(\vartheta_i)$ + higher regularity,

  with $\Phi^{(i)} \in C^\infty([0,\phi_i])^m$, and generally, $\Re\lambda^{(i)} > 0$ only.

Problem: $\mathbf{x} \mapsto (r_i)^\lambda \in H^s(G)$ for $s < \Re\lambda + 1$ only.

$\Rightarrow \rho = \frac{s-1}{2} < \frac{\Re\lambda}{2}$ instead of $\frac{p}{2}$, for uniform refinement.

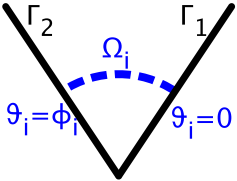
Outline

1. Asymptotics of $\mathbf{u}(\mathbf{x}, t)$ towards corners.
2. Local mesh refinements.
3. Semidiscrete convergence result.
4. Experiments.

Singular exponents and functions for $\mathcal{A} = \Delta$

Assumption: Wave equation, i.e. $m = 1$, $\mathcal{A} = \Delta$.

- Corner $\mathbf{c}_i$: Two edges $\Gamma_{1,2}$, can belong to Γ_D or Γ_N.
- Ω_i: Arc around $\mathbf{c}_i$, $\Omega_i \simeq (0, \phi_i)$.
- Boundary evaluation $\mathfrak{N}_i : H^2(\Omega) \to \mathbb{C}^2$, according to bdry. conditions at $\Gamma_{1,2}$.



Singular exponents: $\lambda_n^{(i)} = \sqrt[+]{\hat{\lambda}_n^{(i)}}$, where

$$-\partial_{\vartheta_i}^2 \Phi^{(i)}(\vartheta_i) = \hat{\lambda}^{(i)} \Phi^{(i)}(\vartheta_i), \quad \text{and} \quad \mathfrak{N}_i \Phi^{(i)} = (0,0).$$

Singular functions: $S_{n,i}(r_i, \vartheta_i) := r_i^{\lambda_n^{(i)}} \Phi_n^{(i)}(\vartheta_i)$, for $n \in \mathbb{N}$.

Singular exponents and functions: Values

Assumption: $m = 1$, $\mathcal{A} = \Delta$.

Consider different cases of $\Gamma_{1,2} \subseteq \Gamma_D$ or Γ_N.

pure Dirichlet: If $\Gamma_1 \cup \Gamma_2 \subset \Gamma_D$, for $n \in \mathbb{N}$,

$$\lambda_n^{(i)} = n\frac{\pi}{\phi_i}, \quad \text{and} \quad \Phi_n^{(i)}(\vartheta_i) := \sin(\lambda_n^{(i)}\, \vartheta_i).$$

pure Neumann: If $\Gamma_1 \cup \Gamma_2 \subset \Gamma_N$, for $n \in \mathbb{N}_0$,

$$\lambda_n^{(i)} = n\frac{\pi}{\phi_i}, \quad \text{and} \quad \Phi_n^{(i)}(\vartheta_i) := \cos(\lambda_n^{(i)}\, \vartheta_i).$$

Dirichlet-Neumann*: If $\Gamma_1 \subset \Gamma_D$, $\Gamma_2 \subset \Gamma_N$, for $n \in \mathbb{N}$,

$$\lambda_n^{(i)} = (n - 1/2)\,\frac{\pi}{\phi_i}, \quad \text{and} \quad \Phi_n^{(i)}(\vartheta_i) := \sin\left(\lambda_n^{(i)}\, \vartheta_i\right),$$

(or $\cos$, for Neumann-Dirichlet).

Singular exponents and functions: General case

General $\mathcal{A}$: More complicated eigenvalue problems to solve.

- Find normal eigenvalues $\lambda \in \mathbb{C}$ of

$$\mathfrak{A}(\lambda) := \left\{ r^{1-\lambda} \mathcal{A}[r^{\lambda} \cdot], r^{1-\lambda} \mathfrak{N}[r^{\lambda} \cdot] \right\}.$$

- Eigenvalues $\left(\lambda_n^{(i)}\right)_n$ isolated in $\mathbb{C}$.
- $\lambda_n^{(i)}$ with partial multiplicities $\kappa_J^{(i)} \leq \cdots \leq \kappa_1^{(i)}$, system of Jordan chains $\left\{ \mathbf{\Phi}_{n,(i)}^{(0,j)}, \ldots, \mathbf{\Phi}_{n,i}^{(\kappa_j - 1, j)}, j = 1, \ldots, J \right\}$.

$$\mathbf{S}_{n,i}^{k,j}(r_i, \vartheta_i) = r^{\lambda_n^{(i)}} \sum_{j=1}^{J} \sum_{k=0}^{\kappa_j - 1} \sum_{q=0}^{k} (\log(r))^q \, \mathbf{\Phi}_{n,i}^{(k-q,j)}(\vartheta_i)$$

Weighted Sobolev spaces

Although $\mathbf{x} \mapsto r^\lambda \notin H^2(G)$ for $\Re\lambda \leq 1$, but

$$\|r^\gamma \, \mathrm{D}^2 \mathbf{S}_{n,i}^{k,j}\|_{L^2(G)^m} < \infty, \quad \text{for } \gamma \in [0,1),\ \gamma > 1 - \Re\lambda_n^{(i)} .$$

The regularity scale for solutions $\mathbf{u}(\mathbf{x}, t)$ is (given $\gamma > 0, \beta \in \mathbb{R}$),

$$\|\mathbf{v}\|^2_{V^s_{\beta,\gamma}(G)^m} := \sum_{j,k=0}^{s} \int_I \int_G e^{-2\gamma t} \; r^{2(\beta+k-s)} \; \left|\partial_t^j \mathrm{D}^k \mathbf{v}(\mathbf{x})\right|^2 \, d\mathbf{x} .$$

Lemma

Given $q, s, s' > 0$, $\beta < -q$, $q + 1 \geq s + s'$. *Then, for all* $\gamma > 0$,

$$V^{q+1}_{\beta+q,\gamma}(G) \hookrightarrow H^s(I, H^{s'}(G)) .$$

Decomposition Theorem

[Kokotov, Plamenevskiĭ '99, '04]

Cut-off: $\chi_i \in C^\infty([0,\infty))$, with $\chi_i(r_i) = \begin{cases} 1 & \text{if } r < R_i/2 \\ 0 & \text{if } r > R_i \end{cases}$.

$\Rightarrow$ For all $s, s' \in \mathbb{N}_0$, $\mathbf{u}(\mathbf{x},t)$ admits a decomposition:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{reg}^{s,s'}(\mathbf{x},t) + \sum_{i\le M} \chi_i(r_i)\, \mathbf{u}_{sing,i}^{s,s'}(\mathbf{x},t),$$

$$\text{with } \mathbf{u}_{sing,i}^{s,s'}(\mathbf{x},t) := \sum_{n=1}^{N(i;s,s')} \sum_{k,j} c_{n,i}^{k,j}(\mathbf{x},t)\, \mathbf{S}_{n,i}^{k,j}(r_i,\vartheta_i),$$

where $c_{n,i}^{k,j} \in H^\alpha(I, C^\infty(\bar{G}))$, $\mathbf{u}_{reg}^{s,s'} \in H^s(I; H^{s'}(G))^m$,

generally, $\alpha < 3$, hence $\mathbf{u} \notin C^2(\bar{I}, L^2(G))^m$!

Regularity of $\mathbf{u}^{s,s'}_{sing,i}(\mathbf{x},t)$ I

Singularity in t: In general, $c^{k,j}_{n,i}(\mathbf{x},t) \notin C^2(\bar{I}, C^\infty(\bar{G}))$.

$\rightarrow$ ongoing research.

Theorem (Kokotov, Plamenevskiĭ)

Whenever $\mathbf{u}_0, \mathbf{u}_1 \in C^\infty_0(G)^m$ and $\mathbf{f} \in C^\infty_0(I, C^\infty_0(G))^m$, we have that $c^{k,j}_{n,i}(\mathbf{x},t)$ is $C^\infty(\bar{I}, C^\infty(\bar{G}))$.

Regularity of $\mathbf{u}_{sing,i}^{s,s'}(\mathbf{x},t)$ II

Recall: $\mathbf{u}_{sing,i}^{s,s'}(\mathbf{x},t) = \sum c_{n,i}^{k,j}(\mathbf{x},t)\mathbf{S}_{n,i}(r_i,\vartheta_i)$.

Singularity in $\mathbf{x}$:

- $c_{n,i}^{k,j}$ smooth in $\mathbf{x}$.
- $\mathbf{x} \mapsto r_i^\lambda \log(r_i)^q \in H^s(G_i)$ for $s < \Re\lambda + 1$.

  Example: $\mathcal{A} = \Delta, \partial G = \Gamma_D$: Worst case $\lambda_{\min} = {}^{\pi}/\max(\phi_i)$.
- But $r_i^\lambda \log(r_i)^q \in C^\infty((\varepsilon, R_i))$ for $\varepsilon > 0$.

"Singularity concentrated in $r = 0$!"

$\Rightarrow$ Can use locally refined meshes to recuperate optimal convergence rates.

Regularity of $\mathbf{u}^{s,s'}_{sing,i}(\mathbf{x},t)$ III

Concentration of singularities is not according to the intuition of "propagation".

- Fourier transform the time variable $t \mapsto \sigma$.
- Get a parametric problem

$$-\sigma^2\hat{\mathbf{u}}(\mathbf{x},\sigma) - \mathcal{A}[\hat{\mathbf{u}}](\mathbf{x},\sigma) = \hat{\mathbf{f}}(\mathbf{x},\sigma)$$

- Singularities for such parametric problems are concentrated in the corners, for all σ.

[Nazarov and Plamenevskiĭ, 1994]

Local mesh refinement: Overview

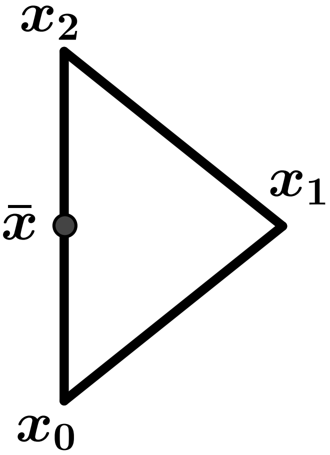
- Mesh family $\{\mathcal{M}_h\}_{h>0}$ on $G \Rightarrow V_h := S^{p,1}(\mathcal{M}_h, G)^m$.
- Refinement parameter $N_h := \dim(V_h) \simeq \#\mathcal{M}_h$.
- Separate errors

$$\|\mathbf{u}(\cdot,t) - \mathbf{u}_{FE,p}(\cdot,t)\| \leq \inf_{\mathbf{v}\in V_h} \|\mathbf{u}_{reg}^{s,s'}(\cdot,t) - \mathbf{v}\| + \sum_{i\leq M} \inf_{\mathbf{v}\in V_h} \|\chi_i \mathbf{u}_{sing,i}^{s,s'}(\cdot,t) - \mathbf{v}\| \,.$$

- Choose s, s' s.t. $\mathbf{u}_{reg}^{s,s'}(\cdot,t) \in H^{p+1}(G)^m$.
- Find $\{\mathcal{M}_h\}_h$, s.t. $\mathbf{u}_{reg}^{s,s'}(\cdot,t)$ and all $\chi_i \mathbf{u}_{sing,i}^{s,s'}(\cdot,t)$ are approximated with optimal rates.

Newest vertex bisection

- New vertex $\bar{\mathbf{x}} := \frac{1}{2}(\mathbf{x}_0 + \mathbf{x}_2)$.
- Children $T_1 := (\mathbf{x}_2, \bar{\mathbf{x}}, \mathbf{x}_1)$ and $T_2 := (\mathbf{x}_1, \bar{\mathbf{x}}, \mathbf{x}_2)$.
- Children bisected through $\bar{\mathbf{x}}$.
  $\Rightarrow$ "Newest vertex bisection" (NVB) .



Given $\mathcal{M} \subset \mathcal{T}_0$, there is a conforming refinement of $\mathcal{T}_0$ s.t. at least all $T \in \mathcal{M}$ are bisected!

(see e.g. [Nochetto et al. '09])

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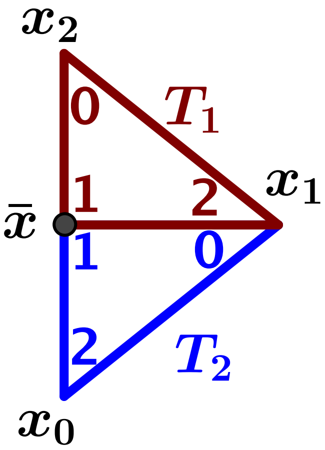

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(see e.g. [Nochetto et al. '09])

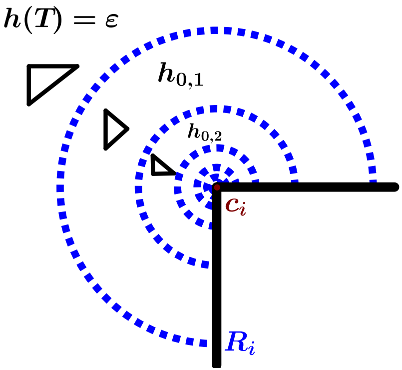
Local bisection refinement towards $\mathbf{c}_i$

General procedure:

Regular, conforming initial triangulation $\mathcal{T}_0$.

Tolerance $\varepsilon > 0$.

1. Refine until $h \leq \varepsilon$ **everywhere.**
2. For $l = 1 : K$
   bisect all elements T, with $\mathrm{dist}(\mathbf{c}_i, T) \leq 2^{-l} R_i$, and $h(T) > h_{0,l}$.



How to choose parameters $K, h_{0,l}$?

Local mesh refinement [Gaspoz/Morin '08]

Let $\mathbf{v}(x) = \mathbf{v}_{reg} + \sum_{i \leq M} \chi_i \mathbf{v}_i$, satisfy componentwise

$$|\mathrm{D}^k\,(\mathbf{v}_i)_j\,(\mathbf{x})| \leq c\, r^{\Re\lambda_i - k} \quad \forall\,|\alpha| \in \{0, 1, p+1\}, \mathbf{x} \in G^\circ,$$

for some $\lambda_i \in \mathbb{C}$, $\Re\lambda_i > 0$.

Set $\varepsilon > 0$, $\lambda \leq \min_i \Re\lambda_i$, and following parameters:

- K, such that $2^{-\lambda K/2(p+1)} \leq \varepsilon < 2^{-\lambda(K-2)/2(p+1)}$,
- $h_{0,l} \leq \varepsilon\, 2^{l(\lambda-p-1)/2(p+1)}$

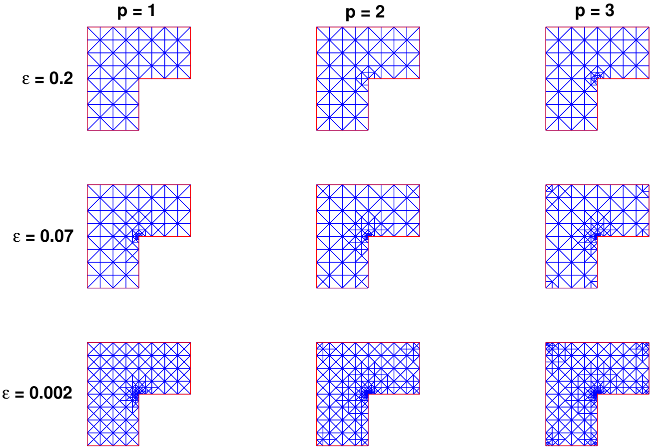
Yields mesh family $\{\mathcal{M}_h\}_h$, for which $\exists c, C > 0$ such that for $\varepsilon \to 0$,

$$\min_{\mathbf{w} \in V_h} |\mathbf{v} - \mathbf{w}|_{H^1(G)^m} \leq c\, N^{-p/2}, \text{ and}$$

$$\#\mathcal{M}_\varepsilon - \#\mathcal{M}_0 \leq C\, \varepsilon^{-2}\,.$$

Local mesh refinement

$h_0 = 0.2, \lambda = \frac{2}{3}$



Semidiscrete Convergence

Meshes, $p \in \mathbb{N} \Rightarrow \{V_h\}_{h>0}$, with $N_h := \dim(V_h) \xrightarrow{h\to 0} \infty$.

Semidiscrete formulation:

Find $\mathbf{u}_h \in C^0(\bar{I}, V_h)$, s.t. $\forall \mathbf{v} \in V_h$, $t \in I$:

$$\partial_t^2 (\mathbf{u}_h(\cdot,t), \mathbf{v}) + a(\mathbf{u}_h(\cdot,t), \mathbf{v}) = (\mathbf{f}(\cdot,t), \mathbf{v}) \quad + \text{initial cond.}$$

Well-known: If $\mathbf{u} \in C^2(\bar{I}, H^{p+1}(G))$, for all $t \in I$,

$$\begin{aligned} \|\mathbf{u}(\cdot,t) - \mathbf{u}_h(\cdot,t); H^1(G)\| + \|\partial_t \mathbf{u}(\cdot,t) - \partial_t \mathbf{u}_h(\cdot,t); L^2(G)\| \\ \leq \text{ Error in initial conditions} \\ + c\, N_h^{-\frac{p}{2}} \Big[\|\mathbf{u}(\cdot,t); H^{p+1}(G)^m\| + \|\partial_t \mathbf{u}(\cdot,t); H^{p+1}(G)^m\| \\ + \int_0^t \|\partial_t^2 \mathbf{u}(\cdot,s); H^{p+1}(G)^m\|\, ds\Big]. \end{aligned}$$

Main result

Let $p \in \mathbb{N}$, and assume that $\mathbf{f} \in C_0^\infty(I, C_0^\infty(G)^m)$, $\mathbf{u}_0, \mathbf{u}_1 \in C_0^\infty(G)^m$.

Then, there is a constant $c > 0$, such that

$$\|\mathbf{u}(\cdot,t) - \mathbf{u}_h(\cdot,t); H^1(G)^m\| + \|\partial_t \mathbf{u}(\cdot,t) - \partial_t \mathbf{u}_h(\cdot,t); L^2(G)^m\|$$
$$\leq \text{ Error in initial cond.}$$
$$+ c\, N^{-p/2} \left[\textbf{Weighted} \text{ norms of } \partial_t^j \mathbf{u}(\cdot,t),\ j = 0,1,2 \right].$$

Proof sketch

- Choose (s,s'), such that $\mathbf{u}_{reg}^{s,s'} \in C^2(\bar{I}, H^{p+1}(G))^m$.
- Split the solution $\mathbf{u} = \mathbf{u}_{reg}^{s,s'} + \sum_{i\leq M} \chi_i\, \mathbf{u}_{sing,i}^{s,s'}$, which yields

$$\|\partial_t^j \mathbf{u}(\cdot,t) - \mathbf{u}_h(\cdot,t)\|_{H^1(G)^m} \leq c \Bigg\{ \underbrace{\min_{\mathbf{w}\in V_h} \|\partial_t^j \mathbf{u}_{reg}(\cdot,t) - \mathbf{v}\|_{H^1(G)^m}}_{=O\left(N^{-p/2}\right)\ \text{OK}}$$

$$+ \sum_{i\leq M} \min_{\mathbf{w}\in V_h} \|\partial_t^j \mathbf{u}_{sing,i}^{s,s'}(\cdot,t) - \mathbf{w}\|_{H^1(G)} \Bigg\}.$$

- Approximation property for locally refined Meshes
  $\Rightarrow$ singular terms OK.

Convergence test 1: Linear FEM on L-Shaped Domain

- L-Shaped domain, $\phi_1 = {}^{3\pi}/_2$, and $\phi_i = {}^{\pi}/_2$, $i = 2, \ldots, 6$.
- $m = 1$, scalar Wave equation, i.e. $\mathcal{A} = \Delta$.
- pure (nonhomogeneous, smooth) Dirichlet conditions

$$\Rightarrow \min_i \lambda_n^{(i)} = \begin{cases} \frac{2}{3} & i = 1 \\ 2 & i > 1 \,. \end{cases}$$

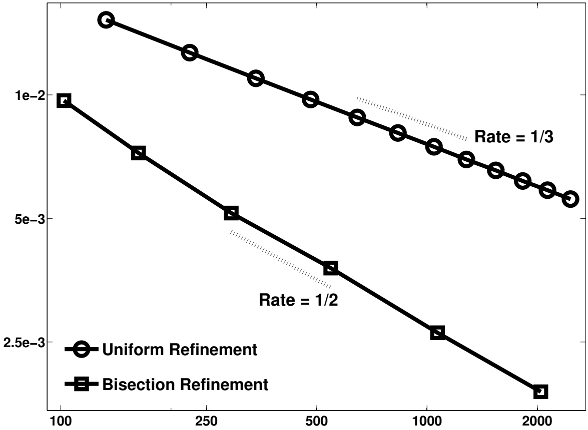
- Linear FEM in space, Crank-Nicolson with $\Delta t = 10^{-6}$ to "cancel" error in time.
- Exact solution: Term with least regularity $= u^{3,2}_{sing,i}$.

$$u(\mathbf{x}, t) := \sin(\pi t) r^{2/3} \sin({}^{2\vartheta}/_3) \in C^\infty(\bar{I}; H^{5/3-\delta}(G)),$$

for arbitrarily small $\delta > 0$.
- With uniform refinement, expect convergence rate $\rho < \frac{1}{3}$.

Convergence test 1

$\|u - u_h; L^2(I; H^1(G))\|$ vs. $\#\mathcal{M}$.



Convergence test 2

Quadratic FEM on L-Shaped Domain

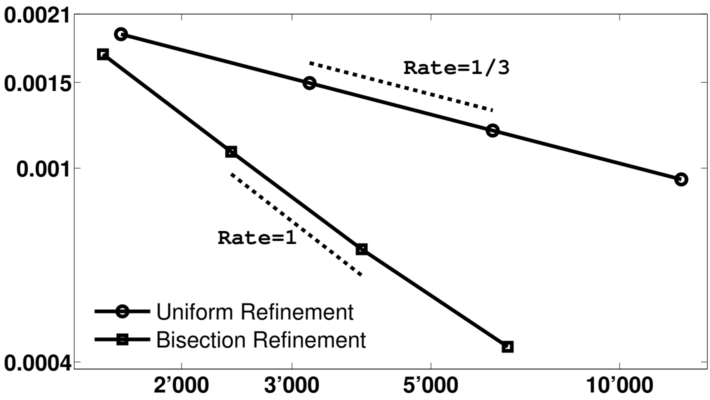
- L-Shaped domain, $\mathcal{A} = \Delta$, $m = 1$.
- pure (nonhomogeneous, smooth) Dirichlet conditions.
- Quadratic FEM in space, Crank-Nicolson in time.
- Exact solution:

$$u(\mathbf{x}, t) := \sin(\pi t) r^{2/3} \sin(2\vartheta/3)$$

- With uniform refinement, expect convergence rate $\rho < \frac{1}{3}$.

Convergence test 2

$\|u - u_h; L^2(I; H^1(G))\|$ vs. $\#\mathcal{M}$.

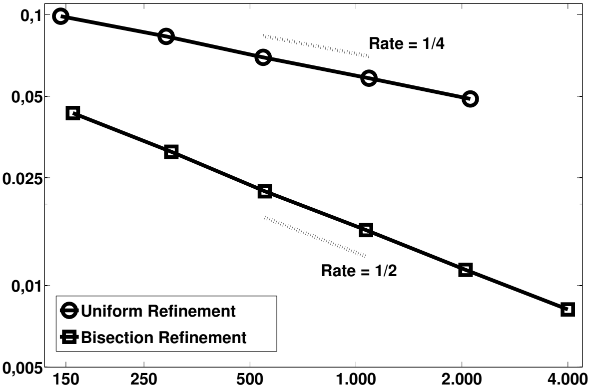


Convergence test 3: Linear FEM on cracked Domain

- $G := [-1,1]^2 \setminus ([0,1] \times \{0\})$, $\phi_1 = 2\pi$, and $\phi_i = \pi/2$, $i = 2, \dots, 6$.
- $m = 1$, scalar Wave equation, i.e. $\mathcal{A} = \Delta$.
- pure (nonhomogeneous, smooth) Dirichlet conditions
$$\Rightarrow \min_i \lambda_n^{(i)} = \begin{cases} \frac{1}{2} & i = 1 \\ 2 & i > 1 \end{cases}.$$
- Linear FEM in space, Crank-Nicolson in time.
- Exact solution: Term with least regularity $= u^{3,2}_{sing,1}$.
$$u(\mathbf{x}, t) := \sin(\pi t) r^{1/2} \sin(\vartheta/2) \in C^\infty(\bar{I}; H^{3/2-\delta}(G)),$$
for arbitrarily small $\delta > 0$.
- With uniform refinement, expect convergence rate $\rho < \frac{1}{4}$.

Convergence test 3

$\|u - u_h; L^2(I; H^1(G))\|$ vs. $\#\mathcal{M}$.



Conclusion

Regularity Generally, solutions to linear, second-order hyperbolic systems on polygonal domains exhibit corner singularities in space and time.

Restrictions on data The C^2-regularity in time needed for a suitable semidiscrete convergence result is only given, if data is C_0^∞.

Method of lines, FEM of degree p To recuperate the lost optimal H^1-convergence rate $O(N^{-p/2})$ in space, locally refined mesh families can be used to discretize in space, as for elliptic PDEs.

Preprint: SAM Report 2013-11, `http://www.sam.math.ethz.ch`

Thank you for your attention