Approximation of Geometric Data

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Outline

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Goal: Solving PDE's an optimization problems where we seek a function with values in a riemannian Manifold.

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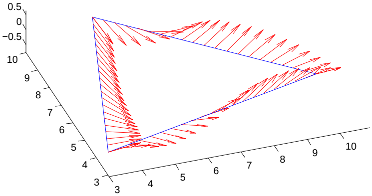
$$f : \Omega \subset \mathbb{R}^3 \to S^2 = \{x \in \mathbb{R}^3 | \, \|x\|_2 = 1\}$$

Boundary value problem with sphere-valued function

Given $\Omega \subset \mathbb{R}^2$ and $g : \delta\Omega \to S^2$ we are interested in minimizing the energy

$$f_{opt} = \underset{f:\Omega \to S^2}{\operatorname{argmin}} \int_\Omega \|\nabla f(x)\|^2 dx$$

subject to $f = g$ on $\delta\Omega$. Here $\|\nabla f(x)\|^2 = \sum_{i=1}^3 \sum_{j=1}^2 \left(\frac{df_i(x)}{dx_j}\right)^2$.



Other Applications

- Rigid Body Motion
- Image Processing
- Diffusion Tensor Imaging
- ...

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How to solve optimization problems numerically?

- Characterize the solution u as the minimum of some functional J on a function space V.
- Consider a subspace V_h of functions which can be handled by a computer, i.e. each function is characterized by finitely many values.
- Find the minimum u_h of the functional on this subspace.

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Theorem ((Linear) Céa)

Let **a** *be a coercive bilinear form,* **L** *a linear form and* u *and* u_h *the solution of*

$$a(u,v) = L(v) \qquad \forall v \in V \ (resp. V_h)$$

then

$$\|u - u_h\| \leqslant C \ \underset{w \in V_h}{argmin} \|w - u\| .$$

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Theorem (Grohs, Hardering, Sander 2012)

If J is elliptic along geodesic homotopies.

$$D(u, u_h) \leqslant C \ \underset{w \in V_h}{argmin} D(w, u)$$

where D is a distance function.

In order to bound the error between the exact solution and the approximation by geodesic finite elements we need to bound the best approximation error of the "finite element space".

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If Q^h is exact for constant functions we have

$$\sum_{i \in I} \psi_i(x) = 1.$$

hence for all $x \in \mathbb{R}^n$, $Q^h f(x)$ is an weighted average of function values at the grid points.

For points $x_1, \ldots, x_n \in M$ and weights $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with sum 1 we define the Riemannian average by

$$av_M((x_i)_{i=1}^n, (\lambda_i)_{i=1}^n) = \underset{x^* \in M}{argmin} \sum_{i=1}^n \lambda_i d^2(x^*, x_i).$$

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If $M = \mathbb{R}^m$ the Riemannian average reduces to the affine combination. Hence av_M is a generalization of affine combinations to the manifold-valued setting.

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The "finite element space" is

$$\{x \mapsto av_M((c_i)_{i\in I}, (\psi_i(h^{-1}x))_{i\in I}) | c_i \in M \text{ s.t. av well defined}\}.$$

Each function can be represented by the values $(c_i)_{i\in I}$.

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Theorem (Jackson)

If $M = \mathbb{R}^m$, $f \in C^k$ and Q^h is exact for polynomials of degree smaller than k then

$$\left\| D^l(Q^h f(x) - f(x)) \right\| \leqslant Ch^{k-l}.$$

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Does the same result hold for manifold-valued functions?

Theorem (Grohs, Hardering, Sander)

For Lagrange-Interpolation (i.e. $\psi_i(x_j) = \delta_{ij}$) the above statement is true for $l = 1$ and the L_∞ as well as the L_2-norm.

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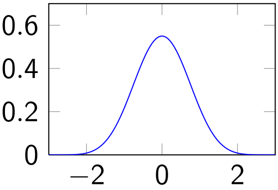
For odd $m \in \mathbb{N}$ there exists $B_m : \mathbb{R} \to \mathbb{R}$ such that

- $B_m \in C^{m-1}$
- $\text{supp}(B_m) = [-(m-1)/2, (m-1)/2]$
- $B_m|_k^{k+1} \in \Pi_m$ for all $k \in \mathbb{N}$



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Spline functions are only exact for polynomials of degree 1.
But we can consider linear combinations of the shifted spline function which are exact for polynomials up to degree n.

$$\phi_m(x) = \sum_{j=-(m-1)/2}^{(m-1)/2} a_j B_m(x-j).$$

E.g. if $m = 3$ then $(a_{-1}, a_0, a_1) = (-1/6, 4/3, -1/6)$ if $m = 5$ then $(a_{-2}, \dots, a_2) = (13/240, -7/15, 73/40, -7/15, 13/240)$

If $M = \mathbb{R}^n$:

$$\begin{aligned}
Q^h f(x) &= \sum_i \phi_m(h^{-1}x - i) f(hi) && (1) \\
&= \sum_{i,j} a_j B_m(h^{-1}x - i - j) f(hi) && (2) \\
&= \sum_k \left(\sum_j a_j f(h(k-j)) \right) B_m(h^{-1}x - k) && (3)
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Numerical experiment

$$\mathbb{R} \to S^2 : x \mapsto \begin{pmatrix} \cos(\cos(x)+1)\cos(\cos(x)) \\ \cos(\cos(x)+1)\sin(\cos(x)) \\ \sin(\cos(x)+1) \end{pmatrix}.$$

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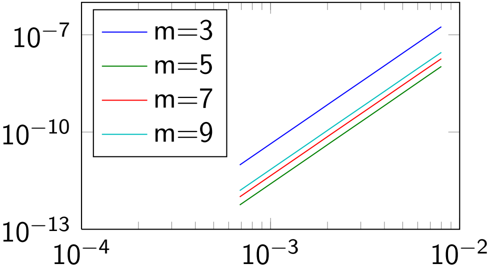

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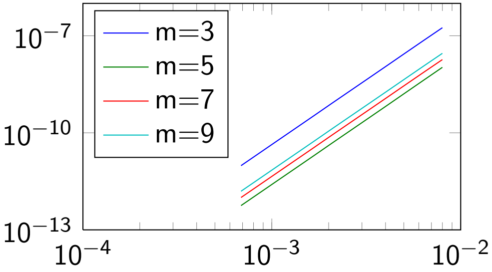

Only order 4 approximation!

For other approximations using subdivisions a similar behavior has been observed. [G. Xie and T. Yu]

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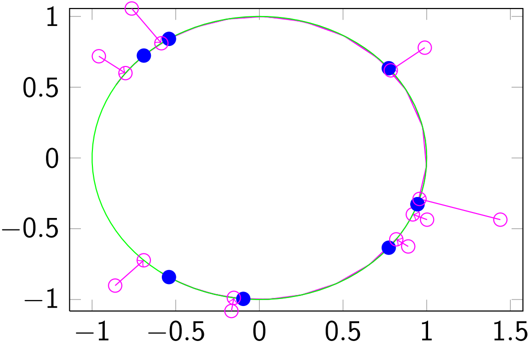
Are there any higher order approximations?

Consider $f : \mathbb{R} \to S^1$



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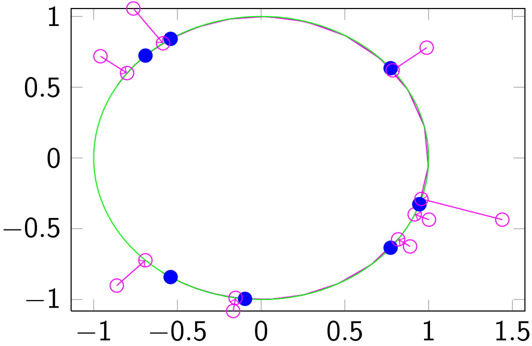

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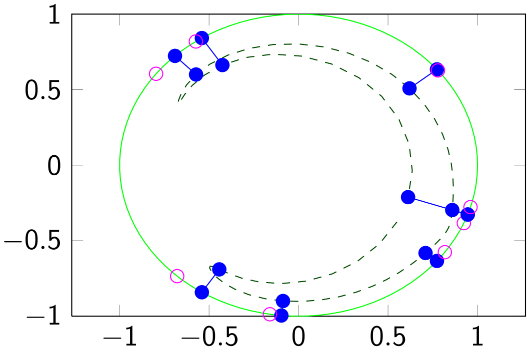
Orthonormal Perturbations



Consider $\overline{f}(x) = g_h(x) f(x)$ with $g_h \colon \mathbb{R} \to \mathbb{R}_+$ such that

$$\sum_j a_j \overline{f}(h(k-j) \in M.$$

Consider a 1-periodic function $f : \mathbb{R} \to S^m$.
Let $n = 1/h$ and $z_1, \ldots, z_n \in \mathbb{R}$ such that

$$c_k := \sum_j a_j (z_{k-j} f(h(k-j))) \in S^m \tag{4}$$

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Theorem

For h small enough there is exactly one solution $z_1, \ldots, z_n$.

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We can define a 1-periodic function $g_h : \mathbb{R} \to \mathbb{R}_+$ and

$$\overline{f}_h(x) = f(x) g_h(x) \tag{5}$$

such that

$$\sum_j a_j \overline{f}(x + h(k-j)) \in S^m \tag{6}$$

Theorem

The function g_h has the same smoothness as f and its derivatives are uniformly bounded independent of h.

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Choosing $c_k = \sum_j a_j \overline{f}(h(k-j)) \in S^m$ *and define* $Q^h f(x) = P_M \left(\sum_k B_m(h^{-1}x - k) c_k \right)$ *will yield an order* $m+1$*-approximation.*

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Proof:

$$\left\| f(x) - P_M\left(\sum_k B_m(h^{-1}x-k)c_k\right)\right\| \tag{7}$$

$$\leqslant \left\| P_M(\overline{f}(x)) - P_M\left(\sum_i \phi_m(h^{-1}x-i)\overline{f}(hi)\right)\right\| \tag{8}$$

$$\leqslant C_1 \left\| \overline{f}(x) - \left(\sum_i \phi_m(h^{-1}x-i)\overline{f}(hi)\right)\right\| \tag{9}$$

$$\leqslant C_2 h^{\alpha} \tag{10}$$

How to compute c_k?

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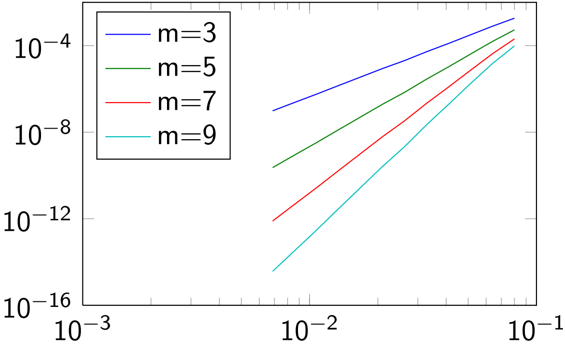
Let $f_1, \dots, f_d$ be the values of f at the points $h, 2h, \dots$ and let $g_1, \dots, g_d$ be our unknowns such that $g_h(hi) = g_i$. As before let $c_i = \sum_k a_k f_{i-k} g_{i-k}$. We find the solution by solving the system of equations

$$\langle c_i, c_i \rangle - 1 = 0 \qquad \forall i \in \{1, \dots, n\}$$

using Newton's Method.

Numerical Test: $\mathbb{R} \to S^2 : x \mapsto \begin{pmatrix} \cos(4\cos(x))\cos(5\cos(x)) \\ \cos(4\cos(x))\sin(5\cos(x)) \\ \sin(4\cos(x)) \end{pmatrix}.$



Some Remarks

- Generalization to Manifolds which are given as $M = \{x \in \mathbb{R}^K : g(x) = 0\}$ where $g: \mathbb{R}^K \to \mathbb{R}^L$ and Dg is of maximal rank is possible.
- Existence of quasiinterpolation operator already sufficient for analysis of numerical solution of PDEs

Thank You for your attention!
Any Questions?