Modern Numerical Methods with Medical Applications
Part II: Multigrid Iteration

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La Habana, June 2023

1 Motivation

The FE stiffness matrix L_h is a sparse matrix of size $n_h \times n_h$, where n_h is very large. Typical value $n_h \gtrsim 1.000.000$.

For its solution one needs methods with a cost being (almost) **linear** in n_h.

Direct methods (Gauss elimination, Cholesky decomposition) cost up to $O(n_h^3)$ operations, traditional iterative methods cost $O(n_h h^{-1})$ to $O(n_h h^{-2})$ operations.

The **multigrid method** applies to rather general discretisations of elliptic PDEs and has linear cost.

Literature:

W.Hackbusch: Multi-Grid Methods and Applications. Springer 1985 and 2003
—: Iterative Solution of Large Sparse Systems of Equations, 2nd ed., Springer 2016

1.1 Linear Iterations

1.1.1 Notations

Linear system: $Lu = f$ with $L \in \mathbb{R}^{n \times n}$, $u, f \in \mathbb{R}^n$.
A general one-step method

$$u^j \mapsto u^{j+1} := \Phi(u^j, f)$$

is called a linear iteration if Φ is linear in both arguments:

$$\Phi(u, f) = Mu + Nf.$$

Consistency: The exact solution $u^* = L^{-1}f$ should be a fixed point of Φ for all f :

$$L^{-1}f = u^* = \Phi(u^*, f) = ML^{-1}f + Nf$$

This implies $L^{-1} = ML^{-1} + N$, i.e.,

$$M + NL = I \qquad \text{(consistency condition)}.$$

Then Φ becomes

$$\Phi(u, f) = u - N\,(Lu - f)\,.$$

1.1.2 Convergence

The iteration error $u^j - u^*$ (u^* solution of $Lu = f$) satisfies

$$u^{j+1} - u^* = M\left(u^j - u^*\right) \quad \text{and therefore} \quad \left\|u^{j+1} - u^*\right\| \leq \|M\| \left\|u^j - u^*\right\|.$$

$\|M\| < 1$ is sufficient for convergence $u^j \to u^*$ ($\|M\|$: contraction number).

A necessary and sufficient condition for convergence is

$$\rho(M) < 1$$

where

$$\rho(M) := \max\{|\lambda| : \lambda \text{ eigenvalue of } M\}$$

is the *spectral radius*.

LEMMA: a) $\rho(M) \leq \|M\|$. b) $M = M^{\mathsf{H}} \Rightarrow \rho(M) = \|M\|$.

1.1.3 Classical Examples of Iterations

Jacobi iteration: $N = D^{-1}$ ($D = \text{diag}(M)$),
i.e. $u^j \mapsto u^{j+1} := u^j - D^{-1}(Lu^j - f)$.

procedure Jacobi(u, f); **array** u, u^{old}, f; **integer** i, j;
begin $u^{old} := u$; **for all** i **do** $u[i] := u^{\text{old}}[i] - \left(\sum_j \left(L[i,j]u^{\text{old}}[j]\right) - f[i]\right)/L[i,i]$
end;

Gauss-Seidel iteration:
procedure GS(u, f); **array** u, f; **integer** i, j;
for $i := 1$ **to** n **do** $u[i] := u[i] - \left(\sum_j (L[i,j]u[j]) - f[i]\right)/L[i,i]$;

$\Rightarrow N = (L_{lower} + D)^{-1} L_{upper}$ where $L = L_{lower} + D + L_{upper}$ (lower triangular / upper triangular part)

Assume $L = L^{\mathsf{H}}$. Then
Jacobi converges if $2D > L > 0$;
Gauss-Seidel converges if $L > 0$.

1.1.4 Speed of Convergence

Discretisation of second order pde with step size h.

$\Rightarrow$ condition number $\|M\|\,||M^{-1}|| = O(h^{-2})$

3D case: $n \sim h^{-3}$.

Jacobi and Gauss-Seidel: contraction number is $1 - O(h^2)$

SOR (successive overrelaxation): $1 - O(h)$

1.1.5 Cost of the iterative scheme

1 iteration step costs $O(n)$ operations (sparse matrix!)

Assume that we want $\left\|u^j - u^*\right\| \cong \varepsilon$ starting from $u^0 := 0$.

$(1 - O(h^\kappa))^m = \varepsilon$ requires $m = O(h^{-\kappa} |\log \varepsilon|)$ iterations

Cost of Jacobi or Gauss-Seidel: $O(nh^{-2} |\log \varepsilon|) = O(h^{-5} |\log \varepsilon|) = O(n^{5/3} |\log \varepsilon|)$

Cost of SOR: $O(nh^{-1} |\log \varepsilon|) = O(h^{-4} |\log \varepsilon|) = O(n^{4/3} |\log \varepsilon|)$

Optimal case would be a contraction number $\zeta < 1$ *independent* of h. Then the cost is $O(n |\log \varepsilon|)$. This is the case of the *multigrid method*.

We shall even obtain $O(n)$ for $\varepsilon =$ discretisation error $= h^\kappa$.

The multigrid approach is based on two ingredients:

- smoothing property
- coarse-grid correction

1.2 Smoothing Property

1D Example: $-u'' = f$ in [0,1] and $u(0) = u(1) = 0$

Discretisation: $h^{-2}\left[-u_h(x-h) + 2u_h(x) - u_h(x+h)\right] = f(x)$

This yields the system $L_h u_h = f_h$ with $u_h = [u_h(h), u_h(2h), u_h(3h), \ldots, u_h(1-h)]^{\mathsf{T}}$ and the sparse matrix

$$L_h = h^{-2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n_h \times n_h} \quad \text{with } n_h = \frac{1}{h} - 1.$$

The eigenvalue problem

$$L_h e_h = \lambda_h e_h$$

is solved by the eigenvectors

$$e_h^\mu(x) = \sin(\mu\pi x) \qquad \text{for } x = h, 2h, 3h, \ldots, 1-h$$

and eigenvalues

$$\lambda_h^\mu = 2\,(1 - \cos\pi h\mu) \qquad \text{for } \mu = 1, \ldots, n_h = \frac{1}{h} - 1$$

Jacobi iteration:

$$u_h^j \longmapsto u_h^{j+1} = u_h^j - D_h^{-1}\left(L_h u_h^j - f_h\right) = M_h u_h^j + D_h^{-1} f_h$$

with the iteration matrix $M_h := I - D_h^{-1} L_h$, where $D_h = \mathrm{diag}(L_h)$.

Damped Jacobi iteration:

$$u_h^j \longmapsto u_h^{j+1} = u_h^j - \frac{1}{2} D_h^{-1}\left(L_h u_h^j - f_h\right) \;\Rightarrow$$

iteration matrix $M_h := I - \frac{1}{2} D_h^{-1} L_h$.

In this case, M_h is symmetric $\Rightarrow \|M_h\| = \rho(M_h) := \max\{|\text{eigenvalues of } M_h|\}$.

Eigenvalues of the iteration matrix: $\lambda_\mu = 1 - 4\omega \sin^2(\mu\pi h/2)$, $1 \le \mu \le n_h$,

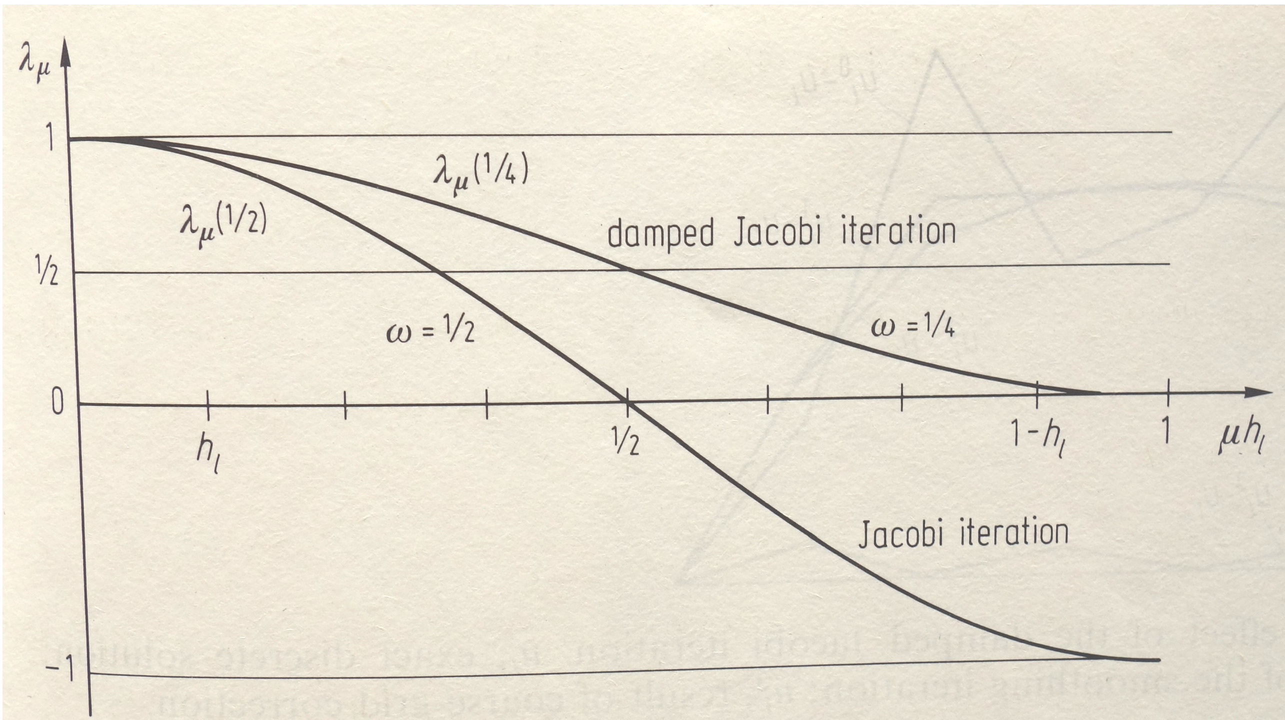
with $\omega = \begin{cases} 1/2 & \text{for standard Jacobi} \\ 1/4 & \text{for damped Jacobi} \end{cases}$

Splitting of $V_h := \mathbb{R}^{n_h} = V_{\text{low}} \oplus V_{\text{high}}$

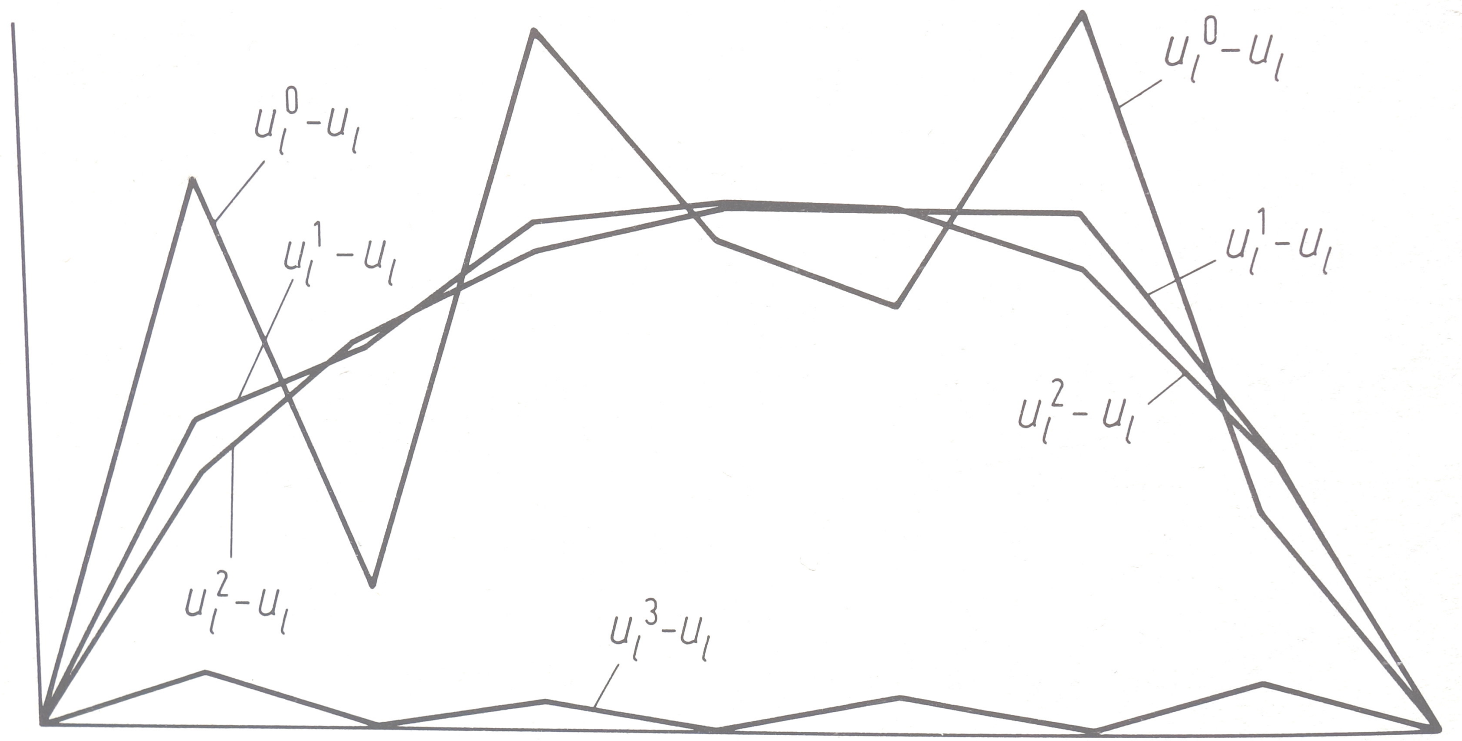
into $\begin{cases} \text{low-frequency part } V_{\text{low}} := \text{span}\{e_h^\mu : 1 \le \mu \le n_h/2\}, \\ \text{high-frequency part } V_{\text{high}} := \text{span}\{e_h^\mu : n_h/2 < \mu \le n_h\} \end{cases}$

Conclusion for damped Jacobi:

Errors in V_{high} are reduced by a factor $\frac{1}{2}$ per iteration.



Smoothing Effect: After few steps of the damped Jacobi iteration the low-frequency part is dominating $\Rightarrow$ The iteration error $u_h^j - u_h$ is smooth:



A smooth error with step size h can be well approximated by a grid function with step size $2h$!

Often used smoothing iteration: **Gauss-Seidel iteration**

1.3 Coarse-Grid Correction

Actual approximation for step size h: $\bar{u}_h$

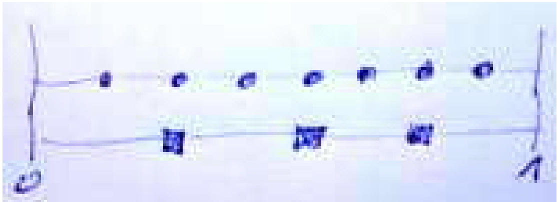
Its *defect* is

$$d_h := L_h \bar{u}_h - f_h.$$

The solution of $L_h v_h = d_h$ is the *exact correction*: $u_h = \bar{u}_h - v_h$.

Coarse-grid equation:

$$L_{2h} v_{2h} = d_{2h} \qquad \text{with} \qquad d_{2h}(x) := \frac{1}{4} d_h(x-h) + \frac{1}{2} d_h(x) + \frac{1}{4} d_h(x+h),$$



short: $d_{2h} = r d_h$ (restriction $r : V_h \to V_{2h}$)

Interpolation of v_{2h}: $p v_{2h}$ (prolongation $p : V_{2h} \to V_h$, Exercise: $p = 2r^{\mathsf{T}}$)

Approximate correction: $u_h^{\text{new}} = \bar{u}_h - p v_{2h}$.

2 Two-Grid Iteration

u_ℓ^j given iterate (ℓ: *level number* corresponding to $h = h_\ell$, $h_{\ell-1} := 2h_\ell$)

smoothing step: ν steps of a smoothing iteration (e.g. damped Jacobi):

$$\bar{u}_\ell := \mathcal{S}_\ell^\nu(u_\ell^j, f_\ell)$$

coarse-grid correction:

$d_\ell := L_\ell \bar{u}_\ell - f_\ell$	defect
$d_{\ell-1} := r d_\ell$	restriction of the defect
$v_{\ell-1} := L_{\ell-1}^{-1} d_{\ell-1}$	exact solution of the coarse-grid equation
$u_\ell^{j+1} := \bar{u}_\ell - p v_{\ell-1}$	correction of $\bar{u}_\ell$

The **two-grid iteration** is defined by $u_\ell^j \longmapsto u_\ell^{j+1}$.
$\Rightarrow$ error reduction independent of h_ℓ :

$$\left\|u_\ell^{j+1} - u_\ell\right\| \le \rho \left\|u_\ell^j - u_\ell\right\| \quad \text{with } \rho < 1 \text{ for all } \ell$$

weak point: $L_{\ell-1}^{-1}$

Notation:

Here: grid sizes h and $2h$.

We can continue: $4h$, $8h$, ...

New notation: ℓ denotes the level.
For $\ell = 0$ we have a coarsest grid size h_0 and set

$$h_\ell = 2^\ell h_0.$$

For h_ℓ the index h in L_h, u_h, f_h is replaced by ℓ : $L_\ell u_\ell = f_\ell$.

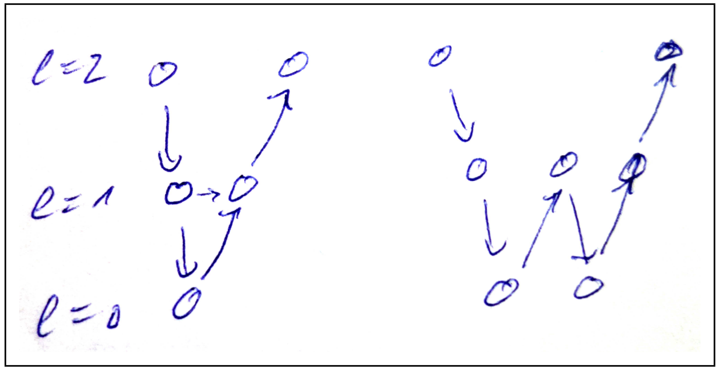
Example: $\Omega = (0, 1)$, $h_0 = 1/2 \Rightarrow x = 1/2$ is the only grid point $\Rightarrow L_0$ is 1×1,
$L_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ with $n_\ell = 2^{1+\ell} - 1$.

Algorithmic notation of the two-grid method:

procedure $TGM(\ell, u, f)$; **integer** ℓ; **array** u, f;
if $\ell = 0$ **then** $u := L_0^{-1} * f$ **else**
begin array v, d;
$u := \mathcal{S}_\ell^\nu(u, f)$; $\quad d := r * (L_\ell * u - f)\,$; $\quad v := L_{\ell-1}^{-1} * d$; $\quad u := u - p * v$
end;

3 Multi-Grid Iteration

procedure $MGM(\ell, u, f)$; **integer** ℓ; **array** u, f;
if $\ell = 0$ **then** $u := L_0^{-1} * f$ **else**
begin array v, d;
$u := \mathcal{S}_\ell^\nu(u, f)$; $\quad d := r * (L_\ell * u - f)\,$;
$v := 0$; $\quad$ **for** $j = 1(1)\gamma$ **do** $MGM(\ell - 1, v, d)$;
$u := u - p * v$
end;



V-cycle: $\gamma = 1$, W-cycle: $\gamma = 2$

Application to FE Equations

Simplest situation:
The finite-element subspaces satisfy

$$\mathcal{H}_{\ell-1} \subset \mathcal{H}_\ell$$

(nested FE spaces).

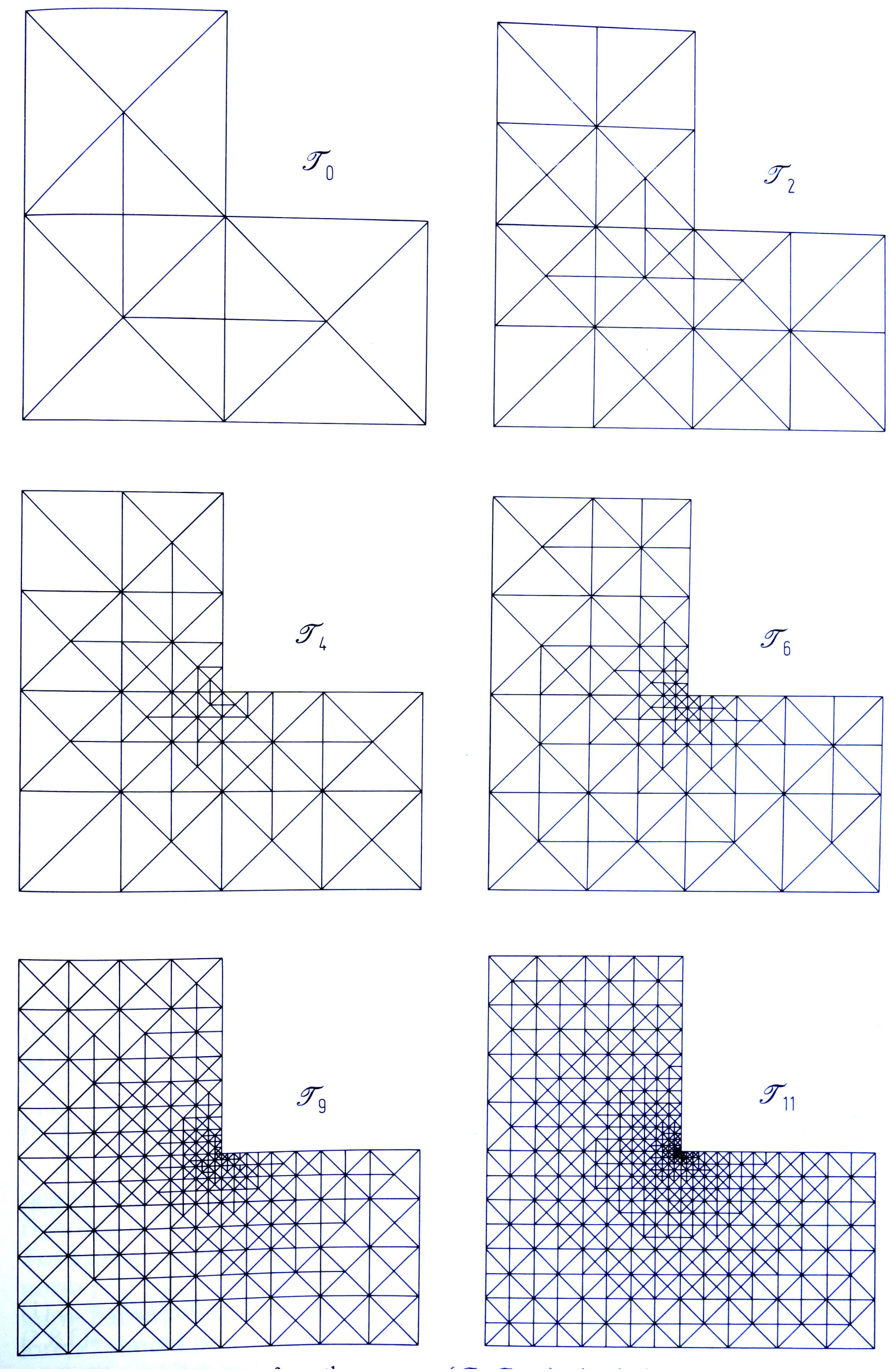
Then:

$p : \mathcal{H}_{\ell-1} \to \mathcal{H}_\ell$ identity,

r : transposed of p,

$L_{\ell-1} = r L_\ell \, p.$

More difficult:
Given a finest FE space,
how to find coarser ones?

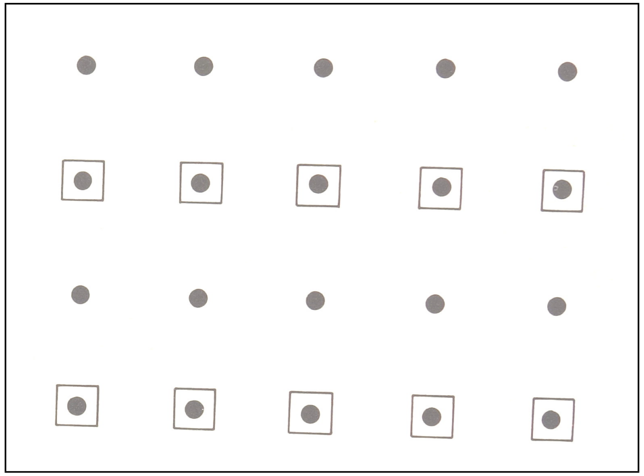


Applicability:

In principle, the multigrid iteration works for discretisations of elliptic PDEs.

The error reduction per iteration is independent of the grid size, but may depend on other parameters, e.g., on the **anisotropy.** Standard example:

$$-\varepsilon u_{xx} - u_{yy} = f \qquad \text{for small } \varepsilon > 0.$$



Remedy: Coarsening only in y-direction:

In the general case, varying anisotropy directions etc., the **construction of the coarser grid** is nontrivial.

3.1 Algebraic Multigrid Iteration

J.W. Ruge, K. Stüben: Algebraic multigrid (AMG). In: Multigrid Methods, Vol. 5 of Frontiers in Applied Mathematics (ed. S. McCormick), SIAM Philadelphia, pp. 73-130, 1986

Coarsening: Let ω_ℓ be the FE grid points (nodal points) at level ℓ corresponding to the FE space H_ℓ. Define a suitable splitting

$$\omega_\ell = \omega_F \,\dot{\cup}\, \omega_C$$

into sets of fine-grid nodes (ω_F) and coarse-grid nodes (ω_C).
$\omega_{\ell-1} := \omega_C$ defines the nodal values of the FE space $H_{\ell-1}$.

Prolongation: $p : H_{\ell-1} \to H_\ell$ interpolation at the nodes ω_ℓ.

Restriction: r transposed of p.

Coarse-grid matrix: $L_{\ell-1} = r L_\ell p$.

Smoothing iteration: Gauss-Seidel iteration

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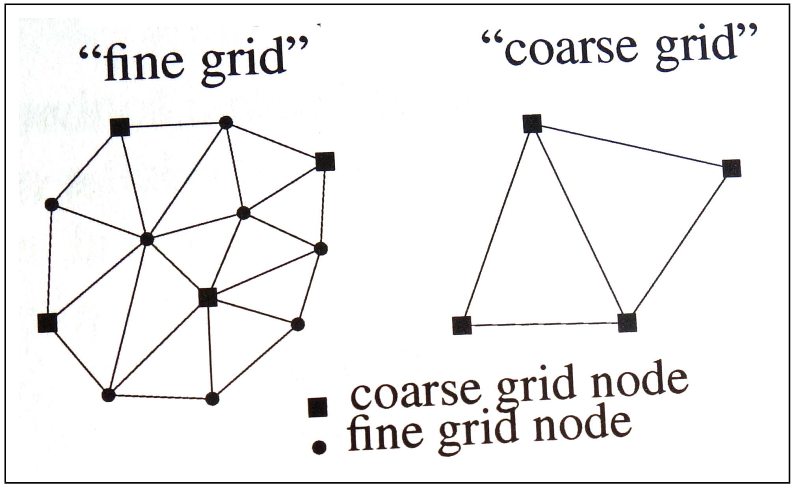

Literature for AMG:

G. Haase, U. Langer, S. Reitzinger, J. Schöberl: A General Approach to Algebraic Multigrid Methods, March 2001 in
`https://www.researchgate.net/publication/2370058_A_General_Approach_to_Algebraic_Multigrid_Methods`

and

C.H. Wolters, M. Kuhn, A. Anwander, S. Reitzinger: A parallel algebraic multigrid solver for finite element method based source localization in the human brain. Computing and Visualization in Science 5(3), pp.165–177 (2002).

4 Nested Iteration

PDE $Lu = f$. Discretizations at all levels $0 \le k \le \ell$ by $L_k u_k = f_k$.

Trivial statements:

1) The iterate u_ℓ^{j+1} of an iteration $u_\ell^j \mapsto u_\ell^{j+1}$ is the better, the better the starting iterate u_ℓ^j is.

2) Solving $L_{\ell-1} u_{\ell-1} = f_{\ell-1}$ (approximately) is cheaper than solving $L_\ell u_\ell = f_\ell$ (lower dimension!).

3) $pu_{\ell-1}$ approximates u_ℓ

Idea: Use $pu_{\ell-1}$ as starting iterate for $u_\ell^j \mapsto u_\ell^{j+1}$.

Nested iteration:

$\tilde{u}_0 := L_0^{-1} f_0$;
for $k := 1\ (1)\ \ell$ **do**
begin $\tilde{u}_k := p\tilde{u}_{k-1}$; **for** $j := 1\ (1)\ i$ **do** $MGM(k, \tilde{u}_k, f_k)$ **end**;

Analysis of the nested iteration:

Assumptions: 1) multigrid convergence:

$$\left\|u_k^{j+1} - u_k\right\| \le \zeta_k \left\|u_k^j - u_k\right\|, \qquad u_k := L_k^{-1} f_k,$$
$$\zeta := \max\{\zeta_k : 1 \le k \le \ell\} < 1.$$

2) interlevel convergence:

$$\|p u_{k-1} - u_k\| \le C_1 h_k^\kappa \qquad (1 \le k \le \ell),$$

3) $\quad C_2 := C_{20} \cdot C_{21} \quad$ with $\quad \|p\| \le C_{20}, \quad h_{k-1}/h_k \le C_{21}.$

Theorem: Under the assumption from above and $C_2 \zeta^i < 1$, the nested iteration yields $\tilde{u}_k$ with

$$\|\tilde{u}_k - u_k\| \le \frac{\zeta^i}{1 - C_2 \zeta^i} C_1 h_k^\kappa \qquad (1 \le k \le \ell).$$

Proof: Exercise

Cost of the Multi-Grid Iteration:

Assume

operation	cost
$\mathcal{S}_\ell(u_\ell, f_\ell)$	$\le C_S n_\ell$
$r\,(L_\ell u_\ell - f_\ell)$	$\le C_D n_\ell$
$u_\ell - p u_{\ell-1}$	$\le C_C n_\ell$
$L_0^{-1} f_0$	$\le C_0$

$$\frac{n_{\ell-1}}{n_\ell} \le C_H, \quad \vartheta := \gamma C_H < 1$$

(standard value: $C_H = 2^{-d}$ for problems in $\mathbb{R}^d$).

Then: $MGM(\ell, \cdot, \cdot)$ requires $C_\ell n_\ell$ operations, where

$$C_\ell < \frac{\nu C_S + C_D + C_C}{1 - \vartheta} + \vartheta^{\ell-1} \frac{C_0}{n_1}.$$

Proof: Exercise

Cost of the nested iteration:

Using $C_\ell \lesssim \frac{\nu C_S + C_D + C_C}{1-\vartheta}$, the cost of the nested iteration with parameter i is bounded by

$$\sum_{k=1}^{\ell} i C_k n_k \le i \sum_{k=1}^{\ell} C_H^{\ell-k} C_\ell n_\ell < \frac{i}{1 - C_H} C_\ell n_\ell.$$

5 Convergence Analysis of the Two-Grid Iteration

Any linear iteration solving $L_\ell u_\ell := f_\ell$ is of the form

$$u_\ell^{j+1} = \Phi(u_\ell^j, f_\ell) = M_\ell u_\ell^j + N_\ell f_\ell \qquad \text{with } M_\ell + N_\ell L_\ell = I.$$

M_ℓ is called the iteration matrix.

Let S_ℓ be the iteration matrix of the smoothing iteration.

Exercise: The iteration matrix of the two-grid iteration with ν smoothing iteration steps is

$$M_\ell(\nu) := \left[I - pL_{\ell-1}^{-1} r L_\ell\right] S_\ell^\nu.$$

Hint: Let M'_ℓ and M''_ℓ be the respective iteration matrices of two linear iterations

$$u_\ell^j \mapsto u_\ell^{j+1} = \Phi'(u_\ell^j, f_\ell) \quad \text{and} \quad v_\ell^j \mapsto v_\ell^{j+1} = \Phi''(v_\ell^j, f_\ell).$$

Then the product of both iterations is $\Phi = \Phi'' \circ \Phi'$ with

$$w_\ell^j \mapsto w_\ell^{j+1} := \Phi''(\Phi'(w_\ell^j, f_\ell), f_\ell) \quad \text{and iteration matrix} \quad M_\ell := M'_\ell M''_\ell.$$

Simplified Convergence Analysis

$$M_\ell(\nu) = \left[I - pL_{\ell-1}^{-1} r L_\ell\right] S_\ell^\nu = \left[L_\ell^{-1} - pL_{\ell-1}^{-1} r\right] \left[L_\ell S_\ell^\nu\right].$$

Smoothing property:

$$\|L_\ell S_\ell^\nu\| \leq \eta(\nu) h_\ell^{-2} \quad \text{for all } \nu \geq 1 \text{ and } \ell \geq 1 \text{ with } \eta(\nu) \to 0 \text{ as } \nu \to \infty.$$

Approximation property: $\left\| L_\ell^{-1} - pL_{\ell-1}^{-1} r \right\| \leq C_A h_\ell^2$.

Combination of both inequalities yields

$$\|M_\ell(\nu)\| \leq C_A \, \eta(\nu)$$

and for sufficiently large ν we have

$$\|M_\ell(\nu)\| \leq \zeta < 1 \qquad \text{implying} \qquad \left\| u_\ell^{j+1} - u_\ell \right\| \leq \zeta \left\| u_\ell^j - u_\ell \right\|.$$

5.0.1 Smoothing Property

Example: L_ℓ symmetric with diagonal $D_\ell = 4h_\ell^{-2} I$ (5-point discretisation) and $\|L_\ell\| \leq 8h_\ell^{-2}$

Damped Jacobi iteration $u_\ell^j \mapsto u_\ell^{j+1} = u_\ell^j - \frac{1}{2} D_\ell^{-1} \left(L_\ell u_\ell^j - f_\ell \right)$.

Iteration matrix: $S_\ell = I - \omega L_\ell$ with $\omega = \frac{1}{2}\left(4h_\ell^{-2}\right)^{-1} = \frac{1}{8}h_\ell^2$.

Euclidean norm: $\left\| L_\ell S_\ell^\nu \right\| = \| L_\ell (I - \omega L_\ell)^\nu \|$,
eigenvalues of L_ℓ between 0 and $8h_\ell^{-2} = 1/\omega$, $\Rightarrow$

$$\begin{aligned} \|L_\ell S_\ell^\nu\| &\leq \max\left\{\lambda (1 - \omega\lambda)^\nu : 0 \leq \lambda \leq \frac{1}{\omega}\right\} \underset{\mu:=\omega\lambda}{=} \frac{1}{\omega} \max\left\{\mu (1-\mu)^\nu : 0 \leq \mu \leq 1\right\} \\ &= 8h_\ell^{-2}\eta_0(\nu). \end{aligned}$$

Exercise: $\eta_0(\nu) := \max\left\{\mu (1-\mu)^\nu : 0 \leq \mu \leq 1\right\}$ satisfies

$$\eta_0(\nu) = \frac{1}{\mathrm{e}\nu} + O(\nu^{-2}), \qquad \eta_0(\nu) \leq \frac{3/8}{\nu + 1/2} \text{ for } \nu \geq 1.$$

5.0.2 Approximation Property

PDE: $Lu = f$, nested FEM $\to L_\ell u_\ell = f_\ell$ and $L_{\ell-1}u_{\ell-1} = f_{\ell-1}$ with $f_\ell := R_\ell f$ and

$u_\ell \in U_\ell$ finite-element coefficients, $P_\ell u_\ell \in L^2(\Omega)$ corresponding finite-element function, $P_\ell : U_\ell \to L^2(\Omega)$,
Similarly: $u_{\ell-1}$ and $P_{\ell-1}u_{\ell-1} \in L^2(\Omega)$. Then

$$P_{\ell-1} = P_\ell\, p : U_{\ell-1} \to L^2(\Omega).$$

The adjoint mappings are $R_\ell = P_\ell^*$, $r = p^*$. In particular, $f_\ell := R_\ell f$ and $f_{\ell-1} := R_{\ell-1} f$, $R_{\ell-1} = rR_\ell$

Under suitable conditions (smooth coefficients, Ω convex): If $f \in L^2(\Omega)$, then $u \in H^2(\Omega)$ and

$$\left\| \left(L^{-1} - P_\ell L_\ell^{-1} R_\ell \right) f \right\|_{L^2} = \left\| L^{-1} f - P_\ell L_\ell^{-1} f_\ell \right\|_{L^2} = \| u - P_\ell u_\ell \|_{L^2} \le C h_\ell^2 \, \| f \|_{L^2}.$$

Repeated: $\left\| \left(L^{-1} - P_\ell L_\ell^{-1} R_\ell \right) f \right\|_{L^2} \leq C h_\ell^2 \left\| f \right\|_{L^2}.$

Similarly, $\left\| \left(L^{-1} - P_{\ell-1} L_{\ell-1}^{-1} R_{\ell-1} \right) f \right\|_{L^2} \leq C h_{\ell-1}^2 \left\| f \right\|_{L^2}.$

Triangle inequality:

$$\left\| \left(P_\ell L_\ell^{-1} R_\ell - P_{\ell-1} L_{\ell-1}^{-1} R_{\ell-1} \right) f \right\|_{L^2} \leq C \left(h_{\ell-1}^2 + h_\ell^2 \right) \left\| f \right\|_{L^2}.$$

$$h_{\ell-1} \leq c h_\ell \quad \Rightarrow \quad \left\| P_\ell L_\ell^{-1} R_\ell - P_{\ell-1} L_{\ell-1}^{-1} R_{\ell-1} \right\|_{L^2 \longleftarrow L^2} \leq C \left(c^2 + 1 \right) h_\ell^2 \quad \Rightarrow$$

$$P_\ell L_\ell^{-1} R_\ell - P_{\ell-1} L_{\ell-1}^{-1} R_{\ell-1} = P_\ell L_\ell^{-1} R_\ell - P_\ell p L_{\ell-1}^{-1} r R_\ell = P_\ell \left(L_\ell^{-1} - p L_{\ell-1}^{-1} r \right) R_\ell.$$

$$\left\| P_\ell u_\ell \right\|_{L^2} \geq C_P \left\| u_\ell \right\| \quad \Rightarrow$$

$$\begin{aligned} \left\| L_\ell^{-1} - p L_{\ell-1}^{-1} r \right\| &\leq C_P^{-2} \left\| P_\ell \left(L_\ell^{-1} - p L_{\ell-1}^{-1} r \right) R_\ell \right\|_{L^2 \longleftarrow L^2} \\ &\leq C_P^{-2} C \left(c^2 + 1 \right) h_\ell^2 = C_A h_\ell^2 \qquad \text{with } C_A := C_P^{-2} C \left(c^2 + 1 \right). \end{aligned}$$

$\Rightarrow$ Approximation Property

6 Adjoint and Symmetric Iterations

Any linear iteration solving $Lu := f$ is of the form $u^{j+1} = \Phi(u^j, f, L)$ with

$$\Phi(u, f, L) := Mu + Nf = u - N(Lu - f) \qquad \text{since } M + NL = I.$$

Here, N depends on L. Notation: $N = N[L]$.

DEFINITION: (a) Given a linear iteration $\Phi(\cdot, \cdot, L)$, the corresponding adjoint iteration is defined by

$$\Phi^*(u, f, L) := u - \left(N[L^{\mathsf{H}}]\right)^{\mathsf{H}} (Lu - f).$$

(b) A linear iteration Φ is symmetric, if $\Phi = \Phi^*$ (i.e., $N[L] = N[L^{\mathsf{H}}]^{\mathsf{H}}$).

EXERCISE: (a) The adjoint iteration of the Gauss-Seidel iteration is the backward Gauss-Seidel iteration.
(b) The product $\Phi^* \circ \Phi$ is a symmetric iteration.
(c) If Ψ is a symmetric iteration, then $\Phi^* \circ \Psi \circ \Phi$ is symmetric.
(d) Φ symmetric and L symmetric matrix $\Longrightarrow$ also N is symmetric.
(e) Φ symmetric and L positive definite $\Longrightarrow$ $L^{1/2} M L^{-1/2}$ is symmetric [essential for application of conjugate gradient methods!]
(f) Assume $p = r^{\mathsf{H}}$ and $L_{\ell-1} = r L_\ell p$. Prove: The coarse-grid iteration is a symmetric iteration.

6.1 Symmetric Multigrid Iteration

The smoothing iteration $\mathcal{S}_\ell$ is now denoted as pre-smoothing $\mathcal{S}_{\ell,\mathsf{pre}}$, while

$$\mathcal{S}_{\ell,\mathsf{post}} := \mathcal{S}_{\ell,\mathsf{pre}}^*$$

is used as post-smoothing (e.g., forward and backward Gauss-Seidel iteration).

procedure $MGM(\ell, u, f)$; **integer** ℓ; **array** u, f;
if $\ell = 0$ **then** $u := L_0^{-1} * f$ **else**
begin array v, d;
$u := \mathcal{S}_{\ell,\mathsf{pre}}^{\nu}(u, f)$; $d := r * (L_\ell * u - f)$;
$v := 0$;   **for** $j = 1(1)\gamma$ **do** $MGM(\ell - 1, v, d)$;
$u := u - p * v$;
$u := \mathcal{S}_{\ell,\mathsf{post}}^{\nu}(u, f)$
end;

Prove: This MGM is a symmetric iteration (suited for cg methods).

Without symmetry: combination with generalised cg methods possible.