A general inverse problem

Given corrupted data f, find u that solves

$$f = T(u) + \varepsilon$$

where T is a forward operator that models the relation between u and f, and ε is a noise component.

- If T has an unbounded inverse, the problem is ill-posed (non-uniqueness, unstable inversion,…); noise has also to be modeled.
- The problem has to be regularized by adding **a-priori** information about the solution.

Variational approach

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- The parameters λ are key to obtain good reconstruction results.
- The result heavily depends on the **prior** (regularity of the image, basis function representation, sparsity, etc.), the **data model** (physical model T, statistics, etc.) and the **(hyper)parameters**.

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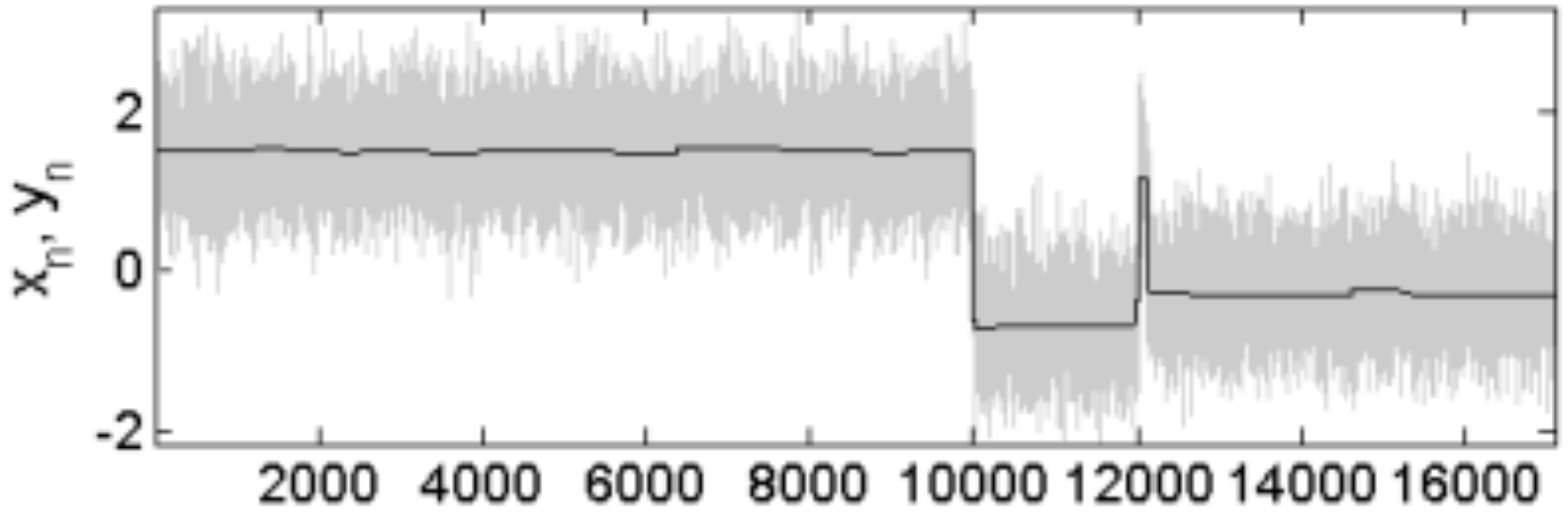

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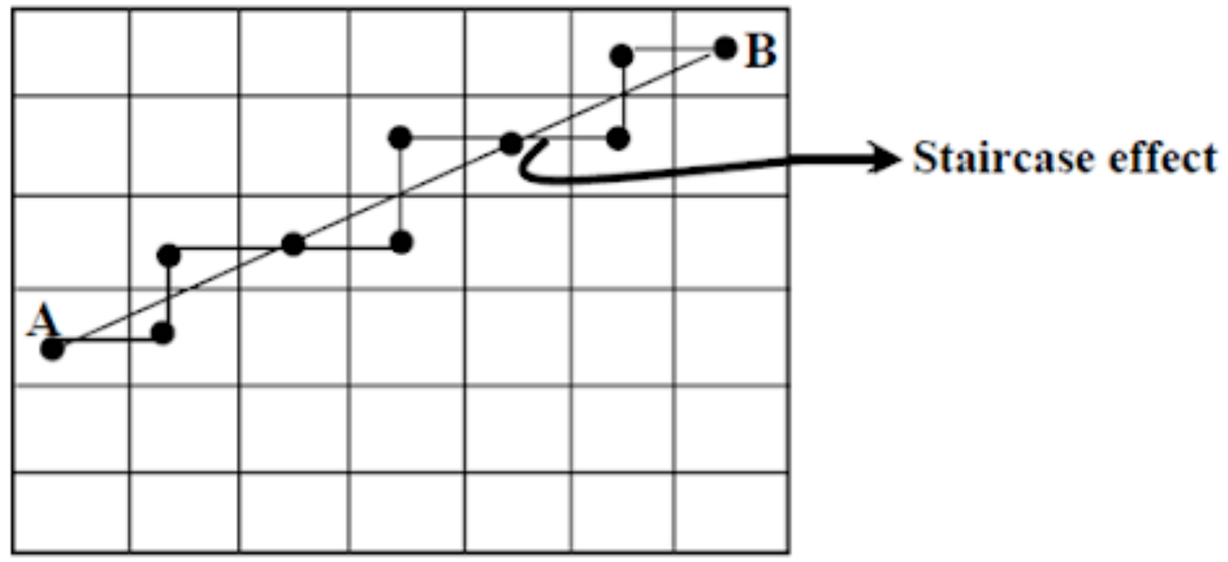

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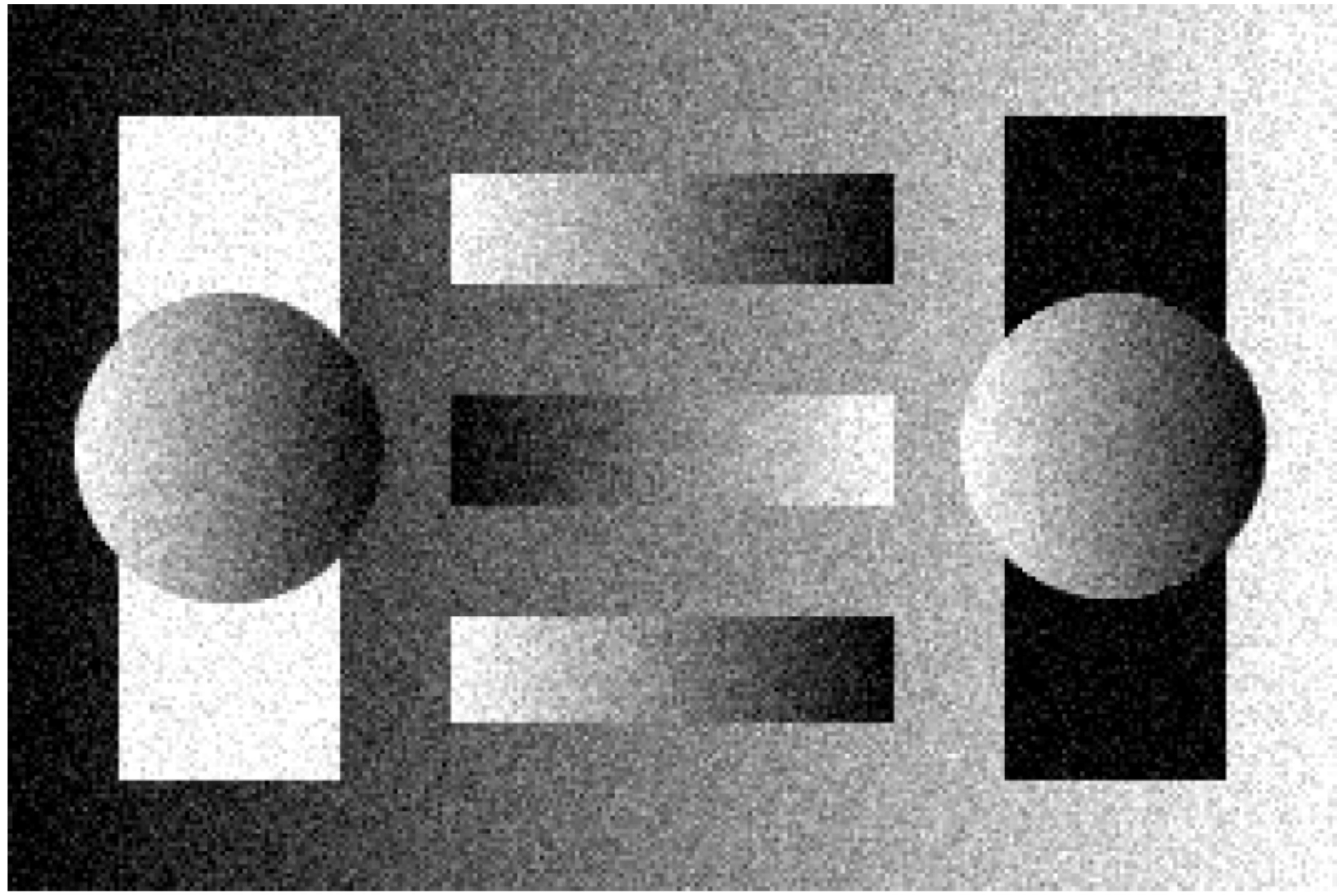
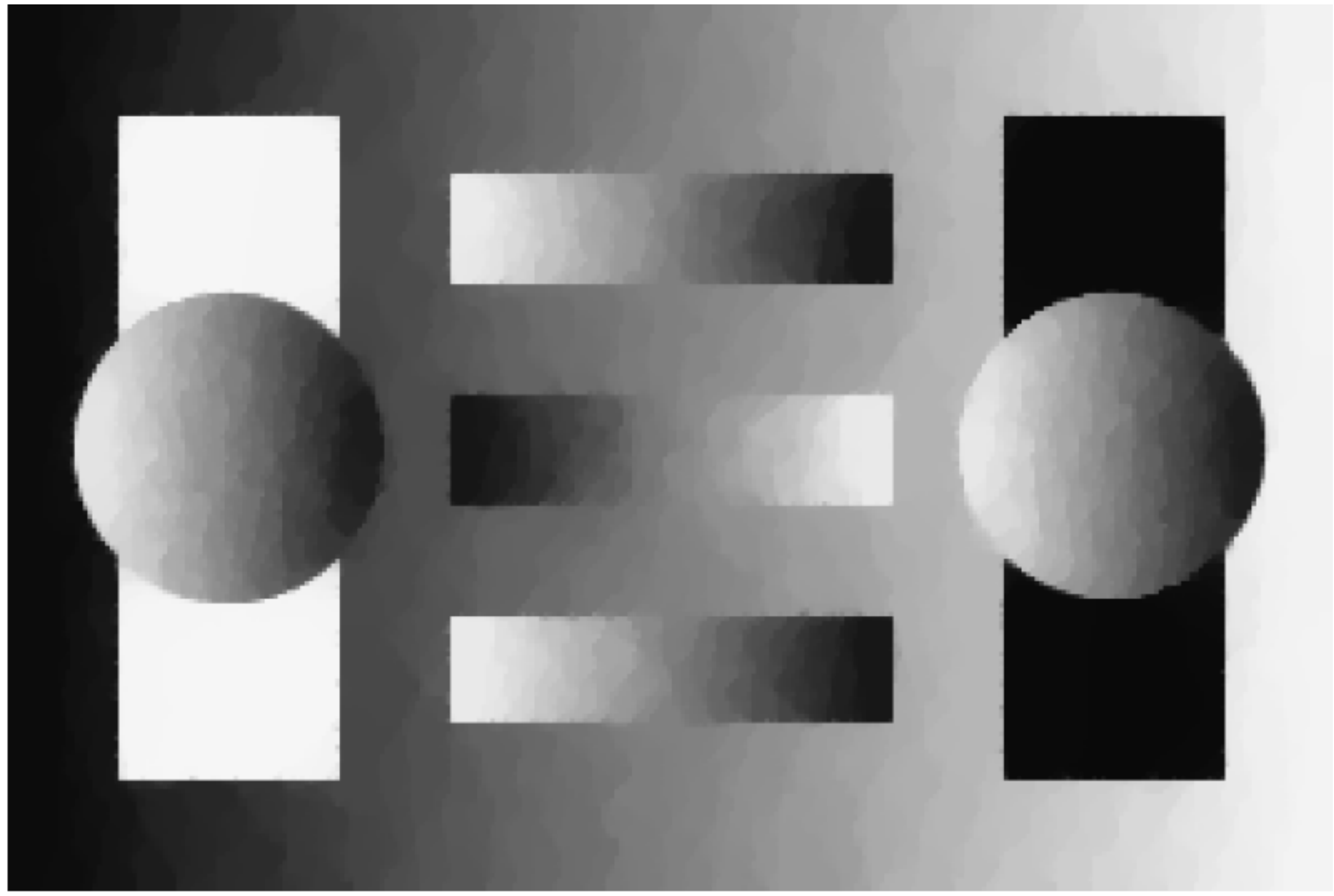
Most imaging regularizers are nonsmooth sparsity-based

Noisy image

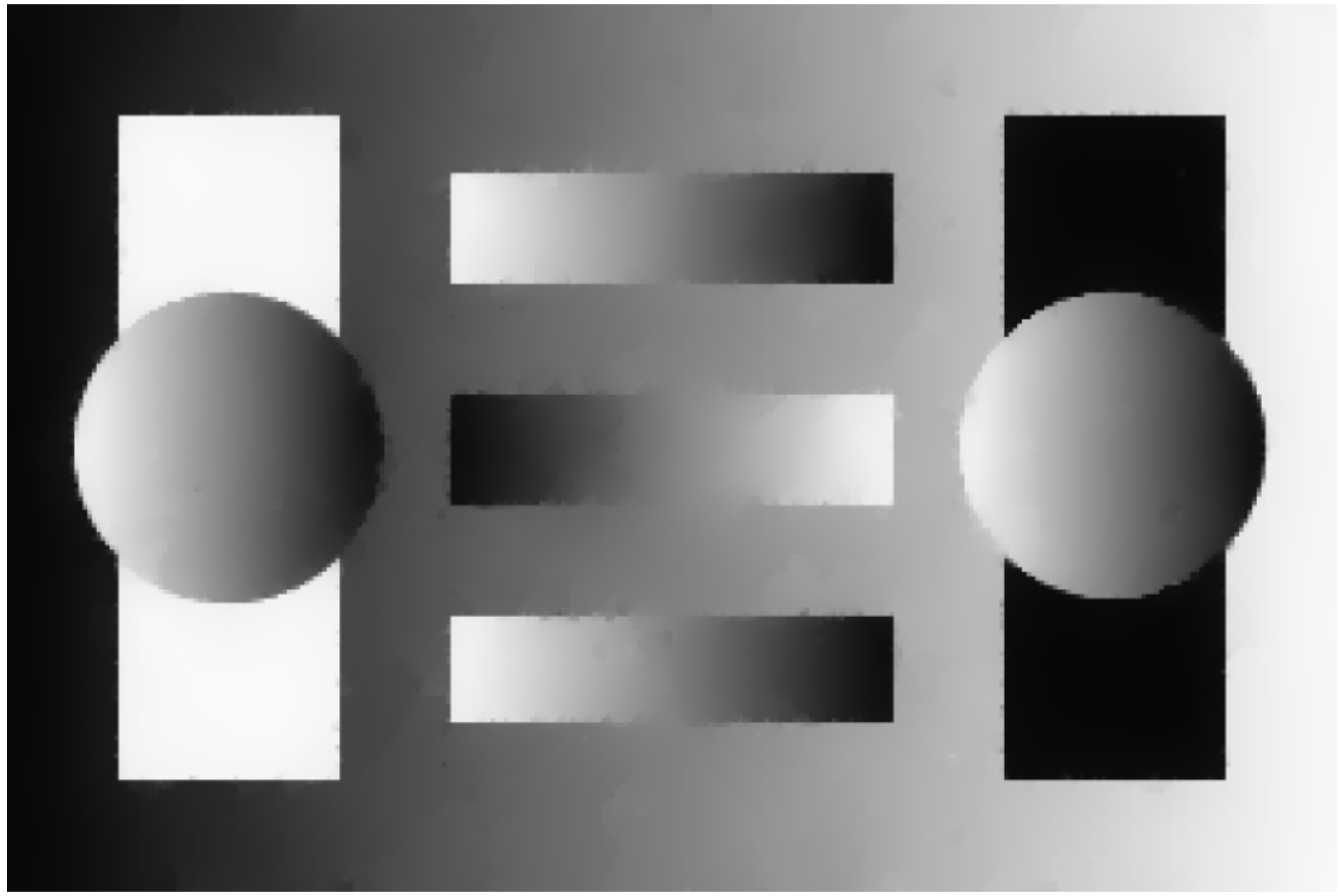


TV denoised image



TV2 denoised image

TGV2 denoised image



Magnetic Resonance Imaging

Magnetic Resonance Imaging

- **What is MRI?** Magnetic resonance imaging (MRI) is a type of scan that uses strong magnetic fields and radio waves to produce detailed images of the inside of the body. An MRI scanner is a large tube that contains powerful magnets. You lie inside the tube during the scan.

Magnetic Resonance Imaging

Magnetic Resonance Imaging

- In MRI, measurements are modeled as samples of the Fourier transform (points in so-called k-space) of the signal that is to be recovered and taking measurements is a time-intensive procedure.
- Keeping acquisition times short is important to ensure patient comfort and to mitigate motion artefacts, and it increases patient throughput,
- Variational model:

$$\min_u \sum_i |(\mathcal{S}Fu)_i - y_i|^2 + \lambda \cdot \mathrm{TV}(u),$$

where $\mathcal{S}$ is the subsambling operator, F the Fourier transform, y are the subsampled measurements.

Magnetic Resonance Imaging

Importance of sparse prior

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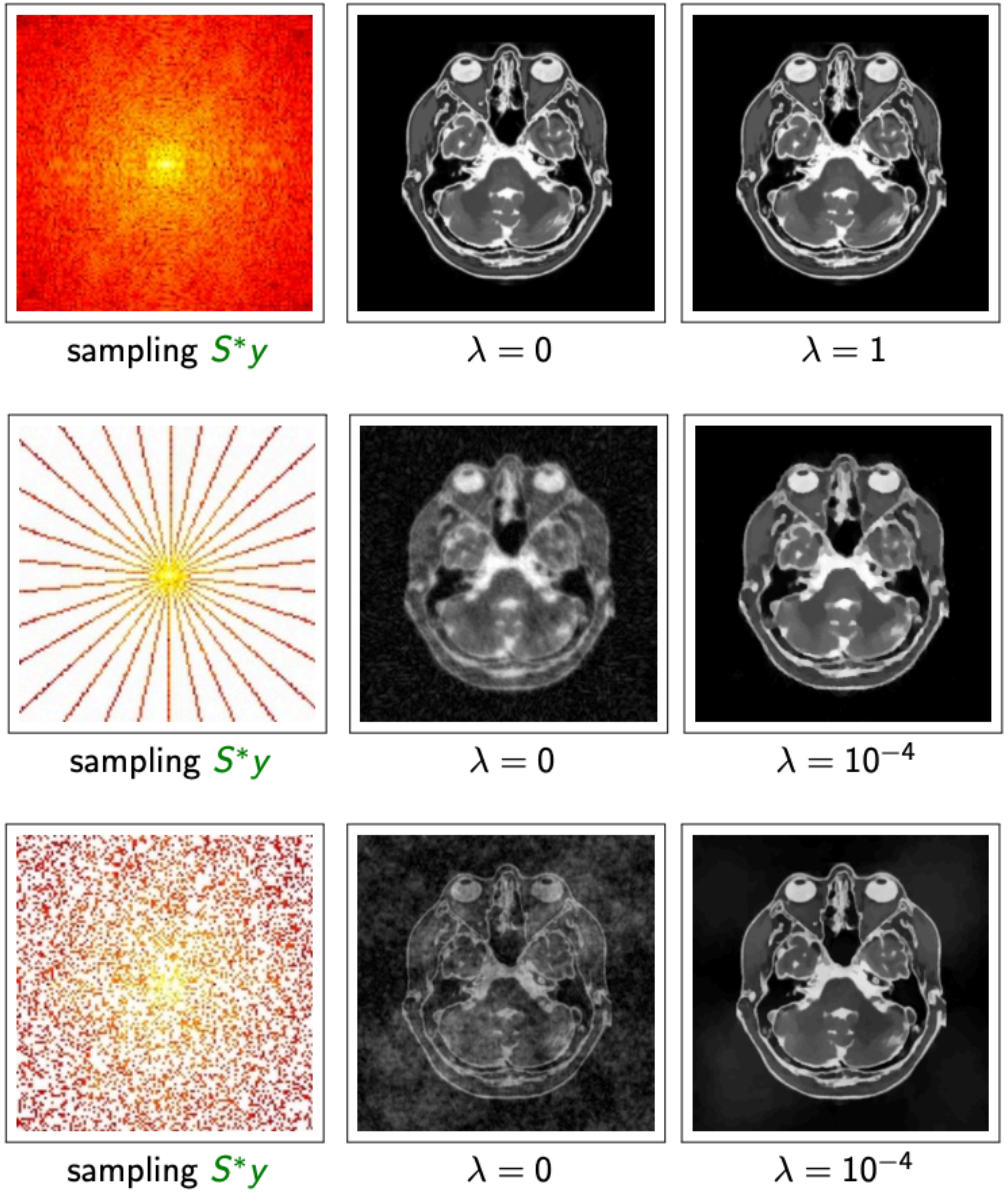

Magnetic Resonance Imaging

Importance of sparse prior

Other medical imaging applications

- X-ray computed tomography (CT): a narrow beam of x-rays is aimed at a patient and quickly rotated around the body, producing signals that are processed by the computer to generate cross-sectional images.

- Positron Emission Tomography (PET): Positron emission tomography (PET) is a type of nuclear medicine procedure that measures metabolic activity of the cells of body tissues.

- Medical optical imaging: uses light and special properties of photons to obtain detailed images of organs, tissues, cells and even molecules. Examples include optical microscopy, spectroscopy, endoscopy, and optical coherence tomography.

- Electrical Impedance Imaging (EIT): is an imaging technique that reconstructs images of a specific region in the human body based on the electrical conductivity of biological tissue.