Multiobjective approximation models arising in radiotherapy treatment in medicine and healthcare logistics

Part 1: Multiobjective approximation models arising in healthcare logistics

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1. Introduction

1.1 Examples for multiobjective location problems arising in healthcare logistics

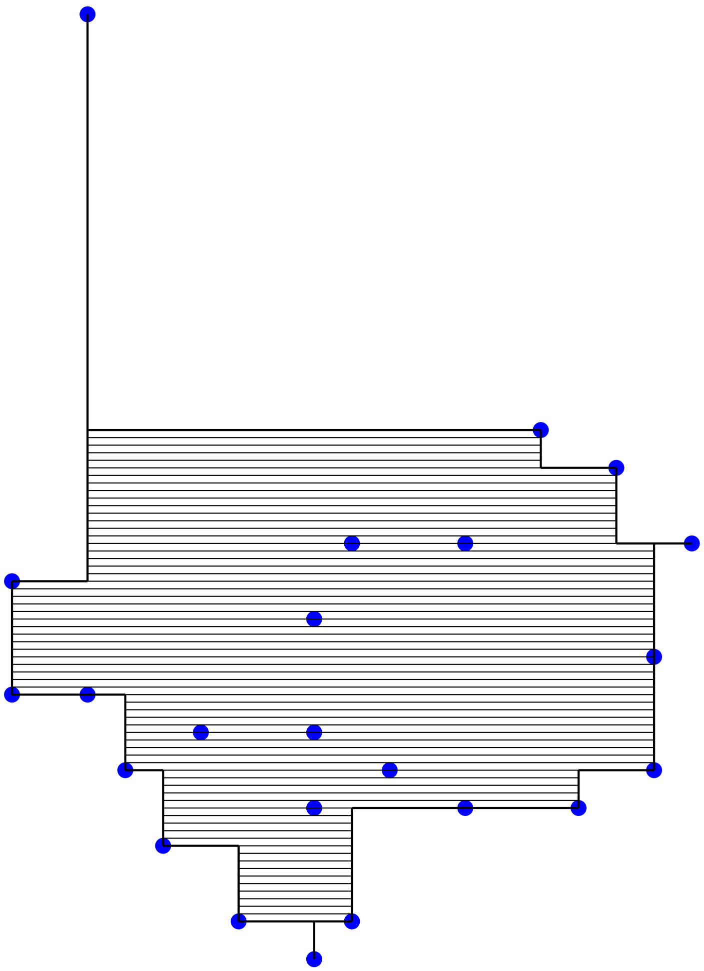
Example : Location for a new Vaccination Center in a district of the town Havana.

Consider a certain district of the town Havana with 23 existing facilities:
Apartment blocks located at $a^1 = (3.5, 3.5)$, $a^2 = (4.5, 4)$, $a^3 = (4, 2.5)$, $a^4 = (5, 1.5)$, $a^5 = (6, 1)$, $a^6 = (6.5, 1.5)$, $a^7 = (6, 4)$, $a^8 = (7, 3.5)$, $a^9 = (8, 3)$, $a^{10} = (9.5, 3)$, $a^{11} = (10.5, 3.5)$, $a^{12} = (10.5, 5)$, $a^{13} = (6, 5.5)$, $a^{14} = (6.5, 6.5)$, $a^{15} = (11, 6.5)$, $a^{16} = (10, 7.5)$, $a^{17} = (9, 8)$,
schools located at $a^{18} = (2, 4.5)$, $a^{19} = (2, 6)$,
day nurseries located at $a^{20} = (3, 4.5)$, $a^{21} = (6, 3)$ and
outpatient health center located at $a^{22} = (8, 6.5)$, $a^{23} = (3, 13.5)$.

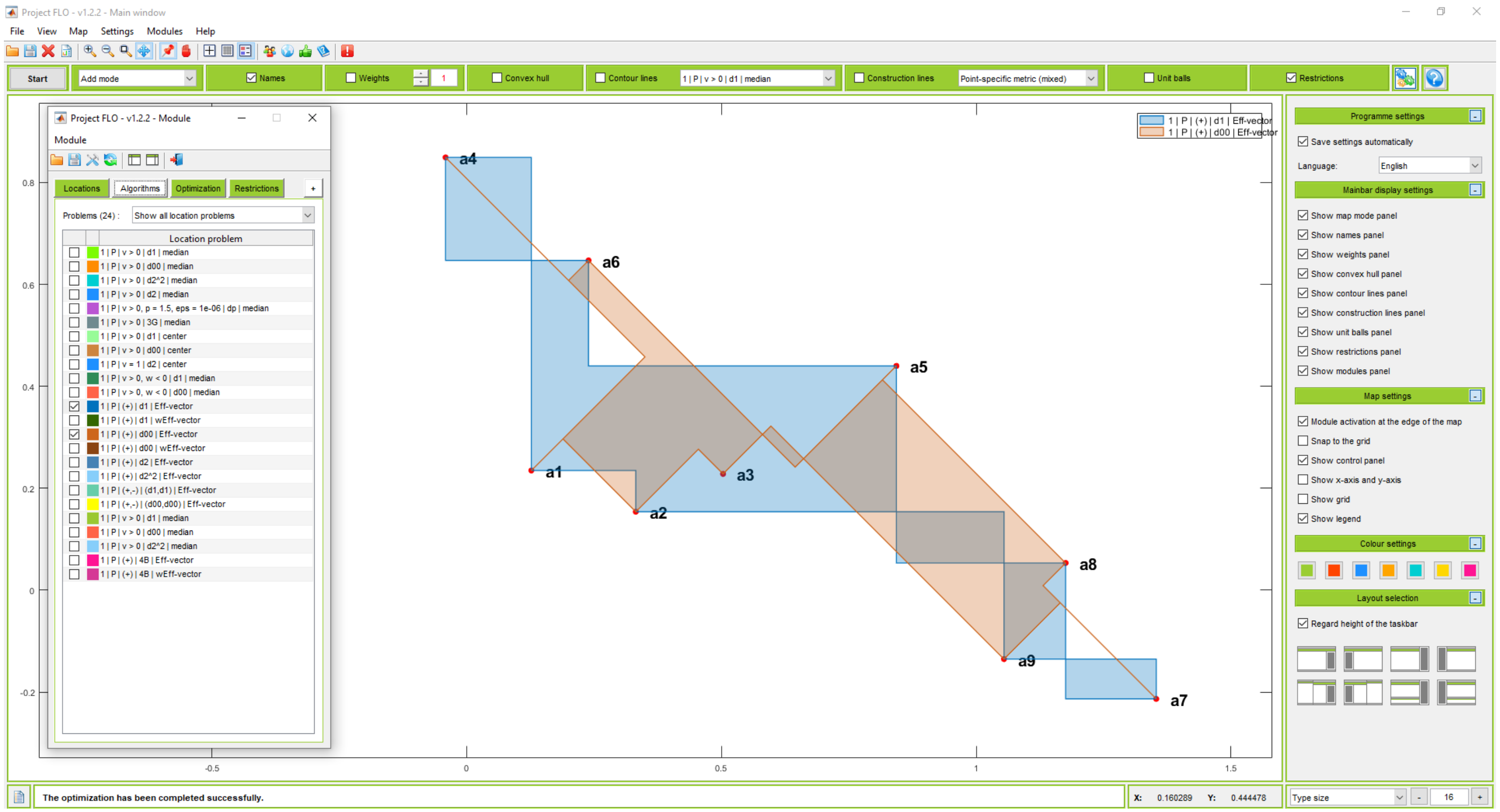
We note that in the blocks located at a^3 and a^{17} there live a great number of children aged of 5 to 12. The decision makers in the public health department of Havana are looking for a location $x \in \mathbb{R}^2$ for a new Vaccination Center such that the distances between the existing facilities $a^1, \ldots, a^{23} \in \mathbb{R}^2$ and the location for the new Vaccination Center $x \in \mathbb{R}^2$ are to be minimized in the sense of multiobjective optimization. So, we study the following multiobjective location problem:

$$\begin{pmatrix} \|x - a^1\| \\ \|x - a^2\| \\ \cdots \\ \|x - a^{23}\| \end{pmatrix} \longrightarrow \min_{x \in \mathbb{R}^2}, \qquad \text{(POLP)}$$

where $\| \cdot \|$ denotes a norm in $\mathbb{R}^2$.



Solution set of (POLP) with Manhattan norm ($\|x\|_1 := |x_1| + |x_2|$, $x \in \mathbb{R}^2$).



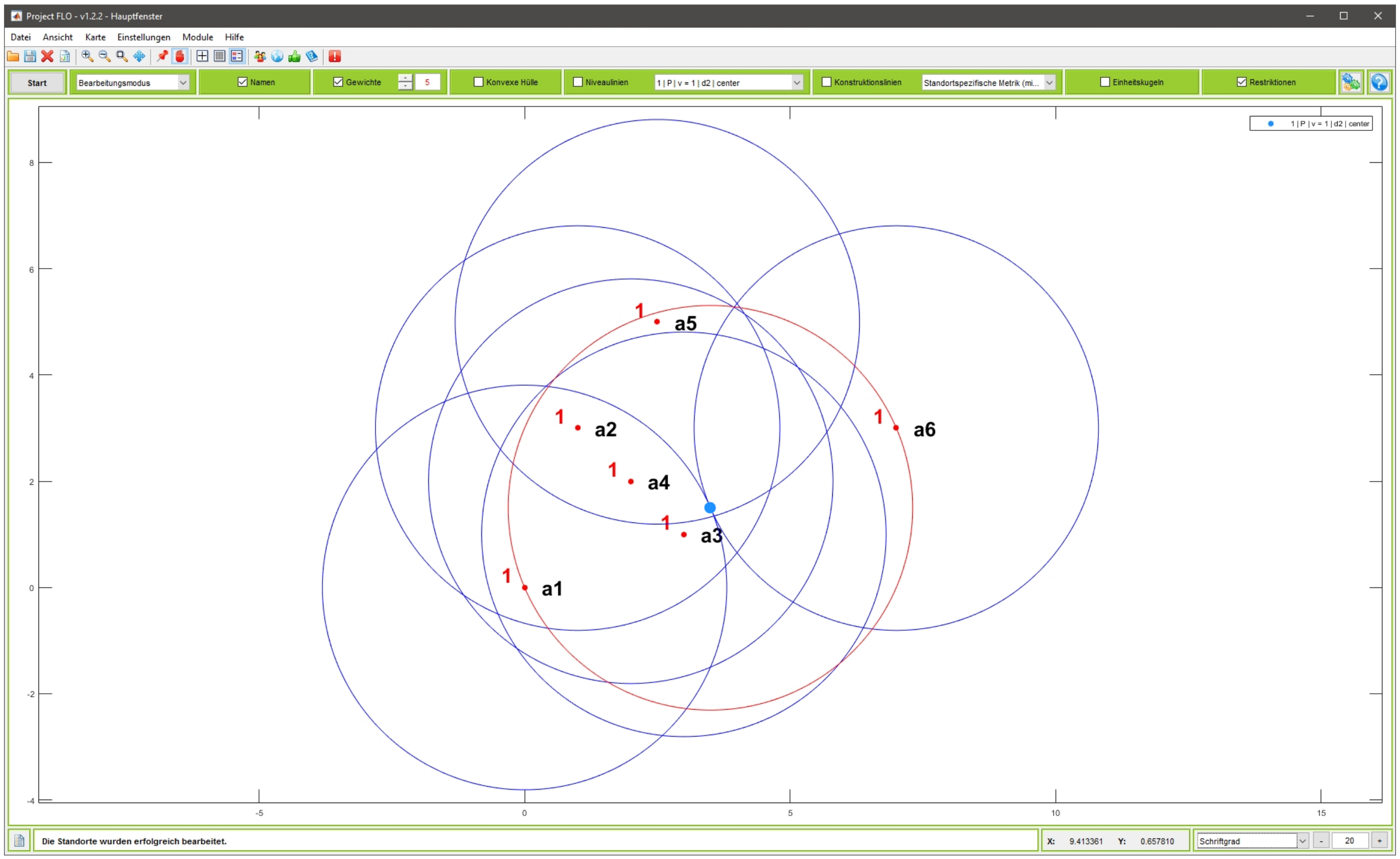
The solution set of a multiobjective location problem with existing facilities $a^1, \ldots, a^9 \in \mathbb{R}^2$ and the maximum norm ($\|x\|_{\max} := \max\{|x_1|, |x_2|\}$) (in red color) as well as with the Manhattan norm ($\|x\|_1 := |x_1| + |x_2|$) (in blue color) generated using the software FLO (https://project-flo.de).

Example: Establishment of an Emergency Ward (location for a rescue helicopter):

In the Vinales Valley in Cuba, a location for a rescue helicopter is to be determined in such a way that it can quickly reach the potential locations (resorts and villages located in a^i, $i = 1, \ldots, 6$) in an emergency case. The location to be determined should be chosen in such a way that, even in the worst case, i.e., if the furthest location reports an emergency call, this location can be reached as quickly as possible. This problem is modeled by a location problem with the center target function. *Center-Problem: Minimize the maximum distance between the new location $x\mathbb{R}^2$ and existing facilities $a^1, \ldots, a^6 \in \mathbb{R}^2$.* Here, we use the Euclidean norm $||\cdot||_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $||x - a^i||_2 := \left(\Sigma_{j=1}^{n}(x_j - a_j^i)^2\right)^{\frac{1}{2}}$.

Centerproblem with Euclidean norm:

$$\min_{x\in\mathbb{R}^2} \max_{i=1,\ldots 6} \{\lambda_i ||x - a^i||_2\}. \tag{1}$$



Location for a rescue helicopter (blue point) generated as solution of the Center-Problem $\max_{i=1,\ldots 6}\{\lambda_i ||x - a^i||_2\} \longrightarrow \min_{x\in\mathbb{R}^2}$ (with the existing facilities a^i, weights $\lambda_i = 1$ (red numbers), $i = 1, \ldots, 6$) and the level lines.

Constrained point-objective location problems

Let m points $a^1, \ldots, a^m \in \mathbb{R}^n$ be a priori given. The distance from the new facility $x \in \mathbb{R}^n$ to a given existing facility $a^i \in \mathbb{R}^n$ will be measured by the metric induced by the Euclidean norm $||\cdot||_2 : \mathbb{R}^n \to \mathbb{R}$, i.e., we have

$$||x - a^i||_2 := \left(\sum_{j=1}^{n} (x_j - a_j^i)^2 \right)^{\frac{1}{2}}.$$

The constrained point-objective location problem involving the Euclidean norm:

$$\begin{cases} f(x) = (||x - a^1||_2, \ldots, ||x - a^m||_2) \to \min \quad \text{w.r.t. } \mathbb{R}^m_+ \\ x \in X, \end{cases} \qquad (\mathrm{POLP}_X)$$

where the feasible set X is a nonempty and closed set in $\mathbb{R}^n$.

1.2 The concept of Pareto efficiency

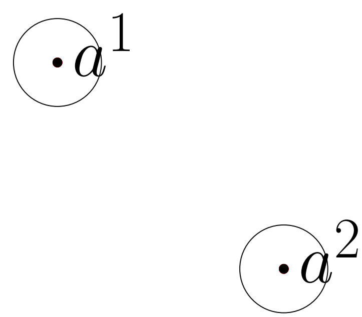
A point $x \in X$ is called Pareto efficient solution for (POLP_X) if

$$\nexists\, x' \in X \;\; \text{s.t.} \;\; \begin{cases} \forall\, i \in I_m: \; ||x' - a^i||_2 \leq ||x - a^i||_2, \\ \exists\, j \in I_m: \; ||x' - a^j||_2 < ||x - a^j||_2, \end{cases}$$

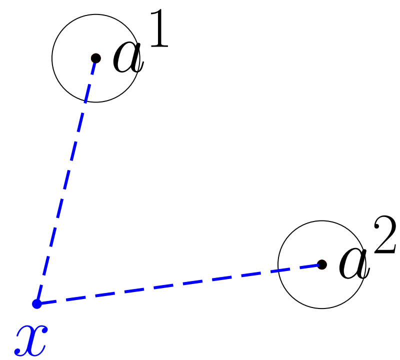
where $I_m := \{1, 2, \cdots, m\}$.

The set of all Pareto efficient solutions is denoted by $\mathrm{Eff}(X \mid f)$. We have

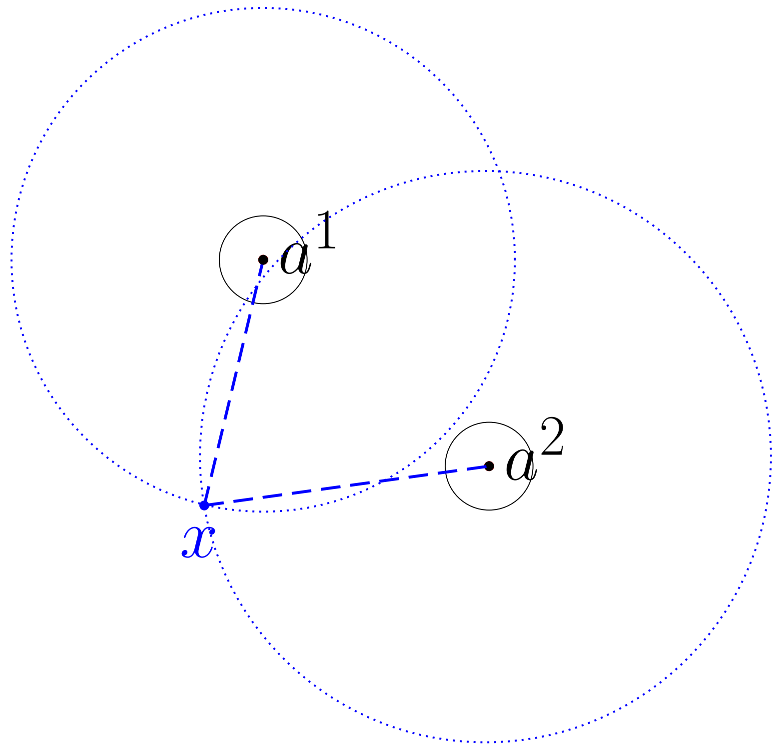
$$\mathrm{Eff}(X \mid f) = \{x \in X \mid f[X] \cap (f(x) - \mathbb{R}^m_+ \setminus \{0\}) = \emptyset\}.$$



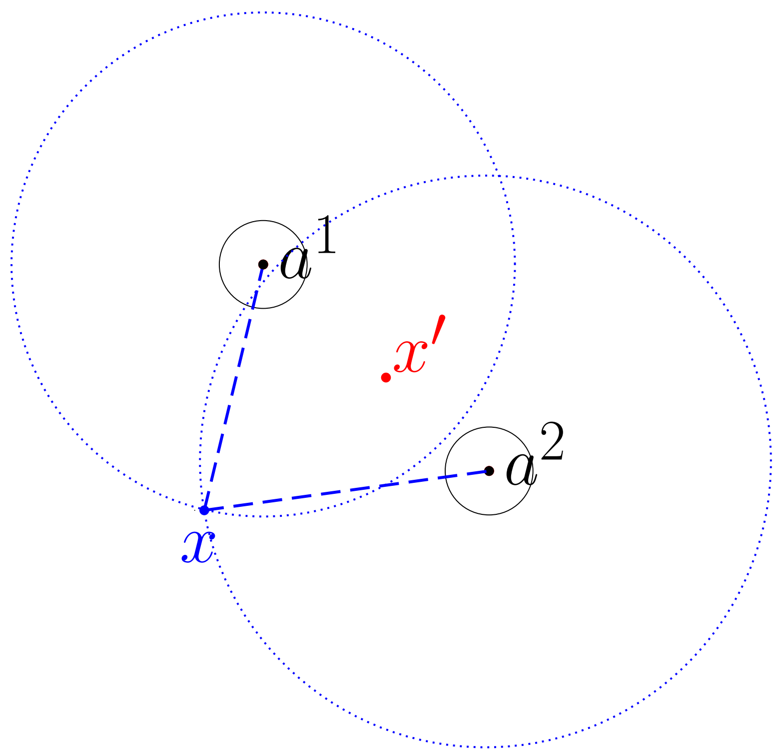
Example: (POLP_X) with $X = \mathbb{R}^2$: Construction of the efficient set of (POLP_X) with the Euclidean norm.



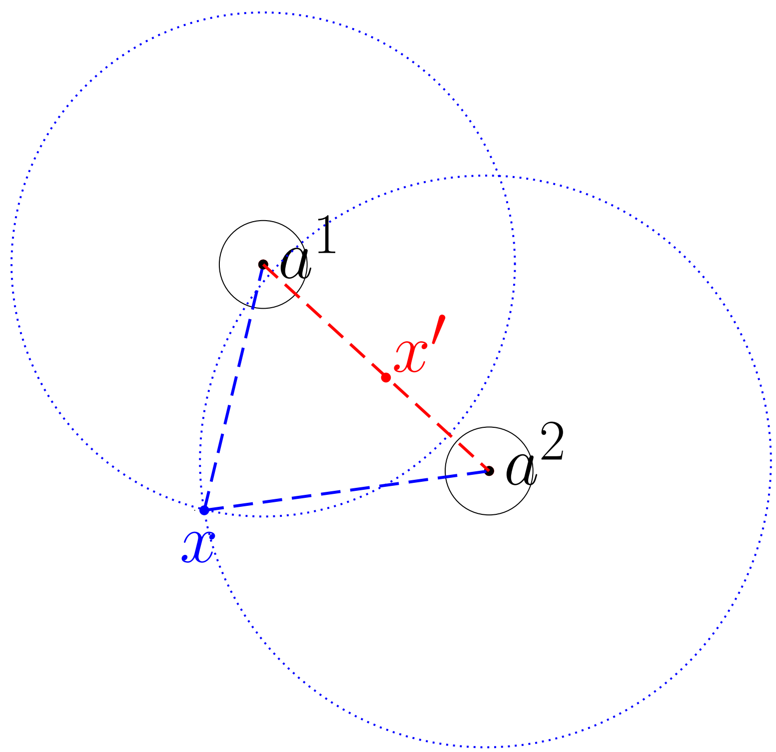
Is x a Pareto efficient solution?



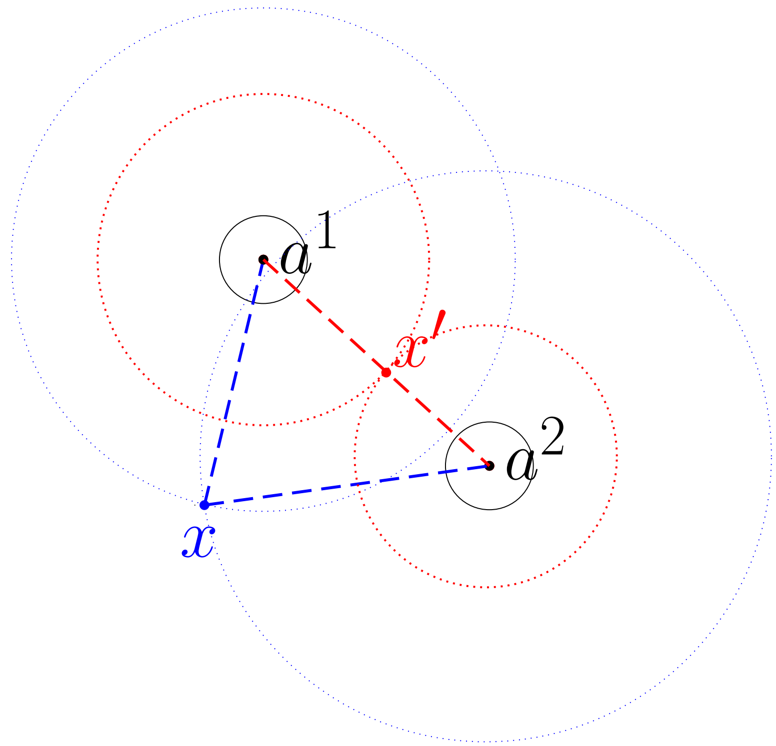
x is not Pareto efficient.



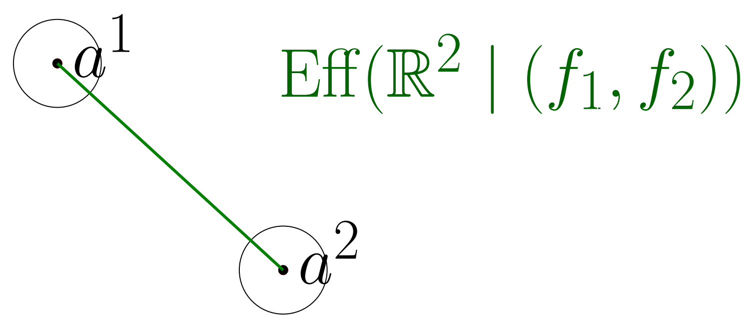
x' dominates x.



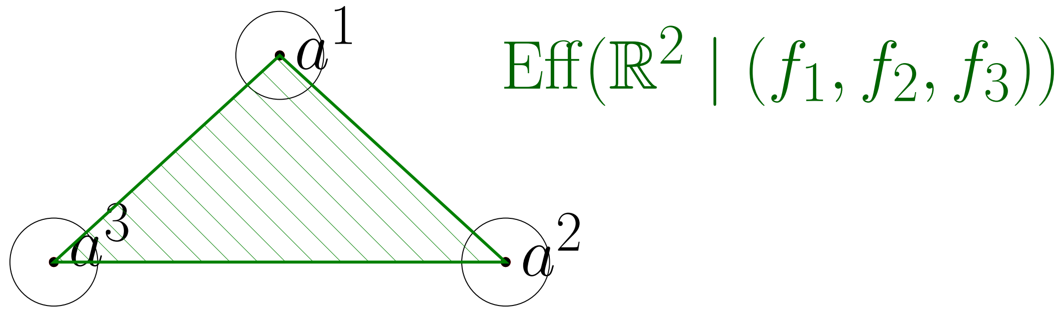
x' dominates x.



x' is Pareto efficient.



The set of efficient solutions $\mathrm{Eff}(\mathbb{R}^2 \mid (f_1, f_2))$.



The set of efficient solutions $\mathrm{Eff}(\mathbb{R}^2 \mid (f_1, f_2, f_3))$.

1.3 Projection property for the set of Pareto efficient solutions

Consider the location problem (POLP_X), i.e.,

$$\begin{cases} f(x) = (||x - a^1||_2, \ldots, ||x - a^m||_2) \to \min \quad \text{w.r.t. } \mathbb{R}^m_+ \\ x \in X. \end{cases}$$

Then, for any nonempty, closed, convex set $X \subseteq \mathbb{R}^n$, we have

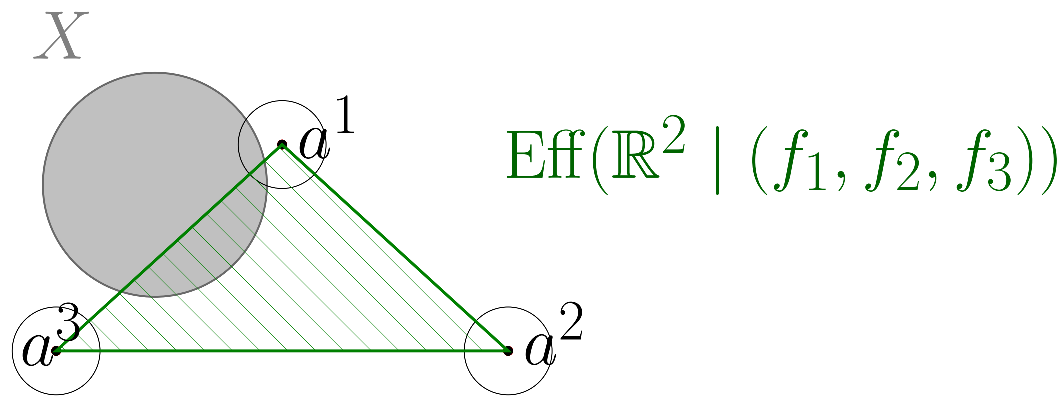
$$\mathrm{Eff}(X \mid f) = \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{Eff}(\mathbb{R}^n \mid f)) = \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{conv}\{a^1, \cdots, a^m\})$$

(see, e.g., Ndiaye and Michelot, 1998).

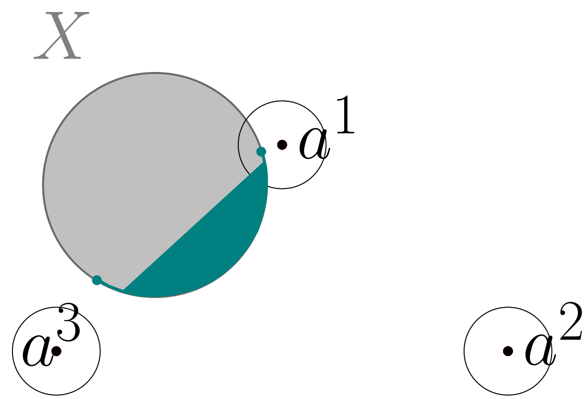
Remark[Ndiaye and Michelot, 1998] The projection property, i.e.,

$$\mathrm{Eff}(X \mid f) = \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{Eff}(\mathbb{R}^n \mid f))$$

fails in general if we replace the Euclidean norm by a not strictly convex norm (e.g, the Manhattan norm or the Maximum norm).



Example: (POLP_X) with closed, convex constraints.



$$\mathrm{Eff}(X \mid (f_1, f_2, f_3)) = \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{conv}\{a^1, a^2, a^3\})$$

Example: The set of efficient solutions $\mathrm{Eff}(X \mid (f_1, f_2, f_3))$.

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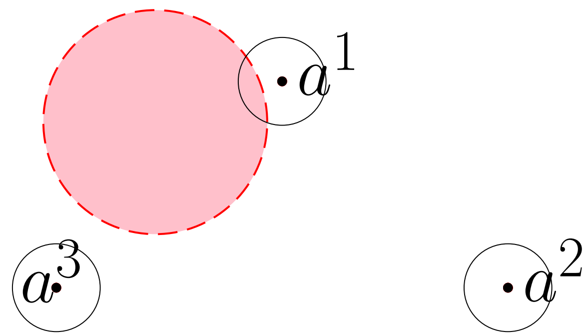
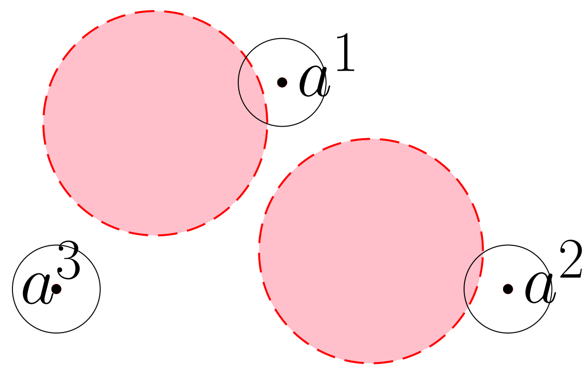
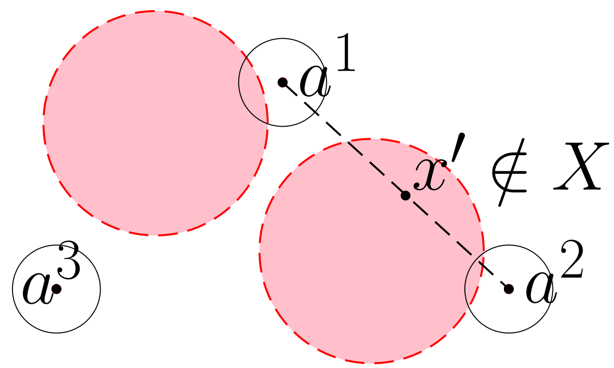


Example: (POLP_X) with nonconvex constraints.



Example: (POLP_X) with nonconvex constraints.



Example: (POLP_X) with nonconvex constraints.

1.4 Basic constrained models and solution concepts

Consider a constrained multi-objective optimization problem with m objective functions $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$:

$$\begin{cases} f(x) = (f_1(x), \ldots, f_m(x)) \to \min \quad \text{w.r.t. } \mathbb{R}^m_+ \\ x \in X, \end{cases} \tag{$\mathcal{P}_X$}$$

where the feasible set X of the problem $(\mathcal{P}_X)$ is a closed set with $\emptyset \neq X \subsetneq \mathbb{R}^n$.

Definition 1. *The set of Pareto efficient solutions of problem $(\mathcal{P}_X)$ with respect to $\mathbb{R}^m_+$ is defined by*

$$\operatorname{Eff}(X \mid f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - \mathbb{R}^m_+ \setminus \{0\}) = \emptyset\}$$

while that of weakly Pareto efficient solutions is given by

$$\operatorname{WEff}(X \mid f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - \operatorname{int} \mathbb{R}^m_+) = \emptyset\}.$$

Definition 2. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function, X be a nonempty set in $\mathbb{R}^n$ and $s \in \mathbb{R}$. Then the lower-level set of h, the level line of h and the strict lower-level set of h to the level s is defined by:*

$$L_{\leq}(X, h, s) := \{x \in X \mid h(x) \leq s\};$$
$$L_{=}(X, h, s) := \{x \in X \mid h(x) = s\};$$
$$L_{<}(X, h, s) := \{x \in X \mid h(x) < s\}.$$

Note that $L_{\sim}(X, h, s) = L_{\sim}(\mathbb{R}^n, h, s) \cap X$ holds for all $\sim \in \{\leq, =, <\}$.

Let $x^0 \in X$ and $f = (f_1, \ldots, f_m)^T$ with $f_i : \mathbb{R}^n \to \mathbb{R}$ for all $i \in I_m$.

Define:

$$S_{\leq}(X, f, x^0) := \bigcap_{i \in I_m} L_{\leq}(X, f_i, f_i(x^0));$$
$$S_{=}(X, f, x^0) := \bigcap_{i \in I_m} L_{=}(X, f_i, f_i(x^0));$$
$$S_{<}(X, f, x^0) := \bigcap_{i \in I_m} L_{<}(X, f_i, f_i(x^0)).$$

Lemma 3 (Nickel (1995), Ehrgott (2005)). *Let* $x^0 \in X$ *and* $f : \mathbb{R}^n \to \mathbb{R}^m$. *Then the following hold:*

$$x^0 \in \mathrm{Eff}(X \mid f) \iff S_{\leq}(X, f, x^0) \subseteq S_{=}(X, f, x^0);$$
$$x^0 \in \mathrm{WEff}(X \mid f) \iff S_{<}(X, f, x^0) = \emptyset;$$

Corollary 4. *It hold*

$$X \cap \mathrm{Eff}(\mathbb{R}^n \mid f) \subseteq \mathrm{Eff}(X \mid f);$$
$$X \cap \mathrm{WEff}(\mathbb{R}^n \mid f) \subseteq \mathrm{WEff}(X \mid f);$$

In order to operate with certain generalized-convexity and semi-continuity notions, we define, for any $(x^0, x^1) \in \mathbb{R}^n \times \mathbb{R}^n$, the function $l_{x^0,x^1} : [0,1] \to \mathbb{R}^n$,

$$l_{x^0,x^1}(\lambda) := (1-\lambda)x^0 + \lambda x^1 \quad \text{for all } \lambda \in [0,1].$$

1.5 Main question of the talk

How is it possible to use techniques and results derived for unconstrained multi-objective optimization problems in order to develop algorithms for solving constrained multi-objective optimization problems?

The results of this talk are based on the articles:

- Günther C, Tammer C (2016) *Relationships between constrained and unconstrained multi-objective optimization and application in location theory.* Mathematical Methods of Operations Research 84(2):359–387;

- Günther C, Tammer C (2018) *On generalized-convex constrained multi-objective optimization.* Pure and Applied Functional Analysis, Volume 3, Number 3, 2018;

- Günther C (2018) *Pareto efficient solutions in multi-objective optimization involving forbidden regions.* Revista de Investigacion Operacional, Volume 39, Issue 3, pp 353-390, 2018.

2. On generalized-convex constrained multi-objective optimization

Consider a nonempty, convex set Y in $\mathbb{R}^n$. $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I_m$, is called

- convex on Y, if for all $x, x' \in Y$ and all $\lambda \in [0, 1]$ we have

$$f_i((1-\lambda)x + \lambda x') \leq (1-\lambda)f_i(x) + \lambda f_i(x').$$

- quasi-convex on Y, if for all $x, x' \in Y$ and all $\lambda \in [0, 1]$ we have

$$f_i((1-\lambda)x + \lambda x') \leq \max\{f_i(x), f_i(x')\}.$$

- semi-strictly quasi-convex on Y, if for all $x, x' \in Y$ with $f_i(x) \neq f_i(x')$, and all $\lambda \in (0, 1)$ we have

$$f_i((1-\lambda)x + \lambda x') < \max\{f_i(x), f_i(x')\}.$$

Lemma 5 ((Giorgi (2004)). *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function and X be a convex set in $\mathbb{R}^n$. Then the following statements are equivalent:*

1°. *h is quasi-convex on X.*

2°. *$L_{\leq}(X, h, s)$ is convex for all $s \in \mathbb{R}$.*

Lemma 6. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function and X be a convex set. Then the following statements are equivalent:*

1°. *h is semi-strictly quasi-convex on X.*

2°. *For all $s \in \mathbb{R}$, $x^0 \in L_{=}(X, h, s)$, $x^1 \in L_{<}(X, h, s)$ it holds*

$$\forall \lambda \in (0, 1] : \quad l_{x^0, x^1}(\lambda) \in L_{<}(X, h, s).$$

Lemma 7. *Let* $X \subseteq \mathbb{R}^n$ *be a convex and closed set with* $\tilde{x} \in \operatorname{int} X$, $x^0 \in X$. *Then, for* $h(x) := \mu_B(x - \tilde{x})$, $\mu_B(z) := \inf\{\lambda > 0 | z \in \lambda B\}$, $B := -\tilde{x} + X$:

1°. *It holds* $L_\sim(\mathbb{R}^n, h, h(x^0)) \subseteq X$ *for all* $\sim \in \{\leq, =, <\}$.

2°. *If* $x^0 \in \operatorname{bd} X$, *then*

$$L_\leq(\mathbb{R}^n, h, h(x^0)) = X;$$
$$L_<(\mathbb{R}^n, h, h(x^0)) = \operatorname{int} X;$$
$$L_=(\mathbb{R}^n, h, h(x^0)) = \operatorname{bd} X.$$

3. Relationships between $(\mathcal{P}_X)$ and $(\mathcal{P}_{\mathbb{R}^n})$

Proposition 8 (Günther & Tammer, 2016, 2018). *Let X be a nonempty subset of $\mathbb{R}^n$. Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $\mathbb{R}^n$. Then, we have*

$$(\operatorname{int} X) \setminus \operatorname{Eff}(\mathbb{R}^n \mid f) \subseteq (\operatorname{int} X) \setminus \operatorname{Eff}(X \mid f).$$

Corollary 9 (Günther & Tammer, 2016, 2018). *Let X be a nonempty subset of $\mathbb{R}^n$. Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $\mathbb{R}^n$. Then,*

$$X \cap \operatorname{Eff}(\mathbb{R}^n \mid f) \subseteq \operatorname{Eff}(X \mid f) \subseteq [X \cap \operatorname{Eff}(\mathbb{R}^n \mid f)] \cup \operatorname{bd} X.$$

4. The penalized multi-objective optimization problem

We consider the penalized multi-objective optimization problem

$$\begin{cases} f^{\oplus}(x) := (f_1(x), \ldots, f_m(x), \phi(x)) \to \min \quad \text{w.r.t. } \mathbb{R}^{m+1}_{+} \\ x \in \mathbb{R}^n, \end{cases} \qquad (\mathcal{P}^{\oplus}_{\mathbb{R}^n})$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is the penalization function. Furthermore, assume:

A1 If $x' \in \operatorname{bd} X$, then

$$L_{\leq}(\mathbb{R}^n, \phi, \phi(x')) := \{x \in \mathbb{R}^n \mid \phi(x) \leq \phi(x')\} = X.$$

A2 If $x' \in \operatorname{bd} X$, then

$$L_{=}(\mathbb{R}^n, \phi, \phi(x')) := \{x \in \mathbb{R}^n \mid \phi(x) = \phi(x')\} = \operatorname{bd} X.$$

Examples for the penalization function ϕ

1°. Assume that X is convex and closed with $d \in \operatorname{int} X$. Then, $\phi(\cdot) := \mu_B(\cdot - d)$ fulfills Assumptions **A1** and **A2**, where μ_B is a Minkowski gauge associated to the set $B := -d + X$, i.e.,

$$\mu_B(x) := \inf\{\lambda \geq 0 \,|\, x \in \lambda \cdot B\} \quad \text{for all } x \in \mathbb{R}^n.$$

2°. Assume that $D \subsetneq \mathbb{R}^n$ is convex and closed with $d \in \operatorname{int} D$. Now, we consider $X := \mathbb{R}^n \setminus \operatorname{int} D$. Then, the function $\phi(\cdot) := -\mu_B(\cdot - d)$, where μ_B is a Minkowski gauge associated to the set $B := -d + D$, fulfills Assumptions **A1** and **A2**.

3°. Assume that X is a closed set with $\emptyset \neq X \subsetneq \mathbb{R}^n$. Based on the distance function $d_X : \mathbb{R}^n \to \mathbb{R}$ with respect to X,

$$d_X(x) := \inf\{||x - z|| \mid z \in X\} \quad \text{for all } x \in \mathbb{R}^n,$$

one can consider the so-called signed distance function (by Hiriart-Urruty, 1979) $\triangle_X : \mathbb{R}^n \to \mathbb{R}$ that is defined by

$$\triangle_X(x) := d_X(x) - d_{\mathbb{R}^n \setminus X}(x) = \begin{cases} d_X(x) & \text{for } x \in \mathbb{R}^n \setminus X, \\ -d_{\mathbb{R}^n \setminus X}(x) & \text{for } x \in X. \end{cases}$$

Then, $\phi := \triangle_X$ satisfies Assumptions **A1** and **A2**.

Main result:

Theorem 10 (Günther & Tammer, 2016, 2018)**.** *Suppose that* $\phi : \mathbb{R}^n \to \mathbb{R}$ *fulfills Assumptions* $\mathbf{A1}$ *and* $\mathbf{A2}$*. Then, the following assertions hold:*

1°. *We have*

$$[X \cap \mathrm{Eff}(\mathbb{R}^n \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus})\right] \subseteq \mathrm{Eff}(X \mid f).$$

2°. *In the case* $\mathrm{int}\, X \neq \emptyset$*, suppose additionally that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is component-wise semi-strictly quasi-convex on* $\mathbb{R}^n$*. Then,*

$$[X \cap \mathrm{Eff}(\mathbb{R}^n \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus})\right] \supseteq \mathrm{Eff}(X \mid f).$$

Proof:

1°. By Corollary 4 we know that $X \cap \mathrm{Eff}(\mathbb{R}^n \mid f) \subseteq \mathrm{Eff}(X \mid f)$ is true. Now, let $x^0 \in \mathrm{bd}\, X \cap \mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus})$. By Lemma 3 (for the problem $(\mathcal{P}^{\oplus}_{\mathbb{R}^n})$ instead of $(\mathcal{P}_X)$) and $\mathbf{A1}$ and $\mathbf{A2}$ it follows

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(\mathbb{R}^n, f, x^0) \cap X \\ &= S_{\leq}(\mathbb{R}^n, f, x^0) \cap L_{\leq}(\mathbb{R}^n, \phi, \phi(x^0)) \\ &\subseteq S_{=}(\mathbb{R}^n, f, x^0) \cap L_{=}(\mathbb{R}^n, \phi, \phi(x^0)) \\ &= S_{=}(\mathbb{R}^n, f, x^0) \cap \mathrm{bd}\, X \\ &\subseteq S_{=}(\mathbb{R}^n, f, x^0) \cap X \\ &= S_{=}(X, f, x^0). \end{aligned}$$

Consequently, applying Lemma 3 we get $x^0 \in \mathrm{Eff}(X \mid f)$.

2°. Let $x^0 \in \mathrm{Eff}(X \mid f)$. The first case is that $x^0 \in X \cap \mathrm{Eff}(\mathbb{R}^n \mid f)$ holds. Now, in the second case suppose that $x^0 \in X \setminus \mathrm{Eff}(\mathbb{R}^n \mid f)$. Hence, by Corollary 9 we assume that $x^0 \in \mathrm{bd}\, X$. Now, by Lemma 3 and **A1** and **A2** it follows

$$\begin{aligned} S_{\leq}(\mathbb{R}^n, f, x^0) \cap L_{\leq}(\mathbb{R}^n, \phi, \phi(x^0)) &= S_{\leq}(\mathbb{R}^n, f, x^0) \cap X \\ &= S_{\leq}(X, f, x^0) \\ &\subseteq S_{=}(X, f, x^0) \\ &= S_{=}(\mathbb{R}^n, f, x^0) \cap X. \end{aligned}$$

Furthermore, it holds

$$S_{=}(\mathbb{R}^n, f, x^0) \cap X = S_{=}(\mathbb{R}^n, f, x^0) \cap \mathrm{bd}\, X. \tag{2}$$

In order to show the validity of (2) it is sufficient to prove $S_{=}(\mathbb{R}^n, f, x^0) \cap \mathrm{int}\, X = \emptyset$. Indeed, if we suppose that there exist some $x^1 \in \mathrm{int}\, X$ with $x^1 \in S_{=}(\mathbb{R}^n, f, x^0)$, then we have to distinguish two cases:

Case 1: If $x^1 \in X \setminus \mathrm{Eff}(\mathbb{R}^n \mid f)$ holds, then by Proposition 8 it follows $x^1 \in X \setminus \mathrm{Eff}(X \mid f)$. Since $x^1 \in S_=(X, f, x^0)$, this implies $x^0 \in X \setminus \mathrm{Eff}(X \mid f)$, in contradiction to the assumption $x^0 \in \mathrm{Eff}(X \mid f)$.
Case 2: If we assume $x^1 \in \mathrm{Eff}(\mathbb{R}^n \mid f)$, then by $x^1 \in S_=(\mathbb{R}^n, f, x^0)$ we conclude $x^0 \in \mathrm{Eff}(\mathbb{R}^n \mid f)$, a contradiction to $x^0 \in X \setminus \mathrm{Eff}(\mathbb{R}^n \mid f)$.

Hence, the equality (2) is true.

Taking into account that $x^0 \in \mathrm{bd}\, X$, we obtain by **A1** and **A2**,

$$S_=(\mathbb{R}^n, f, x^0) \cap \mathrm{bd}\, X = S_=(\mathbb{R}^n, f, x^0) \cap L_=(\mathbb{R}^n, \phi, \phi(x^0)).$$

In view of Lemma 3 for the problem $(\mathcal{P}^{\oplus}_{\mathbb{R}^n})$ instead of $(\mathcal{P}_X)$, we infer that $x^0 \in \mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus})$ holds. □

5. Applications

5.1 The class of point-objective location problems

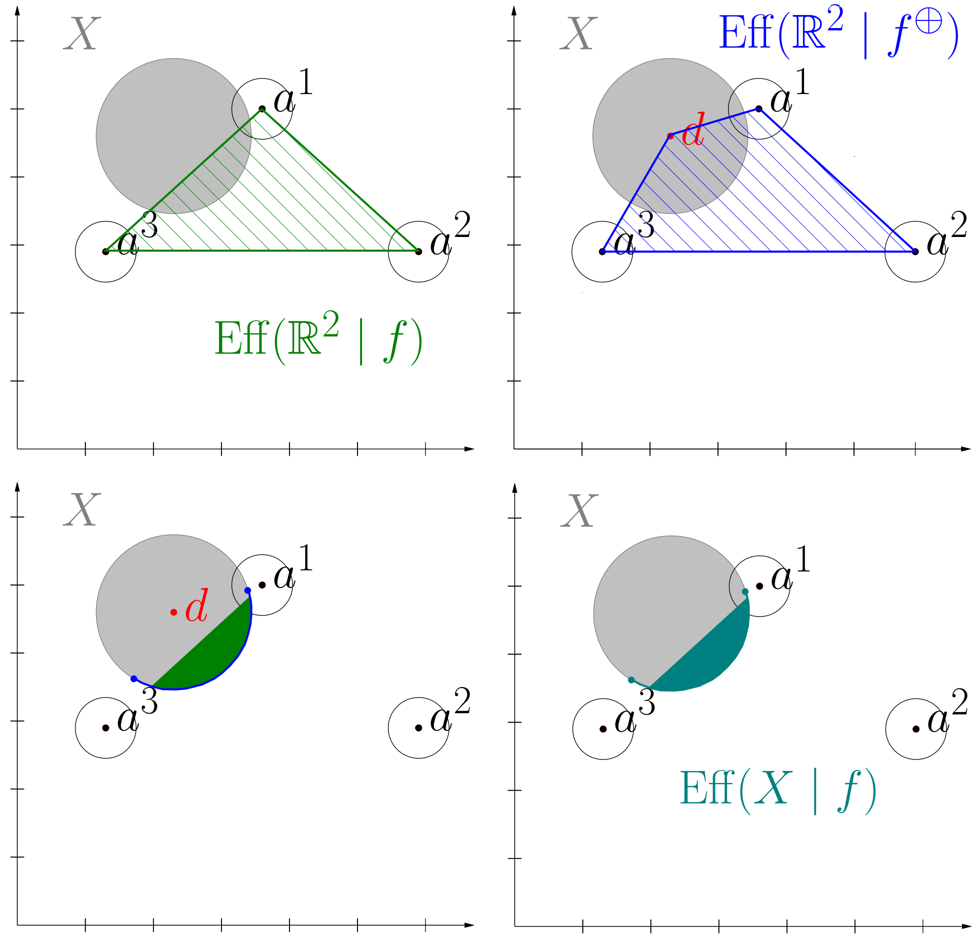
Let m points $a^1, \dots, a^m \in \mathbb{R}^n$ be a priori given. The distance from the new facility $x \in \mathbb{R}^n$ to a given existing facility $a^i \in \mathbb{R}^n$ will be measured by the metric induced by the Euclidean norm $|| \cdot ||_2 : \mathbb{R}^n \to \mathbb{R}$, i.e., we have

$$||x - a^i||_2 := \left(\sum_{j=1}^{n} (x_j - a^i_j)^2 \right)^{\frac{1}{2}}.$$

The constrained point-objective location problem involving the Euclidean norm is given by

$$\begin{cases} f(x) = (||x - a^1||_2, \dots, ||x - a^m||_2) \to \min \quad \text{w.r.t. } \mathbb{R}^m_+ \\ x \in X, \end{cases} \qquad (\text{POLP}_X)$$

where the feasible set X is a nonempty and closed set in $\mathbb{R}^n$.



Example: Construction of the set $\mathrm{Eff}(X \mid (f_1, f_2, f_3))$

5.2 (POLP_X) involving two forbidden regions

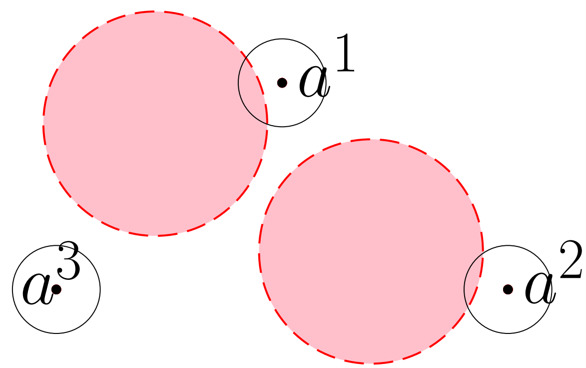
Consider the problem (POLP_X), i.e.,

$$f(x) = (||x - a^1||_2, \ldots, ||x - a^m||_2) \to \min \quad \text{w.r.t. } \mathbb{R}^m_+,$$

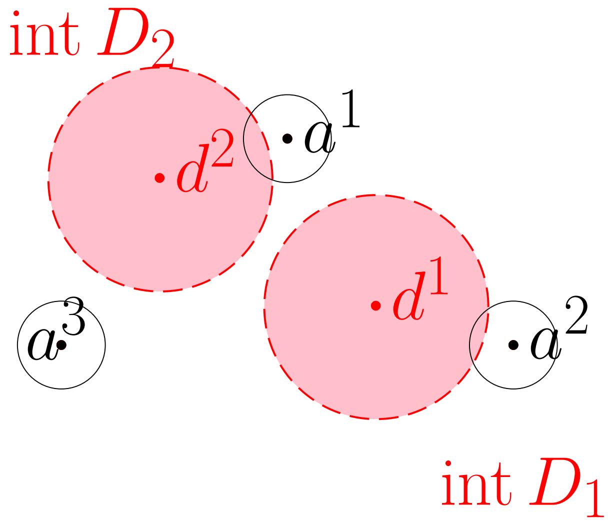
$x \in X$,

and assume that

$$D_i := \overline{B}_{||\cdot||_2}(d^i, r_i) \text{ with } d^i \in \mathbb{R}^n,\ r_i > 0,\ i \in I_l,\ l \in \mathbb{N};$$
$$X := \bigcap_{i \in I_l} X_i \text{ with } X_i := \mathbb{R}^n \setminus \operatorname{int} D_i, i \in I_l.$$



a^1
a^3
a^2



int D_2
a^1
d^2
d^1
a^3
a^2
int D_1

The family of penalized problems:

For any $i \in I_l$, consider a penalized multi-objective optimization problem

$$\begin{cases} f^{\oplus i}(x) = (||x - a^1||_2, \ldots, ||x - a^m||_2, \phi_i(x)) \rightarrow \min \quad \text{w.r.t. } \mathbb{R}^{m+1}_+ \\ x \in \mathbb{R}^n, \end{cases}$$

where the penalization function is given by $\phi_i(\cdot) := -||x - d^i||_2$.

Remark 1. *According to Jourani, Michelot and Ndiaye (2009), this problem can be seen as a point-objective location problem involving attraction and repulsion points.*

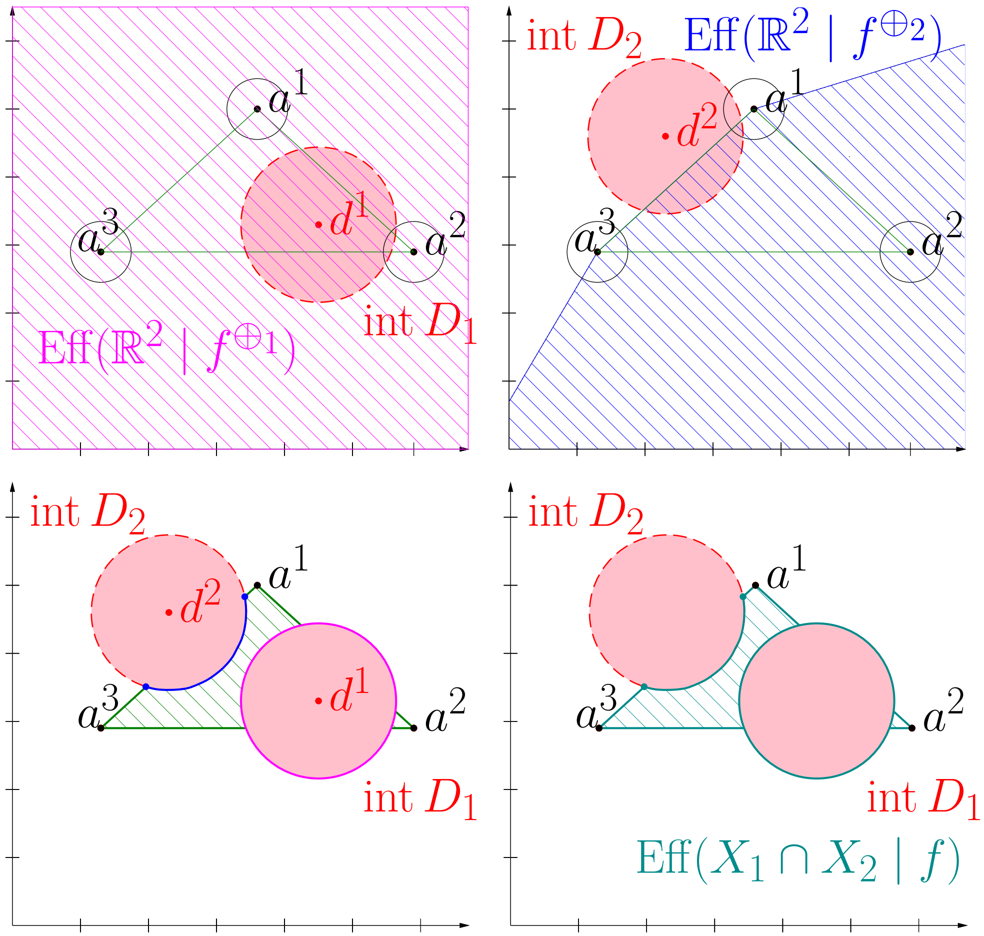
Relationships between $\mathrm{Eff}(X \mid f)$ and $\mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus_i})$, $i \in I_l$:

Lemma 11 (Günther, 2018). 1°. *We have*

$$\mathrm{Eff}(X \mid f) \supseteq \left[X \cap \mathrm{conv}\{a^1, \cdots, a^m\}\right] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus_i})\right].$$

2°. *Assume that the interiors of D_i, $i \in I_l$, are pairwise disjoint. Then,*

$$\mathrm{Eff}(X \mid f) = \left[X \cap \mathrm{conv}\{a^1, \cdots, a^m\}\right] \cup \left[\bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{Eff}(\mathbb{R}^n \mid f^{\oplus_i})\right].$$



Example: Construction of the set $\mathrm{Eff}(X \mid f)$

6. Necessary optimality conditions

(Günther, Tammer, Yao (2018)) An operator ∂ which associates with every $h : \mathbb{R}^n \to \mathbb{R}$ and every $x \in \mathbb{R}^n$ a subset $\partial h(x) \subseteq \mathbb{R}^n$, such that the following axioms are satisfied:

H1 If h is convex, then ∂h coincides with the Fenchel subdifferential, i.e.,

$$\partial h(x) = \{y^* \in \mathbb{R}^n \mid \forall x' \in \mathbb{R}^n : \ \langle y^*, x' - x\rangle + h(x) \leq h(x')\}.$$

H2 If h is locally Lipschitz continuous, and $\overline{x}$ is a local minimum point for h over $\mathbb{R}^n$, then

$$0 \in \partial h(\overline{x}).$$

H3 If $\eta : Y \to \mathbb{R}$ is convex and $\psi \in \mathcal{F}(\mathbb{R}^n, Y)$, then for every $x \in \mathbb{R}^n$,

$$\partial(\eta \circ \psi)(x) \subseteq \cup_{y^* \in \partial\eta(\psi(x))} \partial(y^* \circ \psi)(x).$$

Theorem 12. *Let* $X \subset \mathbb{R}^n$ *be a closed set with* $\operatorname{int} X \neq \emptyset$*. Assume that* ∂ *satisfies* **H1, H2, H3**, $\phi : \mathbb{R}^n \to \mathbb{R}$ *fulfills* **A1** *and* **A2***. Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be component-wise semi-strictly quasi-convex, let* $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$, $f^{\oplus} \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^{m+1})$, f *and* ϕ *locally Lipschitz continuous at* $\overline{x}$*. Take some* $\overline{x} \in \mathrm{WEff}(X \mid f)$*. Then,*

1°. *If* $\overline{x} \in \operatorname{int} X$, *then for every* $\varepsilon > 0$ *there exist* $y^* \in \mathbb{R}^m_+$ *and* $k \in \operatorname{int} \mathbb{R}^m_+$ *with* $\|y^*\|_{\mathbb{R}^m} < \varepsilon$ *and* $\langle y^*, k \rangle = 1$ *such that*

$$0 \in \partial(y^* \circ f)(\overline{x}) = \partial \left(\sum_{i \in I_m} y_i^* f_i \right)(\overline{x}).$$

2°. *If* $\overline{x} \in \operatorname{bd} X$, *then for every* $\varepsilon > 0$ *there exist* $u^* := (y^*, s^*) \in \mathbb{R}^m_+ \times \mathbb{R}_+$ *and* $k \in \operatorname{int} \mathbb{R}^{m+1}_+$ *with* $\|u^*\|_{\mathbb{R}^{m+1}} < \varepsilon$ *and* $\langle u^*, k \rangle = 1$ *such that*

$$0 \in \partial(u^* \circ f^{\oplus})(\overline{x}) = \partial \left(s^* \phi + \sum_{i \in I_m} y_i^* f_i \right)(\overline{x}).$$

7. Conclusions

Fields of application

- **Multiobjective location and approximation problems arising in health-care logistics**.

- **Economics:** Considering models in utility theory (Cobb-Douglas-function).

- **Bioinformatics:** Considering entropy maximization models (based on entropies by Shannon, Tsallis and Renyi) for DNA sequence analysis.

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