Institut für Mathematik
Universität Zürich

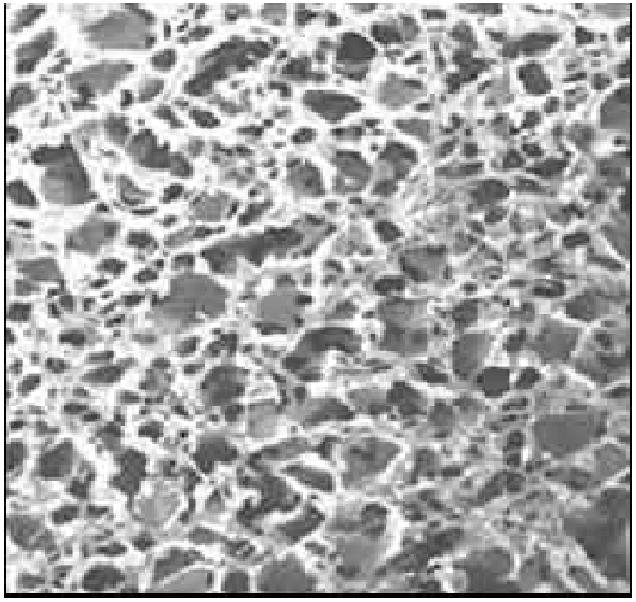
computational mathematics

Composite Finite Elements

Stefan A. Sauter

Introductory Example: Numerical Modelling of Trabecular Bones

Trabecluar Bone is the material microstructure of vertebral bodies.



Affected by osteoporosis by elderly humans.

3.79 millions osteoporotic fractures in Europe in 2000

Direct cost: 31.7 billion Euros.Goal: Reliable numerical simulation.
Numerical challenge: Model contains very different scales
Conventional Finite Elements need a huge number of degrees of freedom for the fine scale resolution.
Accuracy requirements are moderate.

Linearized Elasticity

Find $u \in \mathbf{H}^1_D(\Omega)$ such that

$$\int_\Omega \mathbb{L}\varepsilon(u) : \varepsilon(v) = \int_\Omega f v \qquad \forall v \in \mathbf{H}^1_D(\Omega)\,,$$

where

$\varepsilon(u)$ is the symmetric strain tensor

$$\varepsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^\mathsf{T}\right)$$

$\mathbb{L}\varepsilon$ is the stress tensor

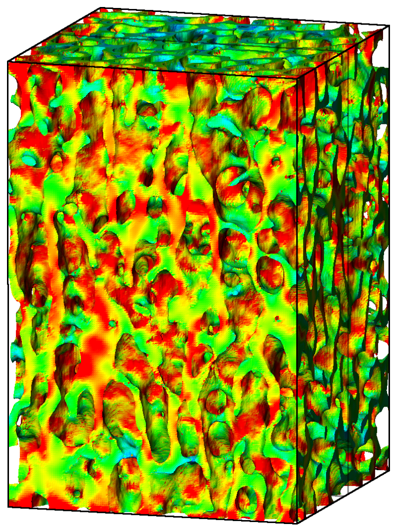
$$\sigma = \mathbb{L}\varepsilon = 2\mu\varepsilon + \lambda\, \mathsf{trace}(\varepsilon)\, \mathsf{Id}\,.$$

The Lamé coefficients μ, λ have discontinuities at the trabeculae boundaries.

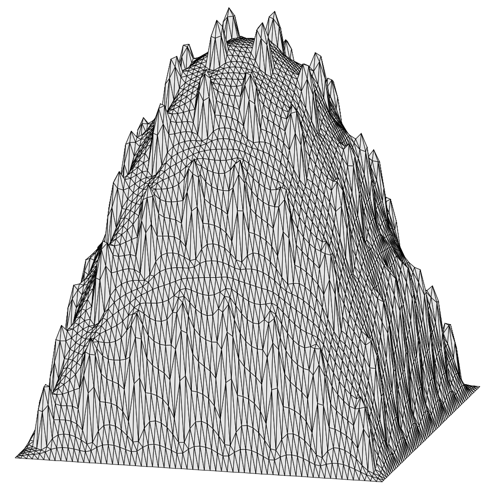
Boundary value problems with multiple scales

Goal: Flexible and adaptive modeling of problems with different scales such as:

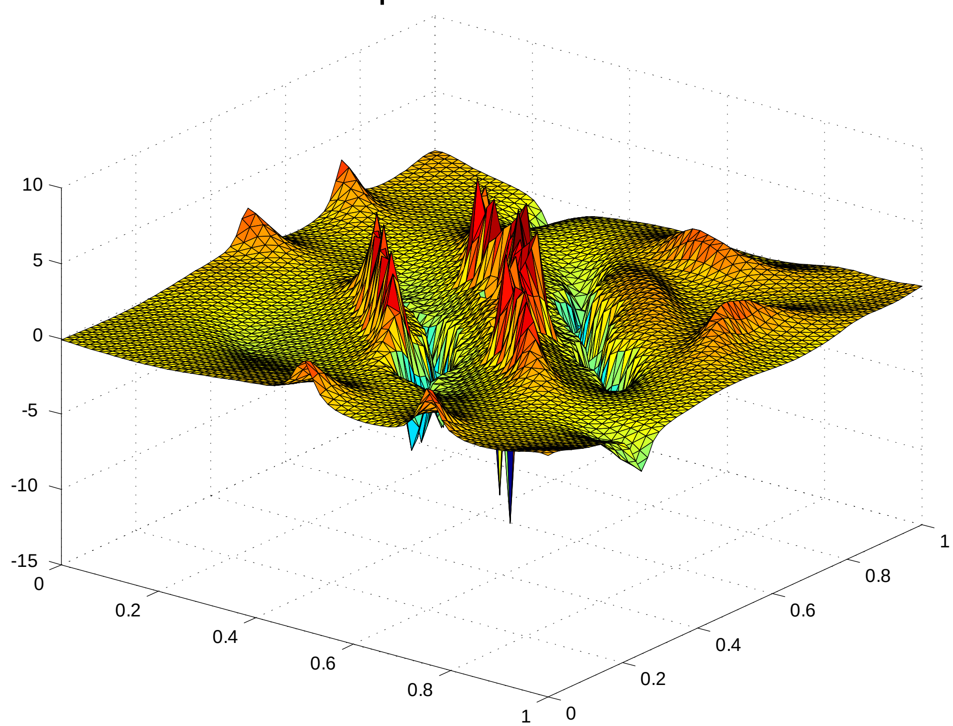
- scales in the physical geometry
- scales in the data
- scales introduced by singular perturbations



von Mises stresses in porcine trabecular bone



Diffusion problem in heterogeneous media



Oscillatory solution of a non-linear Helmholtz equation

Model Problems

a) Poisson-type scalar elliptic equations.

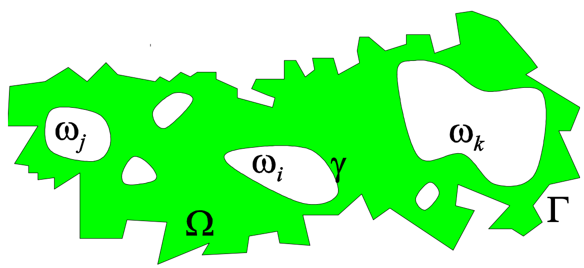
$\Omega \subset \mathbb{R}^d$: bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$,

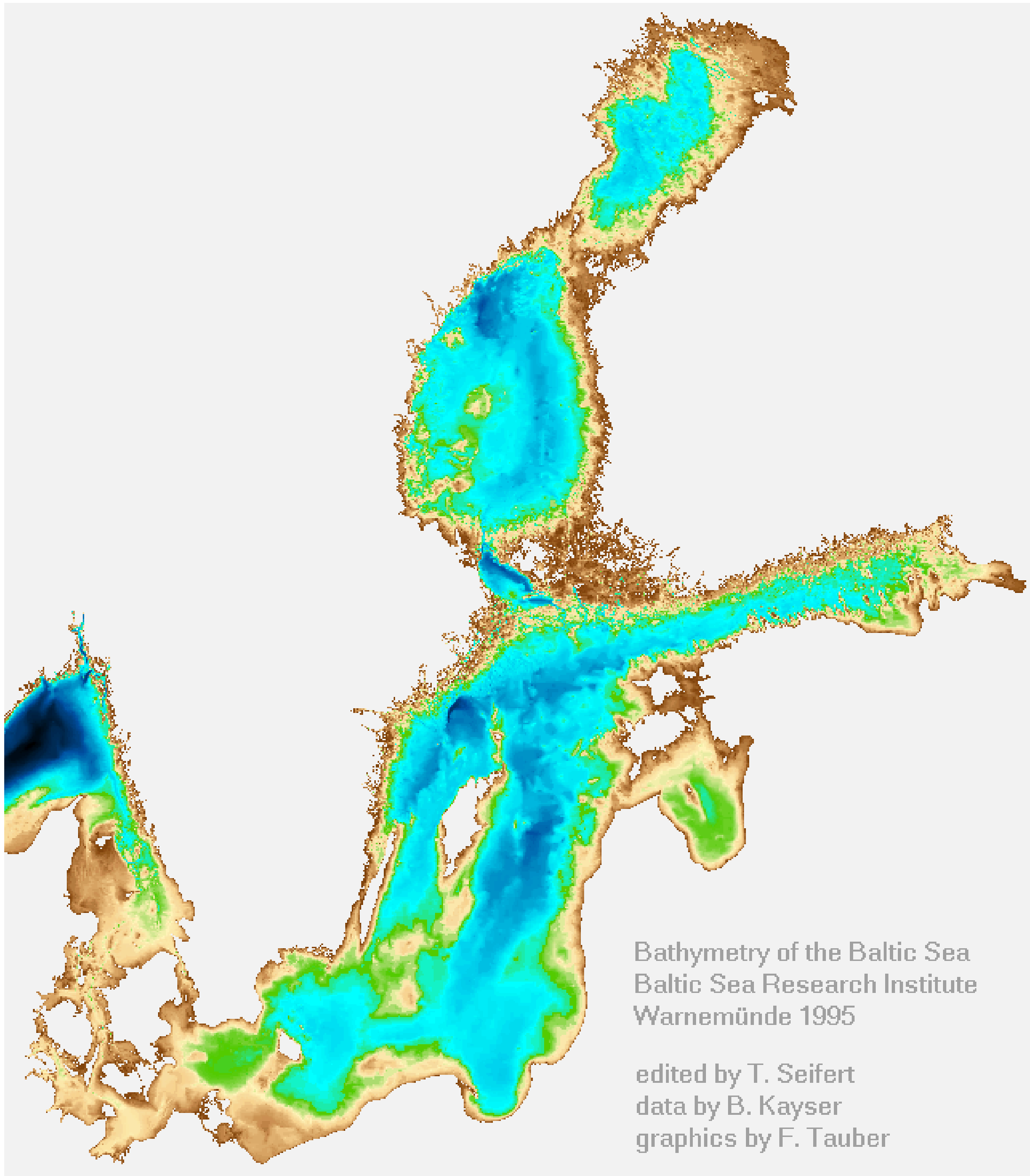
$\mathcal{H} = (\omega_i)_{i=1}^q$: disjoint inclusions in Ω,

$\gamma = \bigcup_{i=1}^q \partial\omega_i$:skeleton of the inclusions,

$$a \in L^\infty(\Omega): \text{ diffusion coefficients}$$

$$\forall \omega \in \mathcal{H} : a|_\omega = a_\omega \in \mathbb{R}_{>0}$$




Bathymetry of the Baltic Sea
Baltic Sea Research Institute
Warnemünde 1995
edited by T. Seifert
data by B. Kayser
graphics by F. Tauber

Continuous Problem:

Find $u \in H^1_D(\Omega)$ such that

$$\int_\Omega \langle a\nabla u, \nabla v\rangle = \int_\Omega fv \qquad \forall v \in H^1_D(\Omega).$$

Number of inclusions is huge and cannot be resolved by finite elements.

Goal: Define low-dimensional finite element spaces in a hierarchical way.

b) Stokes equation.

$$\begin{aligned} -\Delta \mathbf{u} \quad + \nabla p \quad &= \mathbf{f} \\ \operatorname{div} \mathbf{u} \qquad\qquad &= 0 \end{aligned}$$

Boundary conditions:

$$\text{Dirichlet:} \qquad \mathbf{u} = 0 \qquad \text{on } \Gamma_D$$

$$\text{Slip conditions:} \quad \left\{ \begin{array}{l} \langle \mathbf{u}, \mathbf{n} \rangle = 0 \\ \sigma_{\mathbf{n}} - \langle \mathbf{n}, \sigma_{\mathbf{n}} \rangle \, \mathbf{n} = \mathbf{0} \end{array} \right\} \quad \text{on } \Gamma_{\text{slip}}$$

with the stress tensor $\sigma(\mathbf{u}, p) = 2\varepsilon(\mathbf{u}) - p\mathbf{I}$.

Sobolev spaces for velocity:

$$\mathbf{H}^1_D(\Omega) := \left\{ \mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} = 0 \qquad \text{on } \Gamma_D \right\},$$

$$\mathbf{H}^1_{\text{slip}}(\Omega) := \left\{ \mathbf{u} \in \mathbf{H}^1(\Omega) : \langle \mathbf{u}, \mathbf{n} \rangle = 0 \qquad \text{on } \Gamma_{\text{slip}} \right\}.$$

c) Highly indefinite Helmholtz equation.

Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \left(\langle \nabla u, \nabla \overline{v} \rangle - k^2 u \overline{v} \right) + \mathrm{i}\, k \int_{\partial\Omega} u \overline{v} = \int_\Omega f \overline{v} \qquad \forall v \in H^1(\Omega)\,.$$

The solutions become highly oscillatory for large wave numbers $k \geq k_0 > 0$.

Composite Finite Elements

Concept:

a. Introduce *dependent* nodes in vicinity of critical regions, i.e., boundary of the domain and/or of the inclusions.

b. Define the values in the *dependent* nodes via a suitable extrapolation operator which reflects the characteristic behavior of the solution.

c. The number and location of the *dependent* nodes are determined via an a-posteriori error estimator.

Construction of Two-Scale Discretization I

- $\mathcal{G}_H$: Overlapping finite element mesh determining the degrees of freedoms. The mesh width H is related to the desired accuracy.

- $\mathcal{G}_{H,h}$: Finite element mesh which arises by refining recursively all simplices which overlap the boundary of the domain and/or the inclusions. The mesh width which characterizes the boundary resolution is h.

- $\mathcal{G}_{H,h}^{\mathsf{dep}} \subset \mathcal{G}_{H,h}$: Simplices which are generated by the boundary resolution.

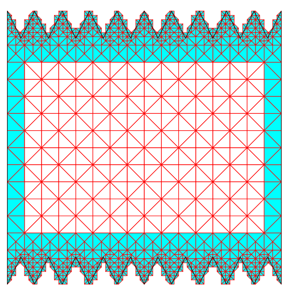
- $\mathcal{G}_{H,h}^{\mathsf{dof}} := \mathcal{G}_{H,h} \backslash \mathcal{G}_{H,h}^{\mathsf{dep}}$: Simplices which have a proper distance to the boundary.

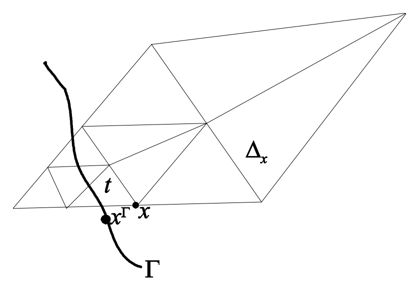
Construction of Two-Scale Discretization II

$\Theta_{H,h}$: Total set of nodal points in the two-scale mesh $\mathcal{G}_{H,h}$

$\Theta_{H,h}^{\mathsf{dep}}$: Set of *dependent* nodes in the vicinity of the critical regions.

$\Theta_{H,h}^{\mathsf{dof}}$: Set of degrees of freedom.





For $x \in \Theta_{H,h}^{\mathsf{dep}}$, let x^{Γ} denote a nearest point on the boundary/interface and let Δ_x denote a nearest simplex in $\mathcal{G}_{H,h}^{\mathsf{dof}}$.

Construction of Composite Finite Element Spaces:

a) Poisson-type problem with Neumann boundary conditions

Space of standard finite elements at proper distance to the boundary/interface is defined by

$$\Omega_{H,h}^{\mathsf{dof}} := \mathsf{int} \bigcup_{\tau \in \mathcal{G}_{H,h}^{\mathsf{dof}}} \tau.$$

$$S_{H,h}^{\mathsf{dof}} := \left\{ u \in C^0 \left(\Omega_{H,h}^{\mathsf{dof}} \right) \mid \forall \tau \in \mathcal{G}_{H,h}^{\mathsf{dof}} : u|_\tau \in \mathbb{P}_1 \right\}.$$

Extrapolation Operator:

Let $u \in S_{H,h}^{\mathsf{dof}}$.

- For nodal points $x \in \Theta_{H,h}$, the extrapolation operator is given by

$$(\mathcal{E}_N u)(x) := \begin{cases} u(x) & x \in \Theta_{H,h}^{\mathsf{dof}}, \\ u_{\Delta_x}(x) & x \in \Theta_{H,h}^{\mathsf{dep}}, \end{cases}$$

  where u_{Δ_x} is the affine extension of the restriction $u|_{\Delta_x}$ to $\mathbb{R}^d$.

- For $x \in \Omega$, the extrapolation operator is the piecewise affine interpolation of the nodal values

$$\mathcal{E}_N u := (\mathcal{I}\mathbf{u})|_{\Omega}\,, \qquad \text{where } \mathbf{u} := ((\mathcal{E}_N u)(x))_{x \in \Theta_{H,h}}\,.$$

The space of composite finite elements is given by, locally, extending standard finite element functions from the interior to a near-boundary zone

$$S_{H,h} := \mathcal{E}_N S_{H,h}^{\mathsf{dof}}.$$

The basis functions are

- as for standard finite elements in the interior
- "smeared" from the interior to the boundary in a near-boundary zone.
- The system matrix is assembled by standard finite element technology while the supports of the modified basis functions are slightly increased leading to a slightly increased sparsity pattern.

b) Poisson-type problem with Dirichlet boundary conditions

The concept for the construction is as for the natural boundary conditions while the definition extrapolation operator employs a suitable weight.

- For nodal points $x \in \Theta_{H,h}$, the extrapolation operator is given by

$$(\mathcal{E}_D u)(x) := \begin{cases} u(x) & x \in \Theta_{H,h}^{\mathsf{dof}}, \\ u_{\Delta_x}(x) - u_{\Delta_x}\left(x^{\Gamma}\right) & x \in \Theta_{H,h}^{\mathsf{dep}}. \end{cases}$$

- For $x \in \Omega$, the extrapolation operator is the piecewise affine interpolation of the nodal values

$$\mathcal{E}_D u := (\mathcal{I}\mathbf{u})|_{\Omega}, \qquad \text{where } \mathbf{u} := ((\mathcal{E}_D u)(x))_{x \in \Theta_{H,h}}.$$

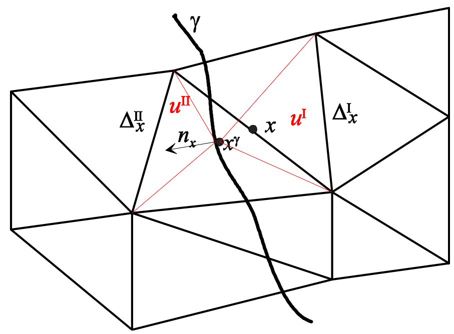
c) Poisson-type problem with discontinuous coefficients

Let γ denote the interfaces between the inclusions. For a dependent node $x \in \Theta_{H,h}^{\mathsf{dep}}$, let $x^\gamma \in \gamma$ denote a nearest interface point and let Δ_x^{I} , Δ_x^{II} denote nearest simplices on each side of the interface. We define

$$\left(\mathcal{E}_{\mathsf{jump}} u\right)(x) := \left(\mathcal{E}_N u\right)(x) - \left(\mathcal{E}_N u\right)(x^\gamma) + c^\gamma,$$

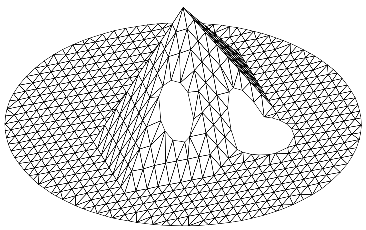
where c_x^{I}, c_x^{II} satisfy

$$\begin{gathered} u_{\Delta_x^{\mathsf{I}}}(x^\gamma) - c_x^{\mathsf{I}} = u_{\Delta_x^{\mathsf{II}}}(x^\gamma) - c_x^{\mathsf{II}} =: c^\gamma \\ a^{\mathsf{I}} \partial_{\mathbf{n}_x} u^{\mathsf{I}}(x^\gamma) = a^{\mathsf{II}} \partial_{\mathbf{n}_x} u^{\mathsf{II}}(x^\gamma) \end{gathered}$$

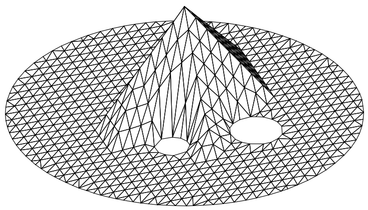


If nearest simplices $\Delta_x^{\mathsf{I}}, \Delta_x^{\mathsf{II}}$ on different sides of γ do **not** exist because the inclusions are **too** small, the construction can be generalized to a multiscale agglomeration approach (work in progress).

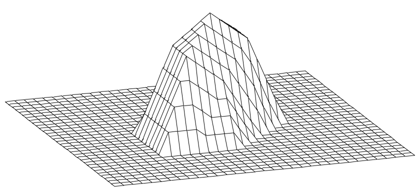
If there is a huge number of small holes as in porous media the definition of the values in the dependent points should be done by solving local problems in a hierarchical way (see S., Warnke)



Basis for Neumann bc



Basis for Dirichlet bc



Basis for problems with discontinuous coefficients

d) Lamé Equations for Problems in Linear Elasticity

The extension operator for problems in elasticity with *mixed boundary conditions* is defined as the componentwise application of the (previous) extension operators for Poisson-type problems.

e) Stokes Problem

We consider the mini-element $\left(S^{1,0} \oplus B\right)^d \times \left(S^{1,0} \cap L_0^2(\Omega)\right)$. The linear part of the velocity field $\mathbf{u} \in \left(S^{1,0} \oplus B\right)^d$ is denoted by $\mathbf{u}^{\text{lin}}$.

Dirichlet boundary conditions: The extension operator $\mathcal{E}_D$ is applied componentwise to the linear part of $\mathbf{u}$ to define values of the velocity in the dependent nodes.

Slip boundary conditions: The normal component is extended as in the case of Dirichlet boundary conditions while the tangential part is extended as for Neumann boundary conditions

$$\left(\mathcal{E}_{\text{slip}}\mathbf{u}\right)(x) := \begin{cases} \mathbf{u}(x) & x \in \Theta_{H,h}^{\text{dof}} \\ (\mathcal{E}_N \mathbf{u})(x) - (\mathcal{E}_N \mathbf{u}(x))_{\mathbf{n}(x^\Gamma)} + (\mathcal{E}_D \mathbf{u}(x))_{\mathbf{n}(x^\Gamma)} & x \in \Theta_{H,h}^{\text{dep}}. \end{cases}$$

A Priori Analysis for Problems in Linear Elasticity

The composite finite element space for Dirichlet boundary conditions is non-conforming $\mathbf{S}^{\mathsf{CFE}} \not\subset \mathbf{H}^1_D(\Omega)$.

Theorem 1. [Rech, Smolianski, S., NuMath] *Let* $h \sim H^{3/2}$*. Then, the bilinearform* $a(\cdot,\cdot)$ *is coercive on* $\mathbf{S}^{\mathsf{CFE}} \subset \mathbf{H}^1_D(\Omega)$*, i.e., there exists* $\gamma > 0$ *such that*

$$a(u,u) \geq \gamma \|u\|^2_{H^1(\Omega)} \qquad \forall u \in \mathbf{S}^{\mathsf{CFE}}.$$

In particular, the Galerkin discretization has a unique solution.

Theorem 2. *Let* $u \in H^1_D(\Omega) \cap H^{1+s}(\Omega)$, $0 \leq s \leq 1$*. Suppose that the following conditions are satisfied*

$$|\tau \cap \Gamma| \leq C h_\tau \qquad \forall \tau \in \mathcal{G}_H$$

and $h \lesssim H^{3/2}$*. Then, the Galerkin solution* $u_S \in \mathbf{S}^{\mathsf{CFE}}$ *exists and fulfills for sufficiently small* H

$$\|u - u_S\|_{m,\Omega} \leq C H^{1+s-m} \|u\|_{H^{1+s}(\Omega)},$$

for $m = 0, 1$ *and* C *is independent of* H *and* h*.*

Proof. M. Rech, thesis, 2006 and Rech, Smolianski, S., NuMath, 2006.

A Posteriori Error Estimation

The accuracy in the *approximation* of the Dirichlet boundary conditions can be controlled by an a-posteriori error estimator which takes into account the effect of approximate Dirichlet boundary conditions.

The error estimator consists of three terms. For any $v \in H^1(\Omega)$, we set

$$m_0^2(v) := \inf_{\tilde{v} \in H_D^1(\Omega)} a(v - \tilde{v}, v - \tilde{v}),$$

$$m_d^2(v, y^\star) := \int_\Omega \left(\varepsilon(v) - \mathbb{L}^{-1} y^\star\right) : (\mathbb{L}\varepsilon(v) - y^\star),$$

$$m_f^2(y^\star) := \|\mathrm{div}\, y^\star + f\|_{L^2(\Omega)}.$$

Theorem 3. *The error of the nonconforming approximation $v \in \mathbf{S}^{\mathsf{CFE}}$ of the exact solution u can be estimated by*

$$a\left(u-v, u-v\right)^{1/2} \leq 2m_0\left(v\right) + m_d\left(v, y^\star\right) + C_\Omega m_f\left(y^\star\right) \qquad \forall y^\star \in H\left(\Omega, \mathsf{div}\right), \tag{1}$$

where C_Ω is the constant in Friedrichs' inequality for the domain Ω, i.e.,

$$C_\Omega := \sup_{w \in H^1_D(\Omega)\setminus\{0\}} \frac{\|w\|_{L^2(\Omega)}}{\|\nabla w\|_{L^2(\Omega)}}.$$

Numerical Experiments

Convergence Rates for the Galerkin Solution

$\Omega = (-1,1)^2$ with a non-aligned rotated overlapping mesh.

level	dof	error **one-scale** CFE	$\frac{error(\ell-1)}{error(\ell)}$	error **two-scale** CFE	$\frac{error(\ell-1)}{error(\ell)}$
3	162	0.34297		0.24245	
4	834	0.18394	1.865	0.12183	1.990
5	3738	0.12994	1.416	0.05171	2.356
6	15786	0.09284	1.340	0.02155	2.340
7	64626	0.06402	1.450	0.00897	2.402

Effectivity Index

$$eff := \frac{\text{estimated energy error}}{\text{energy error}}.$$

level	dof	energy error	$\frac{error(\ell-1)}{error(\ell)}$	estimated error	effectivity index
3	243	0.234		1.345	5.74
4	1251	0.110	2.127	0.692	6.23
5	5607	0.0486	2.263	0.406	8.36
6	23679	0.0165	2.945	0.173	10.46

Analysis for Stokes Flow

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{array}{lcl} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & (\mathbf{f}, \mathbf{v}) \\ b(\mathbf{u}, q) & = & 0 \end{array} \qquad \forall (\mathbf{v}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega),$$

where

$$a(\mathbf{u}, \mathbf{v}) := 2\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \quad \text{and} \quad b(\mathbf{v}, q) := -\int_\Omega q \, \mathsf{div}\, \mathbf{v}.$$

Finite element approximation

Find $(\mathbf{u}, p) \in \mathbf{X}_H \times M_H$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \\ b(\mathbf{u}, q) &= 0 \end{aligned} \qquad \forall (\mathbf{v}, p) \in \mathbf{X}_H \times M_H,$$

where $(\mathbf{X}_H, M_H)$ is the classical mini-element space, i.e.,

$\mathbf{X}_H$: pw. linear functions enriched by cubic bubble functions

M_H: continuous, pw linear functions with integral mean zero.

Classical Convergence Results

- Coercivity: $\forall \mathbf{u}, \mathbf{v} \in \mathbf{X}_H \qquad a(\mathbf{u}, \mathbf{u}) \geq \alpha \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}$,

- Stability:

$$\inf_{p_H \in M_H} \sup_{\mathbf{v}_H \in \mathbf{X}_H \setminus \{0\}} \frac{b(\mathbf{v}_H, p_H)}{\|\mathbf{v}_H\|_{\mathbf{H}^1(\Omega)} \|p_H\|_{L^2(\Omega)}} \geq \beta > 0.$$

- Regularity: $(\mathbf{u}, p) \in \mathbf{H}^{1+r}(\Omega) \times \mathbf{H}^r(\Omega)$ for some $0 < r \leq 2$.

Then: $\|\mathbf{u} - \mathbf{u}_H\|_{\mathbf{H}^1(\Omega)} + \|p - p_H\|_{L^2(\Omega)} \leq C_{\mathbf{f}} H^r.$

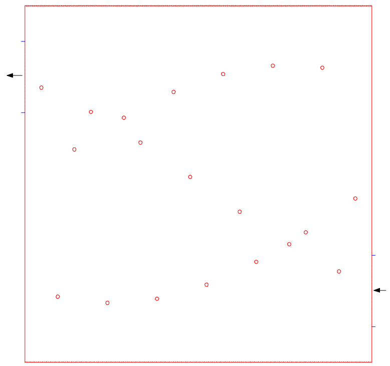
For complicated domains, composite finite elements violate the boundary conditions.

This leads to a *non-conforming method:*

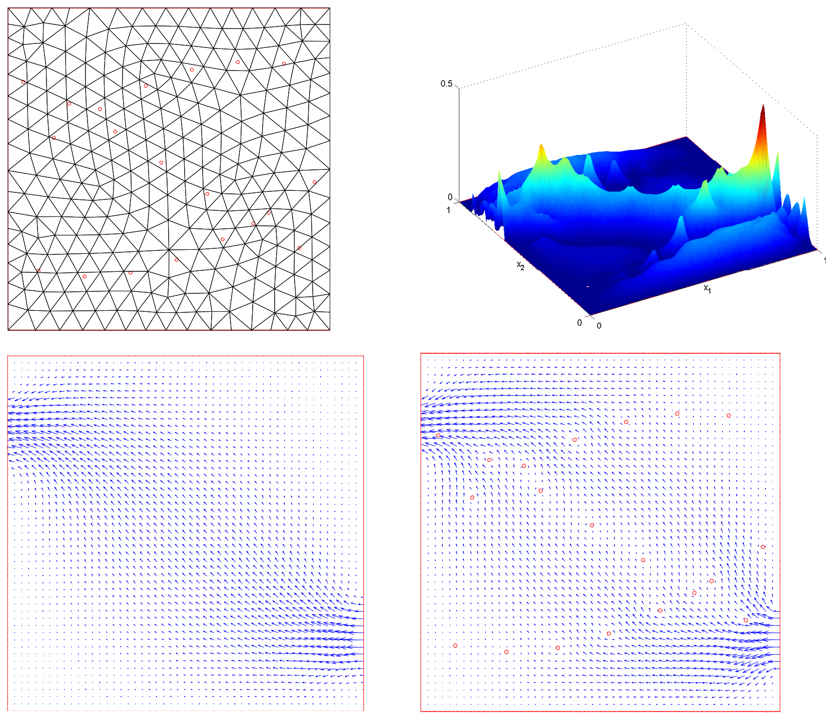
$$\mathbf{X}_H \times \mathbf{M}_H \subsetneq \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \text{ **but** } \mathbf{X}_H \times \mathbf{M}_H \subset \mathbf{H}^1(\Omega) \times L^2(\Omega).$$

Then: $\|\mathbf{u} - \mathbf{u}_H\|_{\mathbf{H}^1(\Omega)} + \|p - p_H\|_{L^2(\Omega)} \leq C_{\mathbf{f}} \left(H^r + \sup_{\mathbf{v}_H \in \mathbf{X}_H \setminus \{0\}} \frac{\|\mathbf{v}_H\|_{\mathbf{L}^2(\partial\Omega)}}{\|\mathbf{v}_H\|_{\mathbf{H}^1(\Omega)}} \right).$

Model Problem: Domain which contains small holes.



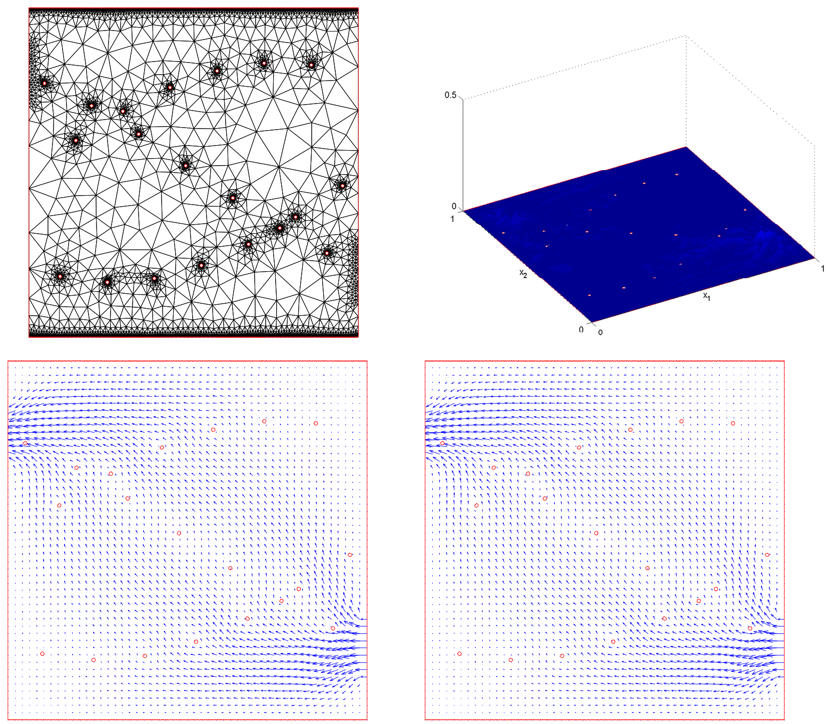
Most Naive Approach: Holes are neglected (due to limited computer capacity) and a quasi-uniform mesh is employed.



Left-top: Quasi-uniform mesh. Right-top: Absolute velocity error.

Left bottom: Numerical solution on quasi-uniform mesh. Right-bottom: Exact solution

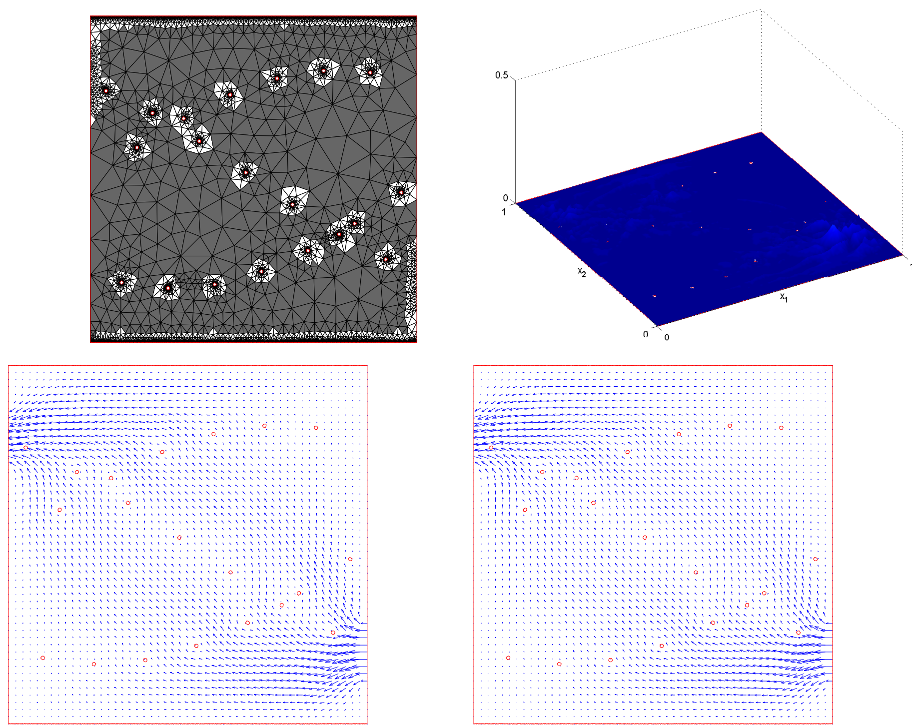
Resolving the holes by an adaptive mesh requires $8 - 10$ times more nodal points.



Left-top: Resolving adaptive mesh. Right-top: Absolute velocity error.

Left bottom: Numerical solution on resolving mesh. Right-bottom: Exact solution

Composite Mini Element. (Number of unknowns is comparable as for quasi-uniform mesh)



Left-top: CME mesh. Right-top: Absolute velocity error.

Left bottom: Numerical solution on CME mesh. Right-bottom: Exact solution

Convergence Results:

Theorem 4. [Stability, Peterseim/S. SIAM J. Numer. Anal.] *The composite mini element space satisfies the* inf-sup-*condition*

$$\inf_{p\in M_{H,h}^{\mathsf{CME}}\setminus\{0\}} \sup_{\mathbf{u}\in\mathbf{X}_{H,h}^{\mathsf{CME}}\setminus\{0\}} \frac{b\left(\mathbf{u},p\right)}{\|p\|_{L^2(\Omega)}\,\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}} \geq \beta > 0.$$

Theorem 5. [Approximation Property, Peterseim/S. SIAM J. Numer. Anal.] *The composite mini element space satisfies the standard approximation property* **already** *on the non-resolved scales.*

$$\inf_{\mathbf{v}\in\mathbf{X}_{H,h}^{\mathsf{CME}}} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^1(\Omega)} + \inf_{q\in M_{H,h}^{\mathsf{CME}}} \|p-q\|_{L^2(\Omega)} \lesssim H\left(\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)}\right).$$

Theorem 6. *The discretization with the composite mini element has a unique solution* $\left(\mathbf{u}_{H,h}, p_{H,h}\right)$.

If $(\mathbf{u}, p) \in \mathbf{H}^{1+r}(\Omega) \times H^{r}(\Omega)$ *for some* $r \in \left]\frac{1}{2}, 1\right]$ *then*

$$\left\|\mathbf{u} - \mathbf{u}_{H,h}\right\|_{\mathbf{H}^1(\Omega)} + \left\|p - p_{H,h}\right\|_{L^2(\Omega)} \lesssim \left(H^r + \rho_{\mathsf{nc}}(H,h)\right) \|\mathbf{f}\|_{\mathbf{H}^{r-1}(\Omega)},$$

with the measure of non-conformity

$$\rho_{\mathsf{nc}}(H,h) := \sup_{\mathbf{v}_{H,h} \in \mathbf{X}_{H,h}^{\mathsf{CME}} \backslash \{0\}} \frac{\left\|\mathbf{v}_{H,h}\right\|_{L^2(\partial\Omega)}}{\left\|\mathbf{v}_{H,h}\right\|_{H^1(\Omega)}} \lesssim \sqrt{H} \max_{\tau \in \mathcal{G}^{\mathsf{dof}}} \max_{\substack{t \in \mathcal{G}^{\mathsf{slave}}(\tau) \\ t \cap \partial\Omega \neq \emptyset}} \frac{\operatorname{diam} t}{\operatorname{diam} \tau}.$$

If the small and coarse scales h *and* H *satisfy* $h \sim H^{r+1/2}$*, the asymptotic convergence rates* **already** *hold on the coarse scales.*