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Proof of Galerkin orthogonality

u solves $\quad a(u,v) = f(v) \quad \forall v \in V_0$

u_S solves $\quad a(u_S,v) = f(v) \quad \forall v \in S \subset V_0$

$$a(u-u_S, v) = 0 \qquad \forall v \in S$$

Quasi-Optimality:

$$c\,\|u-u_S\|_V^2 \overset{\text{coercivity}}{\leq} a(u-u_S, u-u_S)$$
$$= a(u-u_S, u) - \underbrace{a(u-u_S, \underset{\in S}{u_S})}_{0}$$
$$\overset{\text{Gal.-orth.}}{=} a(u-u_S, u) - a(u-u_S, \underset{\in S}{v})$$
$$= a(u-u_S, u-v)$$
$$\leq C\,\|u-u_S\|_V\,\|u-v\|_V$$

$$\Rightarrow \quad \|u-u_S\|_V \leq \frac{C}{c} \inf_{v\in S} \|u-v\|_V \qquad \square$$

Convergence rates under the assumption that u is regular; $u \in H^1_0(\Omega) \cap H^{p+1}(\Omega)$

$$\inf_{v\in S} \|u-v\|_V \leq \|u - \mathrm{Int}^p_{\mathcal{T}}(u)\|_V$$

$$\inf_{v \in S} \|u - v\|_V \le \|u - I_{h,g}(u)\|_V$$

$$\le C_{int}\, h^p \, \|u\|_{H^{p+1}(\Omega)}$$

⇝ Convergence with rates:

$$\|u - u_S\|_V \le \frac{C}{c}\, C_{int}\, h^p \, \|u\|_{H^{p+1}(\Omega)}$$