Course Introduction Fall 2024

Spin Geometry

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Course Description

Spin geometry lies at the crossroad of several subfields[1] of modern mathematics and theoretical physics. It provides a beautiful geometric framework for the study of Dirac operators and forms the mathematical foundation for supersymmetric field theories.

The first half of this course will focus on the definitions and properties of spinors and Dirac operators on curved spaces. The second half will be dedicated to the Atiyah–Singer Index Theorem, including its statement and proof(s).

[1] topological K-theory, geometric analysis, gauge theory, supergravity, etc.

Course Plan

Week 1 Dirac operators and spinor fields on $\mathbb{R}^n$

Weeks 2-3 Laplace operators on Riemannian manifolds

- Hodge/Bohner Laplacians on Riemannian manifolds
- Weitzenböck formula and Bochner's vanishing theorem

Weeks 4-9 Dirac operators on (spin) Riemannian manifolds

- Clifford algebras, spin groups, and Dirac bundles
- Principal bundles, characteristic classes, and Chern-Weil theory
- Spin structures on Riemannian manifolds and the obstruction to their existence
- Dirac operators on spin manifolds
- Weitzenböck formula and Bochner's vanishing theorem

Course Plan

Weeks 10-11 Elliptic operators

- The analytic index of a Fredholm operator
- Elliptic operators on compact manifolds are Fredholm
- Elliptic complexes and Hodge theorem*

Weeks 12-15 Index theorem and its proof(s)

- The topological index of a Dirac operator
- Heat kernel and its asymptotic expansion
- McKean–Singer formula
- Symbol calculus and Getzler's proof
- The supersymmetric quantum mechanics proof*

Week 16 Applications*

- Seiberg–Witten theory*
- Chiral anomalies*

More Information

Prerequisite of the course:

I will assume your familiarity with basic notions of Riemannian geometry, such as a Riemannian manifold, its tangent bundle, and the Levi-Civita connection on its tensor bundle.

Homework:

There will be no homework; however, I will suggest one or two problems for you to solve after class as optional exercises.

Assessment:

There will be an oral exam, and your grade will be determined by your comprehension of core concepts (spin structures, Dirac operators, the statement of the index theorem) in spin geometry demonstrated during this examination.