SYMMETRIC LADDERS AND G-BILIAISON

ELISA GORLA

Abstract: We study the family of ideals generated by minors of mixed size contained in a ladder of a symmetric matrix from the point of view of liaison theory. We prove that they can be obtained from ideals of linear forms by ascending G-biliaison. In particular, they are glicci.

Introduction

Ideals generated by minors have been studied extensively. They are a central topic in commutative algebra, where they have been investigated mainly using Gröbner bases and combinatorial techniques (see among others [10], [18], [1], [2], [21], [17]). They are also relevant in algebraic geometry, since many classical varieties such as the Veronese and the Segre variety are cut out by minors. Degeneracy loci of morphisms between direct sums of line bundles over projective space have a determinantal description, as do the Schubert varieties.

In this paper, we study ideals of minors in a symmetric matrix from the point of view of liaison theory. In particular, we consider ideals generated by minors of mixed size which are contained in a symmetric ladder. Cogenerated ideals in a ladder of a symmetric matrix belong to the family that we study. The family of cogenerated ideals is a natural one to study from the combinatorial point of view (see [7] or [8]). However, from the point of view of liaison theory it is more natural to study a larger class of ideals, as they naturally arise during the linkage process. We call them symmetric mixed ladder determinantal ideals.

In Section 1 we set the notation and define symmetric mixed ladder determinantal ideals (Definition 1.3). In Example 1.5 (3) we discuss why cogenerated ladder determinantal ideals of a symmetric matrix are a special case of symmetric mixed ladder determinantal ideals. In Proposition 1.7 we show that symmetric mixed ladder determinantal ideals are prime and Cohen-Macaulay. In Proposition 1.8 we express their heigth as the cardinality of a suitable subladder.

In Section 2 we review the notion of G-biliaison, stating the definition and main result in the algebraic language (see Definition 2.2 and Theorem 2.4). In Theorem 2.3 we prove that symmetrix mixed ladder determinantal ideals can be obtained from ideals of linear forms by ascending G-biliaison. In particular, they are glicci (Corollary 2.5).

Key words and phrases. G-biliaison, Gorenstein liaison, minor, symmetric matrix, symmetric ladder, complete intersection, Gorenstein ideal, Cohen-Macaulay ideal.

The author was supported by the "Forschungskredit der Universität Zürich" (grant no. 57104101) and by the Swiss National Science Foundation (grant no. 107887).

1. Ideals of minors of a symmetric matrix

Let K be an algebraically closed field. Let $X = (x_{ij})$ be an $n \times n$ symmetric matrix of indeterminates. In other words, the entries x_{ij} with $i \leq j$ are distinct indeterminates, and $x_{ij} = x_{ji}$ for $i > j$. Let $K[X] = K[x_{ij} \mid 1 \leq i \leq j \leq n]$ be the polynomial ring associated to the matrix X. In this paper, we study ideals generated by the minors contained in a ladder of a generic symmetric matrix from the point of view of liaison theory. Throughout the paper, we only consider symmetric ladders. This can be done without loss of generality, since the ideal generated by the minors in a ladder of a symmetric matrix coincides with the ideal generated by the minors in the smallest symmetric ladder containing it.

Definition 1.1. A **ladder** $\mathcal{L}$ of X is a subset of the set $\mathcal{X} = \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i,j \leq n\}$ with the following properties :

(1) if $(i,j) \in \mathcal{L}$ then $(j,i) \in \mathcal{L}$ (i.e. $\mathcal{L}$ is symmetric), and
(2) if $i < h, j > k$ and $(i,j),(h,k) \in \mathcal{L}$, then $(i,k),(i,h),(h,j),(j,k) \in \mathcal{L}$.

We do not make any connectedness assumption on the ladder $\mathcal{L}$. For ease of notation, we also do not assume that X is the smallest symmetric matrix containing $\mathcal{L}$. Let

$$\mathcal{X}^+ = \{(i,j) \in \mathcal{X} \mid 1 \leq i \leq j \leq n\} \quad \text{and} \quad \mathcal{L}^+ = \mathcal{L} \cap \mathcal{X}^+.$$

Since $\mathcal{L}$ is symmetric, $\mathcal{L}^+$ determines $\mathcal{L}$ and viceversa. We will abuse terminology and call $\mathcal{L}^+$ a ladder. Observe that $\mathcal{L}^+$ can be written as

$$\mathcal{L}^+ = \{(i,j) \in \mathcal{X}^+ \mid i \leq c_l \text{ or } j \leq d_l \text{ for } l = 1,\ldots,r \text{ and } i \geq a_l \text{ or } j \geq b_l \text{ for } l = 1,\ldots,u\}$$

for some integers $1 \leq a_1 < \ldots < a_u \leq n$, $n \geq b_1 > \ldots > b_u \geq 1$, $1 \leq c_1 < \ldots < c_r \leq n$, and $n \geq d_1 > \ldots > d_r \geq 1$ with $a_u \leq b_u$, $c_r \leq b_r$, $c_{l+1} \leq d_l$ for $l = 1,\ldots,r-1$, and $a_l \leq b_{l+1}$ for $l = 1,\ldots,u-1$.

We call $(a_1,b_2),\ldots,(a_{u-1},b_u)$ **lower outside corners**, $(a_1,b_1),\ldots,(a_u,b_u)$ **lower inside corners**, $(c_2,d_1),\ldots,(c_r,d_{r-1})$ **upper outside corners**, and $(c_1,d_1),\ldots,(c_r,d_r)$ **upper inside corners**. If $a_u \neq b_u$, then (b_u,b_u) is a lower outside corner and we set $a_{u+1} = b_u$. Similarly, if $c_r \neq d_r$ then (d_r,d_r) is an upper outside corner, and we set $c_{r+1} = d_r$. See also Figure 1. A ladder has at least one upper and one lower outside corner. Moreover, $(a_1,b_1) = (c_1,d_1)$ is both an upper and a lower inside corner.

The **upper border** of $\mathcal{L}^+$ consists of the elements (c,d) of $\mathcal{L}^+$ such that either $c_l \leq c \leq c_{l+1}$ and $d = d_l$, or $c = c_l$ and $d_l \leq d \leq d_{l-1}$ for some l. See Figure 2.

All the corners belong to $\mathcal{L}^+$. In fact, the ladder $\mathcal{L}^+$ corresponds to its set of lower and upper outside (or equivalently lower and upper inside) corners. The upper corners of a ladder belong to its upper border.

Given a ladder $\mathcal{L}$ we set $L = \{x_{ij} \in X \mid (i,j) \in \mathcal{L}^+\}$, and denote by $K[L]$ the polynomial ring $K[x_{ij} \mid x_{ij} \in L]$. For t a positive integer, and $1 \leq r_1 \leq \ldots \leq r_t \leq n$, $1 \leq c_1 \leq \ldots \leq c_t \leq n$ integers, we denote by $[r_1,\ldots,r_t \mid c_1,\ldots,c_t]$ the t-minor $\det(x_{r_i,c_j})$. We let $I_t(L)$ denote the ideal generated by the set of the t-minors of X which involve only indeterminates of L. In particular $I_t(X)$ is the ideal of $K[X]$ generated by the minors of X of size $t \times t$.

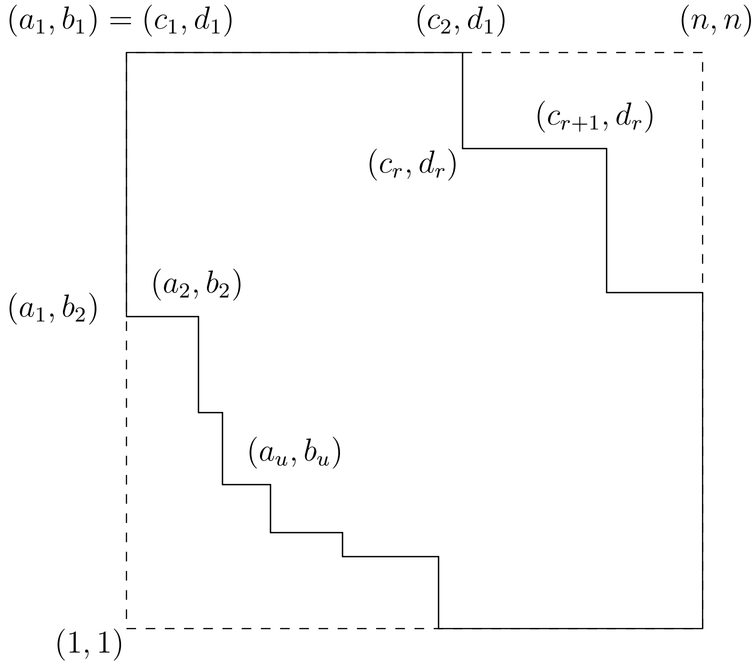

FIGURE 1. An example of ladder with tagged lower and upper corners.

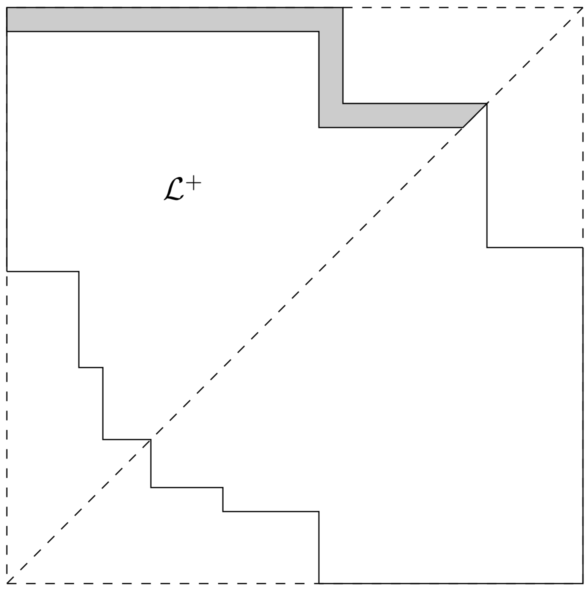

FIGURE 2. The upper border of the same ladder.

In this article, we study the G-biliaison class of a large family of ideals generated by minors in a ladder of a symmetric matrix.

Notation 1.2. Let $\mathcal{L}^+$ be a ladder. For $(c, d) \in \mathcal{L}^+$ let

$$\mathcal{L}^+_{(c,d)} = \{(i, j) \in \mathcal{L}^+ \mid i \leq c,\ j \leq d\}, \qquad L_{(c,d)} = \{x_{ij} \in X \mid (i, j) \in \mathcal{L}^+_{(c,d)}\}.$$

Notice that $\mathcal{L}^+_{(c,d)}$ is a ladder according to Definition 1.1 and

$$\mathcal{L}^+ = \bigcup_{(c,d)\in\mathcal{U}} \mathcal{L}^+_{(c,d)}$$

where $\mathcal{U}$ denotes the set of upper outside corners of $\mathcal{L}^+$.

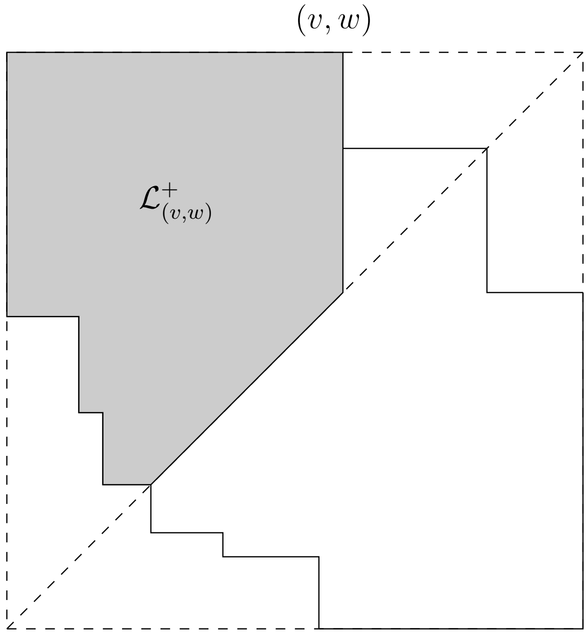

FIGURE 3. The ladder $\mathcal{L}^+$ with a shaded subladder $\mathcal{L}^+_{(v,w)}$.

Definition 1.3. Let $\{(v_1, w_1), \ldots, (v_s, w_s)\}$ be a subset of the upper border of $\mathcal{L}^+$ which contains all the upper outside corners. We order them so that $1 \leq v_1 \leq \ldots \leq v_s \leq n$ and $n \geq w_1 \geq \ldots \geq w_s \geq 1$. Let $t = (t_1, \ldots, t_s)$ be a vector of positive integers. Denote $L_{(v_k,w_k)}$ by L_k. The ideal

$$I_t(L) = I_{t_1}(L_1) + \ldots + I_{t_s}(L_s)$$

is a **symmetric mixed ladder determinantal ideal**. Denote $I_{(t,\ldots,t)}(L)$ by $I_t(L)$. We call $(v_1, w_1), \ldots, (v_s, w_s)$ **distinguished points** of $\mathcal{L}^+$.

Remarks 1.4. (1) Let $\mathcal{M} \supseteq \mathcal{L}$ be two ladders of $\mathcal{X}$, and let M, L be the corresponding sets of indeterminates. We have isomorphisms of graded K-algebras

$$K[L]/I_t(L) \cong K[M]/I_t(L) + (x_{ij} \mid x_{ij} \in M \setminus L) \cong K[X]/I_{2t}(L) + (x_{ij} \mid x_{ij} \in X \setminus L).$$

Here $I_t(L)$ is regarded as an ideal in $K[L], K[M]$, and $K[X]$ respectively. Then the height of the ideal $I_t(L)$ and the property of being prime, Cohen-Macaulay, Gorenstein, Gorenstein in codimension $\leq c$ (see Definition 2.1) do not depend on whether we regard it as an ideal of $K[L], K[M]$, or $K[X]$.

(2) We can assume without loss of generality that for each $l = 1, \ldots, s$ there exists a $k \in \{1, \ldots, u-1\}$ such that

$$t_l \leq \min\{v_l - a_k + 1, w_l - b_{k+1} + 1\}$$

In fact, if $t_l > \min\{v_l - a_k + 1, w_l - b_{k+1} + 1\}$ for all k, then $I_{t_l}(L_l) = 0$. If that is the case, replace L by $M := \cup_{i \neq l} L_i$, eliminate (v_l, w_l) from the distinguished points and remove the l-th entry of t to get a new vector m. Then we obtain a new ladder for which the assumption is satisfied and such that $I_m(M) = I_t(L)$.

(3) We can assume that

$$w_k - w_{k-1} < t_k - t_{k-1} < v_k - v_{k-1}, \quad \text{for } k = 2, \ldots, s.$$

In fact, if $v_k - v_{k-1} \leq t_k - t_{k-1}$, by successively developing a t_k-minor of L_k with respect to the first $v_k - v_{k-1}$ rows we obtain an expression of the minor as a combination of minors of size $t_k - (v_k - v_{k-1}) \geq t_{k-1}$ that involve only indeterminates from L_{k-1}. Therefore $I_{t_k}(L_k) \supseteq I_{t_{k-1}}(L_{k-1})$. Similarly, if $w_k - w_{k-1} \geq t_k - t_{k-1}$, by developing a t_{k-1}-minor of L_{k-1} with respect to the last $w_{k-1} - w_k$ columns we obtain an expression of the minor as a combination of minors of size $t_{k-1} - (w_{k-1} - w_k) \geq t_k$ that involve only indeterminates from L_k. Therefore $I_{t_{k-1}}(L_{k-1}) \subseteq I_{t_k}(L_k)$. In either case, we can remove a part of the ladder and reduce to the study of a proper subladder that corresponds to the same symmetric ladder determinantal ideal.

(4) We can always find $k \in \{1, \ldots, s\}$ such that $v_k > v_{k-1}$ and $w_k > w_{k+1}$. In fact, the two inequalities are satisfied if and only if (v_k, w_k) is an upper outside corner. Notice that if we have distinguished points (v_k, w_k) and (v_{k+1}, w_{k+1}) on the same row or column, then one of the following holds:
   - either $v_k = v_{k+1}$ and $t_k > t_{k+1}$,
   - or $w_k = w_{k+1}$ and $t_{k+1} > t_k$.

   In particular, we can find $k \in \{1, \ldots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$, unless $t_k = 1$ for all k.

The following are examples of determinantal ideals of a symmetric matrix which belong to the class of ideals that we study.

Examples 1.5. (1) If $t = (t, \ldots, t)$ then $I_t(L)$ is the ideal generated by the t-minors of X that involve only indeterminates from L. These ideals have been studied in [4], [5], and [6].

(2) If $\mathcal{L} = \mathcal{X}$, then according to Remarks 1.4 we can assume that $w_l = n$ for all $l = 1, \ldots, s$ and $v_s = n$. From Remark 1.4 (3), we have $t_l > t_{l-1}$ and $v_l > v_{l-1}$ for all l. Then $I_t(L)$ is generated by the t_1-minors of the first v_1 rows, the t_2-minors of the first v_2 rows, ..., the t_s-minors of the whole matrix. This is a simple example of a cogenerated ideal.

(3) The family of symmetric mixed ladder determinantal ideals contains the family of **cogenerated ideals** in a ladder of a symmetric matrix, as defined in [4]. We follow the notation of [4], and assume for ease of notation that $(1, n)$ are is an inside corner of $\mathcal{L}$ (i.e. that X is the smallest matrix containing L). If $\alpha = \{\alpha_1, \ldots, \alpha_t\}$, then $I_\alpha(L) = I_\tau(L)$ where $\{(v_1, w_1), \ldots, (v_s, w_s)\}$ consists of the upper outside corners of $\mathcal{L}$, together with the points of the upper border of $\mathcal{L}$ which belongs to row $\alpha_l - 1$, for all l for which such an intersection point is unique (if for some l the intersection of the row $\alpha_l - 1$ with the upper border of $\mathcal{L}$ consists of more than one

point, then $\mathcal{L}$ has an upper outside corner on the row $\alpha_l - 1$ and we do not add any extra point to the set). For each $k = 1, \ldots, s$, we let $\tau_k = \min\{l \mid \alpha_l > v_k\}$.

(4) Let X be a matrix of size $m \times n$, $m \leq n$, whose entries are indeterminates. Assume that X contains a square symmetric submatrix of indeterminates, and that all the other entries of X are distinct indeterminates. In block notation

$$X = \begin{pmatrix} M & N \\ S & P \end{pmatrix}$$

where S is a symmetric matrix of indeterminates and M, N, P are generic matrices of indeterminates. Let $t \in \mathbb{Z}_+$. Then $I_t(X)$ is a symmetric ladder determinantal ideal generated by the minors of size $t \times t$ contained in a symmetric ladder of

$$\begin{pmatrix} Y & M & N \\ M^t & S & P \\ N^t & P^t & Z \end{pmatrix}$$

where Y, Z are symmetric matrices of indeterminates. This was observed by Conca in [4].

In this section we establish some properties of symmetric mixed ladder determinantal ideals. It is known ([4]) that cogenerated ideals are prime and Cohen-Macaulay. In the sequel we show that the result of Conca easily extends to symmetric mixed ladder determinantal ideals. We exploit a well known localization technique (see [3], Lemma 7.3.3). The same argument was used to prove Lemma 1.19 in [12]. For completeness, we state it for the case of a ladder of a symmetric matrix and we outline the proof. We use the notation of Definitions 1.1 and 1.3. From Remark 1.4 (4) we know that we can always find $k \in \{1, \ldots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$, unless $t = (1, \ldots, 1)$.

Lemma 1.6. *Let $\mathcal{L}$ be a ladder of a symmetric matrix X of indeterminates. $\mathcal{L}$ has a set of distinguished points $\{(v_1, w_1), \ldots, (v_s, w_s)\} \in \mathcal{L}^+$ and $t = (t_1, \ldots, t_s) \in \mathbb{Z}_+^s$. Let $I_t(L)$ be the corresponding symmetric mixed ladder determinantal ideal. Let $k \in \{1, \ldots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$.*

Let $t' = (t_1, \ldots, t_{k-1}, t_k - 1, t_{k+1}, \ldots, t_s)$ and let $\mathcal{L}'$ be the ladder obtained from $\mathcal{L}$ by removing the entries $(v_{k-1}+1, w_k), \ldots, (v_k - 1, w_k), (v_k, w_k), (v_k, w_k - 1) \ldots, (v_k, w_{k+1}+1)$ and the symmetric ones. Let

$$(v_1, w_1), \ldots, (v_{k-1}, v_{k-1}), (v_k - 1, w_k - 1), (v_{k+1}, w_{k+1}), \ldots, (v_s, w_s)$$

be the distinguished points of $\mathcal{L}'$.

Then there is an isomorphism

$$K[L]/I_t(L)[x_{v_k,v_k}^{-1}] \cong K[L']/I_{t'}(L')[x_{v_{k-1}+1,w_k}, \ldots, x_{v_k-1,w_k}, x_{v_k,w_k}^{\pm 1}, x_{v_k,w_k-1}, \ldots, x_{v_k,w_{k+1}+1}].$$

Proof. Under the assumption of the lemma, $\mathcal{L}'$ is a ladder and $I_{t'}(L')$ is a symmetric mixed ladder determinantal ideal. Let

$$A = K[L][x_{v_k,w_k}^{-1}] \text{ and } B = K[L'][x_{v_{k-1}+1,w_k}, \ldots, x_{v_k-1,w_k}, x_{v_k,w_k}^{\pm 1}, x_{v_k,w_k-1}, \ldots, x_{v_k,w_{k+1}+1}].$$

Define a K-algebra homomorphism

$$\begin{array}{rcl} \varphi : A & \longrightarrow & B \\ x_{i,j} & \longmapsto & \left\{ \begin{array}{ll} x_{i,j} + x_{i,w_k} x_{v_k,j} x_{v_k,w_k}^{-1} & \text{if } i \neq v_k, j \neq w_k \text{ and } (i,j) \in \mathcal{L}_{(v_k,w_k)}, \\ x_{i,j} & \text{otherwise.} \end{array} \right. \end{array}$$

The inverse of φ is

$$\begin{array}{rcl} \psi : B & \longrightarrow & A \\ x_{i,j} & \longmapsto & \left\{ \begin{array}{ll} x_{i,j} - x_{i,w_k} x_{v_k,j} x_{v_k,w_k}^{-1} & \text{if } i \neq v_k, j \neq w_k \text{ and } (i,j) \in \mathcal{L}'_{(v_k,w_k)}, \\ x_{i,j} & \text{otherwise.} \end{array} \right. \end{array}$$

It is easy to check that φ and ψ are inverse to each other. Since

$$\varphi(I_{t_k}(L_{(v_k,w_k)})A) = I_{t_k-1}(L'_{(v_k-1,w_k-1)})B$$

we have

$$\varphi(I_t(L)A) = I_{t'}(L')B \quad \text{hence} \quad A/I_t(L)A \cong B/I_{t'}(L')B.$$

□

Using Lemma 1.6 we can establish some properties of symmetric mixed ladder determinantal ideals.

Proposition 1.7. *Symmetric mixed ladder determinantal ideals are prime and Cohen-Macaulay.*

Proof. Let $I_t(L)$ be the symmetric mixed ladder determinantal ideal associated to the ladder $\mathcal{L}$ with distinguished points $(v_1, w_1), \ldots, (v_s, w_s)$ and $t = (t_1, \ldots, t_s)$. Let $t_{\max} = \max\{t_1, ..., t_s\}$. If $t_{\max} = 1$ then $I_t(L)$ is generated by indeterminates, hence it is prime and Cohen-Macaulay. Therefore assume that $t_{\max} \geq 2$ and let $\widehat{\mathcal{L}}$ be the ladder with the same lower outside corners as $\mathcal{L}$, and upper outside corners $(v_k + t_{\max} - t_k, w_k + t_{\max} - t_k)$ for $k = 1, \ldots, s$. Notice that the corners are distinct, and the inequalities of Definition 1.1 are satisfied by Remark 1.4 (3). In other words, for each $k = 2, \ldots, s$ we have

$$w_k + t_{\max} - t_k < w_{k-1} + t_{\max} - t_{k-1} \quad \text{and} \quad v_k + t_{\max} - t_k > v_{k-1} + t_{\max} - t_{k-1}.$$

Let $\widehat{L} = \{x_{ij} \in X \mid (i,j) \in \widehat{\mathcal{L}},\ i \leq j\}$ and let $M = \widehat{L} \setminus L$. Denote by $I_{t_{\max}}(\widehat{L})$ the ideal generated by the minors of size $t_{\max}$ which involve only indeterminates in $\widehat{L}$. By Lemma 1.6 there exists a subset $\{z_1, ..., z_m\}$ of M such that

$$K[\widehat{L}]/I_{t_{\max}}(\widehat{L})[z_1^{-1}, \ldots, z_m^{-1}] \cong K[L]/I_t(L)[M][z_1^{-1}, \ldots, z_m^{-1}].$$

The ring $K[\widehat{L}]/I_{t_{\max}}(\widehat{L})$ is a Cohen-Macaulay domain by Theorem 1.13 in [4]. Therefore $K[L]/I_t(L)[M][z_1^{-1}, \ldots, z_m^{-1}]$ is a Cohen-Macaulay domain. Since M is a set of indeterminates over the ring $K[L]/I_t(L)$ and $z_1, ..., z_m \in M$, then $K[L]/I_t(L)[M]$ is a Cohen-Macaulay domain. Hence $I_t(L)$ is prime and Cohen-Macaulay. □

A standard argument allows us to compute the height of symmetric mixed ladder determinantal ideals. These heights have been computed by Conca in [4] for the family of cogenerated ideals. The arguments in [4] are of a more combinatorial nature, and the height is expressed as a sum of lengths lengths of maximal chains in some subladders. Our formula for the height is very simple. The proof is independent of the results of Conca, and it essentially follows from Lemma 1.6. We use the same notation as in Definitions 1.1 and 1.3, and Lemma 1.6. An example is given in Figure 4.

Proposition 1.8. *Let $\mathcal{L}$ be a ladder with distinguished points $(v_1, w_1), \ldots, (v_s, w_s)$ and let*

$$\mathcal{H}^+ = \{(i,j) \in \mathcal{L}^+ \mid i \leq v_{k-1} - t_{k-1} + 1 \text{ or } j \leq w_k - t_k + 1 \text{ for } k = 2, \ldots, s,$$
$$j \leq w_1 - t_1 + 1,\ i \leq v_s - t_s + 1\}.$$

Let $\mathcal{H} = \mathcal{H}^+ \cup \{(j,i) \mid (i,j) \in \mathcal{H}^+\}$. Then $\mathcal{H}$ is a symmetric ladder and

$$\text{ht } I_t(L) = |\mathcal{H}^+|.$$

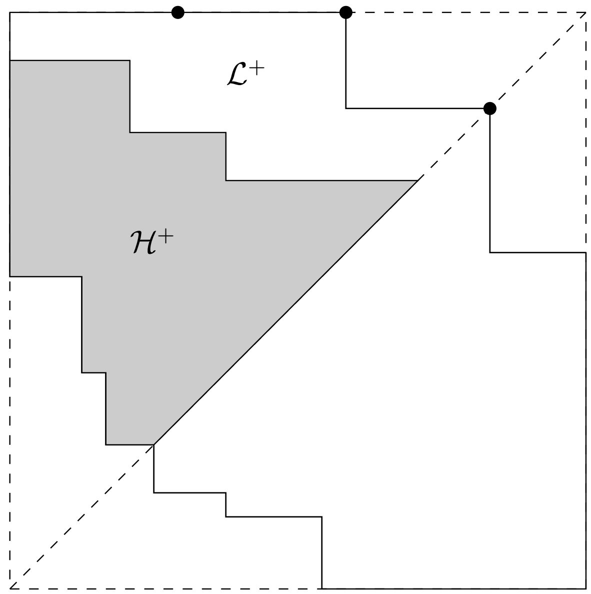

FIGURE 4. An example of $\mathcal{L}^+$ with three distinguished points and $t = (3, 6, 4)$. The corresponding $\mathcal{H}^+$ is shaded.

Proof. Observe that by Remark 1.4 (3)

$$v_k - t_k + 1 > v_{k-1} - t_{k-1} + 1, \quad \text{and} \quad w_k - t_k + 1 < w_{k-1} - t_{k-1} + 1.$$

Therefore $\mathcal{H}$ is a ladder with upper outside corners $\{(v_k - t_k + 1, w_k - t_k + 1) \mid k = 1, \ldots, s\}$ and the same lower outside corners as $\mathcal{L}$. Let $H = \{x_{i,j} \mid (i,j) \in \mathcal{H}^+\}$. We argue by induction on $\tau = t_1 + \ldots + t_s \geq s$. If $\tau = s$, then $t_1 = \ldots = t_s = 1$, and $\mathcal{L} = \mathcal{H}$. Hence

$$I_1(L) = (x_{ij} \mid x_{ij} \in L) = I_1(H) = |\mathcal{H}^+|.$$

Assume now that the thesis holds for $\tau - 1 \geq s$ and prove it for τ. Since $\tau > s$, by Remark 1.4 (4) there exists $k \in \{1, \ldots, s\}$ such that $t_k \geq 2$, $v_k > v_{k-1}$ and $w_k > w_{k+1}$. By Lemma 1.6 we have an isomorphism

$$K[L]/I_t(L)[x_{v_k,w_k}^{-1}] \cong K[L']/I_{t'}(L')[x_{v_{k-1},w_k}, \ldots, x_{v_k-1,w_k}, x_{v_k,w_k}^{\pm 1}, x_{v_k,w_k-1}, \ldots, x_{v_k,w_{k+1}+1}].$$

Since x_{v_k,w_k} does not divide zero modulo $I_{t'}(L')$ and $I_t(L)$, we have

$$\text{ht } I_t(L) = \text{ht } I_{t'}(L').$$

The thesis follows by the induction hypothesis, observing that the same ladder $\mathcal{H}$ computes the height of both $I_{t'}(L')$ and $I_t(L)$. □

2. G-biliaison of symmetric mixed ladder determinantal ideals

In this section we study symmetric mixed ladder determinantal ideals from the point of view of liaison theory. We prove that they belong to the G-biliaison class of a complete intersection. In particular, they are glicci. This is yet another family of ideals of minors for which one can perform a descending G-biliaison to an ideal in the same family, in such a way that one eventually reaches an ideal generated by linear forms. Other families of ideals that were treated with an analogous technique are ideals generated by maximal minors of a matrix with polynomial entries [16], minors of a symmetric matrix with polynomial entries [11], minors of a matrix with polynomial entries [13], minors of mixed size in a ladder of a generic matrix [12], and pfaffians of mixed size in a ladder of a generic skew-symmetric matrix [9].

In [14], [15], [16] Hartshorne developed the theory of generalized divisors, which is a useful language for the study of Gorenstein liaison via the study of G-biliaison classes. In [15] it was shown that even CI-liaison and CI-biliaison generate the same equivalence classes. In [19] they proved that a G-biliaison on an arithmetically Cohen-Macaulay, G_1 scheme can be realized via two G-links. The result was generalized in [16] to G-biliaison on an arithmetically Cohen-Macaulay, G_0 scheme.

In Proposition 1.7 we saw that symmetric mixed ladder determinantal ideals are prime, hence they define reduced and irreducible, projective algebraic varieties. Since we wish to work in the algebraic setting, we state the definition of G-biliaison and the main theorem connecting G-biliaison and G-liaison in the language of ideals.

Definition 2.1. Let $R = K[L]$ and let $J \subseteq R$ be a homogeneous, saturated ideal. We say that J is **Gorenstein in codimension $\leq$ c** if the localization $(R/J)_P$ is a Gorenstein ring for any prime ideal P of R/J of height smaller than or equal to c. We often say that J is G_c. We call **generically Gorenstein**, or G_0, an ideal J which is Gorenstein in codimension 0.

Definition 2.2. ([16], Sect. 3) Let $R = K[X]$ and let I_1 and I_2 be homogeneous ideals in R of pure height c. We say that I_1 is obtained by an **elementary G-biliaison** of height h from I_2 if there exists a Cohen-Macaulay, generically Gorenstein ideal J in R of height $c-1$ such that $J \subseteq I_1 \cap I_2$ and $I_1/J \cong [I_2/J](-h)$ as R/J-modules. If $h > 0$ we speak about **ascending** elementary G-biliaison.

The following theorem gives a connection between G-biliaison and G-liaison.

Theorem 2.3. *[Kleppe, Migliore, Mirò-Roig, Nagel, Peterson* [19]*; Hartshorne* [16]*] Let I_1 be obtained by an elementary G-biliaison from I_2. Then I_2 is G-linked to I_1 in two steps.*

We now show that symmetric mixed ladder determinantal ideals belong to the G-biliaison class of a complete intersection. The idea of the proof is as follows: starting from a symmetric mixed ladder determinantal ideal I, we construct two symmetric mixed ladder determinantal ideals I' and J such that J is contained in $I \cap I'$ and ht $I =$ ht $I' =$ ht $J+1$. We show that I can be obtained from I' by an elementary G-biliaison of height 1 on J.

Theorem 2.4. *Any symmetric mixed ladder determinantal ideal can be obtained from an ideal generated by linear forms by a finite sequence of ascending elementary G-biliaisons.*

Proof. Let $I_t(L)$ be a symmetric mixed ladder determinantal ideal associated to a ladder $\mathcal{L}^+$ with distinguished points $(v_1, w_1), \dots, (v_s, w_s)$. Let $L_k = L_{(v_k,w_k)}$, then

$$I_t(L) = I_{t_1}(L_1) + \cdots + I_{t_s}(L_s) \subseteq K[L].$$

As discussed in Remark 1.4 (1) we will not distinguish between symmetric mixed ladder determinantal ideals and their extensions. Therefore, all ideals will be in $R = K[L]$. If $t_1 = \ldots = t_s = 1$ then $I_t(L)$ is generated by linear forms. Hence let $t_k = \max\{t_1, \dots, t_s\} \geq 2$. From Remark 1.4 (3) we have that $w_{k+1} - w_k < 0 < v_k - v_{k-1}$. In particular (v_k, w_k) is an upper outside corner.

Let $\mathcal{L}'^+$ be the ladder obtained from $\mathcal{L}^+$ by removing the entries $(v_{k-1} + 1, w_k), \dots, (v_k - 1, w_k), (v_k, w_k), (v_k, w_k + 1), \dots, (v_k, w_{k+1} - 1)$. Let $(v_1, w_1), \dots, (v_{k-1}, w_{k-1}), (v_k - 1, w_k - 1), (v_{k+1}, w_{k+1}), \dots, (v_s, w_s)$ be the distinguished points of $\mathcal{L}'^+$, and let $t' = (t_1, \dots, t_{k-1}, t_k - 1, t_{k+1}, \dots, t_s)$. Let $I_{t'}(L')$ be the associated symmetric mixed ladder determinantal ideal. It is easy to check that $\mathcal{L}'^+$ and t' satisfy the inequalities of Definition 1.3 and of Remarks 1.4. By Proposition 1.8, ht $I_t(L) =$ ht $I_{t'}(L') = |\mathcal{H}^+|$ where

$$\mathcal{H}^+ = \{(i,j) \in \mathcal{L}^+ \mid i \leq v_{k-1} - t_{k-1} + 1 \text{ or } j \leq w_k - t_k + 1 \text{ for } k = 2, \dots, s,$$
$$j \leq w_1 - t_1 + 1,\ i \leq v_s - t_s + 1\}.$$

Let $\mathcal{J}^+$ be the ladder obtained from $\mathcal{L}^+$ by removing (v_k, w_k), and let $(v_1, w_1), \dots, (v_{k-1}, w_{k-1}), (v_k - 1, w_k), (v_k, w_k - 1)(v_{k+1}, w_{k+1}), \dots, (v_s, w_s)$ be its distinguished points.

Let $u = (t_1, \dots, t_{k-1}, t_k, t_k, t_{k+1}, \dots, t_s)$. Then

$$I_u(J) = I_{t_1}(L_1) + \ldots + I_{t_{k-1}}(L_{k-1}) + I_{t_k}(J_{(v_k-1,w_k)}) + I_{t_k}(J_{(v_k,w_k-1)}) + I_{t_{k+1}}(L_{k+1}) + \ldots + I_{t_s}(L_s).$$

In other words, $I_u(J)$ is the ideal generated by the minors of $I_t(L)$ that do not involve the indeterminate x_{v_k,w_k}. We claim that $I_u(J) \subseteq I_t(L) \cap I_{t'}(L')$. It is clear that $I_u(J) \subseteq I_t(L)$. The inclusion $I_u(J) \subseteq I_{t'}(L')$ follows from $I_{t_k}(L_{(v_k-1,w_k)}) + I_{t_k}(L_{(v_k,w_k-1)}) \subset I_{t_k-1}(L'_{(v_k-1,w_k-1)})$.

By Proposition 1.8 ht $I_u(J) = |\mathcal{I}^+|$, where $\mathcal{I}^+ = \mathcal{H}^+ \setminus \{(v_k - t_k + 1, w_k - t_k + 1)\}$. Therefore ht $I_u(J) =$ ht $I_t(L) - 1$. The ideal $I_u(J)$ is prime and Cohen-Macaulay by Proposition 1.7. In particular it is generically Gorenstein.

We claim that $I_t(L)$ is obtained from $I_{t'}(L')$ by an elementary G-biliaison of height 1 on $I_u(J)$. This is equivalent to showing that

$$I_t(L)/I_u(J) \cong [I_{t'}(L')/I_u(J)](-1) \tag{1}$$

as $R/I_u(J)$-modules. Denote by K_J the ring of total quotients of $R/I_u(J)$, and by $[\alpha_1, \dots, \alpha_t; \beta_1, \dots, \beta_t]$ the $t \times t$-minor of X which involves rows $\alpha_1, \dots, \alpha_t$ and columns

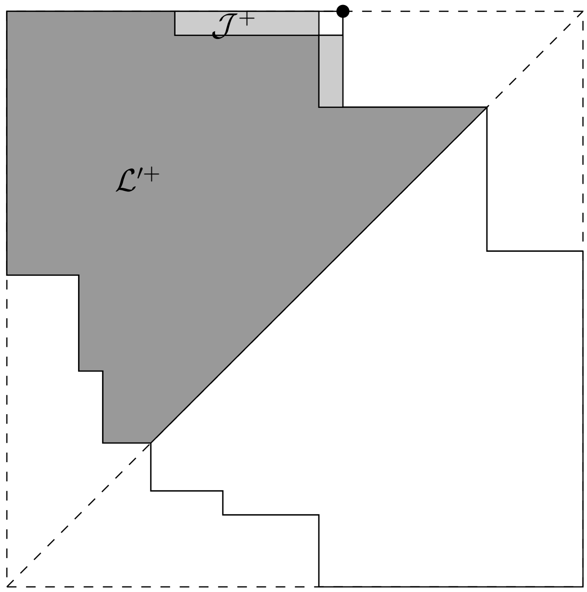

FIGURE 5. An example of $\mathcal{L}^+$ with $\mathcal{L}'^+$ and $\mathcal{J}^+$. The distinguished point (v_k, w_k) is marked. $\mathcal{L}'^+$ is colored in a darker shade and the entries which belong to $\mathcal{J}^+$ but not to $\mathcal{L}'^+$ are colored in a lighter shade.

$\beta_1, \dots, \beta_t$. Multiplication by

$$f = \frac{[v_k - t_k + 1, \dots, v_k - 1; w_k - t_k + 1, \dots, w_k - 1]}{[v_k - t_k + 1, \dots, v_k - 1, v_k; w_k - t_k + 1, \dots, w_k - 1, w_k]} \in K_J$$

yields an isomorphism between $I_t(L)/I_u(J)$ and $[I_{t'}(L')/I_u(J)](-1)$, regarded as $R/I_u(J)$-submodules of K_J. Notice in fact that the ideal

$$I_t(L)/I_u(J) \cong I_{t_k}(L_k)/I_{t_k}(J_{(v_k-1,w_k)}) + I_{t_k}(J_{(v_k,w_k-1)})$$

is generated by the minors of size $t_k \times t_k$ of L_k which involve both row v_k and column w_k, while the ideal

$$I_{t'}(L')/I_u(J) \cong I_{t_k-1}(L'_{(v_k-1,w_k-1)})/I_{t_k}(J_{(v_k-1,w_k)}) + I_{t_k}(J_{(v_k,w_k-1)})$$

is generated by the minors of size $(t_k - 1) \times (t_k - 1)$ of L'_k. By [11], Lemma 2.6 for any choice of minors $[\alpha_1, \dots, \alpha_{t_k-1}, v_k; \beta_1, \dots, \beta_{t_k-1}, w_k]$, $[\gamma_1, \dots, \gamma_{t_k-1}, v_k; \delta_1, \dots, \delta_{t_k-1}, w_k]$ in $I_{t_k}(L_k)$, then the minors $[\alpha_1, \dots, \alpha_{t_k-1}; \beta_1, \dots, \beta_{t_k-1}]$ and $[\gamma_1, \dots, \gamma_{t_k-1}; \delta_1, \dots, \delta_{t_k-1}]$ are in $I_{t_k-1}(L'_k)$ and

$$[\alpha_1, \dots, \alpha_{t_k-1}; \beta_1, \dots, \beta_{t_k-1}] \cdot [\gamma_1, \dots, \gamma_{t_k-1}, v_k; \delta_1, \dots, \delta_{t_k-1}, w_k] =$$

$$[\gamma_1, \dots, \gamma_{t_k-1}; \delta_1, \dots, \delta_{t_k-1}] \cdot [\alpha_1, \dots, \alpha_{t_k-1}, v_k; \beta_1, \dots, \beta_{t_k-1}, w_k]$$

modulo $I_u(J)$. Therefore the isomorphism (1) holds, and $I_t(L)$ and $I_{t'}(L')$ are G-bilinked on $I_u(J)$. Repeating this procedure, one eventually reaches the ideal generated by the entries of the ladder $\mathcal{H}$ defined in Proposition 1.8. Clearly $I_1(H) = (x_{ij} \mid (i,j) \in \mathcal{H}$ is a complete intersection. □

The following is a straightforward consequence of Theorem 2.4, according to Theorem 2.3.

Corollary 2.5. *Every symmetric mixed ladder determinantal ideal* $I_t(L)$ *can be G-linked in* $2(t_1 + \ldots + t_s)$ *steps to a complete intersection of linear forms of the same height. Hence symmetric mixed ladder determinantal ideals are glicci.*

References

[1] W. Bruns, The Eisenbud-Evans generalized principal ideal theorem and determinantal ideals, Proc. Amer. Math. Soc. **83** (1981), no. 1, 19–24.
[2] W. Bruns, U. Vetter, Determinantal rings, Lecture Notes in Mathematics **1327**, Springer-Verlag, Berlin (1988).
[3] W. Bruns, J. Herzog, Cohen-Macaulay rings, Cambridge Studies in Adv. Math. **39**, Cambridge University Press, Cambridge (1993).
[4] A. Conca, Symmetric ladders, Nagoya Math. J. **136** (1994), 35–56.
[5] A. Conca, Divisor class group and canonical class of determinantal rings defined by ideals of minors of a symmetric matrix, Arch. Math. (Basel) **63** (1994), no. 3, 216–224.
[6] A. Conca, Gröbner bases of ideals of minors of a symmetric matrix, J. Algebra **166** (1994), no. 2, 406–421.
[7] C. De Concini, C. Procesi, A characteristic free approach to invariant theory, Advances in Math. **21** (1976), no. 3, 330–354.
[8] C. De Concini, D. Eisenbud, C. Procesi, Hodge algebras, Astérisque **91**, Société Mathématique de France, Paris (1982).
[9] E. De Negri, E. Gorla, G-biliaison of ladder Pfaffian varieties, preprint (2008).
[10] M. Hochster, J. Eagon, Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci, Amer. J. Math. **93** (1971), 1020–1058.
[11] E. Gorla, The G-biliaison class of symmetric determinantal schemes, J. Algebra **310** (2007), no. 2, 880–902.
[12] E. Gorla, Mixed ladder determinantal varieties from two-sided ladders, J. Pure Appl. Alg. **211** (2007), no. 2, 433–444.
[13] E. Gorla, A generalized Gaeta's Theorem, Compositio Math. **144** (2008), no. 3, 689–704.
[14] R. Hartshorne, Generalized divisors on Gorenstein curves and a theorem of Noether, J. Math. Kyoto Univ. **26** (1986), no. 3, 375–386.
[15] R. Hartshorne, Generalized divisors on Gorenstein schemes, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), K-Theory **8** (1994), no. 3, 287–339.
[16] R. Hartshorne, Generalized divisors and biliaison, Illinois J. Math. **51** (2007), no. 1, 83–98.
[17] J. Herzog, N. V. Trung, Gröbner bases and multiplicity of determinantal and Pfaffian ideals, Adv. Math. **96** (1992), no. 1, 1–37.
[18] H. Kleppe, D. Laksov, The generic perfectness of determinantal schemes, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), 244–252, Lecture Notes in Math. **732**, Springer, Berlin (1979).
[19] J. O. Kleppe, J. C. Migliore, R. M. Miró-Roig, U. Nagel, and C. Peterson, Gorenstein liaison, complete intersection liaison invariants and unobstructedness, Mem. Amer. Math. Soc. **154** (2001), no. 732.
[20] J. Migliore, Introduction to Liaison Theory and Deficiency Modules, Birkhäuser, Progress in Mathematics **165** (1998).
[21] B. Sturmfels, Gröbner bases and Stanley decompositions of determinantal rings, Math. Z. **205** (1990), no. 1, 137–144.

Institut für Mathematik, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: elisa.gorla@math.uzh.ch