A generalized Gaeta's Theorem

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Abstract

We generalize Gaeta's Theorem to the family of determinantal schemes. In other words, we show that the schemes defined by minors of a fixed size of a matrix with polynomial entries belong to the same G-biliaison class of a complete intersection whenever they have maximal possible codimension, given the size of the matrix and of the minors that define them.

Introduction

In this paper we study the G-biliaison class of a family of schemes, whose saturated ideals are generated by minors of matrices with polynomial entries. A classical theorem of Gaeta ([9]) states that every codimension 2 scheme defined by the maximal minors of a $t \times (t+1)$ matrix with polynomial entries can be CI-linked in a finite number of steps to a complete intersection. Recently, other families of schemes defined by minors have been studied in the context of liaison theory. The results obtained in this paper are a natural extension of some of the results proven in [20], [14], [10], and [11]. In [20] Kleppe, Migliore, Miró-Roig, Nagel, and Peterson proved that standard determinantal schemes are glicci, i.e. that they belong to the G-liaison class of a complete intersection. We refer to [22] for the definition of standard and good determinantal schemes. Hartshorne pointed out in [14] that the double G-links produced in [20] can indeed be regarded as G-biliaisons. Hence, standard determinantal schemes belong to the G-biliaison class of a complete intersection. In [11] we defined symmetric determinantal schemes as schemes whose saturated ideal is generated by the minors of size $t \times t$ of an $m \times m$ symmetric matrix with polynomial entries, and whose codimension is maximal for the given t and m. In the same paper we proved that these schemes belong to the G-biliaison class of a complete intersection. We recently proved in [10] that mixed ladder determinantal varieties belong to the G-biliaison class of a linear variety, therefore they are glicci. Ladder determinantal varieties are defined by the ideal of $t \times t$ minors of a ladder of indeterminates. We call them mixed ladder determinantal varieties, since we allow minors of different sizes in different regions of the ladder. The results in this paper provide us with yet another family of arithmetically Cohen-Macaulay schemes, for which we can produce explicit G-biliaisons that terminate with a complete intersection. The question that one would hope to answer is *whether every arithmetically Cohen-Macaulay scheme is glicci*. Considerable progress has been made by several authors in showing that special families of schemes are glicci (see e.g. [3], [4], [20], [24], [13], [5], [6], and [19]). On the other side, some results show that the theory will not extend in a straightforward fashion from codimension 2 to higher codimension (see e.g. [13], [23], and [15]).

In this paper, we study a family of schemes that correspond to ideals of minors of fixed size of some matrix with polynomial entries. We call them *determinantal schemes* (see Definition 1.3).

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In Section 1 we establish the setup, and some preliminary results about determinantal schemes. In Remarks 1.6 and Lemma 1.12, we characterize the determinantal schemes that are complete intersections or arithmetically Gorenstein schemes. In Theorem 1.15 and Proposition 1.18 we relate the property of being locally complete intersection outside a subscheme to the height of the ideal of minors of size one less. Section 2 contains results about heights of ideals of minors. It contains material that will be used to obtain the linkage results, but it can be read independently from the rest of the article. In this section we consider an $m\times n$ matrix M, such that the ideal $I_t(M)$ has maximal height $(m-t+1)(n-t+1)$. In Proposition 2.2 we show that deleting a column of M we obtain a matrix O whose ideal of $t\times t$ minors $I_t(O)$ has maximal height $(m-t+1)(n-t)$. In Theorem 2.4, we show that if we apply generic invertible row operations to O and then delete a row, we obtain a matrix N whose ideal of $(t-1)\times(t-1)$ minors has maximal height $(m-t+1)(n-t+1)$. Under the same assumptions, we show that if we apply generic invertible row operations to M and then delete one entry, we obtain a region L whose ideal of $t\times t$ minors has maximal height $(m-t+1)(n-t+1)-1$ (see Corollary 2.9). The consequence which is relevant in terms of the liaison result is that starting from a determinantal scheme X we can produce schemes X' and Y such that X' is determinantal and both X and X' are generalized divisors on Y (see Theorem 2.11). Section 3 contains the G-biliaison results. The main result of the paper is Theorem 3.1, where we show that any determinantal scheme can be obtained from a linear variety by a finite sequence of ascending elementary G-biliaisons. In particular, determinantal schemes are glicci (Corollary 3.2), although they are not always licci.

1. Determinantal schemes

Let X be a scheme in $\mathbb{P}^r=\mathbb{P}^r_K$, where K is an algebraically closed field. Let I_X be the saturated homogeneous ideal associated to X in the polynomial ring $R=K[x_0,x_1,\ldots,x_r]$. For an ideal $I\subset R$, we denote by $H^0_*(I)$ the saturation of I with respect to the maximal ideal $\mathfrak{m}=(x_0,x_1,\ldots,x_r)\subset R$.

Let $\mathcal{I}_X\subset\mathcal{O}_{\mathbb{P}^r}$ be the ideal sheaf of X. Let Y be a scheme that contains X. We denote by $\mathcal{I}_{X|Y}$ the ideal sheaf of X restricted to Y, i.e. the quotient sheaf $\mathcal{I}_X/\mathcal{I}_Y$. For $i\geqslant 0$, we let $H^i_*(\mathbb{P}^r,\mathcal{I})=\oplus_{t\in\mathbb{Z}}H^i(\mathbb{P}^r,\mathcal{I}(t))$ denote the i-th cohomology module of the sheaf $\mathcal{I}$ on $\mathbb{P}^r$. We simply write $H^i_*(\mathcal{I})$ when there is no ambiguity on the ambient space $\mathbb{P}^r$.

Notation 1.1. Let $I\subset R$ be a homogeneous ideal. We let $\mu(I)$ denote the cardinality of a set of minimal generators of I.

In this paper we deal with homogeneous ideals in the polynomial ring R.

Definition 1.2. Let M be a matrix with entries in R of size $m\times n$, where $m\leqslant n$. We say that M is *homogeneous* if its minors of any size are homogeneous polynomials. Notice that M is homogeneous if and only if both its entries and its 2×2 minors are homogeneous.

We always consider homogeneous matrices. We study a family of schemes whose homogeneous saturated ideal I is generated by the $t\times t$ minors of a homogeneous matrix M. In this case we denote I by $I_t(M)$. Notice that for a 1-generic matrix of linear forms, the ideal $I_t(M)$ has ht $I_t(M)\geqslant m+n-1-2(t-1)\geqslant 1$, hence it is always saturated (see [7] for the definition of 1-generic and the proof of the lower bound on the height).

We regard matrices up to invertible transformations, since they do not change the ideal $I_t(M)$. We always assume that the transformations that we consider preserve the homogeneity of the matrix. In the special case $t=1$ it suffices to assume that the entries of the matrix are homogeneous polynomials.

Definition 1.3. Let $X\subset\mathbb{P}^r$ be a scheme. We say that X is *determinantal* if:

i) there exists a homogeneous matrix M of size $m \times n$ with entries in R, such that the saturated ideal of X is generated by the minors of size $t \times t$ of M, $I_X = I_t(M)$, and
ii) X has codimension $(m-t+1)(n-t+1)$.

We refer to [22] for the definition of *standard* and *good determinantal* schemes.

REMARK 1.4. The ideal $I_t(M)$ generated by the minors of size $t \times t$ of an $m \times n$ matrix M has

$$\text{ht } I_t(M) \leqslant (m-t+1)(n-t+1).$$

This is a classical result of Eagon and Northcott. For a proof see Theorem 2.1 in [2]. Therefore the schemes of Definition 1.3 have maximal codimension for fixed m, n, t.

The matrix M defines a morphism of free R-modules

$$\varphi : R^n \longrightarrow R^m.$$

Invertible row and column operations on M correspond to changes of basis in the domain and codomain of φ. The scheme X is the locus where rk $\varphi \leqslant t-1$. So it only depends on the map φ and not on the matrix M chosen to represent it.

EXAMPLES 1.5. Many varieties which are classically studied in algebraic geometry are determinantal. The following and more examples are discussed in Chapter 9 of [12].

i) The Segre variety is an example of determinantal variety. It is the image of the Segre map

$$\sigma : \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{mn-1}$$

hence its saturated ideal is generated by the 2×2 minors of the generic matrix

$$M = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{bmatrix}.$$

ii) Let $M = [x_{i,j}]_{1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant n}$, and let $S \subset \mathbb{P}^{mn-1}$ be the Segre variety with $I_S = I_2(M)$ as in the previous example. The variety V of secant $(t-2)$-planes to S is the determinantal variety corresponding to the ideal $I_t(M)$. V is arithmetically Cohen-Macaulay of codimension $(m-t+1)(n-t+1)$. In [10] we proved that both S and V belong to the G-biliaison class of a complete intersection.
iii) Let $1 \leqslant m < n$, $t \geqslant 2$, and let

$$M = \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-m} \\ x_1 & x_2 & x_3 & \ldots & x_{n-m+1} \\ \vdots & \vdots & \vdots & & \vdots \\ x_m & x_{m+1} & x_{m+2} & \ldots & x_n \end{bmatrix}$$

be a Hankel matrix of size $(m+1) \times (n-m+1)$. The ideal $I_t(M)$ defines the variety of secant $(t-2)$-planes to the rational normal curve $C \subset \mathbb{P}^n$ defined by $I_C = I_2(M)$. Observe that both C and its $(t-2)$-secant variety are determinantal. In fact they are good determinantal, defined by the maximal minors of

$$\begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-1} \\ x_1 & x_2 & & \ldots & x_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-t+1} \\ x_1 & x_2 & x_3 & \ldots & x_{n-t+2} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{t-1} & x_t & x_{t+1} & \ldots & x_n \end{bmatrix}$$

respectively. The fact that good determinantal schemes are glicci is shown in [20], while the fact that they belong to the G-biliaison class of a complete intersection can be found in [14].

The family of determinantal schemes contains well-studied families of schemes, such as complete intersections and standard determinantal schemes.

Remarks 1.6. (i) Complete intersections are a subfamily of determinantal schemes. More precisely, they coincide with the determinantal schemes that have $t = 1$ or $t = m = n$ (see also Lemma 1.12).

(ii) Standard determinantal schemes are a subfamily of determinantal schemes. In fact, a determinantal scheme is standard determinantal whenever $t = m \leqslant n$, that is whenever its saturated ideal is generated by the maximal minors of M.

(iii) Notice that the family of determinantal schemes strictly contains the family of standard determinantal schemes. For example, the schemes of Example 1.5 (ii) are determinantal, but not standard determinantal for $2 \leqslant t \leqslant m-1$. This can be checked e.g. by comparing the number of minimal generators for the saturated ideals of determinantal and standard determinantal schemes.

(iv) The Cohen-Macaulay type of a determinantal scheme as of Definition 1.3 is

$$\prod_{i=1}^{t-1} \frac{\binom{n-i}{t-1}}{\binom{m-i}{t-1}}$$

(see [2]). In particular, a determinantal scheme is arithmetically Gorenstein if and only if $m = n$. Glicciness of arithmetically Gorenstein schemes is established in [6]. In [21] it is shown that the determinantal arithmetically Gorenstein schemes with $t + 1 = m = n$ are glicci. Theorem 3.1 will imply that an arithmetically Gorenstein determinantal scheme belongs to the G-biliaison class of a complete intersection.

In some cases, we will be interested in ideals that are generated by a subset of the minors of M.

Notation 1.7. Let $M = [F_{ij}]_{1\leqslant i\leqslant m,\, 1\leqslant j\leqslant n}$ be an $m \times n$ matrix with entries in the polynomial ring R. Fix a choice of row indexes $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_t \leqslant m$ and of column indexes $1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_t \leqslant n$. We denote by $M_{i_1,\ldots,i_t;j_1,\ldots,j_t}$ the determinant of the submatrix of M consisting of the rows $i_1, \ldots, i_t$ and of the columns $j_1, \ldots, j_t$.

Let L be the subset of M consisting of all the entries except for F_{mn}. We denote by $I_t(L)$ the ideal generated by the minors that involve only entries of L:

$$I_t(L) = (M_{i_1,\ldots,i_t;j_1,\ldots,j_t} \mid i_t \neq m \text{ or } j_t \neq n) \subset I_t(M)$$

Remark 1.8. Let L be the subset of M consisting of all the entries except for F_{mn}. The ideal $I_t(L)$ has height

$$\text{ht } I_t(L) \leqslant (m-t+1)(n-t+1) - 1.$$

This is a special case of Corollary 4.7 of [16].

We now establish some properties of determinantal schemes that will be needed in the sequel. We use the notation of Definition 1.3. We start with a result due to Hochster and Eagon (see [17]). We state only a special case of their theorem that is sufficient for our purposes.

Theorem 1.9. *(Hochster, Eagon) Determinantal schemes are arithmetically Cohen-Macaulay.*

In the sequel, we will also need the following theorem proven by Herzog and Trung. In Corollary 4.10 of [16] they establish Cohen-Macaulayness of a larger family of ideals of minors, but we state their result only for the family of ideals that we are interested in.

THEOREM 1.10. *(Herzog, Trung) Let $U = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, and let V be the subset consisting of the all entries of U except for x_{mn}. Then*

$$I_t(V) := (U_{i_1,\dots,i_t;j_1,\dots,j_t} \mid i_t \neq m \text{ or } j_t \neq n)$$

is a Cohen-Macaulay ideal of height

$$ht\ I_t(V) = (m-t+1)(n-t+1) - 1.$$

We recall that if a scheme defined by the $t \times t$ minors of a matrix of indeterminates is a complete intersection, then it is generated by the entries of the matrix or by its determinant (in the case of a square matrix). In fact, the $t \times t$ minors of $U = [x_{ij}]$ are a minimal system of generators of $I_t(U)$ of cardinality $\binom{m}{t}\binom{n}{t}$ (as they have the same degree and are linearly independent over K). The height of the ideal is $(m-t+1)(n-t+1)$, and the two quantities agree exactly when $t = 1$ or $t = m = n$. We are now going to prove the analogous result for a homogeneous matrix M whose entries are arbitrary polynomials. We also prove a similar result for a subset of the $t \times t$ minors of M. We start by proving an easy numerical lemma.

LEMMA 1.11. *Let m, n, t be positive integers satisfying $2 \leqslant t \leqslant m-1$, $m \leqslant n$. The following inequality holds:*

$$(mn - t^2)(m-1) \cdot \ldots \cdot (m-t+2)(n-1) \cdot \ldots \cdot (n-t+2) > (t!)^2.$$

Proof. Since $t \leqslant m-1 \leqslant n-1$,

$$(m-1) \cdot \ldots \cdot (m-t+2)(n-1) \cdot \ldots \cdot (n-t+2) \geqslant [(t!)/2]^2.$$

Therefore it suffices to show that

$$mn - t^2 > 4.$$

But

$$mn - t^2 \geqslant m^2 - (m-1)^2 = 2m - 1 > 4$$

since $m \geqslant t+1 \geqslant 3$. □

The following lemma is analogous to Lemma 1.16 of [11].

LEMMA 1.12. *Let $M = [F_{ij}]$ be a homogeneous matrix of size $m \times n$ with entries in R or in R_P for some prime P. Let L be the subset consisting of the all entries of M except for F_{mn}.*

(i) If M has no invertible entries and $I_t(M)$ is a complete intersection of codimension $(m-t+1)(n-t+1)$, then $t = 1$ or $t = m = n$.

(ii) If L has no invertible entries and $I_t(L)$ is a complete intersection of codimension $(m-t+1)(n-t+1) - 1$, then $t = 1$ or $t = m = n-1$.

Proof. (i) The $t \times t$ minors of a generic matrix $U = [x_{ij}]$ are a minimal system of generators of $I_t(U)$ since they are K-linearly independent. By Theorem 3.5 in [2] it follows that the minors of the $t \times t$ submatrices of M are a minimal system of generators of $I_t(M)$. If $I_t(M)$ is a complete intersection, then

$$\mu(I_t(M)) = \binom{m}{t}\binom{n}{t} = \text{ht } I_t(M) = (m-t+1)(n-t+1).$$

Computations yield

$$[m \cdot \ldots \cdot (m-t+2)][n \cdot \ldots \cdot (n-t+2)] = [t \cdot \ldots \cdot 2][t \cdot \ldots \cdot 2].$$

Both sides of the equality contain the same number of terms, and $t - i \leqslant m - i \leqslant n - i$ for all $i = 0, \dots, t-2$. So the equality holds if and only if $t = 1$ or $t = m = n$.

(ii) For a generic matrix $M = [x_{ij}]$, the minors of the $t \times t$ submatrices that do not involve the entry x_{mn} are a minimal system of generators of $I_t(L)$. This follows e.g. from the observation that they are linearly independent. By Theorem 3.5 in [2], if we substitute F_{ij} for x_{ij} in a minimal system of generators of $I_t(L)$, we obtain a minimal system of generators for $I_t(L)$ in the case $M = (F_{ij})$ and ht $I_t(L) = (m-t+1)(n-t+1)-1$. In particular, the cardinality of a minimal generating system for $I_t(L)$ is in both cases

$$\mu(I_t(L)) = \binom{m}{t}\binom{n}{t} - \binom{m-1}{t-1}\binom{n-1}{t-1}.$$

If $I_t(L)$ is a complete intersection, then

$$\text{ht } I_t(L) = \binom{m}{t}\binom{n}{t} - \binom{m-1}{t-1}\binom{n-1}{t-1} = (m-t+1)(n-t+1)-1. \tag{1}$$

It follows that

$$(mn-t^2)(m-1)\cdot\ldots\cdot(m-t+1)(n-1)\cdot\ldots\cdot(n-t+1) = (t!)^2[(m-t+1)(n-t+1)-1]$$

By Lemma 1.11 we have that if $t \neq 1, m$, then the left hand side of the equality is greater than $(t!)^2(m-t+1)(n-t+1)$. This is a contradiction, so $t = 1$ or $t = m$. Moreover, if $t = m$ then (1) simplifies to

$$\binom{n}{m} - \binom{n-1}{m-1} = n - m$$

or equivalently to

$$\binom{n-1}{m} = \frac{(n-1)\cdot\ldots\cdot(n-m)}{m!} = n-m.$$

Therefore $m = 1$ or $m = n-1$. Hence either $t = 1$ and $I_t(L)$ is generated by the entries of L, or $t = m = n-1$ and $I_t(L)$ corresponds to a hypersurface (whose equation is the determinant of the first m columns of M). □

Definition 1.13. Let $X \subset \mathbb{P}^r$ be a scheme. We say that X is *generically complete intersection* if it is locally complete intersection at all its components. That is, if the localization $(I_X)_P$ is generated by an R_P-regular sequence for every P minimal associated prime of I_X.

We say that X is *locally complete intersection outside a subscheme of codimension d in $\mathbb{P}^r$* if the localization $(I_X)_P$ is generated by an R_P-regular sequence for every $P \supseteq I_X$ prime of ht $P \leqslant d-1$.

We say that X is *generically Gorenstein*, abbreviated G_0, if it is locally Gorenstein at all its components. That is, if the localization $(I_X)_P$ is a Gorenstein ideal for every P minimal associated prime of I_X.

Remark 1.14. The locus of points at which a scheme fails to be locally complete intersection is closed. Therefore, a scheme of codimension c in $\mathbb{P}^r$ is locally complete intersection outside a subscheme of codimension $c+1$ in $\mathbb{P}^r$ if and only if it is generically complete intersection. Both of these assumptions imply that the scheme is generically Gorenstein.

We now prove two results that relate the height of the ideal of $(t-1)$-minors of M with local properties of the scheme defined by the vanishing of the t-minors of M or L. The notation is as in Definition 1.3.

Theorem 1.15. *Let X be a determinantal scheme with defining matrix M, $I_X = I_t(M)$. Let $c = (m-t+1)(n-t+1)$ be the codimension of X. Assume that X is not a complete intersection, i.e. $t \neq 1$ and t, m, n are not all equal. Let $d \geqslant c+1$ be an integer. Then the following are equivalent:*

i) *X is locally complete intersection outside of a subscheme of codimension d in $\mathbb{P}^r$.*

ii) *ht* $I_{t-1}(M) \geqslant d$.

Proof. (1) $\Longrightarrow$ (2): let $P \supseteq I_t(M)$ be a prime ideal of height $c \leqslant \text{ht } P \leqslant d-1$. In order to prove (2), it suffices to show that $P \not\supseteq I_{t-1}(M)$. Let M_P denote the localization of M at P. The matrix M_P can be reduced after invertible row and column operations to the form

$$M_P = \begin{bmatrix} I_s & 0 \\ 0 & B \end{bmatrix},$$

where I_s is an identity matrix of size $s \times s$, $s \leqslant t$, 0 represents a matrix of zeroes, and B is a matrix of size $(m-s) \times (n-s)$ that has no invertible entries. By assumption, $I_t(M)_P \subseteq R_P$ is a complete intersection ideal. Since $I_t(M_P) = I_{t-s}(B)$ and B has no invertible entries, it follows by Lemma 1.12 that either $t-s=1$, or $t-s=m-s=n-s$. If the latter holds, then $t=m=n$ and X is a hypersurface. Then $t-s=1$ and $I_{t-1}(M_P) = R_P$, so $P \not\supseteq I_{t-1}(M)$.

(2) $\Longrightarrow$ (1): let $P \supseteq I_t(M)$ be a prime of height $c \leqslant \text{ht } P \leqslant d-1$. The thesis is proven if we show that $I_t(M)$ is locally generated by a regular sequence at P. Since ht $P <$ ht $I_{t-1}(M)$, then $P \not\supseteq I_{t-1}(M)$, and the localization M_P of M at P can be reduced, after invertible row and column operations, to the form

$$M_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & B \end{bmatrix},$$

where I_{t-1} is an identity matrix of size $(t-1) \times (t-1)$, 0 represents a matrix of zeroes, and B is a matrix of size $(m-t+1) \times (n-t+1)$. Since $PR_P \supseteq I_t(M_P) = I_1(B)$, we have

$$\mu(I_t(M)_P) \leqslant (m-t+1)(n-t+1) = c = \text{ht } I_t(M)_P.$$

Then $I_t(M)$ is locally generated by a regular sequence at P. □

REMARK 1.16. Assume that X is not a complete intersection. For $d = c+1$, the conclusion of Theorem 1.15 can be restated as: X is generically complete intersection if and only if ht $I_{t-1}(M) >$ ht $I_t(M)$.

The implication (2) $\Longrightarrow$ (1) of Theorem 1.15 clearly holds true without the assumption that X is not a complete intersection. The next example shows that the assumption that X is not a complete intersection is necessary for the implication (1) $\Longrightarrow$ (2).

EXAMPLE 1.17. Let $F \in R$ be a homogeneous form and consider the $t \times t$ matrix

$$M = \begin{bmatrix} F & 0 & \dots & \dots & 0 \\ 0 & F & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & F \end{bmatrix}.$$

Let $X \subseteq \mathbb{P}^r$ be the scheme with $I_X = I_t(M) = (F^t)$. Then X is a hypersurface, hence a complete intersection, therefore locally complete intersection outside any subscheme. However the ideal $I_{t-1}(M) = (F^{t-1})$ defines a hypersurface in $\mathbb{P}^r$, hence ht $I_{t-1}(M) = 1$.

The following proposition gives a sufficient condition for the scheme defined by $I_t(L)$ to be generically complete intersection.

PROPOSITION 1.18. *Let $M = [F_{ij}]$ be a homogeneous matrix of size $m \times n$. Let L be the subset of M consisting of all the entries except for F_{mn}. Let N be the submatrix obtained from M by deleting the last row and column, and let $I_{t-1}(N)$ be the ideal generated by the minors of size $(t-1) \times (t-1)$ of N. Let Y be the scheme corresponding to the ideal $I_t(L)$. Assume that ht $I_t(L) = c-1 = (m-t+1)(n-t+1)-1$ and ht $I_{t-1}(N) = c$. Then Y is generically complete intersection.*

Proof. Let P be a minimal associated prime of $I_Y = I_t(L)$, then $P \not\supseteq I_{t-1}(N)$. Denote by L_P, N_P the localizations of L, N at P. Then $N_P \subseteq L_P$ contains an invertible minor of size $t-1$. We can assume without loss of generality that the minor involves the first $t-1$ rows and columns. After invertible row and column operations (that involve only the first $t-1$ rows and columns) we have

$$L_P = \begin{bmatrix} I_{t-1} & 0 \\ 0 & B \end{bmatrix},$$

where B is the localization at P of the subset obtained by removing the entry in the lower right corner from the submatrix of M consisting of the last $m-t+1$ rows and $n-t+1$ columns. We have

$$\mu((I_Y)_P) = \mu(I_1(B)) \leqslant (m-t+1)(n-t+1) - 1 = \text{ht } (I_Y)_P.$$

Then I_Y is locally generated by a regular sequence at P, i.e. Y is generically complete intersection. □

Remark 1.19. By Proposition 1.18, the condition that ht $I_{t-1}(N) = c$ implies that Y contains a determinantal subscheme X' of codimension 1, whose defining ideal is $I_{X'} = I_{t-1}(N)$. Notice that whenever this is the case, Y is generically complete intersection, hence it is G_0. Under this assumption we have a concept of generalized divisor on Y (see [14] about generalized divisors). Then X' is a generalized divisor on Y. Proposition 1.18 proves that the existence of such a subscheme X' of codimension 1 guarantees that Y is locally a complete intersection. Notice the analogy with standard determinantal ([20]) and symmetric determinantal schemes ([11]).

2. Heights of ideals of minors

In this section we study the schemes associated to the matrix obtained from M by deleting a column, or a column and a generalized row. We assume that the ideal $I_t(M)$ has maximal height according to Remark 1.4. This section can be read independently from the rest of the paper.

As before, let M be a homogeneous matrix of size $m \times n$ with entries in R. Assume that $I_t(M)$ defines a determinantal scheme $X \subset \mathbb{P}^r$ of codimension $c = (m-t+1)(n-t+1)$. We assume that m, n, t are not all equal. In fact, if $m = n = t$ then X is a hypersurface and all the results about the heights are easily verified.

Definition 2.1. Fix a matrix O of size $m \times (n-1)$. Following [22], we call *generalized row* any row of the matrix obtained from O by applying generic invertible row operations. By *deleting a generalized row of* O we mean that we first apply generic invertible row operations to O, and then we delete a row.

A *generalized entry* of O is an entry in a generalized row, i.e. an entry of the matrix obtained from O by applying generic invertible row operations. By *deleting a generalized entry of* O we mean that we first apply generic invertible row operations to O, and then we delete an entry.

We start by deleting a column of M and look at the scheme defined by the $t \times t$ minors of the remaining columns.

Proposition 2.2. *Let $X \subset \mathbb{P}^r$ be a determinantal scheme with associated matrix M, $I_X = I_t(M)$. Let O be the matrix obtained from M by deleting a column. Then $I_t(O)$ is the saturated ideal of a determinantal scheme Z of codimension $(m-t+1)(n-t)$. Moreover, Z is locally complete intersection outside a subscheme of codimension $(m-t+1)(n-t+1)$ in $\mathbb{P}^r$.*

Proof. From the Lemma following Theorem 2 in [1]

$$\text{ht } I_t(M)/I_t(O) \leqslant m-t+1.$$

Hence ht $I_t(O) \geqslant (m-t+1)(n-t+1)-(m-t+1) = (m-t+1)(n-t)$, so equality holds. Then $I_t(O)$ is the saturated ideal of a determinantal scheme Z of codimension $(m-t+1)(n-t)$. Since ht $I_{t-1}(O) \geqslant$ ht $I_t(M) = (m-t+1)(n-t+1)$, by Theorem 1.15 Z is locally complete intersection outside a subscheme of codimension $(m-t+1)(n-t+1)$ in $\mathbb{P}^r$. □

Notation 2.3. We let

$$\varphi : \mathbb{F} \longrightarrow \mathbb{G}$$

be the morphism of free R-modules associated to the matrix O, $\mathbb{F} = R^{n-1}$, $\mathbb{G} = R^m$.

Our goal is to prove that if we delete a generalized row of O, the minors of size $t-1$ of the remaining rows define a determinantal scheme of the same codimension as X. By the upper-semicontinuity principle, it suffices to show that one can apply chosen invertible row and column operations to O, then delete a row, and obtain a matrix whose $t-1$ minors define a determinantal scheme.

Theorem 2.4. *Let O be as in Proposition 2.2. Deleting a generalized row of O, one obtains a matrix N with ht $I_{t-1}(N) = (m-t+1)(n-t+1)$.*

Proof. If $t = m \leqslant n$ then $I_m(O)$ defines a good determinantal scheme, and the result was proven by Kreuzer, Migliore, Nagel, and Peterson in [22]. Assume then that $t < m \leqslant n$, and consider the exact sequence associated to the morphism φ

$$0 \longrightarrow B \longrightarrow \mathbb{F} \xrightarrow{\varphi} \mathbb{G} \longrightarrow \text{Coker } \varphi \longrightarrow 0.$$

Deleting a row of O corresponds to a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & & 0 & & R & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & \mathbb{F} & \xrightarrow{\varphi} & \mathbb{G} & \longrightarrow & \text{Coker } \varphi & \longrightarrow 0 \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & \\
0 & \longrightarrow & B' & \longrightarrow & \mathbb{F} & \xrightarrow{\varphi'} & \mathbb{G}' & \longrightarrow & \text{Coker } \varphi' & \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & & 0 &
\end{array} \tag{2}$$

where φ' is the morphism associated to the submatrix obtained from O after deleting a row (possibly after applying invertible row operations).

We first consider the case when $m < n$. Since $I_m(M)$ defines a standard determinantal scheme and O is obtained from M by deleting a column, then $I_m(O)$ defines a good determinantal scheme (see [20], proof of Theorem 3.6). By Proposition 3.2 in [22], we have that Coker φ is an ideal of positive height in $R/I_m(O)$. Then there is a minimal generator of Coker φ as an R-module that is non zero-divisor modulo $I_m(O)$. Call it f. Denote by s the multiplication map by f:

$$0 \longrightarrow R/I_m(O) \xrightarrow{s} \text{Coker } \varphi \longrightarrow \text{Coker } s \longrightarrow 0. \tag{3}$$

Since $I_m(O) + (f) \subseteq \text{Ann}_R(\text{Coker } s)$, Coker s is supported on a subscheme of codimension at least ht $I_m(O)+1$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ & R & \longrightarrow & R/I_m(O) & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ \mathbb{F} \xrightarrow{\varphi} & \mathbb{G} & \longrightarrow & \text{Coker } \varphi & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ & \mathbb{G}' & \xrightarrow{\beta} & \text{Coker } s & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ & 0 & & 0 & \end{array}$$

Let π denote the morphism $\mathbb{G} \longrightarrow \mathbb{G}'$ in the diagram above, and define $\varphi' = \pi \circ \varphi$. Using the snake lemma, one can check that

$$\mathbb{F} \xrightarrow{\varphi'} \mathbb{G}' \xrightarrow{\beta} \text{Coker } s \longrightarrow 0$$

is exact. Therefore Coker $\varphi' =$ Coker s, and by taking kernels of φ and φ' we produce a diagram as (2).

Let $P \subset R$ be a prime ideal, ht $P \leqslant (m-t+1)(n-t+1)-1$. Since $P \not\supseteq I_{t-1}(O)$, by Proposition 16.3 in [2] $\mu(\text{Coker } (\varphi_P)) \leqslant m-t+1$. We claim that $P \not\supseteq I_{t-1}(N)$. If $P \not\supseteq I_m(O)$, then the claim is proven. Therefore we can assume that $P \supseteq I_m(O)$. Localizing at P the short exact sequence (3) we have that

$$\mu(\text{Coker } (\varphi'_P)) = \mu(\text{Coker } (\varphi_P)) - 1 \leqslant m-t.$$

Here φ_P and φ'_P denote the localization at P of φ and φ', respectively. Then $P \not\supseteq I_{t-1}(N)$, again by Proposition 16.3 in [2]. Therefore the claim is proven, hence ht $I_{t-1}(N) = c$.

Consider now the case $t < m = n$, and consider the morphism $\psi : R^m \longrightarrow R^{m-1}$ defined by the transposed of O. We have ht $I_{m-2}(O) \geqslant$ ht $I_{m-1}(M) = 4 >$ ht $I_{m-1}(O) = 2$. The conditions of Theorem A2.14 in [8] are satisfied, hence Coker $\psi \subseteq R/I_{m-1}(O)$ is an ideal of positive height. One can proceed as in the previous case, constructing an exact sequence

$$0 \longrightarrow R/I_{m-1}(O) \xrightarrow{s} \text{Coker } \psi \longrightarrow \text{Coker } s \longrightarrow 0. \qquad (4)$$

This produces a commutative diagram with exact rows and columns

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ & R & \longrightarrow & R/I_{m-1}(O) & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ \mathbb{F} \xrightarrow{\psi} & \mathbb{G} & \longrightarrow & \text{Coker } \psi & \longrightarrow 0 \\ \| & \downarrow & & \downarrow & \\ \mathbb{F} \xrightarrow{\psi'} & \mathbb{G}' & \longrightarrow & \text{Coker } s & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ & 0 & & 0 & \end{array}$$

Let $P \subset R$ be a prime ideal, ht $P \leqslant (m-t+1)^2 - 1$. Since $P \not\supseteq I_{t-1}(O)$, by Proposition 16.3 in [2] $\mu(\text{Coker } (\psi_P)) \leqslant m-t+1$. We claim that $P \not\supseteq I_{t-1}(N)$, where N is the matrix corresponding to ψ'. If $P \not\supseteq I_{m-1}(O)$, then the claim is proven. Therefore we can assume that $P \supseteq I_{m-1}(O)$. Localizing at P the short exact sequence (4) we have that

$$\mu(\text{Coker } (\psi'_P)) = \mu(\text{Coker } (\psi_P)) - 1 \leqslant m-t.$$

Here ψ_P and ψ'_P denote the localization at P of ψ and ψ', respectively. Then $P \not\supseteq I_{t-1}(N)$, again by Proposition 16.3 in [2]. Therefore the claim is proven, hence ht $I_{t-1}(N) = (m-t+1)^2$. □

The following is a straightforward consequence of Proposition 2.2 and Theorem 2.4.

Corollary 2.5. *Let $X \subset \mathbb{P}^r$ be a determinantal scheme with associated matrix M, $I_X = I_t(M)$. Delete a column of M, then a generalized row, to obtain the matrix N. Then the ideal $I_{t-1}(N)$ defines a determinantal scheme X' of the same codimension as X.*

The next corollary is obtained by repeatedly applying Proposition 2.2 and Theorem 2.4.

Corollary 2.6. *Let M be a homogeneous matrix of size $m \times n$ with entries in R. Assume that ht $I_t(M) = (m-t+1)(n-t+1)$. Delete $t-1$ columns and $t-1$ generalized rows. The remaining entries form a regular sequence.*

Remark 2.7. Under the assumptions of Corollary 2.6 it is clear that for any submatrix H consisting of $n-t+1$ columns of M

$$\text{ht } I_1(H) \geqslant \text{ht } I_t(M) = (m-t+1)(n-t+1).$$

What we prove in Corollary 2.6 is exactly that if we apply generic invertible row operations to M, then pick *any* $n-t+1$ columns as H and delete *any* $t-1$ rows of H, the height of the ideal defined by the entries does not decrease. So after applying generic invertible row operations to M, the matrix has the property that the entries of any submatrix of M of size $(m-t+1)\times(n-t+1)$ form an R-regular sequence.

Apply generic row operations to M and then delete an entry, to obtain the subset $L \subset M$. Then delete the row and the column to which the element in $M \setminus L$ belongs. Call N the matrix obtained. Notice that N is obtained from M by deleting a column and a generalized row. Delete a (generalized) entry of N to obtain the set K. We have $K \subset N \subset L \subset M$. Below we study the relation between the heights of $I_t(M), I_t(L), I_{t-1}(N), I_{t-1}(K)$.

Theorem 2.8. *Let M be a homogeneous matrix of size $m \times n$ with entries in R. Assume that ht $I_t(M) = (m-t+1)(n-t+1)$. Let L be the subset of M obtained by deleting a generalized entry. To fix ideas, assume this is the entry in position (m,n). Let K be the subset obtained from M by deleting the last row and column, and the (generalized) entry in position $(m-1,n-1)$. Then*

$$\text{ht } I_t(L) \geqslant \text{ht } I_{t-1}(K).$$

Proof. By contradiction, suppose that $h = \text{ht } I_t(L) < \text{ht } I_{t-1}(K)$. Let P be a minimal associated prime of $I_t(L)$ of height h. Then $P \not\supseteq I_{t-1}(K)$. Denote by K_P, L_P and M_P the localizations at P of K, L and M. Since $P \not\supseteq I_{t-1}(K)$, then K_P contains an invertible submatrix A of size $(t-1)\times(t-1)$. Since $K_P \subset L_P$, A is a submatrix of L_P which involves neither the last row, nor the last column. Moreover, A cannot involve both row $m-1$ and column $n-1$. To fix ideas, assume that A involves the first $t-1$ rows and columns. By applying invertible row and column operations to M_P, we have

$$M_P \sim \begin{bmatrix} I_{t-1} & 0 \\ 0 & B_P \end{bmatrix}.$$

Notice that the row and column operations can be chosen so that they only affect the rows and columns of A. Therefore B_P is the localization at P of the submatrix B obtained from M by deleting the first $t-1$ rows and columns. The same operations yield

$$L_P \sim \begin{bmatrix} I_{t-1} & 0 \\ 0 & C_P \end{bmatrix}.$$

Here C is obtained from B by removing the entry in the lower right corner, and C_P denotes its localization at P. By Corollary 2.6 the entries of B, hence of C, form a regular sequence in R. Moreover $I_1(C) \subseteq P$, since $P \supseteq I_t(L)$. Therefore the entries of C_P form a regular sequence in R_P, and

$$\text{ht } I_t(L) = \text{ht } I_t(L_P) = \text{ht } I_1(C_P) = c - 1.$$

But this is a contradiction, since ht $I_{t-1}(K) \leqslant c - 1$. $\square$

Corollary 2.9. *Let M, L be as above. If ht $I_t(M) = (m-t+1)(n-t+1)$, then*

$$\text{ht } I_t(L) = (m-t+1)(n-t+1) - 1.$$

Moreover, $I_t(L)$ is generically complete intersection.

Proof. Let L_i be the subset obtained from L by deleting the last i rows and columns and the entry in position $(m-i, n-i)$, $1 \leqslant i \leqslant t-1$. By repeatedly applying Theorem 2.8, one has

$$\text{ht } I_t(L) \geqslant \text{ht } I_{t-1}(L_1) \geqslant \ldots \geqslant \text{ht } I_1(L_{t-1}) = (m-t+1)(n-t+1) - 1. \tag{5}$$

The last equality follows from Corollary 2.6, where we show that the entries of the submatrix of M consisting of the last $m-t+1$ rows and the last $n-t+1$ columns form a regular sequence (see also Remark 2.7). Then ht $I_t(L) = (m-t+1)(n-t+1) - 1$. Let N be obtained from M by deleting the last row and column. By Theorem 2.4 we have ht $I_{t-1}(N) = (m-t+1)(n-t+1)$. Then $I_t(L)$ is generically complete intersection by Proposition 1.18, since

$$\text{ht } I_{t-1}(N) = (m-t+1)(n-t+1) > \text{ht } I_t(L).$$

$\square$

Remark 2.10. As a consequence of Corollary 2.9, we obtain that

$$\text{ht } I_t(M)/I_t(L) \leqslant 1. \tag{6}$$

Since we are working under the assumption that ht $I_t(M) = (m-t+1)(n-t+1)$, the inequality (6) is equivalent to ht $I_t(L) = (m-t+1)(n-t+1) - 1$. We believe that the inequality (6) holds even without the assumption that ht $I_t(M) = (m-t+1)(n-t+1)$, however we were not able to prove this.

Starting from a determinantal scheme X we thus can produce schemes X' and Y such that X' is determinantal and both X and X' are generalized divisors on Y. We summarize these results in the next statement.

Theorem 2.11. *Let X be a determinantal scheme with defining matrix M, $I_X = I_t(M)$. Let $c = (m-t+1)(n-t+1)$ be the codimension of $X \subset \mathbb{P}^r$. Let N be the submatrix obtained from M by deleting a generalized row and a column. Then $I_{t-1}(N)$ is the saturated ideal of a determinantal scheme $X' \subset \mathbb{P}^r$ of codimension c. Let L be the subset obtained from M by deleting the (generalized) entry that belongs to the row and column that appear in M and not in N. Let $I_t(L)$ be the ideal generated by the minors of size $t \times t$ of L. Then $I_t(L)$ is the saturated ideal of an arithmetically Cohen-Macaulay, generically complete intersection scheme Y of codimension $c-1$, that contains X and X' as generalized divisors.*

3. The Theorem of Gaeta for minors of arbitrary size

A classical theorem of Gaeta ([9]) proves that every standard determinantal codimension 2 subscheme of $\mathbb{P}^r$ can be CI-linked in a finite number of steps to a complete intersection. It follows from the Theorem of Hilbert-Burch that every arithmetically Cohen-Macaulay scheme of codimension 2 belongs to the CI-liaison class of a complete intersection. The result was reproven and stated in the language of liaison theory by Peskine and Szpiro in [25]. In Chapter 3 of [20], Gaeta's Theorem is regarded as a statement about standard determinantal schemes of codimension 2, and extended to standard determinantal schemes of arbitrary codimension. With these in mind, we wish to extend the result to the larger class of determinantal schemes. Determinantal schemes include the standard determinantal ones. More precisely, the family of standard determinantal schemes coincides with the determinantal schemes defined by maximal minors (see Remark 1.6 (ii)).

The next theorem generalizes Gaeta's Theorem, Theorem 3.6 of [20], and Theorem 4.1 of [14], to whose argument our proof is inspired. Our result is the analogue of Theorem 2.3 of [11] for a matrix that is not symmetric. A special case of Theorem 3.1 for a matrix of indeterminates follows also from the main result in [10].

Theorem 3.1. *Any determinantal scheme in $\mathbb{P}^r$ can be obtained from a linear variety by a finite sequence of ascending elementary G-biliaisons.*

Proof. Let $X \subset \mathbb{P}^r$ be a determinantal scheme. We use the notation of Definition 1.3. Let $M = [F_{ij}]$ be a homogeneous matrix whose minors of size $t \times t$ define X. Let c be the codimension of X, $c = (m-t+1)(n-t+1)$. If $t = 1$ or $t = m = n$ then X is a complete intersection, therefore we can perform a finite sequence of descending elementary CI-biliaisons to a linear variety. In fact, let X be a complete intersection with $I_X = (F_1, \ldots, F_c)$, $\deg F_i = d_i$, $1 \leqslant d_1 \leqslant \ldots \leqslant d_c$. If $d_c = 1$ then X is a linear variety. Assume that $d_c > 1$ and let $Y \supset X$ be the complete intersection cut out by $F_1, \ldots, F_{c-1}$. X is a hypersurface section divisor on Y. Let $X' \subset Y$ be a hyperplane section divisor on Y, X' is a complete intersection scheme cut out by $L, F_1, \ldots, F_{c-1}$ for some linear form L. We have

$$X \sim d_c X' = X' + (d_c - 1)X'$$

where $\sim$ denotes linear equivalence of generalized divisors on Y. It follows that X is obtained from X' by an ascending elementary CI-biliaison of degree $d_c - 1$. Moreover, X' is a complete intersection cut out by hypersurfaces of degrees $1, d_1, \ldots, d_{c-1}$. Repeating this procedure, we obtain X via at most c ascending CI-biliaisons from a linear variety.

If X is not a complete intersection then $t \geqslant 2$, and $t < m$ if $m = n$. After applying generic invertible row operations to M, we can assume that the subset L and the submatrix N of Theorem 2.11 are obtained from M by deleting the entry in position (m, n), or the last row and column, respectively. Let Y be the scheme with associated saturated ideal

$$I_Y = (M_{i_1,\ldots,i_t;j_1,\ldots,j_t} \mid i_t \neq m \text{ or } j_t \neq n).$$

By Corollary 2.9 (see also Theorem 2.11), Y is arithmetically Cohen-Macaulay and generically complete intersection. In particular, it satisfies the property G_0. The scheme Y has codimension $c-1$, and X is a generalized divisor on Y. Therefore a biliaison on Y is a G-biliaison, in particular it is an even G-liaison (see [20] and [14] for a proof).

Let N be the matrix obtained from M by deleting the last row and column. N is a homogeneous matrix of size $(m-1)\times(n-1)$. Let X' be the scheme cut out by the $(t-1)\times(t-1)$ minors of N. By Corollary 2.5 (and Theorem 2.11) X' is a generalized divisor on Y. We denote by H a hyperplane section divisor on Y. We claim that

$$X \sim X' + aH \quad \text{for some } a > 0,$$

where $\sim$ denotes linear equivalence of generalized divisors on Y. It follows that X is obtained by an ascending elementary G-biliaison from X'. Repeating this argument, after $t-1$ biliaisons we reduce to the case $t = 1$, when the scheme X is a complete intersection. Then we can perform descending CI-biliaisons to a linear variety.

Let $\mathcal{I}_{X|Y}$, $\mathcal{I}_{X'|Y}$ be the ideal sheaves on Y of X and X'. In order to prove the claim we must show that

$$\mathcal{I}_{X|Y} \cong \mathcal{I}_{X'|Y}(-a) \quad \text{for some } a > 0. \tag{7}$$

To keep the notation simple, we denote both an element of R and its image in R/I_Y with the same symbol. By definition, the ideal of Y is generated by the minors of size $t \times t$ of M, except for those that involve both the last row and the last column. Therefore a minimal system of generators of $I_{X|Y}$ is given by

$$I_{X|Y} = (M_{i_1,\ldots,i_{t-1},m;j_1,\ldots,j_{t-1},n} \mid 1 \leqslant i_1 < \ldots < i_{t-1} \leqslant m-1, 1 \leqslant j_1 < \ldots < j_{t-1} \leqslant n-1).$$

A minimal system of generators of $I_{X'|Y} = H^0_*(\mathcal{I}_{X'|Y}) = I_{t-1}(N)/I_Y$ is given by the images in the coordinate ring of Y of the minors of N of size $(t-1) \times (t-1)$

$$I_{X'|Y} = (M_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}} \mid 1 \leqslant i_1 < \ldots < i_{t-1} \leqslant m-1, 1 \leqslant j_1 < \ldots < j_{t-1} \leqslant n-1).$$

Minimality of both systems of generators can be checked with a mapping cone argument, using the fact that the $t \times t$ minors of M, L, N are minimal systems of generators of $I_X, I_Y, I_{X'}$ respectively.

In order to produce an isomorphism as in (7), it suffices to observe that the ratios

$$\frac{M_{i_1,\ldots,i_{t-1},m;j_1,\ldots,j_{t-1},n}}{M_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}}} \tag{8}$$

are all equal as elements of $H^0(\mathcal{K}_Y(a))$, where $\mathcal{K}_Y$ is the sheaf of total quotient rings of Y. Then the isomorphism (7) is simply given by multiplication by that element. Moreover, we can compute the value of a as

$$deg(M_{i_1,\ldots,i_{t-1},m;j_1,\ldots,j_{t-1},n}) - deg(M_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}}) = deg(F_{m,n}).$$

Equality of all the ratios in (8) follows if we prove that

$$M_{i_1,\ldots,i_{t-1},m;j_1,\ldots,j_{t-1},n} \cdot M_{k_1,\ldots,k_{t-1};l_1,\ldots,l_{t-1}} - M_{k_1,\ldots,k_{t-1},m;l_1,\ldots,l_{t-1},n} \cdot M_{i_1,\ldots,i_{t-1};j_1,\ldots,j_{t-1}} \in I_Y$$

for any choice of i, j, k, l. This follows from Lemma 2.4 and Lemma 2.6 in [11]. In those two lemmas, the result is proven in the case $m = n$. The proof however applies with no changes to the situation when $m \neq n$. This completes the proof of the claim and of the theorem. □

Theorem 3.1 together with standard results in liaison theory imply that every determinantal scheme is glicci.

Corollary 3.2. *Every determinantal scheme X can be G-bilinked in $t-1$ steps to a complete intersection, whenever X is defined by the minors of size $t \times t$ of a homogeneous matrix. In particular, every determinantal scheme is glicci.*

We wish to emphasize that for a scheme of codimension 2 the following are all equivalent: being aCM, determinantal, licci, obtained from a linear variety by ascending CI-biliaison. Moreover, in codimension 2 CI-liaison is equivalent to G-liaison, and even CI-liaison is equivalent to CI-biliaison. In higher codimension these concepts diverge (as shown e.g. in [13] and [15]).

In this regard, notice that standard determinantal schemes are in general not licci (i.e. they do not belong to the CI-linkage class of a complete intersection). This follows from the results established in [18] (see Corollary 5.13), and was already shown in [20]. In particular, determinantal scheme are glicci but in general not licci.

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