Homework 1

1. Give an example of a continuous onto and 1-to-1 map which is not a homeomorphism.

2. (a) Show that the collection of sets (a, ∞), $[a, \infty)$, $a \in \mathbb{R}$, $\mathbb{R}$, and $\emptyset$ is a topology on $\mathbb{R}$.

   (b) Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous with respect to this topology if and only if f is increasing (i.e., $x \geq y \Rightarrow f(x) \geq f(y)$).

3. (a) Write explicitly a homeomorphism between the unit square $[0, 1] \times [0, 1]$ and the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$.

   (b) Show that any closed convex polygon is homeomorphic to the unit disk.

4. In this problem you give a topological proof of the fact that there are infinitely many primes (an integer is called *prime* if it is divisible only by itself and 1).

   (a) Consider all arithmetic progressions

   $$S_{a,b} = a + b\mathbb{Z}, \quad a \in \mathbb{Z}, \ \ b \in \mathbb{Z}_{>0}.$$

   Show that these sets form a base.

   (b) Define topology on $\mathbb{Z}$ by this base. Show that each of the sets $S_{a,b}$ is open and closed.

   (c) Show that the set $\{\pm 1\}$ is closed.

   (d) Show that if there are only finitely many primes, then the set $\{\pm 1\}$ is open. Since this is not the case, there are infinitely many primes!

5. (a) Cut the Möbius strip along the middle. What object would you get?

   (b) Cut the Möbius strip along the line which is 1/3 of the width from the edge. What object would you get?

6. Show that $[0, 1)$ and $[0, 1]$ are not homeomorphic, but $[0, 1) \times [0, 1)$ and $[0, 1] \times [0, 1)$ are homeomorphic.