Homework 2

1. Prove or give a counterexample:

   (a) If a topological space X is connected, and every point is contained in an open path-connected set, then X is path-connected.

   (b) Every contractable space is path-connected.

   (c) If f is 1-to-1 continuous map, then preimage of a compact set is compact.

2. Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

3. A topological space (X, τ) is called *locally compact* if every point is contained in an open set with compact closure. Suppose that X is a Hausdorff locally compact space. We define the *one-point compactification* of X:

$$\begin{aligned}\tilde{X} &= X \cup \{\infty\},\\ \tilde{\tau} &= \{U : U \text{ is open in } X\} \cup \{(X - C) \cup \{\infty\} : C \text{ is compact in } X\}.\end{aligned}$$

   (a) Check that $(\tilde{X}, \tilde{\tau})$ is a topological space.

   (b) Prove that $(\tilde{X}, \tilde{\tau})$ is compact and Hausdorff.

   (c) Prove that the one-point compactification of the plane is homeomorphic to the 2-dimensional sphere.

4. Let $f : S^n \to S^n$ be a continuous map ($S^n \subset \mathbb{R}^{n+1}$ is the n-dimensional unit sphere). Show that if $f(x) \neq x$ for every $x \in S^n$, then f is homotopic to the "reflection" map $x \mapsto -x$.

5. Show that the Möbius leaf is homotopy equivalent to the annulus

$$\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}.$$