Homework 3

1. Prove that the fundamental group of n-dimensional sphere S^n, $n \geq 2$, is trivial. You may assume that any loop $f : [0,1] \to S^n$ is equivalent to a loop which is not onto.

2. Find the fundamental group of $\mathbb{R}^3 - L$ where L is a line in $\mathbb{R}^3$.

3. (fundamental theorem of algebra) In this problem we prove that any polynomial
$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n, \quad n \geq 1,$$
with complex coefficients has a root in $\mathbb{C}$.

   (a) Consider the family of maps
   $$f_{r,n} : S^1 \to \mathbb{C} - \{0\} : z \mapsto r z^n, \quad r > 0, \ \ n \in \mathbb{Z}.$$
   Describe which of the maps in this family are homotopic.

   (b) Show that if $r > \max\{1, |a_1| + \cdots + |a_n|\}$, then $P(z) \neq 0$ for $|z| = r$.

   (c) Using (b), show that for large r, the map
   $$z \mapsto P(rz) : S^1 \to \mathbb{C} - \{0\}$$
   is homotopic to $f_{r^n,n}$.

   (d) Suppose that $P(z) \neq 0$ for every $z \in \mathbb{C}$. Then show that the map
   $$z \mapsto P(rz) : S^1 \to \mathbb{C} - \{0\}$$
   is homotopic to the constant map
   $$z \mapsto a_n.$$
   Deduce a contradiction.

4. Let $\phi : D \to D$ be a continuous map where D is an *open* disk in $\mathbb{R}^2$. Does the Brouwer fixed point theorem hold in this case?