Homework 4

1. Show that cylinder $S^1 \times \mathbb{R}$ with one point removed and torus $\mathbb{T}^2$ with one point removed are homotopy equivalent. Compute the fundamental group of these spaces.

2. (a) Find all covering spaces of degree 2 (i.e., such that every point has exactly 2 preimages) of the figure ∞.

   (b) How many subgroup of index 2 does the free group F_2 have?

3. (a) Let X be a finite connected graph (i.e., 1-dimensional cell complex) with v vertices and e edges. Compute the fundamental group of X.

   (b) Using the theory of covering spaces, show that a subgroup of index m in the free group F_r is free of finite rank and compute its rank.

4. Let F_2' be the commutant subgroup of F_2, i.e.,

$$F_2' = \langle [x, y] : x, y \in F_2 \rangle .$$

   (a) Construct the covering space of the figure ∞ that corresponds to this subgroup.

   (b) Show that F_2' is a free subgroup of infinite rank and write explicitly its free generating set.

5. Show the boundary ∂M of the Möbius band M is not a retract of M. (Hint: compute the map of the fundamental groups that corresponds to the map $\partial M \to M$.)