Homework 6

In this homework set, you may use that

- $\pi_n(S^n)$ *is infinite cyclic group generated by the homotopy class of the identity map* $S^n \to S^n$.
- *For* $k > n$, $\pi_k(S^n)$ *is finite except when* $n = 2i$ *and* $k = 4i - 1$.

1. Prove that that the spaces $S^2 \vee S^1$ and $\vee_{i=1}^{\infty} S^2$ have the same n-homotopy groups for $n \geq 2$. Are those spaces homotopy equivalent?

2. Prove that the Hopf fiber bundle $S^3 \to S^2$ is not trivial.

3. Prove that there is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ for $n > k > 0$.

4. (a) An exact sequence of homomorphisms of abelian groups

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

   is called *split* if there exists a homomorphism $\gamma : C \to B$ such that $\beta\gamma = id_C$. Prove that for split sequences as above, $B \simeq A \oplus C$.

   (b) Let $S^{n-1} = S^n \cap \{x_{n+1} = 0\} \subset S^n$, $n \geq 4$. Using the long exact sequence of homotopy groups, compute $\pi_k(S^n, S^{n-1})$ for $k = 2, \ldots, n$.

5. (a) Prove the Brouwer fixed point theorem: every continuous map $f : \bar{D}^n \to \bar{D}^n$ has a fixed point.

   (b) (weak Perron-Frobeneous theorem) Let A be a nondegenerate $(n \times n)$-matrix with nonnegative entries. Using the Brouwer theorem, prove that A has a positive eigenvalue with eigenvector whose coordinates are nonnegative. (Hint: Consider a map $f : R \times [a, b] \to R \times [a, b]$ where R is a closed region in $\mathbb{R}^n$ and $a, b > 0$.)