Homework 1

1. A polyhedron is called regular is every face has the same number of edges, and every vertex meets the same number of edges.

   (a) Let P be a convex regular polyhedron such that each face has p edges and every vertex meets q edges. Prove that

   $$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}$$

   where e is the number of edges.

   (b) Prove that there are at most five convex regular polyhedra.

2. (a) Show that the collection of sets (a, ∞), $[a, \infty)$, for $a \in \mathbb{R}$, $\mathbb{R}$, and $\emptyset$ is a topology on $\mathbb{R}$.

   (b) Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous with respect to this topology if and only if f is increasing (i.e., $x \geq y \Rightarrow f(x) \geq f(y)$).

3. (a) Write explicitly a homeomorphism between the unit square $[0, 1] \times [0, 1]$ and the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$.

   (b) Show that any closed convex polygon is homeomorphic to the unit disk.

4. In this problem you'll give a topological proof of the fact that there are infinitely many primes (an integer is called *prime* if it is divisible only by itself and 1).

   (a) Consider all arithmetic progressions

   $$S_{a,b} = a + b\mathbb{Z}, \quad a \in \mathbb{Z}, \ \ b \in \mathbb{Z}_{>0}.$$

   Construct a topology on $\mathbb{Z}$ using the sets $S_{a,b}$ (check the axioms!). Show that each of the sets $S_{a,b}$ is open and closed.

   (b) Show that the set $\{\pm 1\}$ is closed.

   (c) Show that if there are only finitely many primes, then the set $\{\pm 1\}$ is open. Since this is not the case, there are infinitely many primes!

5. (a) Cut the Möbius strip along the middle. What object would you get?

   (b) Cut the Möbius strip along the line which is 1/3 of the width from the edge. What object would you get?