Homework 2

1. Find a topological space and a compact subset whose closure is not compact.

2. A topological space (X, τ) is called *locally compact* if every point is contained in an open set with compact closure. Suppose that X is a Hausdorff locally compact space. We define the *one-point compactification* of X:

$$\begin{aligned} \tilde{X} &= X \cup \{\infty\}, \\ \tilde{\tau} &= \{U : U \text{ is open in } X\} \cup \{(X - C) \cup \{\infty\} : C \text{ is compact in } X\}. \end{aligned}$$

   (a) Check that $(\tilde{X}, \tilde{\tau})$ is a topological space.

   (b) Prove that $(\tilde{X}, \tilde{\tau})$ is compact and Hausdorff.

   (c) Prove that the one-point compactification of the plane is homeomorphic to the 2-dimensional sphere.

   (d) Prove that if X and Y are homeomorphic, then their one-point compactifications are homeomorphic too.

3. Show that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

4. Show that $[0, 1)$ and $[0, 1]$ are not homeomorphic, but $[0, 1) \times [0, 1)$ and $[0, 1] \times [0, 1)$ are homeomorphic.

5. Prove or give a counterexample:

   (a) If a topological space X is connected, and every point is contained in an open path-connected set, then X is path-connected.

   (b) If f is 1-to-1 continuous map, then preimage of a compact set is compact.