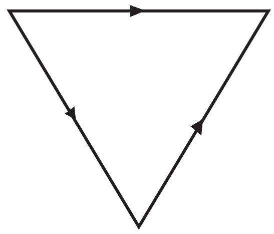
Homework 3

1. Prove that every contractible space is path-connected.

2. The *dunce hat* is obtained from a triangle with sides identified as follows:



   Prove that the dunce hat is contractible.

3. What is the fundamental group of $\mathbb{R}^3 - L$ where L is a line in $\mathbb{R}^3$?

4. Which of the following spaces are homotopy equivalent

   - the Möbius band,
   - the annulus $\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$,
   - $\mathbb{R}^3 - \{0\}$,
   - the torus $\mathbb{T}^2$?

5. (fundamental theorem of algebra) Show that any polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

   with complex coefficients has a complex root.

   Hint: Assume that there are no roots and consider maps $f_t(z) = \frac{p(tz)}{|p(tz)|}$, $z \in S^1$, $t > 0$. Show that this maps are homotopic to a constant map and to the map $z \mapsto z^n$, which gives a contradiction.