# Homework 4

1. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$.

2. We say that a topological space $X$ has the fixed-point property if every continuous function $f : X \to X$ has a fixed point. Which of the following spaces have the fixed-point property?

   - the sphere $S^n$,
   - the square $[0, 1] \times [0, 1]$,
   - the open unit ball in $\mathbb{R}^n$,
   - the letter T.

3. Let $M$ be the Möbius band and let $\partial M$ be the boundary of $M$.

   (a) Consider the maps $f_0, f_1 : S^1 \to M$ such that $f_0(t)$ makes the complete loop along $\partial M$ and $f_1(t)$ makes the complete loop along the circle that cuts $M$ in the middle. Prove that $f_0$ is not homotopic to $f_1$.

   (b) Prove that there is no continuous map $f : M \to \partial M$ that leaves all points on $\partial M$ fixed.

4. Consider a map $f : S^1 \to S^1$. We take a point $x_0 \in S^1$ and choose the simple loops $\alpha$ and $\beta$ with the same orientation that generate $\pi_1(S^1, x_0)$ and $\pi_1(S^1, f(x_0))$ respectively. Then

$$[f \circ \alpha] = [\beta]^n \text{ in } \pi_1(S^1, f(x_0))$$

   for some integer $n$. This integer is called the *degree* of the map $f$.

   (a) Show that $\deg(f)$ is independent of the choice of $x_0$.

   (b) Show that for $f, g : S^1 \to S^1$, $\deg(f \circ g) = \deg(f) \deg(g)$.

   (c) Show that $f, g : S^1 \to S^1$ are homotopic if and only if $\deg(f) = \deg(g)$.

   (d) Every map $f : S^1 \to S^1$ with $\deg(f) \neq 1$ has a fixed point.

5. (a) Find all covering spaces of degree 2 (i.e., such that every point has exactly 2 preimages) of the figure $\infty$.

   (b) How many subgroup of index 2 does the free group $F_2$ have?

   (c) Match the subgroups in (b) with the covering spaces in (a).