MATH 109A. AUTUMN 2006.

Homework 5

1. Describe the simply connected covering spaces and the corresponding covering maps for

    - the torus $\mathbb{T}^2$,
    - the Klein bottle,
    - the projective space $\mathbb{R}P^n$, $n \geq 1$,
    - $\mathbb{R}^n \backslash \{0\}$, $n \geq 2$.

2. Compute the fundamental group of the projective space $\mathbb{R}P^n$, $n \geq 1$.

3. Show that the fundamental group of the Klein bottle is not abelian and it contains a subgroup of index 2 isomorphic to $\mathbb{Z}^2$.

4. Using the theory of covering spaces, show that a subgroup of finite index m in the free group of finite rank r is free of finite rank and compute its rank in terms of m and r.

5. Let F_2 be the commutant subgroup of the free group F_2, i.e.,

$$F_2' = \langle [x, y] : x, y \in F_2 \rangle \quad \text{where } [x, y] = x^{-1}y^{-1}xy.$$

    (a) Construct the covering space of the figure ∞ that corresponds to this subgroup.

    (b) Show that F_2' is a free subgroup of infinite rank and write explicitly its free generating set.