Homework set 1 (due Wed., April 5)

1. (a) We define the complex projective space $\mathbb{CP}^n$ as the set of equivalence classes of nonzero vectors in $\mathbb{C}^{n+1}$ with the equivalence relation
$$v \sim w \;\Leftrightarrow\; w = \lambda v \quad \text{for } \lambda \in \mathbb{C} - \{0\}.$$
Equip $\mathbb{CP}^n$ with a structure of smooth manifold.
   (b) Prove that $\mathbb{CP}^1$ is diffeomorphic to the sphere S^2.

2. (a) Let M and N be smooth manifolds and $f : M \to N$ a local diffeomorphism. Prove that if f is one-to-one, then f is a diffeomorphism between M and an open subset of N.
   (b) Construct a local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is not a diffeomorphism onto its image.
   (c) Show that any local diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism onto its image.

3. Construct a submersion $\phi : S^{2n+1} \to \mathbb{CP}^n$ such that for every $x \in \mathbb{CP}^n$, the fiber $\phi^{-1}(x)$ is diffeomorphic to the circle S^1.

4. Consider the map $f : \mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^{nm+n+m}$ defined by
$$([x_i : i = 0, \ldots, n], [y_j : j = 0, \ldots, m]) \mapsto [x_i y_j : i = 0, \ldots, n, j = 0, \ldots, m].$$
Show that the map f is an embedding.