Homework set 2 (due Wed., April 12)

1. Prove that the following sets of $n \times n$ real matrices are submanifolds of $\mathbb{R}^{n^2}$ and determine their dimensions:
   (a) $\mathrm{SL}(n, \mathbb{R}) = \{X : \det X = 1\}$,
   (b) $\mathrm{O}(n, \mathbb{R}) = \{X : {}^tX \cdot X = E\}$.

2. Show that the set of $n \times n$ real matrices of rank $n - 1$ is a submanifold of $\mathbb{R}^{n^2}$. What is its dimension?

3. (a) Let M be a smooth manifold and N its *closed* submanifold. Prove that for any smooth function $f : N \to \mathbb{R}$ has a smooth extension, that is, there exists a smooth function $F : M \to \mathbb{R}$ such that $F|_N = f$.
   (b) Show that without the assumption that N is closed, the previous claim is false.

4. (a) Let M be a smooth manifold, C a closed subet of M, and $U \supset C$ an open subset of M. Prove that there exists a smooth function $f : M \to [0, 1]$ such that $f|_C = 0$ and $f|_{M \setminus U} = 1$.
   (b) A function $f : M \to \mathbb{R}$ is called *proper* if the preimage of every compact subset of $\mathbb{R}$ is compact. Prove that on every smooth manifold M, there exists a smooth proper function.

   This result can useful for proving the Whitney Embedding Theorem for noncompact manifolds.