Homework set 3 (due Wed., April 19)

1. (a) Let $f : M \to M$ be a smooth map. Prove that the submanifolds
$$\Delta = \{(x, x) : x \in M\} \quad \text{and} \quad \Gamma = \{(x, f(x)) : x \in M\}$$
of $M \times M$ are transversal if and only if $(df)_x$ doesn't have eigenvalue 1 for all fixed points x of f.

   (b) Let M be a compact manifold and $f : M \to M$ a smooth map such that $(df)_x$, $x \in M$, don't have eigenvalue 1. Prove that f has only finitely many fixed points.

2. Prove or disprove that the following properties are stable:
   (a) injective,
   (b) surjective,
   (c) diffeomorphism (consider both compact and noncompact cases).

3. Construct a vector field on a sphere of any dimension that has exactly one zero.

4. Let $f : M \to N$ be surjective submersion. A smooth function $g : M \to \mathbb{R}$ *factors through* f if there exists a smooth function $h : N \to \mathbb{R}$ such that $g = h \circ f$. We call a vector field X on M *vertical* if $(df)_p X_p = 0$ for all $p \in M$.
   (a) Check that g factors through f if and only if g is constant on fibers $f^{-1}(q)$, $q \in N$.
   (b) Show that if g factors through f, then for any vertical vector field X, we have $Xg = 0$.
   (c) Let $q \in N$. Prove that if $Xg = 0$ for any vertical vector field, then $g|_{f^{-1}(q)}$ is locally constant.
   (d) Assuming that all fibers $f^{-1}(q)$, $q \in N$, are connected, prove the converse of (b).
   (e) Give a counterexample to (d) if we don't assume that the fibers are connected.