MATH 109C. SPRING 2006.

Homework set 4 (due Wed., April 26)

1. A tangent bundle TM is called *trivial* if there exists a diffeomorphism $f : TM \to M \times \mathbb{R}^d$, $d = \dim M$, such that the following diagram commute:

$$\begin{array}{ccc} TM & \xrightarrow{f} & M \times \mathbb{R}^d \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

and f is linear on the fibers.

   (a) Prove that a tangent bundle is trivial if and only if there exist vector fields $X_1, \dots, X_d$ such that $X_1(p), \dots X_d(p)$ are linearly independent for every $p \in M$.

   (b) Give an example of a tangent bundle which is not trivial.

2. A *Lie group* G is a group with a structure of a smooth manifold such that the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

   (a) Prove that the tangent bundle TG is trivial.

   (b) Prove that the tangent bundle TS^3 is trivial. (Hint: use the group SU(2).)

3. Show that the following set is a submanifold of $\mathbb{C}^{n+1}$

$$\left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} z_i^2 = 1 \right\}$$

and prove that it is diffeomorphic to the tangent bundle TS^n.

4. Give an example of a flow on a projective space $\mathbb{RP}^2$, which has exactly one fixed point and all other orbits are periodic.

5. Let $M = \mathbb{R}^2$ and $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ be a vector field on M. Find the corresponding flow on M.