Homework set 5 (due Wed., May 10)

1. Given a vector field X on a manifold M, we define the contraction map $i_X : \Lambda^k M \to \Lambda^{k-1} M$:

$$(i_X \omega)_p(v_1, \dots, v_{k-1}) = \omega_p(X_p, v_1, \dots, v_{k-1})$$

where $p \in M$ and $v_1, \dots, v_{k-1} \in T_pM$.

   (a) Prove that $i_X \circ i_X = 0$.

   (b) Prove that for $\omega \in \Lambda^r M$ and $\eta \in \Lambda^s M$,

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^r \omega \wedge (i_X \eta).$$

2. Prove that the tangent bundle and the cotangent bundle on a manifold are isomorphic, i.e., there exists a diffeomorphism that respects the projection maps $TM \to M$ and $T^*M \to M$ and is linear on the fibers. Hint: Use existence of a Riemannian metric.

3. Let M be a manifold with a Riemannian metric and $f \in C^\infty(M)$.

   (a) Prove that there exists a unique a vector field $\text{grad}(f)$ on M such that

$$\langle \text{grad}(f)_p, X_p \rangle_p = Xf(p)$$

   for every $p \in M$ and every vector field X on M.

   (b) Compute $\text{grad}(f)$ when M is the Euclidean space with the standard Riemannian metric.

   (c) For $p \in M$, compute

$$\max\{Xf(p) : X - \text{vector field}, \ \|X_p\|_p = 1\}$$

   in terms of $\text{grad}(f)$.

4. Given a nonorientable connected manifold M, we define

$$\tilde{M} = \bigcup_{p \in M} \{\text{orientations of } T_pM\}.$$

Show that $\tilde{M}$ has a structure of smooth manifold such that it is connected, orientable, and the natural projection map $\pi : \tilde{M} \to M$ is a local diffeomorphism.