Homework set 6 (due Wed., May 17)

1. In this problem, we extend the notion of the Lie derivative to differential forms. Let ω be a differential form and X a vector field on a manifold M. The *Lie derivative* of ω along X at $p \in M$ is defined by

$$L_X(\omega)_p = \frac{d}{dt}\phi_t^*\omega_{\phi_t(p)}|_{t=0},$$

where π_t is the local flow corresponding to X. Prove that

   (a) $L_X(f) = X(f)$ for $f \in C^\infty(M) = \Lambda^0 M$.

   (b) $L_X(f\omega) = X(f)\omega + fL_X(\omega)$ for $f \in C^\infty(M)$ and $\omega \in \Lambda^k M$.

   (c) $L_X \circ d = d \circ L_X$ where d denotes the exterior derivative.

   (d) $L_X = i_X \circ d + d \circ i_X$ where $i_X : \Lambda^k M \to \Lambda^{k-1} M$ is the contraction map defined in the previous homework.

2. Consider the differential form

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

defined on $\mathbb{R}^2 - \{0\}$.

   (a) Prove that ω is exact on $\{x > 0\}$.

   (b) Prove that ω is closed, but not exact on $\mathbb{R}^2 - \{0\}$. Hint: compute $\int_{S^1} \omega$.

3. Let ω be a 1-form on a connected manifold M such that for all closed continuous piecewise smooth curves γ, $\int_\gamma \omega = 0$. Prove that ω is exact.

4. Let M and N be compact manifolds without boundary of dimension d, $f_0, f_1 : M \to N$ are smooth maps homotopic via a smooth homotopy, and ω a closed d-form on N. Prove that

$$\int_M f_0^*\omega = \int_M f_1^*\omega.$$